

Generalized Levi Conditions for Weakly Hyperbolic Equations —An Attempt to Treat the Degeneracy with Respect to the Space Variables—

By

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§ 0. Introduction

In the celebrated paper [3], Ivrii and Petkov showed that the effective hyperbolicity of an operator is necessary for the Cauchy problem to be C^∞ -well-posed with arbitrary lower order terms. In the same paper, they also gave general necessary conditions on lower order terms for the C^∞ -well-posedness. (Conditions of this kind are sometimes called “generalized Levi conditions”.) The individual behavior of each characteristic root, however, is not reflected in these conditions. Hence, when some characteristic roots coincide with one another in a variety of ways, their results fail to give the conditions expected to be necessary and sufficient.

In [8], we treated each characteristic root separately, restricting our consideration to the degeneracy with respect to the time variable. We gave some necessary conditions for C^∞ -well-posedness and some estimates for the regularity-loss of solutions. These results were “micro-local” in the sense that conditions were stated in terms of the characteristic root 0 in a fixed cotangential direction. By applying these results after appropriate coordinate transformations, however, we got a necessary and sufficient condition in case of the operators with only finite-order degeneracy w. r. t. the time variable ([9]).

This paper is an attempt to treat the degeneracy with respect to the space variables in a similar way. Though the results in this paper hold for the operators with singular coefficients of the same kind as discussed in [8] and [9], we restrict ourselves to the case of operators with C^∞ -coefficients for simplicity.

We introduce some notations for further explanation.

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$$\begin{aligned}
(t, x) &= (t, x_1, \dots, x_n) = (t, x', x_n) \in \mathbf{R}^{n+1}, \\
(\tau, \xi) &= (\tau, \xi_1, \dots, \xi_n) = (\tau, \xi', \xi_n) \in \mathbf{R}^{n+1}. \\
\partial_t &= \partial/\partial t, \quad \partial_{x_j} = \partial/\partial x_j, \quad D_t = -i\partial_t, \quad D_{x_j} = -i\partial_{x_j}, \\
D_x^\alpha &= D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad \text{where } \alpha = (\alpha_1, \dots, \alpha_n) \text{ is a multi-index.} \\
q_{(\beta, \bar{\beta})}^{(j, \alpha)}(t, x; \tau, \xi) &= (\partial_t^j \partial_x^\alpha \partial_{\xi'}^{\bar{\beta}} q)(t, x; \tau, \xi), \\
\dot{\mathbf{R}}^n &= \mathbf{R}^n \setminus \{0\}, \quad e_n = (0, \dots, 0, 1) \in \mathbf{R}^n.
\end{aligned}$$

Let Ω be an open neighborhood of $(0, 0) \in \mathbf{R}^{n+1}$ and let $\Omega^+ = \{(t, x) \in \Omega; t \geq 0\}$. Let P be a differential operator on Ω with C^∞ -coefficients and set $P = \sum_{h=0}^m p_{m-h}(t, x; D_t, D_x)$, where $p_{m-h}(t, x; \tau, \xi)$ is homogeneous of degree $m-h$ with respect to (τ, ξ) and $p_m(t, x; 1, 0) = 1$ on Ω .

To avoid ambiguity, the definition of well-posedness will be specified as follows.

Definition 0.1. We say that the Cauchy problem for P is *well-posed* on Ω^+ if the following two conditions are satisfied.

a) For any $f \in C^\infty(\Omega)$ such that $f=0$ for $t \leq 0$, there exists $u \in C^\infty(\Omega)$ such that $u=0$ for $t \leq 0$ and $Pu=f$ on Ω .

b) For any $\hat{t} \geq 0$ and any $u \in C^\infty(\Omega)$ such that $u=0$ for $t \leq 0$, if there holds $Pu=0$ for $t \leq \hat{t}$, then there holds $u=0$ for $t \leq \hat{t}$.

Remark 0.2. 1) In [3], [8] etc., certain conditions on dependence domains are assumed. In this paper, we do not make such assumptions. Further, the results in this paper hold under some other definitions of well-posedness. (See Remark 1.3 1).)

2) If the coefficients of P are C^∞ as we have assumed, then this definition is equivalent to the following condition c).

c) For any $\hat{t} \geq 0$, any $f \in C^\infty(\Omega)$ and any $g_j \in C^\infty(\Omega \cap \{t=\hat{t}\})$ ($j=0, 1, \dots, m-1$), there exists a unique $u \in C^\infty(\Omega)$ such that $Pu=f$ for $t \geq \hat{t}$ and $\partial_{t_i}^j u|_{t=\hat{t}} = g_j$ ($j=0, 1, \dots, m-1$).

Since the well-posedness implies hyperbolicity ([10]), we assume throughout this paper that P is hyperbolic on Ω^+ , that is, the equation $p_m(t, x; \tau, \xi) = 0$ has only real roots τ for any $(t, x; \xi) \in \Omega^+ \times \mathbf{R}^n$.

Now, we review the simplest version of the “*Levi condition*” ([7], [11], [12], [2]). Let $p_{m-h}(t, x; \tau, e_n) = a_0^{(h)}(t, x)\tau^{m-h} + \dots + a_{m-h}^{(h)}(t, x)$. If $a_m^{(0)}(t, x) = \dots = a_{m-r+1}^{(0)}(t, x) = 0$ on Ω^+ and the Cauchy problem for P is well-posed on Ω^+ , then there holds $a_{m-h}^{(h)}(t, x) = \dots = a_{m-r+1}^{(h)}(t, x) = 0$ on Ω^+ for $h=1, \dots, r-1$. This is the condition to be called the “*micro-local Levi condition with respect to the characteristic root 0 in e_n -direction*”. If the characteristic roots of P have

constant multiplicities, then the usual Levi condition, which is also a sufficient condition, is obtained by combining the conditions which are obtained as above after transforming P by appropriate coordinate transformations. In this paper, we are concerned with the case where $a_m^{(0)}(t, x), \dots, a_{m-r+1}^{(0)}(t, x)$ vanish at $(0, 0)$ but do not vanish identically near $(0, 0)$. To describe the behavior of $a_j^{(0)}(t, x)$ ($j=0, \dots, m$) near $(0, 0)$, the Newton polygon shall play an important role.

Definition 0.3. We call a subset Δ of \mathbf{R}^2 a *Newton polygon* of size m (m is a non-negative integer) if $\Delta = \{(j, \mu) \in [0, m] \times \mathbf{R}; \mu \geq \Gamma(j)\}$, where $\Gamma(j)$ has one of the following forms.

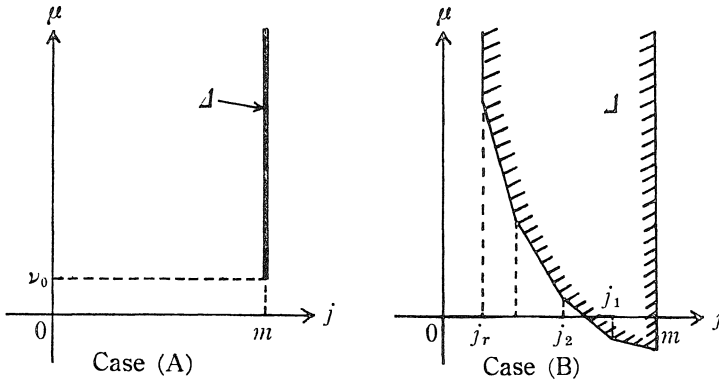


Figure 1.

- (A) $\Gamma(m) = \nu_0$ and $\Gamma(j) = \infty$ for $0 \leq j < m$, where ν_0 is a rational number.
- (B) $\Gamma(j) = \nu_k - \kappa_k j$ for $j_k \leq j \leq j_{k-1}$ ($k=1, \dots, r$) and $\Gamma(j) = \infty$ for $0 \leq j < j_r$, where r is a positive integer, $m = j_0 > j_1 > \dots > j_r \geq 0$ are integers and $\nu_1 < \nu_2 < \dots < \nu_r$, $0 \leq \kappa_1 < \kappa_2 < \dots < \kappa_r$ are rational numbers.

The function $\mu = \Gamma(j)$ ($0 \leq j \leq m$) is called the lower side of Δ .

In §1, we shall give necessary conditions for well-posedness with respect to the characteristic root 0 (Theorem 1.2). This theorem is an extension of Theorems 4.1, 7.1 in [3]. The proof is given in §3. In §2, we shall consider a class of “ t -involutive” operators (Definition 2.1). For such operators, a necessary and sufficient condition is established (Theorem 2.5). The proof is given in §4 by applying Theorem 1.2 after appropriate coordinate transformations.

§1. Generalized Levi Conditions with Respect to the Characteristic Root 0

Fix positive rational numbers q_j ($j=0, 1, \dots, n$). Put $\mathcal{P} = \{(j, q_0 k + \langle q, \beta \rangle) \in \mathbf{R}^2; p_{m, (k, \beta)}^{(j, 0)}(0, 0; 0, e_n) \neq 0\}$, where $\langle q, \beta \rangle = \sum_{j=1}^n q_j \beta_j$. Let Δ be the closed convex

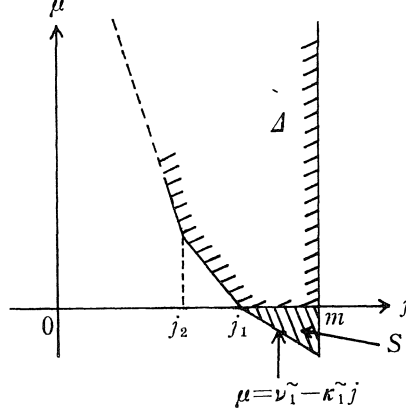


Figure 2

hull of the set $\{(j, \mu) \in \mathbf{R}^2; \mu \geq \mu' \text{ for some } (j, \mu') \in \mathcal{P}\}$ in \mathbf{R}^2 . This is a Newton polygon. Since the coefficients are C^∞ and $p_m(t, x; 1, 0) = 1$, there holds $(m, 0) \in \Delta \subset \{(j, \mu) \in \mathbf{R}^2; \mu \geq 0\}$. For this Δ , we get ν_0 (case (A)) or r, j_k, κ_k and ν_k ($1 \leq k \leq r$) (case (B)) as in Definition 0.3. In the case (B) with $\kappa_1 = 0$ (hence $\nu_1 = 0$), we may modify Δ as follows. Take an arbitrary rational number $\tilde{\kappa}_1$ such that $0 \leq \tilde{\kappa}_1 \leq \kappa_2$ and put $\tilde{\nu}_1 = \tilde{\kappa}_1 j_1$. Let S be the closed triangle with the vertexes $(m, 0)$, $(j_1, 0)$, $(m, \tilde{\nu}_1 - \tilde{\kappa}_1 m)$ and let $\Delta^\sim = \Delta \cup S$. This is another Newton polygon (Figure 2). In the case (B) with $\kappa_1 > 0$, we make no modification, that is, put $\tilde{\kappa}_1 = \kappa_1$ and $\Delta^\sim = \Delta$. In the case (A), put $\tilde{\kappa}_1 = \infty$ and $\Delta^\sim = \Delta$. We make the following assumptions on q_j ($0 \leq j \leq n$), $\tilde{\kappa}_1$ and p_m . In other words, we consider only $q_j, \tilde{\kappa}_1$ that satisfy the following conditions.

$$[A] \quad \begin{cases} 1) & q_0 + \tilde{\kappa}_1 > q_j \quad (j=1, 2, \dots, n). \\ 2) & \text{If } (j+|\alpha|, q_0(k+|\alpha|) + \langle q, \beta - \alpha \rangle) \in \Delta^\sim, \text{ then } p_m^{(j, \alpha)}(k, \beta)(0, 0; 0, e_n) = 0. \end{cases}$$

Remark 1.1. i) For $\alpha = 0$, the condition 2) is satisfied by the definition of Δ^\sim .

ii) If $q_0 \geq q_j$ ($j=1, 2, \dots, n$), then the condition 2) follows from the hyperbolicity of p_m . (The proof is given in Appendix 1.)

iii) Assume 1). Then, the condition 2) for such α as satisfy $\alpha_n = 0$ implies the condition 2) for all α . Especially, the condition 2) is necessarily satisfied if $n=1$.

Theorem 1.2. *Assume the condition [A]. If the Cauchy problem for P is well-posed on Ω^+ , then the following holds:*

$$\text{If } (j+|\alpha|+h, q_0(k+|\alpha|+h) + \langle q, \beta - \alpha \rangle) \in \Delta^\sim, \text{ then } p_m^{(j, \alpha)}(k, \beta)(0, 0; 0, e_n) = 0.$$

Remark 1.3. 1) The assumption of well-posedness is used only to derive the energy inequality (3-3) in §3. Hence, any definition of well-posedness that

implies this inequality, such as the H^∞ -well-posedness, can be adopted.

2) Though the definition of the well-posedness in [3] is not the same as ours, we can easily derive Theorem 4.1 and 7.1 in [3] from the proof of the above theorem. Especially, we can omit the assumption that P satisfies Condition $(U_{\Gamma(x, \gamma)})$ and can relax the condition $p \geq q$ to $p+1 > q$ in Theorem 4.1 in [3].

3) Theorem 3.1 in [5] can be essentially restated as follows in our setting. (The statement of Theorem 3.1 in [5] is wrong (perhaps misprinted). The condition (N) must be “for $|\alpha| + \langle \beta - \alpha, p \rangle < r(1 - p_0) - (m - s)$ ”.)

Theorem. (Theorem 3.1 in [5]) *For a rational $\kappa > 0$, put $\Delta_{r, \kappa} = \{(j, \mu); r \leq j \leq m, \mu \geq 0\} \cup \{(j, \mu); 0 \leq j \leq r, \mu \geq \kappa(r - j)\}$ and $\widetilde{\Delta}_{r, \kappa} = \{(j, \mu); 0 \leq j \leq m, \mu \geq \kappa(r - j)\}$ ($1 \leq r \leq m$). Assume that $q_0 + \kappa > q_j$ ($j = 1, \dots, n$), $p_m^{(r, 0)}(0, 0; 0, e_n) \neq 0$ and the following condition :*

If $(j + |\alpha|, q_0(k + |\alpha|) + \langle q, \beta - \alpha \rangle) \in \Delta_{r, \kappa}$, then $p_m^{(j, \alpha)}(0, 0; 0, e_n) = 0$.

If the Cauchy problem for P is well-posed on Ω^+ , then the following holds.

If $(j + |\alpha| + h, q_0(k + |\alpha| + h) + \langle q, \beta - \alpha \rangle) \in \widetilde{\Delta}_{r, \kappa}$, then $p_m^{(j, \alpha)}(0, 0; 0, e_n) = 0$ ($h = 1, \dots, m$).

Thus, our theorem is essentially an extension of Theorem 3.1 in [5].

Example 1.4. In the followings, let $n = 1$ and assume that the Cauchy problem is well-posed in a neighborhood of $(0, 0)$.

1) Let $m = 3$ and $p_3 = (\tau - x\xi)(\tau - x^2\xi)\tau$. Applying Theorem 1.2 with $q_0 = \varepsilon$ (arbitrary positive number) and $q_1 = 1$, we get the condition that $p_2 = a(t, x)\tau^2 + xb(t, x)\tau\xi + x^2c(t, x)\xi^2$ and $p_1 = d(t, x)\tau + xe(t, x)\xi$ near $(0, 0)$ for some C^∞ -functions a, b, c, d, e . This is also a sufficient condition for well-posedness. (See Example 2.7.)

2) Let $m = 3$ and $p_3 = (\tau - x\xi)(\tau - x^2\xi)\{\tau - (x + x^2)\xi\}$. Applying Theorem 1.2 with $q_0 = \varepsilon$ (arbitrary positive number) and $q_1 = 1$, we get the condition that $p_2 = a(t, x)\tau^2 + xb(t, x)\tau\xi + x^2c(t, x)\xi^2$ and $p_1 = d(t, x)\tau + xe(t, x)\xi$ near $(0, 0)$ for some C^∞ -functions a, b, c, d, e . However, this is not sufficient for well-posedness. This condition does not reflect the contact-order of characteristic roots $x\xi$ and $(x + x^2)\xi$. If we apply Theorem 1.2 after transforming P by the coordinate transformation $s = t, y = xe^t$, then we get a further condition $a(t, 0) + b(t, 0) + c(t, 0) = 1$. These conditions are sufficient for well-posedness. (The sufficiency follows from Theorem 2.2 in § 2.)

3) Let $m = 4$ and $p_4 = \{\tau^2 - (t^2 + x^2)\xi^2\} \{\tau^2 - (t^2 + x^2)^2\xi^2\}$. Put $p_3 = \sum_{i=0}^3 a_i(t, x)\tau^i\xi^{3-i}$. Applying Theorem 1.2 with $q_0 = q_1 = 1$, we get $a_1(0, 0) = (\partial_t^j \partial_x^k a_0)(0, 0) = 0$ for $j + k \leq 2$. The author does not know whether these conditions are sufficient or not.

**§ 2. A Necessary and Sufficient Condition
to a Class of t -Involutive Operators**

In this section, put $\Omega^+ = [0, T] \times \mathbf{R}^n$ and assume that the coefficients of p_m are bounded on Ω^+ .

Definition 2.1. 1) We say that p_m (or P) is *t-involutive* (resp. *involutive*) if the following two conditions are satisfied. (M. Zeman [18], K. Yamamoto [17].)

i) p_m has the smooth real characteristic roots. That is, there exist $\lambda_j(t, x; \xi) \in C^\infty(\Omega^+ \times \dot{\mathbf{R}}^n)$ ($j=1, \dots, m$) such that λ_j are real-valued and $p_m(t, x; \tau, \xi) = \prod_{j=1}^m (\tau - \lambda_j(t, x; \xi))$.

ii) For any $1 \leq j, k \leq m$, there exists $A_{j,k}(t, x; \xi) \in C^\infty(\Omega^+ \times \dot{\mathbf{R}}^n)$ such that

$$(2-1) \quad \{\tau - \lambda_j(t, x; \xi), \tau - \lambda_k(t, x; \xi)\} = \frac{1}{t} A_{j,k}(t, x; \xi) (\lambda_j(t, x; \xi) - \lambda_k(t, x; \xi))$$

$$(\text{resp.} = A_{j,k}(t, x; \xi) (\lambda_j(t, x; \xi) - \lambda_k(t, x; \xi))),$$

where $\{\cdot, \cdot\}$ is the Poisson bracket w. r. t. $(t, x; \tau, \xi)$.

2) Put $\partial_j = D_t - \lambda_j(t, x; D_x)$. For $J \subset I = \{1, \dots, m\}$, put $\pi_J = \prod_{j \in J} \partial_j$, where the order of the product is arbitrarily fixed. Let $|J|$ denote the number of the elements of J .

A sufficient condition for well-posedness of the Cauchy problem to t -involutive operators is given by K. Yamamoto [17]. (Cf. M. Zeman [18], H. Uryu [15], H. Kumano-go [6].) Let $\mathcal{S}^p(V)$ denote the class of all classical pseudo-differential operators of order p on an open subset V of \mathbf{R}^n .

Theorem 2.2. Assume that p_m is t -involutive and that for any $J \subseteq I$, there exists $A_J(t, x; D_x) \in C^\infty([0, T]; \mathcal{S}^0(\mathbf{R}^n))$ such that

$$(2-2) \quad P = \pi_I + \sum_{J \subseteq I} t^{|J|-m} A_J(t, x; D_x) \pi_J.$$

Then, the Cauchy problem for P is well-posed on Ω^+ .

Remark 2.3. 1) In [17], there is proved only the H^∞ -well-posedness under the assumption that all the coefficients belong to $\mathcal{G}^\infty(\Omega^+) = \{f \in C^\infty(\Omega^+); \partial_t^\alpha \partial_x^\beta f \text{ is bounded on } \Omega^+ \text{ for any } (j, \alpha)\}$. However, we can prove that there exists a finite propagation speed (, that is, there holds the condition (U) with $\mu_0 = 1$ of Definition 1.1 in [9]). (See Appendix 2.) Hence, the C^∞ -well-posedness holds under our assumption.

2) The condition that P can be written as (2-2) is not always necessary for well-posedness as is shown in the following example.

Example 2.4. Consider $P=D_t^2-(x^2+y^2)^2D_x^2-a(x, y)D_x$ in \mathbf{R}^3 . The above condition is $\frac{a(x, y)}{x^2+y^2} \in C^\infty(\mathbf{R}^2)$, though the Cauchy problem for P is well-posed if and only if $a(0, 0)=\partial_x a(0, 0)=\partial_y a(0, 0)=0$. (The “only if” part follows from Theorem 1.2. The “if” part follows from Theorem 1 in [13].)

The purpose of this section is to give an answer to the question when the condition that P can be written as (2-2) is necessary for well-posedness. We make the following assumption [B] on p_m .

- [B] $\left\{ \begin{array}{l} \text{For any } (t, \hat{x}) \in \Omega^+, \text{ there exist a neighborhood } U \text{ of } (t, \hat{x}), \text{ a non-negative} \\ \text{integer } n_1, C^\infty\text{-functions } C_l(t, x) \text{ } (l=1, \dots, n_1) \text{ on } U \text{ and non-negative} \\ \text{integers } \kappa(l; j, k) \text{ } (l=0, 1, \dots, n_1; 1 \leq j, k \leq m) \text{ such that the following} \\ \text{three conditions are satisfied.} \\ \text{i) } C_l(t, \hat{x})=0 \text{ } (l=1, \dots, n_1). \\ \text{ii) } \{(\text{grad}_x C_l)(t, \hat{x}); l=1, \dots, n_1\} \text{ are linearly independent.} \\ \text{iii) For any } 1 \leq j, k \leq m, \text{ there holds either of the followings.} \\ \text{a) } \lambda_j(t, x; \xi) - \lambda_k(t, x; \xi) = t^{\kappa(0; j, k)} \prod_{l=1}^{n_1} C_l(t, x)^{\kappa(l; j, k)} \Phi_{j, k}(t, x; \xi), \\ \text{where } \Phi_{j, k} \in C^\infty(U \times \mathbf{R}^n) \text{ and } \Phi_{j, k}(t, x; \xi) \neq 0 \text{ on } U \times \mathbf{R}^n. \\ \text{b) } \lambda_j(t, x; \xi) = \lambda_k(t, x; \xi) \text{ on } U \times \mathbf{R}^n. \end{array} \right.$

Theorem 2.5. Assume that p_m is t -involutive and that p_m satisfies the assumption [B]. The Cauchy problem for P is well-posed on Ω^+ if and only if P can be written as (2-2).

Remark 2.6. Assume that $\kappa(0; j, k)=0$ for any j, k . Then, p_m is t -involutive if and only if p_m is involutive. Further, in this case, P can be written as (2-2) if and only if P can be written as follows.

$$(2-2)' \quad P = \pi_I + \sum_{j \in I} A_j(t, x; D_x) \pi_J,$$

where $A_j(t, x; D_x) \in C^\infty([0, T]; S^0(\mathbf{R}^n))$.

In some cases, the condition that P can be written as (2-2) (or (2-2)') is reduced to a simpler condition. (Example 1.4, 1), 2).)

Example 2.7. (Cf. [14]) Let $\sigma(x) \in C^\infty(\mathbf{R}^n)$ satisfy $\text{grad}_x \sigma(x) \neq 0$ if $\sigma(x) = 0$. Let r be an integer such that $0 \leq r \leq m$ and let $\kappa(1) \leq \kappa(2) \leq \dots \leq \kappa(m-r)$ be non-negative integers. Assume the following three conditions.

- $$\left\{ \begin{array}{l} \text{i) } p_m = \tau^r \prod_{j=1}^{m-r} (\tau - \lambda_j(t, x; \xi)) \text{ and } \lambda_j(t, x; \xi) = \sigma(x)^{\kappa(j)} \tilde{\lambda}_j(t, x; \xi), \text{ where} \\ \quad \tilde{\lambda}_j \in C^\infty(\Omega^+ \times \dot{R}^n) \text{ and } \tilde{\lambda}_j(t, x; \xi) \neq 0 \text{ on } \Omega^+ \times \dot{R}^n \text{ (} j=1, \dots, m-r \text{).} \\ \text{ii) If } j \neq k, \sigma(x) \neq 0 \text{ and } \xi \neq 0, \text{ then } \lambda_j(t, x; \xi) \neq \lambda_k(t, x; \xi). \\ \text{iii) If } j \neq k \text{ and } \kappa(j) = \kappa(k), \text{ then } \tilde{\lambda}_j(t, x; \xi) \neq \tilde{\lambda}_k(t, x; \xi) \text{ on } \Omega^+ \times \dot{R}^n. \end{array} \right.$$

Let $P = \sum_{j+|\alpha| \leq m} a_{j,\alpha}(t, x) D_x^\alpha (a_{j,\alpha} \in \mathcal{D}^\infty(\Omega^+))$ and put $\Gamma(j) = \sum_{l=1}^{m-j} \kappa(l)$ for $j=r, r+1, \dots, m$. The Cauchy problem for P is well-posed on Ω^+ if and only if there holds

$$(2-3) \quad \begin{cases} a_{j,\alpha}(t, x) = \sigma(x)^{\Gamma(m-|\alpha|)} b_{j,\alpha}(t, x) \\ \quad \text{for some } b_{j,\alpha} \in C^\infty(\Omega^+), \text{ if } |\alpha| \leq m-r, \\ a_{j,\alpha}(t, x) = 0 \quad \text{on } \Omega^+, \text{ if } |\alpha| > m-r. \end{cases}$$

The “only if” part is easily proved by Theorem 1.2. The “if” part is proved by showing that p_m is involutive and P can be written as (2-2)'. (Cf. [16; § 1].)

§ 3. Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. This theorem follows from the following proposition.

Proposition 3.1. *Let κ, ν be positive rational numbers such that $q_0 + \kappa > q_j$ ($1 \leq j \leq n$). Assume the following three conditions.*

- 1) $p_m^{(j, \alpha)_{\hat{k}, \hat{\beta}}}(0, 0; 0, e_n) = 0$, if $q_0(k + |\alpha|) + \langle q, \beta - \alpha \rangle < \nu - \kappa(j + |\alpha|)$.
- 2) There exist $\hat{k}, \hat{\beta}, \hat{j}$ such that $p_m^{(j, \alpha)_{\hat{k}, \hat{\beta}}}(0, 0; 0, e_n) \neq 0$ and $q_0 \hat{k} + \langle q, \hat{\beta} \rangle = \nu - \kappa \hat{j}$.
- 3) Either of the followings is satisfied.

Case (a) There holds $\nu = \kappa \hat{j}$ (, hence $\hat{k} = 0, \hat{\beta} = 0$).

Case (b) There exist rational numbers κ^*, ν^* such that $0 \leq \kappa^* < \kappa, \nu^* - \kappa^* \hat{j} = \nu - \kappa \hat{j}$ and

$$(3-1) \quad p_m^{(j, \alpha)_{\hat{k}, \hat{\beta}}}(0, 0; 0, e_n) = 0, \\ \text{if } q_0(k + |\alpha| + h) + \langle q, \beta - \alpha \rangle < \nu^* - \kappa^*(j + |\alpha| + h).$$

Under this assumption, if the Cauchy problem for P is well-posed on Ω^+ , then there holds the following.

$$(3-2) \quad p_m^{(j, \alpha)_{\hat{k}, \hat{\beta}}}(0, 0; 0, e_n) = 0, \\ \text{if } q_0(k + |\alpha| + h) + \langle q, \beta - \alpha \rangle < \nu - \kappa(j + |\alpha| + h).$$

Proof of Theorem 1.2 via Proposition 3.1. If Δ has a form of type (A), the assumption of Proposition 3.1 with Case (a) in 3) holds for any $\kappa > 0$ with $\hat{j} = m, \nu = \kappa m, \hat{k} = 0, \hat{\beta} = 0$. Hence, we get the desired result. Consider the case of type (B). If $\kappa_1 \sim \kappa_1 > 0$, then put $\hat{j} = m$, and if $\kappa_1 = 0$, then put $\hat{j} = j_1$. Let $\nu_1 \sim \kappa_1 \hat{j}$. In the case that $\kappa_1 \sim > 0$, put $\kappa = \kappa_1 \sim$ and $\nu = \nu_1 \sim$. Then, we can apply Proposition

3.1 with Case (a) in 3) and we get (3-2). In the case that $\kappa_1^{\sim}=0$, put $\kappa=\varepsilon$ and $\nu=\varepsilon\hat{j}$ for an arbitrary positive number ε . We can also apply Proposition 3.1 and we get (3-2) for any $\kappa>0$ and $\nu=\kappa\hat{j}$. Hence, we get (3-2) for $\kappa=\nu=0$. Thus, we get (3-2) for $\kappa=\kappa_1^{\sim}$, $\nu=\nu_1^{\sim}$. Next, we shall prove (3-2) for $\kappa=\kappa_i$, $\nu=\nu_i$ under the assumption that there holds (3-2) for $\kappa=\kappa_{i-1}$, $\nu=\nu_{i-1}$. (If $i=2$, then regard κ_1 as κ_1^{\sim} and ν_1 as ν_1^{\sim} .) If $i=2$ and $\kappa_1^{\sim}=\kappa_2$, then we have nothing to prove. In the other cases, take $\hat{j}=j_{i-1}$, $\kappa=\kappa_i$, $\nu=\nu_i$, $\kappa^*=\kappa_{i-1}$, $\nu^*=\nu_{i-1}$. We can apply Proposition 3.1 with Case (b) in 3) and we get (3-2) for $\kappa=\kappa_i$, $\nu=\nu_i$. Thus, we have (3-2) for $\kappa=\kappa_i$, $\nu=\nu_i$ ($i=2, \dots, r$). The conclusion of Theorem 1.2 is just (3-2) for $\kappa=\kappa_1^{\sim}$, $\nu=\nu_1^{\sim}$ and $\kappa=\kappa_i$, $\nu=\nu_i$ ($i=2, \dots, r$). \square

In the following proof, we shall write $\rho^a=\exp_{\rho}a$, when a is complicated.

Proof of Proposition 3.1. This proposition is proved by modifying the method used by Ivrii [4], [5]. Since, the Cauchy problem for P is well-posed on Ω^+ , we have the following energy inequality.

Lemma 3.2. *For any compact set $K\subset\Omega^+$, there exist constants C, L such that the following holds. For any $(\hat{t}, \hat{x})\in K$ and for any $u\in C_0^\infty(K)=\{v\in C_0^\infty(\mathbb{R}^{n+1})\}$; $\text{supp } v\subset K$, there holds.*

$$(3-3) \quad |u(\hat{t}, \hat{x})| \leq C \sup_{p+|\beta|\leq L} \sup_{(t,x)\in K, t\leq\hat{t}} |(\partial_t^p \partial_x^\beta Pu)(t, x)|.$$

We may assume that Ω^+ is star-shaped, that is, for any $(t, x)\in\Omega^+$ and any $0\leq\lambda\leq 1$, there holds $(\lambda t, \lambda x)\in\Omega^+$. For a positive λ , consider the coordinate transformation:

$$(3-4) \quad \begin{cases} s=t \exp_{\rho}(\lambda q_0) \\ y_j=x_j \exp_{\rho}(\lambda q_j) \quad (j=1, \dots, n) \end{cases} \quad (\rho\geq 1).$$

By this coordinate transformation, the operator P is transformed to another operator P_{ρ} . We have

$$\begin{aligned} p_{\rho, m-h}(s, y; \sigma, \eta) = & \sum_{k, \beta, j, \alpha'} K_h(k, \beta, j, \alpha') s^k y^\beta \sigma^j \eta'^{\alpha'} \eta_n^{m-h-j-|\alpha'|} \\ & \times \exp_{\rho}[-\lambda\{q_0 k + \langle q, \beta \rangle - q_0 j - \langle q', \alpha' \rangle - q_n(m-h-j-|\alpha'|)\}], \end{aligned}$$

where $K_h(k, \beta, j, \alpha')=(k! \beta! j! \alpha'!)^{-1} p_{m-h, (k, \beta)}^{(j, \alpha', 0)}(0, 0; 0, e_n)$. ($\eta'=(\eta_1, \dots, \eta_{n-1})$, $q'=(q_1, \dots, q_{n-1})$, etc.) For a real number γ , a C^∞ -function φ and a positive number $\delta<1$, set $E=\exp\{i(\gamma y_n \rho + \varphi \rho^\delta)\}$. These are determined later. We have

$$\begin{aligned} E^{-1} \circ P_{\rho} \circ E = & \sum_{h, k, \beta, j, \alpha'} K_h(k, \beta, j, \alpha') s^k y^\beta \\ & \times \exp_{\rho}[-\lambda\{q_0 k + \langle q, \beta \rangle - (q_0 - q_n)j - \langle q' - q_n, \alpha' \rangle - q_n(m-h)\} \\ & + \delta(j + |\alpha'|) + m - h - j - |\alpha'|] \\ & \times \{(\partial_s \varphi)^j (\partial_{y'} \varphi)^{\alpha'} \gamma^{m-h-j-|\alpha'|} + R_{\rho, h, k, \beta, j, \alpha'}(s, y; D_s, D_y)\}, \end{aligned}$$

where R_ρ are differential operators of order m whose coefficients tend to 0 in $C^\infty(\Omega^+)$ as $\rho \rightarrow \infty$. Denote the exponent of ρ by $d_h(k, \beta, j, \alpha')$. That is,

$$\begin{aligned} d_h(k, \beta, j, \alpha') = & (\lambda q_0 - \lambda q_n + \delta - 1)j + \langle \lambda q' - \lambda q_n + \delta - 1, \alpha' \rangle \\ & - \lambda q_0 k - \lambda \langle q, \beta \rangle + (\lambda q_n + 1)(m - h). \end{aligned}$$

Put $d_{\max} = d_0(\hat{k}, \hat{\beta}, \hat{j}, 0) = (\lambda q_0 - \lambda q_n + \lambda \kappa + \delta - 1)\hat{j} - \lambda \nu + (\lambda q_n + 1)m$. Now, let ε be the maximum of ε' that satisfies the following. (The maximum does exist if $\varepsilon = -\infty$ is admitted.)

There exist $h \geq 1$, k, β, j, α' such that

$$\begin{cases} q_0 k + \langle q, \beta \rangle + \langle q_0 + \kappa - q', \alpha' \rangle + \varepsilon'(q_0 + \kappa)h = \nu - \kappa j \\ \text{and } K_h(k, \beta, j, \alpha') \neq 0. \end{cases}$$

If $\varepsilon \leq 1$, then the conclusion of Proposition 3.1 holds.

Assume that $\varepsilon > 1$. Let $\mathcal{M} = \{(h, k, \beta, j, \alpha'); q_0 k + \langle q, \beta \rangle + \langle q_0 + \kappa - q', \alpha' \rangle + \varepsilon(q_0 + \kappa)h = \nu - \kappa j \text{ and } K_h(k, \beta, j, \alpha') \neq 0\}$. Note that $(0, \hat{k}, \hat{\beta}, \hat{j}, 0) \in \mathcal{M}$ and that there exists $(h, k, \beta, j, \alpha') \in \mathcal{M}$ such that $h \geq 1$. We have

$$\begin{aligned} d_{\max} - d_h(k, \beta, j, \alpha') & = \lambda \{q_0 k + \langle q, \beta \rangle + q_0 + \kappa - q', \alpha'\} + \varepsilon(q_0 + \kappa)h - \nu + \kappa j \\ & \quad + (\hat{j} - j - |\alpha'|)(\lambda q_0 - \lambda q_n + \lambda \kappa + \delta - 1) + (\lambda q_n + 1 - \lambda \varepsilon q_0 - \lambda \varepsilon \kappa)h. \end{aligned}$$

Since $\varepsilon > 1$ and $q_0 + \kappa > q_n$, we can take $\lambda > 0$ and $0 < \delta < 1$ such that $\lambda q_0 - \lambda q_n + \lambda \kappa + \delta - 1 = 0$ and $\lambda q_n + 1 - \lambda \varepsilon(q_0 + \kappa) = 0$. We take such λ and δ . If $K_h(k, \beta, j, \alpha') \neq 0$, then $d_{\max} \geq d_h(k, \beta, j, \alpha')$ and the equality holds only if $(h, k, \beta, j, \alpha') \in \mathcal{M}$.

Put $\Phi(s, y, \gamma; \sigma, \eta') = \sum_{(h, k, \beta, j, \alpha') \in \mathcal{M}} K_h(k, \beta, j, \alpha') s^h y^\beta \sigma^j \eta'^{\alpha'} \gamma^{\hat{j} - j - |\alpha'| - h}$. Then, we have

$$(3-5) \quad E^{-1} \circ P_\rho \circ E = \rho^{d_{\max}} \gamma^{m - \hat{j}} \{ \Phi(s, y, \gamma; \partial_s \varphi, \partial_y \varphi) + R_\rho^\sim(s, y; D_s, D_y) \},$$

where R_ρ^\sim is a differential operator of order m whose coefficients tend to 0 in $C^\infty(\Omega^+)$ as $\rho \rightarrow +\infty$.

Lemma 3.3. *If $(h, k, \beta, j, \alpha') \in \mathcal{M}$, then there hold either (i) $h=0, j=\hat{j}$ or (ii) $h+j < \hat{j}$.*

Proof. Case (a). If $\nu = \kappa \hat{j}$, then

$$\begin{aligned} 0 & = q_0 k + \langle q, \beta \rangle + \langle q_0 + \kappa - q', \alpha' \rangle + \varepsilon(q_0 + \kappa)h - \nu + \kappa j \\ & = q_0 k + \langle q, \beta \rangle + \langle q_0 + \kappa - q', \alpha' \rangle + \kappa(j + h - \hat{j}) + (\varepsilon q_0 + \varepsilon \kappa - \kappa)h \\ & \geq \kappa(j + h - \hat{j}). \end{aligned}$$

Hence, there holds $h + j \leq \hat{j}$ and the equality holds only if $k=0, \beta=0, \alpha'=0, h=0, j=\hat{j}$.

Case (b). We have

$$\begin{aligned} \nu - \kappa j &= q_0 k + \langle q, \beta \rangle + \langle q_0 + \kappa^* - q', \alpha' \rangle + (q_0 + \kappa^*)h + (\kappa - \kappa^*)|\alpha'| \\ &+ \{(\varepsilon - 1)q_0 + \varepsilon\kappa - \kappa^*\}h \geq \nu^* - \kappa^* j + (\kappa - \kappa^*)|\alpha'| \\ &+ \{(\varepsilon - 1)q_0 + \varepsilon\kappa - \kappa^*\}h. \end{aligned}$$

Since $\nu - \kappa \hat{j} = \nu^* - \kappa^* \hat{j}$ and $\kappa > \kappa^* \geq 0$, we have $(\kappa - \kappa^*)(\hat{j} - j - h) \geq (\kappa - \kappa^*)|\alpha'| + (\varepsilon - 1)(q_0 + \kappa)h \geq 0$. Hence, we have $j + h \leq \hat{j}$ and the equality holds only if $\alpha' = 0$, $h = 0$, $j = \hat{j}$. \square

Lemma 3.4. (Cf. [5; Lemma 1.2]) *There exist $(\hat{s}, \hat{y}) \in \Omega^+$, $\hat{\eta}' \in \mathbb{R}^n \setminus \{0\}$, $\hat{\gamma} \in \mathbb{R}$, $\hat{\sigma} \in \mathbb{C}$ such that $\Phi(\hat{s}, \hat{y}, \hat{\gamma}; \hat{\sigma}, \hat{\eta}') = 0$ and $\text{Im } \hat{\sigma} < 0$.*

Proof. Take $\theta = \max\left\{\frac{(\varepsilon - 1)(q_0 + \kappa)h}{\hat{j} - j}; (h, k, \beta, j, \alpha') \in \mathcal{M}, j < \hat{j}\right\}$ (> 0) and put $\mathcal{M}' = \{(h, k, \beta, j, \alpha') \in \mathcal{M}; \theta(\hat{j} - j) = (\varepsilon - 1)(q_0 + \kappa)h\}$. For a positive parameter ω , we have

$$\begin{aligned} &\Phi(s\omega^{-q_0}, y\omega^{-q}, \gamma\omega^{\kappa+q_0}; \sigma\omega^{q_0+\theta}, \eta'\omega^{q'}) \\ &= \exp_{\omega}\{-\nu + (\kappa + q_0 + \theta)\hat{j}\} \{\Phi_1(s, y, \gamma; \sigma, \eta') + o(1)\} \quad (\omega \rightarrow \infty), \end{aligned}$$

where

$$\Phi_1(s, y, \gamma; \sigma, \eta') = \sum_{\mathcal{M}'} K_h(k, \beta, j, \alpha') s^k y^\beta \sigma^j \eta'^{\alpha'} \gamma^{\hat{j} - j - 1\alpha' - h}.$$

Consider

$$\Phi_1(s, y, \gamma; \sigma, \gamma\eta') = \sum_{j, h} A_{j, h}(s, y, \eta') \sigma^j \gamma^{\hat{j} - j - h}.$$

There exist $(s^{\sim}, y^{\sim}) \in \Omega^+$, $\eta'^{\sim} \in \mathbb{R}^n \setminus \{0\}$, j^* , h^* such that $A_{\hat{j}, 0}(s^{\sim}, y^{\sim}, \eta'^{\sim}) \neq 0$, $A_{j^*, h^*}(s^{\sim}, y^{\sim}, \eta'^{\sim}) \neq 0$, $j^* + h^* < \hat{j}$, $h^* \geq 1$. The root of the equation $\Phi_2(\sigma, \gamma) = \Phi_1(s^{\sim}, y^{\sim}, \gamma; \sigma, \gamma\eta'^{\sim}) = 0$ have the Puiseux expansion with respect to $\gamma \in \mathbb{C}$. Let $\sigma = a\gamma^\mu$, then

$$\Phi_2(\sigma, \gamma) = \sum_{j, h} A_{j, h}(s^{\sim}, y^{\sim}, \eta'^{\sim}) \sigma^j \gamma^{j\mu + \hat{j} - j - h}.$$

If we take $\mu = \max\left\{\frac{\hat{j} - j - h}{\hat{j} - j}; A_{j, h}(s^{\sim}, y^{\sim}, \eta'^{\sim}) \neq 0, j < \hat{j}\right\}$, then $0 < \mu < 1$ and $\Phi_2(\sigma, \gamma) = 0$ has a root $\sigma = a\gamma^\mu\{1 + o(1)\}$ as $|\gamma| \rightarrow \infty$ with $a \neq 0$. If we take a suitable branch of γ^μ , then there exist $\gamma^{\sim} \in \mathbb{R}$, $\sigma^{\sim} \in \mathbb{C}$ such that $\Phi_2(\sigma^{\sim}, \gamma^{\sim}) = 0$ and $\text{Im } \sigma^{\sim} < 0$. Taking $\hat{s} = s^{\sim}\omega^{-q_0}$, $\hat{y} = y^{\sim}\omega^{-q}$, $\hat{\gamma} = \gamma^{\sim}\omega^{\kappa+q_0}$, $\hat{\eta}' = \eta'^{\sim}\omega^{q'}$ for sufficiently large $\omega \in \mathbb{R}$, we have the desired result. \square

There exists a neighborhood V of $(\hat{s}, \hat{y}, \hat{\eta}')$ in $\Omega^+ \times \mathbb{R}^n$ such that $\Phi(s, y, \hat{\gamma}; \sigma, \eta') = 0$ has a root σ such that $\text{Im } \sigma < 0$ for any $(s, y, \eta') \in V$. There exist a non-empty open set $V_1 \subset V$, a positive number θ and a real-analytic function $F(s, y, \eta')$ on V_1 such that $\text{Im } F(s, y, \eta') < -\theta$ on V_1 and $\sigma = F(s, y, \eta')$ is a root of $\Phi(s, y, \hat{\gamma}; \sigma, \eta') = 0$ with constant multiplicity d (≥ 1). Retake and fix a point $(\hat{s}, \hat{y}, \hat{\eta}')$ in V_1 . There exist a neighborhood W of (\hat{s}, \hat{y}) and a real

analytic function φ on W such that

$$(3-6) \quad \begin{cases} \partial_s \varphi = F(s, y, \partial_y \varphi) \\ \varphi(\hat{s}, y) = \langle \hat{\eta}', y' \rangle + i |y - \hat{y}|^2 \end{cases} \quad \text{on } W.$$

Note that

$$(3-7) \quad (\partial_s^l \partial_y^\nu \Phi)(s, y, \hat{\eta}; \partial_s \varphi, \partial_y \varphi) \begin{cases} = 0 \text{ on } W, & \text{if } l + |\nu| < d \\ \neq 0 \text{ on } W, & \text{if } l = d, \nu = 0 \end{cases}$$

and

$$(3-8) \quad \text{Im } \varphi(s, y) \geq |y - \hat{y}|^2 + \theta(\hat{s} - s) \quad \text{for } s \leq \hat{s}, (s, y) \in W.$$

Taking such φ as the function in E , we have from (3-5),

$$E^{-1} \circ P_\rho \circ E = \rho^{d \max \hat{\eta}^m - \hat{j}} R_\rho^\sim(s, y; D_s, D_y).$$

We shall show that R_ρ^\sim has a suitable structure.

Lemma 3.5. *There exists a differential operator*

$$Q_0 = \sum_{l+|\nu| \leq d} \sum_{j \in J_0} a_{j, l, \nu}(s, y) \rho^{j \delta'} D_s^l D_y^\nu$$

such that

- (i) J_0 is a set of a finite number of rational numbers,
- (ii) δ' is a rational number such that $0 \leq \delta' < \delta$,
- (iii) $a_{j, l, \nu}$ are analytic functions on W , $a_{0, 0, 0}(s, y) \neq 0$ on W and $a_{j, l, \nu}(s, y) = 0$ on W if $j + l + |\nu| > d$,
- (iv) for any N , the operator $R_\rho^\sim = \hat{\eta}^{m - \hat{j}} \rho^{d \delta} R_\rho^\sim - Q_0$ can be written as

$$R_\rho^\sim = \sum_{h=1}^{N+1} \rho^{-\delta h} R_h(s, y; D_s, D_y, \rho),$$

where R_h is a differential operator of order $\leq d + h$ whose coefficients are bounded in $C^\infty(W)$ as $\rho \rightarrow \infty$. Further, the coefficients of R_h ($h \leq N$) are analytic on W .

Proof. For an arbitrary function $w(s, y)$, we have by Leibniz rule,

$$P_\rho(E \cdot w) = \sum_{l+|\nu| \leq m} \frac{1}{l! \nu!} D_s^l D_y^\nu(w) (\partial_s^l \partial_y^\nu P_\rho)(E),$$

where $\partial_s^l \partial_y^\nu P_\rho$ is a differential operator with the symbol $\partial_s^l \partial_y^\nu p_\rho(s, y; \sigma, \eta)$. We have

$$E^{-1} (\partial_s^l \partial_y^\nu P_\rho)(E) = l! \nu! \rho^{d \max - (l+|\nu|) \delta} \hat{\eta}^{m - \hat{j}} \{ (\partial_s^l \partial_y^\nu \Phi)(s, y, \hat{\eta}; \partial_s \varphi, \partial_y \varphi) + o(1) \},$$

by the repeated use of Corollary A.4 in [2]. Thus, we have

$$E^{-1} P_\rho(E \cdot w) = \rho^{d \max - d \delta} \sum_{l+|\nu| \leq m} \sum \omega D_s^l D_y^\nu(w) a_{l, \nu, \omega}(s, y) \rho^\omega.$$

Here, the second sum is taken for rational numbers ω such that $\omega \leq (d - |\nu| - l) \delta$.

By (3-7), if $l+|\nu|<d$ and $\omega=(d-|\nu|-l)\delta$, then $a_{i,\nu,\omega}=0$ on W . Further, there holds $a_{d,0,0}(s, y)\neq 0$ on W . Let $\delta'=\max\left[\left\{\frac{\omega}{d-|\nu|-l}; d>|\nu|+l, a_{i,\nu,\omega}\neq 0\right\}\cup\{0\}\right]$.

We have $0\leq\delta'<\delta$ and we can write

$$E^{-1}P_\rho(E\cdot w) = \rho^{d_{\max}-d\delta}\left\{\sum_{l+|\nu|\leq d}\sum a_{j,l,\nu}(s, y)\rho^{j\delta'}D_s^l D_y^\nu(w)+R_\rho^\approx(s, y; D_s, D_y)(w)\right\},$$

where the second sum is taken for $j\in J_0$ such that $j\leq d-l-|\nu|$, and $J_0, a_{j,l,\nu}, R_\rho^\approx$ have the desired properties. \square

From now on, regard (\hat{s}, \hat{y}) as the origin $(0, 0)$. Under this situation, from Lemma 3.2, we have the following.

Proposition 3.6. *For any compact set $K\subset W$, there exist positive constants L, H, C such that there holds*

$$(3-9) \quad |u(0, 0)|\leq C\rho^H \sup_{p+|\beta|\leq L} \sup_{\substack{s\leq 0 \\ (s, y)\in K}} |(\partial_s^p \partial_y^\beta P_\rho u)(s, y)|$$

for any $u\in C_0^\infty(K)$.

We want to construct an asymptotic solution u of $P_\rho u=0$ that violates the above inequality.

Definition 3.7. ([4]) Let d be a fixed positive integer. For a formal power series $g=g(\omega; z)$ of $z=(z_0, z')\in\mathbb{C}^{n+1}$ with coefficients depending on a parameter ω , put

$$\left(\begin{array}{l} \Psi(g)(\omega; \tau, \xi)=\sum_\alpha \frac{1}{\alpha!} \tau^{\alpha_0} \xi^{|\alpha'|} |(\partial_z^\alpha g)(\omega; 0)|, \\ \Phi(g)(\omega; \tau, \xi)=\sum_{k+|\beta|\leq d-1} \sum_\alpha \frac{1}{\alpha!} \tau^{\alpha_0} \xi^{|\alpha'|} |(\partial_z^{\alpha+\beta} g)(\omega; 0)|\omega^k, \end{array} \right.$$

which are formal power series of (τ, ξ) .

The following is given in [4].

Proposition 3.8. *Let M be a positive integer, consider an operator $Q=Q(\omega; z, \partial_z)=\sum b_{\alpha,j}\omega^j \partial_z^\alpha$ on a neighborhood U of $0\in\mathbb{C}^{n+1}$, where $b_{\alpha,j}$ are holomorphic on U , $b_{(d,0,\dots,0),0}=1$ and the sum is taken for a finite number of (α, j) such that α is a multi-index, j is a rational number and $j+|\alpha|\leq d$. There exist positive constants K, ε_1 and ε_2 depending only on Q such that for any a, b which satisfy $b\leq\varepsilon_1$ and $0<a\leq\varepsilon_2 b$, there holds the following. If $v=v(\omega; s, y)$ and $g=g(\omega; s, y)$ satisfy*

$$(3-10) \quad \begin{cases} Qv = g, \\ \Psi(g) \ll \left(1 - \frac{\tau}{a} - \frac{\xi}{b}\right)^{-n} \exp(K\omega\tau), \\ \Phi(v)_{|\tau=0} \ll \omega^{-1} \left(1 - \frac{\xi}{b}\right)^{-n}, \end{cases}$$

then v satisfies

$$(3-11) \quad \Phi(v) \ll \omega^{-1} \left(1 - \frac{\tau}{a} - \frac{\xi}{b}\right)^{-n} \exp(K\omega\tau),$$

where $A \ll B$ means that B is a majorant series of A .

Let $u(\omega; s, y)$ be a formal power series of (s, y) with coefficients depending on ω . The following lemma is easily proved.

Lemma 3.9. (1) Consider an operator $R = \sum_{k \leq H, \beta} a_{k, \beta}(\omega; s, y) \partial_s^k \partial_y^\beta$, where $a_{k, \beta}(\omega; s, y) = \sum_{l, \gamma} a_{k, \beta}^{(l, \gamma)}(\omega) s^l y^\gamma \ll \sum_{l, \gamma} A_{k, \beta}^{(l, \gamma)} s^l y^\gamma$ for any ω and $\sum_{l, \gamma} A_{k, \beta}^{(l, \gamma)} s^l y^\gamma$ is a convergent power series. If $\Psi(u) \ll \left(1 - \frac{\tau}{a} - \frac{\xi}{b}\right)^{-n} \exp(K\omega\tau)$, then there exist positive constants C, a', b' such that $\Psi(Ru) \ll C\omega^H \left(1 - \frac{\tau}{a'} - \frac{\xi}{b'}\right)^{-n} \exp(K\omega\tau)$.

(2) If $\Psi(u) \ll A \left(1 - \frac{\tau}{a} - \frac{\xi}{b}\right)^{-n} \exp(K\omega\tau)$, then for a sufficiently small neighborhood V of $(0, 0)$ and for any (p, β) , there exists a constant $C_{p, \beta}$ such that

$$(3-12) \quad |\partial_s^p \partial_y^\beta u(\omega; s, y)| \leq C_{p, \beta} \omega^p \exp(K\omega|s|) \quad \text{on } V.$$

Now, we shall solve $(Q_0 + R_\rho^\approx)u_\rho = 0$ asymptotically. Consider $\rho^{\delta'}$ as ω . First, take $u_0(\rho; s, y)$ such that $Q_0 u_0 = 0$ and $\partial_s^j u_0|_{s=0} = \delta_{0, j}$ for $0 \leq j \leq d-1$ ($\delta_{0, j}$ is the Kronecker's δ). Since $\Phi(u_0)_{|\tau=0} \ll \rho^{(d-1)\delta'}$, there holds

$$\Phi(u_0) \ll \left(1 - \frac{\tau}{a_0} - \frac{\xi}{b_0}\right)^{-n} \rho^{(d-1)\delta'} \exp(K\rho^{\delta'}\tau)$$

for some positive constants a_0, b_0, K . Assume that u_0, \dots, u_h ($0 \leq h \leq N-1$) are taken such that

$$Q_0 u_j = - \sum_{l=1}^j \rho^{-l\delta} R_l u_{j-l},$$

$$\Phi(u_j) \ll C_j \left(1 - \frac{\tau}{a_j} - \frac{\xi}{b_j}\right)^{-n} \rho^{(d-1)\delta' - j(\delta - \delta')} \exp(K\rho^{\delta'}\tau),$$

($j=0, 1, \dots, h$),

where $a_0 \geq a_1 \geq \dots \geq a_h > 0$ and $b_0 \geq b_1 \geq \dots \geq b_h > 0$. Then, there exist positive constants $A_{h, l}, a_{h+1} (\leq a_h), b_{h+1} (\leq b_h)$ such that $a_{h+1} \leq \varepsilon_2 b_{h+1}$ and

$$\Psi(R_l u_{h+1-l}) \ll A_{h,l} \left(1 - \frac{\tau}{a_{h+1}} - \frac{\xi}{b_{h+1}}\right)^{-n} \rho^{(d+l)\delta' - (h+1-l)(\delta-\delta')} \exp(K\rho^{\delta'}\tau).$$

Hence, there exists a constant C_{h+1} such that

$$\Psi\left(\sum_{l=1}^{h+1} \rho^{-l\delta} R_l u_{h+1-l}\right) \ll C_{h+1} \left(1 - \frac{\tau}{a_{h+1}} - \frac{\xi}{b_{h+1}}\right)^{-n} \rho^{(d+h+1)\delta' - (h+1)\delta} \exp(K\rho^{\delta'}\tau).$$

By Cauchy-Kowalevskaya theorem and Proposition 3.8, there exists u_{h+1} such that

$$\begin{cases} Q_0 u_{h+1} = -\sum_{l=1}^{h+1} \rho^{-l\delta} R_l u_{h+1-l}, \\ \partial_s^j u_{h+1}|_{s=0} = 0 \quad (j=0, \dots, d-1), \\ \Phi(u_{h+1}) \ll C_{h+1} \left(1 - \frac{\tau}{a_{h+1}} - \frac{\xi}{b_{h+1}}\right)^{-n} \rho^{(d+h)\delta' - (h+1)\delta} \exp(K\rho^{\delta'}\tau). \end{cases}$$

Thus, we can take u_0, \dots, u_N such that

$$\begin{aligned} \Psi(u_j) &\ll C_j \left(1 - \frac{\tau}{a_j} - \frac{\xi}{b_j}\right)^{-n} \rho^{-j(\delta-\delta')} \exp(K\rho^{\delta'}\tau), \\ E^{-1}P_\rho\left(E \sum_{j=0}^N u_j\right) &= \rho^{d_{\max} - d\delta} \left\{ \sum_{h=1}^N \sum_{j=N+1-h}^N \rho^{-h\delta} R_h u_j + \rho^{-(N+1)\delta} R_{N+1} \left(\sum_{j=0}^N u_j\right) \right\}. \end{aligned}$$

For any $h=1, \dots, N+1$ and any (p, β) , there exists a constant $C_{h,p,\beta}$ such that

$$|\partial_s^p \partial_y^\beta R_h u_j(s, y)| \leq C_{h,p,\beta} \rho^{(d+h+p)\delta' - j(\delta-\delta')} \exp(K\rho^{\delta'} |s|)$$

near $(0, 0)$. Hence, there exist a neighborhood W_1 of $(0, 0)$ and constants $C_{p,\beta}^{(N)}$ such that

$$\begin{aligned} \left| \partial_s^p \partial_y^\beta P_\rho \left(E \cdot \sum_{j=0}^N u_j \right) (s, y) \right| &\leq C_{p,\beta}^{(N)} \rho^{d_{\max} + p + \beta - d\delta + d\delta' - (N+1)(\delta-\delta')} \\ &\quad \times \exp\{-\operatorname{Im} \varphi(s, y) \rho^\delta + K\rho^{\delta'} |s|\} \quad \text{on } W_1. \end{aligned}$$

Since $-\operatorname{Im} \varphi(s, y) \leq \theta |s| - |y - \hat{y}|^2 \leq -\theta |s| - |y - \hat{y}|^2$ for $s \leq 0$, we have

$$(3-13) \quad \left| \partial_s^p \partial_y^\beta P_\rho \left(E \cdot \sum_{j=0}^N u_j \right) (s, y) \right| \leq C_{p,\beta}^{(N)} \rho^{d_{\max} + p + \beta - d\delta + d\delta' - (N+1)(\delta-\delta')}$$

on $W_1 \cap \{s \leq 0\}$ for sufficiently large ρ .

On the other hand, let W_2 be a relatively compact neighborhood of $(0, 0)$ such that $\overline{W_2} \subset W_1$. There exists a positive number ε such that for any (p, β) , there holds

$$(3-14) \quad \left| \partial_s^p \partial_y^\beta \left(E \cdot \sum_{j=0}^N u_j \right) (s, y) \right| \leq A_{p,\beta}^{(N)} \rho^{p+\beta} \exp(-\varepsilon \rho^\delta)$$

on $(W_1 \setminus W_2) \cap \{s \leq 0\}$ for some constant $A_{p,\beta}^{(N)}$.

Now, we take $\chi(s, y) \in C_0^\infty(W_1)$ such that $\chi(s, y) = 1$ on W_2 . Then, by (3-13),

(3-14), the asymptotic solution $u(s, y) = \chi(s, y) \cdot E \cdot \sum_{j=0}^N u_j$, violates the inequality (3-9) as $\rho \rightarrow +\infty$ for sufficiently large N .

§ 4. Proof of Theorem 2.5

In this section, assume that p_m is t -involutive and satisfies the assumption [B]. Further, assume that the Cauchy problem for P is well-posed on Ω^+ . Essentially, the proof of Theorem 2.5 goes along the same line as the proof of Main Theorem in [9]. For simplicity, we assume that $\lambda_j \neq \lambda_k$ if $j \neq k$ and that $\kappa(0; j, k) = 0$ for any j, k (, hence p_m is involutive). Fix an arbitrary $(\hat{t}, \hat{x}) \in \Omega^+$. We have only to show that there exists a neighborhood V of (\hat{t}, \hat{x}) such that P can be written as (2-2)' on V . The general case is proved by taking the arguments in [9] into account and modifying the following arguments.

Fix $1 \leq \hat{l} \leq n_1$ until Proposition 4.11. Since $\text{grad}_x C_{\hat{l}}(\hat{t}, \hat{x}) \neq 0$, we can retake a coordinate on a neighborhood U_1 of (\hat{t}, \hat{x}) such that $C_{\hat{l}}(t, x) = x_1$.

Fix an arbitrary $z^{\sim} = (t^{\sim}, x^{\sim}) \in U_1$ such that $x_1^{\sim} = 0$ and $C_l(t^{\sim}, x^{\sim}) \neq 0$ for all $l \neq \hat{l}$. Put $J[\hat{l}] = \{j \in \{1, \dots, m\}; \text{there exists } k \neq j \text{ such that } \kappa(\hat{l}; j, k) \neq 0\}$.

Lemma 4.1. *If $j \in J[\hat{l}]$, then $\partial_{\xi_1} \lambda_j(t, x; \xi)$ is divisible by x_1 .*

Proof. There holds $\lambda_j(t, x; \xi) - \lambda_k(t, x; \xi) = x_1^{\kappa} \phi(t, x; \xi)$, where $\kappa = \kappa(\hat{l}; j, k) > 0$ and ϕ is a C^∞ -function such that $\phi(t, x; \xi) \neq 0$ if $C_l(t, x) \neq 0$ for all $l \neq \hat{l}$. Since

$$\begin{aligned} \{\tau - \lambda_j(t, x; \xi), \tau - \lambda_k(t, x; \xi)\} &= \{\tau - \lambda_j, \lambda_j - \lambda_k\} \\ &= x_1^{\kappa} \partial_t \phi - \sum_{i=1}^n (\partial_{\xi_i} \lambda_j) x_1^{\kappa} \partial_{x_i} \phi - (\partial_{\xi_1} \lambda_j) \kappa x_1^{\kappa-1} \phi + \sum_{i=1}^n (\partial_{x_i} \lambda_j) x_1^{\kappa} \partial_{\xi_i} \phi \end{aligned}$$

is divisible by x_1^{κ} , the function $\partial_{\xi_1} \lambda_j$ is divisible by x_1 . □

For $\hat{k} \in J[\hat{l}]$, the following holds.

Lemma 4.2. *There exists a nowhere dense subset S of \mathbf{R}^n such that for any $\hat{\xi} \in \mathbf{R}^n \setminus S$, there exists a coordinate transformation $T = T(z^{\sim}, \hat{k}, \hat{\xi})$ which satisfies the followings.*

(1) T transforms z^{\sim} to $(0, 0)$ and a neighborhood W of z^{\sim} onto a neighborhood W^{\sim} of $(0, 0)$.

(2) T has a form $s = t - t^{\sim}$, $y = f(t, x) = (f_1(t, x), \dots, f_n(t, x))$. There hold $\text{grad}_x f_{n+1} \big|_{t=t^{\sim}} = \hat{\xi}$ and that $f_1(t, x) = x_1 g(t, x)$, where $g \in C^\infty(W)$ and $g(t, x) \neq 0$ on W .

(3) Put $\lambda_j^{\sim}(s, y; \eta) = \lambda_j(t, x; \partial_x \langle f(t, x), \eta \rangle) - \partial_t \langle f(t, x), \eta \rangle$, where $\langle f(t, x), \eta \rangle = \sum_i f_i(t, x) \eta_i$. There holds $\lambda_{\hat{k}}^{\sim}(s, y; e_n) = 0$ on W^{\sim} .

Proof. We can take h_n such that $\partial_t h_n = \lambda_{\hat{k}}(t, x; \hat{\xi} + \partial_x h_n)$ and $h_n \big|_{t=t^{\sim}} = 0$. Put $h_j = 0$ for $1 \leq j \leq n-1$.

First, consider the case $n \geq 2$. Put $S = \{\xi \in \mathbb{R}^n; \xi_j = 0, \text{ for } 2 \leq j \leq n\}$. If $\hat{\xi} \in \mathbb{R}^n \setminus S$, then we can take an invertible matrix A such that ${}^t A e_n = \hat{\xi}$ and $(Ax)_1 = x_1$. Putting $f(t, x) = Ax + h(t, x)$, we get the desired result.

Next, consider the case $n = 1$. Put $S = \{0\}$. If $\hat{\xi} \in \mathbb{R} \setminus \{0\}$, then put $f(t, x) = \hat{\xi}x + h_n(t, x)$. Since $\partial_t f = \lambda_{\hat{k}}(t, x; \partial_x f) = \partial_x f \cdot \partial_{\xi} \lambda_{\hat{k}}(t, x; \partial_x f)$, $f|_{t=\tau} = \hat{\xi}x$ and $\partial_{\xi} \lambda_{\hat{k}}(t, x; \xi)$ is divisible by x , it is easy to see that $f(t, x) = xg(t, x)$ for a non-zero C^∞ -function g . Hence, we have the desired result. \square

Note that $p_m^\sim = \prod_{j=1}^m (\sigma - \lambda_j^\sim)$ is the principal symbol of the operator P^\sim which is transformed from P by T . Put $C_l^\sim(s, y) = C_l(t, x)$. We may assume that $C_l^\sim(s, y) \neq 0$ on W^\sim for all $l \neq \hat{l}$.

Fix $1 \leq \hat{k} \leq m$ and $\hat{\xi} \in \mathbb{R}^n \setminus S$. Denoting $\kappa(j) = \kappa(\hat{l}; j, \hat{k})$, reorder $\{\kappa(j); j \neq \hat{k}\}$ as $\kappa(\rho_1) \leq \dots \leq \kappa(\rho_{m-1})$. Let $\Delta = \Delta\langle \hat{l}; \hat{k} \rangle$ be the Newton polygon of size m with the lower side $\Gamma(j) = \Gamma\langle \hat{l}; \hat{k} \rangle(j) = \sum_{i=1}^{m-j} \kappa(\rho_i)$ ($j = 1, \dots, m$) and $\Gamma(0) = \infty$. (This is the Newton polygon determined by $(\kappa(\rho_1), \dots, \kappa(\rho_{m-1}), \infty)$ in the terminology of § 3 in [9].) For this Newton polygon, we get $\kappa_1, \dots, \kappa_r$ as in Definition 0.3. Note that κ_j are integers in this case. If $\kappa_1 = 0$, then modify Δ by setting $\kappa_1^\sim = 1$ as in § 1. Denote this Newton polygon by $\Delta^\sim = \Delta^\sim\langle \hat{l}; \hat{k} \rangle$ and its lower side by $\mu = \Gamma^\sim(j) = \Gamma^\sim\langle \hat{l}; \hat{k} \rangle(j)$. Now, put $q_1 = 1$ and $q_j = \varepsilon$ (arbitrary positive number) for $j \neq 1$. It is easy to see that $\Delta = \Delta\langle \hat{l}; \hat{k} \rangle$ is the Newton polygon drawn from P^\sim as in § 1.

Lemma 4.3. *The assumption [A] in § 1 is satisfied for P^\sim and Δ^\sim .*

Proof. The condition 1) of [A] is trivial. We have $p_m^\sim(s, y; \sigma, \eta) = \prod_{k=1}^m (\sigma - \lambda_k^\sim(s, y; \eta))$, $\lambda_{\hat{k}}^\sim(s, y; e_n) \equiv 0$, $\lambda_k^\sim(s, y; e_n) = y_1^{\varepsilon(k)} \phi_k(s, y; \eta)$. Further, by Lemma 4.1, if $\kappa(k) > 0$, then $\partial_{\eta_1} \lambda_k^\sim(s, y; \eta) = y_1 \phi_k(s, y; \eta)$. (Here, ϕ_k, ϕ_k are C^∞ -functions.) Hence, it is easy to see that $\partial_{\eta}^j \partial_{\eta}^\alpha p_m^\sim(s, y; 0, e_n) = y_1^{\Gamma^\sim(j+|\alpha|)+\alpha_1} \times \Phi_{j,\alpha}(s, y)$, where $\Phi_{j,\alpha}$ is a C^∞ -function. (Here, regard y_1^∞ as 0.) Thus, if $q_0(k+|\alpha|) + \langle q, \beta - \alpha \rangle < \Gamma^\sim(j+|\alpha|)$, then $\beta_1 < \Gamma^\sim(j+|\alpha|) + \alpha_1$, hence $\partial_{\eta}^* \partial_{\eta}^j \partial_{\eta}^\alpha p_m^\sim(0, 0; 0, e_n) = 0$. This is the condition 2). \square

Proposition 4.4. *For $h = 1, \dots, m$, there holds $p_m^{\sim(\hat{l}, \alpha)}(0, 0; 0, e_n) = 0$ if $\beta_1 < \Gamma^\sim(j+|\alpha|+h) + \alpha_1$.*

Proof. By Theorem 1.2, we have $p_m^{\sim(\hat{l}, \alpha)}(0, 0; 0, e_n) = 0$ if $q_0(k+|\alpha|+h) + \langle q, \beta - \alpha \rangle < \Gamma^\sim(j+|\alpha|+h)$. Since $q_0(k+|\alpha|+h) + \langle q, \beta - \alpha \rangle = \beta_1 - \alpha_1 + \varepsilon(k+h+|\beta| - \beta_1 + \alpha_1)$ and $\varepsilon > 0$ is arbitrary, we have the desired result. \square

Now, we set up some classes of operators corresponding to a Newton polygon. Let Δ be a Newton polygon of size m with the lower side $\Gamma(j)$ (see § 0). From now on, assume that all the Newton polygon that we treat satisfy

$\Gamma(m-1)-\Gamma(m)\geq 1$, that is, $\kappa_1\geq 1$ in Definition 0.3. Further, we always extends $\Gamma(j)$ as $\Gamma(j)=\Gamma(m)-j+m$ for $j\geq m$.

Definition 4.5. (Cf. [9; Definition 3.6]) (1) Let p be an integer such that $0\leq p\leq m$. We say $A=a(s, y; D_y)\in\mathcal{S}^p\langle\mathcal{A}\rangle$ if A satisfies the following conditions.

- (i) $A\in C^\infty([0, \varepsilon]; \mathcal{S}^p(G))$, where ε is a positive number, G is a neighborhood of $y=0$ and $[0, \varepsilon]\times G\subset W^\sim$.
- (ii) Let $a(s, y; \eta)\sim\sum_{l=0}^{\infty} a_{p-l}(s, y; \eta)$ be the asymptotic expansion of the symbol a of A , where a_{p-l} is homogeneous of degree $p-l$ w. r. t. η . Then, there holds $a_{p-l, (\kappa, \beta)}^{(\alpha)}(0, 0; e_n)=0$ if $\beta_1<\Gamma(m-p+l+|\alpha|)+\alpha_1$.

(2) Let H be an integer such that $0\leq H\leq m$. We say $Q\in W_H\langle\mathcal{A}\rangle$ if $Q=\sum_{h=H}^m a_h(s, y; D_y)D_s^{m-h}$, where $a_h(s, y; D_y)\in\mathcal{S}^{h-H}\langle\mathcal{A}\rangle$.

Proposition 4.4 means that $P^\sim\in W_0\langle\mathcal{A}^\sim\langle\hat{l}; \hat{k}\rangle\rangle$. We shall show some basic properties of the above classes of operators.

Definition 4.6. (1) For a positive integer κ , let $\mathcal{A}[\kappa]$ be the Newton polygon of size 1 with the lower side $\mu=\kappa-\kappa j$ ($0\leq j\leq 1$). Let $\mathcal{A}[0]$ be the Newton polygon of size 1 with the lower side $\mu=-j$ ($0\leq j\leq 1$). Put $\mathcal{A}[\infty]=\{(1, \mu)\in\mathbf{R}^2; \mu\geq 0\}$.

(2) Let \mathcal{A}_i ($i=1, 2$) be Newton polygons of size m_i with the lower side $\Gamma_i(j)$ ($0\leq j\leq m_i$). The Newton polygon $\mathcal{A}=\mathcal{A}_1+\mathcal{A}_2$ is the one of size m_1+m_2 with the lower side $\Gamma(j)=\min\{\Gamma_1(d)+\Gamma_2(e); d+e=j, 0\leq d\leq m_1, 0\leq e\leq m_2\}$ ($0\leq j\leq m_1+m_2$). (Cf. Definition 3.1 and Remark 3.2 in [9].)

Lemma 4.7. (1) Let $\lambda(s, y; D_y)\in C^\infty([0, \varepsilon]; \mathcal{S}^1(G))$ and let $\lambda_0(s, y; \eta)$ be its principal symbol. Let κ be a positive integer or ∞ . If $\lambda_0(s, y; e_n)=y_1^\kappa\phi(s, y)$ and $\partial_{\eta_1}\lambda_0(s, y; \eta)=y_1\psi(s, y; \eta)$ near $(0, 0)$ for some C^∞ -function ϕ, ψ , then $\lambda(s, y; D_y)\in\mathcal{S}^1\langle\mathcal{A}[\kappa]\rangle$ and $D_s-\lambda(s, y; D_y)\in W_0\langle\mathcal{A}[\kappa]\rangle$. Here, if $\kappa=\infty$, then regard y_1^∞ as 0.

(2) If $\lambda(s, y; D_y)\in C^\infty([0, \varepsilon]; \mathcal{S}^1(G))$, then $\lambda(s, y; D_y)\in\mathcal{S}^1\langle\mathcal{A}[0]\rangle$ and $D_s-\lambda(s, y; D_y)\in W_0\langle\mathcal{A}[0]\rangle$.

(3) Let \mathcal{A}_i ($i=1, 2$) be Newton polygons. If $\mathcal{A}_i\in\mathcal{S}^{p_i}\langle\mathcal{A}_i\rangle$ ($i=1, 2$), then $\mathcal{A}_1\mathcal{A}_2\in\mathcal{S}^{p_1+p_2}\langle\mathcal{A}_1+\mathcal{A}_2\rangle$. If $Q_i\in W_{H_i}\langle\mathcal{A}_i\rangle$ ($i=1, 2$), then $Q_1Q_2\in W_{H_1+H_2}\langle\mathcal{A}_1+\mathcal{A}_2\rangle$.

Proof. (1) Assume $\kappa<\infty$. From the assumption, it is easy to see that $\lambda_{0, (\kappa, \beta)}^{(\alpha)}(0, 0; e_n)=0$ if $\beta_1<\Gamma(|\alpha|)+\alpha_1$, where $\Gamma(j)=\kappa-\kappa j$ for $0\leq j\leq 1$ and $\Gamma(j)=1-j$ for $j\geq 1$. Since $\Gamma(1+|\alpha|)+\alpha_1\leq 0$ for any α , we have the desired result. If $\kappa=\infty$, the proof is similar and easier.

(2) is obvious.

(3) Let m_i be the size of \mathcal{A}_i and $\Gamma_i(j)$ ($0\leq j\leq m_i$) be the lower side of \mathcal{A}_i ($i=1, 2$). It is easy to see that if we extend Γ 's as already stated, then the

lower side of $\mathcal{A}=\mathcal{A}_1+\mathcal{A}_2$ is $\Gamma(j)=\min\{\Gamma_1(d)+\Gamma_2(e); d+e=j, d\geq 0, e\geq 0\}$ for $j\geq 0$. (Remember that we have assumed $\Gamma_i(m_i-1)-\Gamma_i(m_i)\geq 1$.) Let $a_i\sim\sum_{l=0}^{\infty}a_{i,p_i-l}$ be the asymptotic expansion of the symbol of A_i ($i=1, 2$). The asymptotic expansion of the symbol of A_1A_2 is $a\sim\sum_{L=0}^{\infty}a_{p_1+p_2-L}$, where

$$a_{p_1+p_2-L}(s, y; \eta)=\sum_{l+h+|\delta|=L}(\text{const.})a_{1,p_1-l}^{(\delta)}(s, y; \eta)a_{2,p_2-h,(0,\delta)}(s, y; \eta).$$

Hence,

$$a_{p_1+p_2-L,(k,\beta)}^{(\alpha)}=\sum_{l+h+|\delta|=L}\sum_{\gamma\leq\alpha, p\leq k, \omega\leq\beta}(\text{const.})a_{1,p_1-l,(p,\omega)}^{(\delta+\gamma)}a_{2,p_2-h,(k-p,\delta+\beta-\omega)}^{(\alpha-\gamma)}.$$

By the assumption, there holds $a_{1,p_1-l,(p,\omega)}^{(\delta+\gamma)}(0, 0; e_n)=0$ if $\omega_1<\Gamma_1(m_1-p_1+l+|\delta+\gamma|)+\delta_1+\gamma_1$, and there holds $a_{2,p_2-h,(k-p,\delta+\beta-\omega)}^{(\alpha-\gamma)}(0, 0; e_n)=0$ if $\delta_1+\beta_1-\omega_1<\Gamma_2(m_2-p_2+h+|\alpha-\gamma|)+\alpha_1-\gamma_1$. If $\omega_1\geq\Gamma_1(m_1-p_1+l+|\delta+\gamma|)+\delta_1+\gamma_1$ and $\delta_1+\beta_1-\omega_1\geq\Gamma_2(m_2-p_2+h+|\alpha-\gamma|)+\alpha_1-\gamma_1$, then $\beta_1\geq\Gamma(m_1+m_2-p_1-p_2+L+|\alpha|)+\alpha_1$. Hence, we get $a_{p_1+p_2-L,(k,\beta)}^{(\alpha)}(0, 0; e_n)=0$ if $\beta_1<\Gamma(m_1+m_2-p_1-p_2+L+|\alpha|)+\alpha_1$.

Thus, there holds $A_1A_2\in\mathcal{S}^{p_1+p_2}\langle\mathcal{A}_1+\mathcal{A}_2\rangle$. Next, let $Q_i=\sum_{h=H_i}^{m_i}a_h^{(i)}(s, y; D_y)D_s^{m_i-h}$ ($i=1, 2$). We have $Q_1Q_2=\sum_{L=H_1+H_2}^{m_1+m_2}\sum_{h+k+l=L}a_h^{(1)}(s, y; D_y)(D_s^l a_k^{(2)})(s, y; D_y)D_s^{m_1+m_2-L}$. Since $a_h^{(1)}(s, y; D_y)\in\mathcal{S}^{h-H_1}\langle\mathcal{A}_1\rangle$ and $(D_s^l a_k^{(2)})(s, y; D_y)\in\mathcal{S}^{k-H_2}\langle\mathcal{A}_2\rangle$, we get $Q_1Q_2\in\mathcal{W}_{H_1+H_2}\langle\mathcal{A}_1+\mathcal{A}_2\rangle$. \square

So far, we have fixed $z\sim, \hat{k}, \hat{\xi}$. We shall unite the above results for every $z\sim, \hat{k}, \hat{\xi}$.

Definition 4.8. Let H be an integer such that $0\leq H\leq m$. We say $Q\in\mathcal{W}_H(\dot{I})$ if the following conditions are satisfied.

- (i) $Q=\sum_{h=H}^m a_h(t, x; D_x)D_t^{m-h}$, where $a_h(t, x; D_x)\in C^\infty([0, T]; \mathcal{S}^{h-H}(\mathbf{R}^n))$.
- (ii) For any $z\sim=(t\sim, x\sim)\in U_1$ such that $x_1\sim=0$ and $C_l(t\sim, x\sim)\neq 0$ for $l\neq \dot{I}$, any $\hat{k}\in J[\dot{I}]$ and any $\hat{\xi}\in\mathbf{R}^n\setminus\mathcal{S}$, let $Q\sim$ be the operator transformed from Q by $T(z\sim, \hat{k}, \hat{\xi})$. Then, $Q\sim\in\mathcal{W}_H\langle\mathcal{A}\sim\langle\dot{I}, \hat{k}\rangle\rangle$.

By Proposition 4.4, we have the following theorem.

Theorem 4.9. *There holds $P\in\mathcal{W}_0(\dot{I})$.*

On the other hand, we have the following lemma.

Lemma 4.10. *Let $A\in C^\infty([0, T]; \mathcal{S}^0(\mathbf{R}^n))$ and $J\subset I$. Then, there holds $A\pi_J\in\mathcal{W}_{m-1,J_1}(\dot{I})$.*

Proof. Fix $z\sim, \hat{k}, \hat{\xi}$, and let $\partial_k\sim$ (resp. $A\sim, \pi_J\sim$) be the operator transformed from ∂_k (resp. A, π_J) by $T(z\sim, \hat{k}, \hat{\xi})$. By Lemma 4.7 (1), (2), we have

$\partial_k^\sim \in W_0 \langle \mathcal{A}[\kappa(\hat{l}; k, \hat{k})] \rangle$, where $\kappa(\hat{l}; \hat{k}, \hat{k}) = \infty$. Note that $1 \in W_1 \langle \mathcal{A}[\kappa] \rangle$ for any κ ($0 \leq \kappa \leq \infty$), and $A^\sim \in W_0 \langle \mathcal{A}_0 \rangle$, where $\mathcal{A}_0 = \{(0, \mu); \mu \geq 0\}$. Since $\mathcal{A}^\sim \langle \hat{l}; \hat{k} \rangle = \mathcal{A}[\kappa(\hat{l}; 1, \hat{k})] + \cdots + \mathcal{A}[\kappa(\hat{l}; m, \hat{k})]$, we have $A^\sim \pi_j^\sim \in W_{m-1, j, 1} \langle \mathcal{A}^\sim \langle \hat{l}; \hat{k} \rangle \rangle$ by Lemma 4.7 (3). Hence, there holds $A\pi_j \in W_{m-1, j, 1}(\hat{l})$. \square

The following is the most important property of $W_H(\hat{l})$.

Proposition 4.11. *Let $Q = \sum_{h=H}^m a_h(t, x; D_x) D_t^{m-h} \in W_H(\hat{l})$ ($H \geq 1$) and let its principal symbol be $q(t, x; \tau, \xi) = \sum_{h=H}^m a_h^0(t, x; \xi) \tau^{m-h}$, where a_h^0 is homogeneous of degree $h-H$ w. r. t. ξ . Then, for any $1 \leq k \leq m$ and $0 \leq r \leq m-H$, there holds*

$$(\partial_t^r q)(t, x; \lambda_k(t, x; \xi), \xi) \in x_1^{\Gamma \langle \hat{l}; k \rangle (r+H)} \times C^\infty(V \times \hat{\mathbf{R}}^n),$$

where V is a neighborhood of (\hat{t}, \hat{x}) .

In the original coordinate, there holds

$$(4-1) \quad (\partial_t^r q)(t, x; \lambda_k(t, x; \xi), \xi) \in C_l(t, x)^{\Gamma \langle \hat{l}, k \rangle (r+H)} \times C^\infty(V \times \hat{\mathbf{R}}^n).$$

Proof. If $k \in J[\hat{l}]$, then $\Gamma \langle \hat{l}, k \rangle(j) = 0$ ($1 \leq j \leq m$), hence there is nothing to prove. Assume that $k \in J[\hat{l}]$. Fix $z^\sim, \hat{\xi}$ and let Q^\sim be the operator transformed from Q by $T(z^\sim, k, \hat{\xi})$. Since $Q^\sim \in W_H \langle \mathcal{A}^\sim \langle \hat{l}; k \rangle \rangle$, its principal symbol $q^\sim(s, y; \sigma, \eta)$ satisfies that $q^\sim_{(\alpha, \beta)}(0, 0; 0, e_n) = 0$ if $\beta_1 < \Gamma^\sim \langle \hat{l}; k \rangle (r+H)$. Since $q^\sim(0, y; \sigma, e_n) = q(t^\sim, x; \sigma + \lambda_k(t^\sim, x; \hat{\xi}), \hat{\xi})$ and $\Gamma \langle \hat{l}; k \rangle(j) = \max(0, \Gamma^\sim \langle \hat{l}; k \rangle(j))$, we have $\partial_{x_1}^j \{(\partial_t^r q)(t^\sim, x; \lambda_k(t^\sim, x; \hat{\xi}), \hat{\xi})\}_{|x=x^\sim} = 0$ if $j < \Gamma \langle \hat{l}; k \rangle (r+H)$. Since $\hat{\xi} \in \hat{\mathbf{R}}^n \setminus \mathcal{S}$ and $(t^\sim, x^\sim) \in U_1$ are arbitrary as long as $x_1^\sim = 0$ and $C_l(t^\sim, x^\sim) \neq 0$ for $l \neq \hat{l}$, we have the desired result. \square

So far, we have fixed \hat{l} arbitrarily. If $Q \in W_H(l)$ for $l=1, \dots, n_1$ ($H \geq 1$), then for a sufficiently small neighborhood V_1 of (\hat{t}, \hat{x}) there holds

$$(4-2) \quad (\partial_t^r q)(t, x; \lambda_k(t, x; \xi), \xi) \in \prod_{l=1}^{n_1} C_l(t, x)^{\Gamma \langle l; k \rangle (r+H)} \times C^\infty(V_1 \times \hat{\mathbf{R}}^n)$$

for any $1 \leq k \leq m, 0 \leq r \leq m-H$,

since $\{\text{grad}_x C_l(t, x); l=1, \dots, n_1\}$ is linearly independent.

Lemma 4.12. *For any $1 \leq \hat{k} \leq m$, there exists an ordering $(\sigma_1, \dots, \sigma_{m-1})$ of $\{1, \dots, m\} \setminus \{\hat{k}\}$ such that for any $1 \leq l \leq n_1$, there holds $\kappa(l; \hat{k}, \sigma_1) \leq \kappa(l; \hat{k}, \sigma_2) \leq \cdots \leq \kappa(l; \hat{k}, \sigma_{m-1})$.*

Proof. Put $\rho(l; j, k) = \min\{\kappa(l; \hat{k}, j), \kappa(l; \hat{k}, k)\}$ for $j \neq \hat{k}, k \neq \hat{k}, j \neq k$. For such j, k , by the assumption [B]-iii)-a), there holds $\lambda_j(t, x; \xi) - \lambda_k(t, x; \xi) = (\lambda_{\hat{k}} - \lambda_k) - (\lambda_{\hat{k}} - \lambda_j) = \left(\prod_{l=1}^{n_1} C_l(t, x)^{\rho(l; j, k)} \right) \Psi_{j, k}(t, x; \xi)$, where

$$(4-3) \quad \Psi_{j, k} = \left(\prod_{l=1}^{n_1} C_l(t, x)^{\kappa(l; \hat{k}, k) - \rho(l; j, k)} \right) \Phi_{\hat{k}, k} - \left(\prod_{l=1}^{n_1} C_l(t, x)^{\kappa(l; \hat{k}, j) - \rho(l; j, k)} \right) \Phi_{\hat{k}, j}$$

$$(4-4) \quad = \left(\prod_{l=1}^{n_1} C_l(t, x)^{\kappa(l; j, k) - \rho(l; j, k)} \right) \Phi_{j, k}.$$

Assume that there exist l_1, l_2, j, k such that $\kappa(l_1; \hat{k}, j) > \kappa(l_1; \hat{k}, k)$ and $\kappa(l_2; \hat{k}, k) > \kappa(l_2; \hat{k}, j)$. Note that j, k, \hat{k} are distinct. By (4-3), neither C_{l_1} nor C_{l_2} divide $\Psi_{j, k}$. Hence, by (4-4), if $C_l(t, x) \neq 0$ for $l \neq l_1, l_2$ and $\xi \neq 0$, then $\Psi_{j, k}(t, x; \xi) \neq 0$. On the other hand, by (4-3), if $C_{l_1}(t, x) = C_{l_2}(t, x) = 0$, then $\Psi_{j, k}(t, x; \xi) = 0$. These contradict each other, since there exists (t, x) such that $C_l(t, x) \neq 0$ for $l \neq l_1, l_2$ and $C_{l_1}(t, x) = C_{l_2}(t, x) = 0$. Thus, if $\kappa(l_1; \hat{k}, j) > \kappa(l_1; \hat{k}, k)$ for some l_1 , then $\kappa(l; \hat{k}, j) \geq \kappa(l; \hat{k}, k)$ for any l . The desired result easily follows from this. \square

Remark 4.13. The assumption ii) in [B] can be replaced by the following two conditions.

- (ii)- α) $(\text{grad}_x C_l)(\hat{t}, \hat{x})$ ($l=1, \dots, n_1$) are pairwise linearly independent.
- (ii)- β) There holds the statement of Lemma 4.12.

By using this lemma, the following proposition is proved similarly to Proposition 5.1 in [9].

Proposition 4.14. *Let $q(t, x; \tau, \xi) = \sum_{h=H}^m a_h(t, x; \xi) \tau^{m-h}$ satisfy (4-2), where $a_h \in C^\infty(V_1 \times \hat{\mathbf{R}}^n)$ is homogeneous of degree $h-H$ ($H \geq 1$) w. r. t. ξ . Then, for any $J \subset I$ such that $|J|=m-H$, there exists $A_J(t, x; \xi) \in C^\infty(V_1 \times \hat{\mathbf{R}}^n)$ such that A_J is homogeneous of degree 0 w. r. t. ξ and there holds*

$$q(t, x; \tau, \xi) = \sum_{|J|=m-H} A_J(t, x; \xi) \prod_{j \in J} (\tau - \lambda_j(t, x; \xi)).$$

Theorem 2.5 is proved similarly to §6 in [9], by combining Theorem 4.9, Lemma 4.10, Propositions 4.11 and 4.14.

Appendix 1. Proof of Remark 1.1, ii)

Lemma A.1. *Assume that $p_m(t, x; \tau, \xi)$ is hyperbolic and q_j ($j=0, \dots, n$), ν and κ are positive rational numbers. If $\partial_t^k \partial_x^\beta \partial_\tau^j p_m(0, 0; 0, e_n) = 0$ for any (k, β, j) such that $q_0 k + \langle q, \beta \rangle < \nu - \kappa j$, then $\partial_t^k \partial_x^\beta \partial_\tau^\alpha p_m(0, 0; 0, e_n) = 0$ for any (k, β, j, α) such that $q_0 k + \langle q, \beta \rangle < \nu - \kappa(j + |\alpha|)$.*

Proof. For arbitrary (\hat{t}, \hat{x}) , consider $f(t; \tau, \xi) = p_m(\hat{t} t^{\alpha_0}, \hat{x} t^\alpha; \tau, \xi)$. Then, $\partial_t^j f(t; 0, e_n) \in t^{\nu - \kappa j} \times C^0[0, T]$. We can apply Proposition 3.2 in [8] when we regard $f(t; \tau, \xi)$ as $P_m(t, \hat{x}; \tau, \xi)$ and $\mu_0 = 1$. Thus, we get $\partial_t^j \partial_\tau^\alpha f(t; 0, e_n) \in t^{\nu - \kappa(j + |\alpha|)} \times C^0[0, T]$. Since (\hat{t}, \hat{x}) is arbitrary, this means that $\partial_t^k \partial_x^\beta \partial_\tau^\alpha p_m(0, 0; 0, e_n) = 0$ for any (k, β, j, α) such that $q_0 k + \langle q, \beta \rangle < \nu - \kappa(j + |\alpha|)$. \square

If $q_0 \geq q_j$ ($j=1, \dots, n$), then $q_0(k+|\alpha|) + \langle q, \beta - \alpha \rangle < \nu - \kappa(j+|\alpha|)$ implies $q_0 k + \langle q, \beta \rangle < \nu - \kappa(j+|\alpha|)$. Hence, Remark 1.1, ii) follows from this lemma.

Appendix 2. The Existence of a Finite Propagation Speed

In this appendix, we shall give a rough sketch of the proof that there exists a finite propagation speed, as is stated in Remark 2.3, 1). First, assume that p_m is involutive and has the form of (2-2)' on \mathbf{R}^{n+1} . We have only to show the invariance of the conditions under space-like coordinate transformations. Let $\varphi(x)$ be a C^∞ -function such that $|\text{grad}_x \varphi| < [\sup\{|\lambda_j(t, x; \xi)|; (t, x) \in \mathbf{R}^{n+1}, |\xi|=1, j=1, \dots, m\}]^{-1}$. Consider a coordinate transformation T ; $s=t-\varphi(x)$, $y=x$. Let P^\sim be the operator transformed from P and put $P^\sim = \{p_m(s+\varphi(y), y; 1, -\text{grad}_x \varphi)\}^{-1} P^\sim$. It is easy to see that P^\sim is also involutive. The problem is whether P^\sim can be written in the form of (2-2)'. The main difficulty comes from the fact that $\lambda_j(t, x; D_x)$ is not necessarily a pseudo-differential operator on \mathbf{R}^{n+1} . In fact, $\lambda_j(t, x; D_x)$ is pseudo-local on \mathbf{R}^{n+1} if and only if $\lambda_j(t, x; D_x)$ is a differential operator. (We say that an operator $Q; \mathcal{E}'(\mathbf{R}^{n+1}) \rightarrow \mathcal{D}'(\mathbf{R}^{n+1})$ is pseudo-local if $\text{singsupp } Qu \subset \text{singsupp } u$ for any $u \in \mathcal{E}'(\mathbf{R}^{n+1})$.)

Let $S = \{(t, x; \tau, \xi) \in \mathbf{R}^{n+1} \times \dot{\mathbf{R}}^{n+1}; \tau = \lambda_k(t, x; \xi) \text{ for some } k\}$. Take $A_J \in S^1(\mathbf{R}^{n+1})$ whose principal symbol is $\tau - \lambda_j(t, x; \xi)$ near S . Put $A_J = \prod_{j \in J} A_j$ for $J \subset I$.

Proposition A.2. *The operator P can be written in the form of (2-2)' if and only if it can be written in the following form:*

$$P = A_I + \sum_{J \neq I} B_J A_J + R,$$

where $B_J \in S^0(\mathbf{R}^{n+1})$, $R \in S^m(\mathbf{R}^{n+1})$ and the symbol of R is rapidly decreasing near S .

The "only if" part of this proposition is almost trivial. The "if" part is proved by using the following lemma and noting that P and $A_J(t, x; D_x) \pi_J$ ($J \neq I$) are differential operators w. r. t. t .

Lemma A.3. *Let $\varphi_l(t, x; \tau, \xi) \in C^\infty(\mathbf{R}^{n+1} \times \dot{\mathbf{R}}^{n+1})$ ($l=1, 2$) be homogeneous functions of degree 0 w. r. t. (τ, ξ) such that $\varphi_1=1$ near S , $\varphi_2=1$ near $\text{supp } \varphi_1$ and $\varphi_2=0$ near $\Gamma = \{(t, x; \tau, \xi) \in \mathbf{R}^{n+1} \times \dot{\mathbf{R}}^{n+1}; \xi=0\}$. Put $\Phi_l = \varphi_l(t, x; D_t, D_x) \in S^0(\mathbf{R}^{n+1})$. Assume that $B_J(t, x; D_t, D_x) \in S^0(\mathbf{R}^{n+1})$ ($|J|=m-h$) and the principal symbol of $\sum_{|J|=m-h} B_J(t, x; D_t, D_x) \pi_J \Phi_2$ is a polynomial of τ on $\text{supp } \varphi_1$. Then, there exist $A_J(t, x; D_x) \in C^\infty([0, T]; S^0(\mathbf{R}^n))$ ($|J|=m-h$) and $B_K(t, x; D_t, D_x) \in S^0(\mathbf{R}^{n+1})$ ($|K| < m-h$) such that*

$$\begin{aligned} & \sum_{|J|=m-h} B_J(t, x; D_t, D_x) \pi_J \Phi_1 \\ &= \sum_{|J|=m-h} A_J(t, x; D_x) \pi_J \Phi_1 + \sum_{|K| < m-h} B_K(t, x; D_t, D_x) \pi_K \Phi_1. \end{aligned}$$

The latter condition in Proposition A.2 is clearly invariant under the coordinate transformation T . Thus, we have the desired result in the case of involutive operators on \mathbf{R}^{n+1} . The case of involutive operators on $\Omega^+ = [0, T] \times \mathbf{R}^n$ is easily reduced to the case of \mathbf{R}^{n+1} . The case of t -involutive operators is proved by the same arguments in the proof of Lemma 2.11 in [15]. (See also Theorem 3.3 in [6] or the last part of § 2 in [9].)

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