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Stationary Fourier Hyperprocesses

By

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Introduction

In this paper, we will define stationary Fourier hyperprocesses as an extension of stationary random functions and stationary random distributions in a similar way to Itô [6] and study their properties.

In § 1, we will first introduce some fundamental notions and prepare some notations.

In § 2, we will define the covariance Fourier hyperfunctions of stationary Fourier hyperprocesses, which correspond to Khintchine's covariance functions and Itô's covariance distributions [15], [6].

In § 3, we will prove the spectral decomposition theorem of covariance Fourier hyperfunctions.

In § 4, we will prove the spectral decomposition theorem of stationary Fourier hyperprocesses.

In § 5, we will mention the derivatives of stationary Fourier hyperprocesses.

§ 1. Fundamental Notions and Notations

In this paper we will restrict ourselves to complex valued random variables with mean 0 and finite variance. Let H be the Hilbert space constituted by all such variables. In H, we define the inner product by the following relation:

$$(X, Y)=E(X\cdot \overline{Y}), \qquad \text{for} \quad X, Y\in H,$$

where E denotes the expectation. We will here consider only the strong topology on H. A continuous random process $X(t)$, $-\infty<t<\infty$, is an H-valued continuous function on $\boldsymbol{R}=(-\infty, \infty)$. The set of all continuous processes will be denoted by $C(H)$.

Now we will remember the notions of Fourier hyperfunctions and vector valued Fourier hyperfunctions following Sato [17], Kawai [13], [14], Ito and

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Nagamachi [9], [10], Junker [11], [12], Ito [7], [8]. Let $\boldsymbol{D}=[-\infty, \infty]$ be a directional compactification of $\boldsymbol{R}=(-\infty, \infty)$. Let $\underset{\sim}{\mathcal{A}}$ be the sheaf of germs of rapidly decreasing real analytic functions over $\boldsymbol{D}$. Let $\underset{\sim}{A}=\underset{\sim}{\mathcal{A}}(\boldsymbol{D})$ be the space of all sections of $\underset{\sim}{\mathcal{A}}$ on $\boldsymbol{D}$. $\underset{\sim}{A}$ is endowed with the usual DFS topology. A Fourier analytic functional defined on $\underset{\sim}{A}$ is called a Fourier hyperfunction on $\boldsymbol{D}$ and a Fourier analytic linear mapping from $\underset{\sim}{A}$ to H is called an H-valued Fourier hyperfunction on $\boldsymbol{D}$. We will denote by $\underset{\sim}{A}'$ the space of all Fourier hyperfunctions on $\boldsymbol{D}$ and by $\underset{\sim}{A}'(H)$ the space of all H-valued Fourier hyperfunctions on $\boldsymbol{D}$.

Definition 1.1. A Fourier hyperprocess is defined to be an H-valued Fourier hyperfunction.

Remark. Our concept of Fourier hyperprocesses is a generalization of Okabe's concept of hyperprocesses in [18], Definition 6.1.

Let $\tilde{C}(H)$ be the set of all H-valued continuous functions on $\boldsymbol{R}$ which satisfy the following estimate:

$$\text{for any } \varepsilon>0, \ \sup\{\|X(t)\|e^{-\varepsilon|t|};\ t\in\boldsymbol{R}\}<\infty,$$

where $\|\ \|$ denotes the norm on H. An element of $\tilde{C}(H)$ is called a slowly increasing continuous process. Then $\tilde{C}(H)$ may be considered as a subsystem of $A'(H)$, since we can identify a slowly increasing continuous random process $X(t)$ with the following Fourier hyperprocess $X(\phi)$ determined by it:

$$X(\phi)=\int X(t)\phi(t)dt\equiv\int_{-\infty}^{\infty} X(t)\phi(t)dt, \qquad \text{for} \quad \phi\in\underset{\sim}{A}.$$

The following notations will be often used in the theory of Fourier hyperfunctions. Let $F\in\underset{\sim}{A}'$ or $\underset{\sim}{A}'(H)$ and $\phi\in\underset{\sim}{A}$.

τ_h (shift transformation): $\tau_h\phi(t)=\phi(t+h)$, $\tau_h F(\phi)=F(\tau_{-h}\phi)$.

D (derivative): $D\phi(t)=\phi'(t)$, $DF(\phi)=-F(D\phi)$.

$\check{}$ (inversion): $\check{\phi}(t)=\phi(-t)$, $\check{F}(\phi)=F(\check{\phi})$.

$\bar{}$ (conjugate): $\bar{\phi}(t)=\overline{\phi(t)}$, $\bar{F}(\phi)=\overline{F(\bar{\phi})}$.

$\tilde{}$ ($=\check{\bar{}}=\bar{\check{}}$): $\tilde{\phi}(t)=\overline{\phi(-t)}$, $\tilde{F}(\phi)=\overline{F(\tilde{\phi})}$.

$\hat{}$ (Fourier transformation): $\hat{\phi}(\lambda)=\int e^{-i2\pi\lambda t}\phi(t)dt$, $\hat{F}(\phi)=F(\hat{\phi})$.

The following relation should be noted.

$$(F*\phi)(0)=F(\check{\phi})=\check{F}(\phi), \quad (\phi*\psi)\hat{}=\hat{\phi}\cdot\hat{\psi},\ \hat{\bar{\phi}}=\bar{\hat{\check{\phi}}}.$$

Generalizing Khintchine-Itô's notions of (weakly) stationary processes, we have the following

Definition 1.2. We will call $X\in \underset{\sim}{A}'(H)$ weakly stationary or merely stationary for short if we have, for any $\phi, \psi\in \underset{\sim}{A}$,

$$(\tau_h X(\phi), \tau_h X(\psi))=(X(\phi), X(\psi))$$

and strictly stationary if the joint probability law of

$$(\tau_h X(\phi_1), \cdots, \tau_h X(\phi_n))$$

is independent of h for any n and $\phi_1, \cdots, \phi_n\in \underset{\sim}{A}$.

We shall adopt here the following notations:

S: the totality of stationary Fourier hyperprocesses,
S^0: the totality of slowly increasing stationary processes,
$\bar{S}$: the totality of strictly stationary Fourier hyperprocesses,
$\bar{S}^0$: the totality of slowly increasing strictly stationary processes.

Clearly we have

$$S\supset \bar{S}\cup S^0, \quad S^0\supset \bar{S}^0.$$

Definition 1.3. A Fourier hyperprocess X is called a complex normal Fourier hyperprocess if $X(\phi)$, $\phi\in \underset{\sim}{A}$, constitute a complex normal system and a real normal Fourier hyperprocess if X is real viz. $X=\bar{X}$ and $X(\phi)$, ϕ running over real functions in $\underset{\sim}{A}$, constitute a (real) normal system (see Itô [4], [5] and Hida [3]).

This is an extension of normal processes or Gaussian processes (Doob [1], II, § 3) and complex (or real) normal random distributions (Itô [6]). A (complex as well as real) normal Fourier hyperprocess will be strictly stationary, if it is weakly stationary. The corresponding fact is well-known regarding stationary processes.

We shall here mention a typical example of real stationary Fourier hyperprocesses which are not stationary processes. Let $B(t)$ be a (real) Brownian motion process (Doob [1], p. 97). The derivative (in the sense of Fourier hyperfunctions) of this process $B'\equiv DB$ is a Fourier hyperprocess defined by

$$B'(\phi)=-B(\phi')=\int\phi(t)dB(t)$$

(Wiener integral, see Itô [4]). This is evidently real normal and stationary, since

$$\begin{aligned}(\tau_h B'(\phi), \tau_h B'(\psi))&=(B'(\tau_{-h}\phi), B'(\tau_{-h}\psi))\\ &=\left(\int\phi(t-h)dB(t), \int\psi(t-h)dB(t)\right)\\ &=\int\phi(t-h)\overline{\psi(t-h)}dt=\int\phi(t)\overline{\psi(t)}dt\end{aligned}$$

which shows that $B' \in S$. The fact that $B' \notin S^0$ will be proved in § 2.

§ 2. Covariance Fourier Hyperfunctions

Similarly to Khintchine-Itô's notion of covariance distributions, we will here define the notion of the covariance Fourier hyperfunctions.

Theorem 2.1. *Let $X(\phi)$ be any stationary Fourier hyperprocess. Then there exists one and only one Fourier hyperfunction $\rho \in \underset{\sim}{A}'$ satisfying the relation*

$$(X(\phi), X(\psi)) = \rho(\phi * \tilde{\bar{\psi}}), \qquad \phi, \psi \in A.$$

Definition 2.2. The Fourier hyperfunction ρ in Theorem 2.1 is called the covariance Fourier hyperfunction of X.

Proof of Theorem 2.1. If we put

$$T_\phi(\psi) = (X(\phi), X(\bar{\psi})), \qquad \phi, \psi \in \underset{\sim}{A},$$

then we get a Fourier hyperfunction $T_\phi \in \underset{\sim}{A}'$ for each $\phi \in \underset{\sim}{A}$. Taking into account the fact that $T_\phi(\psi)$ is continuous in $(\phi, \psi) \in \underset{\sim}{A} \times \underset{\sim}{A}$ and by virtue of Kernel Theorem, we will easily see that $\phi \to T_\phi$ is a continuous linear mapping from $\underset{\sim}{A}$ into $\underset{\sim}{A}'$ (see Grothendieck [2], Chap. II, Théorèm 12 and Ito [7]). Furthermore this transformation commutes with the shift transformation:

$$\begin{aligned}(\tau_h T_\phi)(\psi) &= T_\phi(\tau_{-h}\psi) = (X(\phi), X(\overline{\tau_{-h}\psi})) \\ &= (X(\phi), X(\tau_{-h}\bar{\psi})) = (X(\tau_h\phi), X(\bar{\psi})) \\ &= T_{\tau_h\phi}(\psi).\end{aligned}$$

Here we will use the following

Lemma. *A continuous linear mapping $\phi \to T_\phi$ from $\underset{\sim}{A}$ to $\underset{\sim}{A}'$ commutes with the shift transformation if and only if there exists a Fourier hyperfunction T such that $T_\phi = T * \phi$ holds.*

Postponing the proof of this Lemma until the end of the proof of this Theorem, we will continue the proof of the Theorem. Thus by the above Lemma T_ϕ is expressible as a convolution of a Fourier hyperfunction T and ϕ:

$$T_\phi = T * \phi.$$

Hence it follows that

$$\begin{aligned}(X(\phi), X(\psi)) &= T_\phi(\bar{\psi}) = (T * \phi)(\bar{\psi}) \\ &= (T * \phi * \tilde{\bar{\psi}})(0) = \rho(\phi * \tilde{\bar{\psi}}),\end{aligned}$$

where we put $\rho = \check{T}$.

The uniqueness of ρ follows at once from the fact that the set of all elements of the form $\phi * \psi$, $\phi, \psi \in \underset{\sim}{A}$, is dense in $\underset{\sim}{A}$. Q. E. D.

Proof of Lemma. It is evident that a continuous linear mapping $\phi \to T_\phi = T * \phi$ commutes with the shift transformation. Thus we have only to prove that T_ϕ is expressible as a convolution of a Fourier hyperfunction T and $\phi: T_\phi = T * \phi$ if it commutes with the shift transformation.

We will first show that, if $\alpha \in \underset{\sim}{A}$, we have

$$T_{\alpha * \phi} = T_\alpha * \phi .$$

In fact, we have, by the definition of integration,

$$\begin{aligned}\langle \phi, \psi \rangle &= \int \phi(t) \psi(t) dt \\ &= \lim_j \sum_n a_{nj} \psi(h_{nj}) \\ &= \langle \lim_j \sum_n a_{nj} \tau_{h_{nj}} \delta, \psi \rangle\end{aligned}$$

for any $\psi \in \underset{\sim}{A}$ considering ϕ as a Fourier hyperfunction, where $\sum_n$'s are finite sums. Thus we have the relation

$$\phi = \lim_j \sum_n a_{nj} \tau_{h_{nj}} \delta$$

as a Fourier hyperfunction. Therefore we have

$$\begin{aligned}\alpha * \phi(t) &= (\lim_j \sum_n a_{nj} \tau_{h_{nj}} \delta) * \alpha(t) \\ &= \lim_j \sum_n a_{nj} \tau_{h_{nj}} \alpha(t) .\end{aligned}$$

Hence we have, by the assumption,

$$\begin{aligned}T_{\alpha * \phi} &= \lim_j \sum_n a_{nj} \tau_{h_{nj}} T_\alpha \\ &= (\lim_j \sum_n a_{nj} \tau_{h_{nj}} \delta) * T_\alpha \\ &= \phi * T_\alpha = T_\alpha * \phi .\end{aligned}$$

Then if we choose a sequence $\alpha_\nu \in \underset{\sim}{A}$ which converges to δ in $\underset{\sim}{A}'$, the sequence $\alpha_\nu * \phi$ converges to ϕ in $\underset{\sim}{A}$. Hence $T_{\alpha_\nu * \phi}$ converges to T_ϕ in $\underset{\sim}{A}'$. Hence, for the Fourier hyperfunctions T_{α_ν}, their regularizations $T_{\alpha_\nu} * \phi$ converges in $\underset{\sim}{A}'$ for every $\phi \in \underset{\sim}{A}$. Thus T_{α_ν} itself converges in $\underset{\sim}{A}'$. If T is its limit, we have

$$T_\phi = T * \phi , \qquad T \in \underset{\sim}{A}'. \qquad \text{Q. E. D.}$$

Theorem 2.3. *If $X(\phi)$ is a real stationary Fourier hyperprocess, then its covariance Fourier hyperfunction is real, i.e. $\rho = \bar{\rho}$.*

Proof. Let ρ be the covariance Fourier hyperfunction. Then that of $\bar{X}$

will become $\bar{\rho}$, since we have

$$\begin{aligned}(\bar{X}(\phi),\ \bar{X}(\psi)) &= (\overline{X(\bar{\phi})},\ \overline{X(\bar{\psi})}) = \overline{(X(\bar{\phi}),\ X(\bar{\psi}))} \\ &= \overline{\rho(\bar{\phi} * \tilde{\bar{\psi}})} = \overline{\rho(\overline{\phi * \tilde{\psi}})} \\ &= \bar{\rho}(\phi * \tilde{\psi})\,.\end{aligned}$$

Thus $X=\bar{X}$ implies $\rho=\bar{\rho}$. Q. E. D.

Example. The covariance Fourier hyperfunction of B' is Dirac's δ-function, since

$$\begin{aligned}(B'(\phi),\ B'(\psi)) &= \int \phi(t)\overline{\psi(t)}\,dt = \int \phi(t)\tilde{\psi}(-t)\,dt \\ &= (\phi * \tilde{\psi})(0) = \delta(\phi * \tilde{\psi})\,.\end{aligned}$$

Thus we see that $B' \notin S^0$, because, if $B' \in S^0$, then the covariance Fourier hyperfunction would be induced by a slowly increasing continuous function as shown similarly to Itô [6], § 2.

§ 3. Spectral Decomposition of Covariance Fourier Hyperfunctions

Let $X(\phi)$ be any stationary Fourier hyperprocess with the covariance Fourier hyperfunction ρ. Then we have

$$\rho(\phi * \tilde{\phi}) = (X(\phi),\ X(\phi)) \geqq 0\,,$$

which implies that ρ is a positive semidefinite Fourier hyperfunction. Thus, by virtue of Bochner-Nagamachi-Mugibayashi-Junker's Theorem (see Nagamachi-Mugibayashi [16], Theorem 4.1 and Junker [12], Theorem 5.8), we have the following

Theorem 3.1. *ρ is expressible in the form*

$$(*) \qquad \rho(\phi) = \int \hat{\phi}(\lambda)\,d\mu(\lambda)\,, \qquad \phi \in \underset{\sim}{A}$$

in one and only one way, where μ is a nonnegative measure satisfying

$$\int e^{-\varepsilon|\lambda|}\,d\mu(\lambda) < \infty$$

for every $\varepsilon > 0$.

Definition 3.2. We will call the expression $(*)$ the spectral decomposition of ρ and μ the spectral measure of ρ.

Conversely we have

Theorem 3.3. *Any Fourier hyperfunction of the above form $(*)$ is the* *covariance Fourier hyperfunction of a stationary Fourier hyperprocess which is complex normal.*

Proof. Let ρ be a Fourier hyperfunction of the above form. Put

$$\Gamma(\phi, \psi)=\rho(\phi * \tilde{\bar{\psi}}), \qquad \phi, \psi \in \underset{\sim}{A}.$$

Then $\Gamma(\phi, \psi)$ is positive semidefinite in (ϕ, ψ), as we have

$$\sum_{i,j=1}^{n} \Gamma(\phi_i, \phi_j)\xi_i\bar{\xi}_j=\rho(\theta * \tilde{\bar{\theta}})\geqq 0, \qquad \theta=\sum_i \xi_i\phi_i.$$

Therefore we can define a complex normal system $X(\phi)$, $\phi \in \underset{\sim}{A}$, such that $EX(\phi)=0$ and $E(X(\phi)\cdot\overline{X(\psi)})=\Gamma(\phi, \psi)=\rho(\phi * \tilde{\bar{\psi}})$ as in Itô [5] and Hida [3]. It remains only to show that $X(\phi)$ is a Fourier hyperprocess. From the identity:

$$\begin{aligned}\|X(c\phi)-cX(\phi)\|^2&=(X(c\phi), X(c\phi))-c(X(\phi), X(c\phi))\\&\quad-\bar{c}(X(c\phi), X(\phi))+c\bar{c}(X(\phi), X(\phi))\\&=\rho(c\phi * \widetilde{\overline{c\phi}})-c\rho(\phi * \widetilde{\overline{c\phi}})-\bar{c}\rho(c\phi * \tilde{\bar{\phi}})+c\bar{c}\rho(\phi * \tilde{\bar{\phi}})\\&=c\bar{c}\rho(\phi * \tilde{\bar{\phi}})-c\bar{c}\rho(\phi * \tilde{\bar{\phi}})-c\bar{c}\rho(\phi * \tilde{\bar{\phi}})+c\bar{c}\rho(\phi * \tilde{\bar{\phi}})\\&=0,\end{aligned}$$

it follows that $X(c\phi)=cX(\phi)$. By a similar way we can see that $X(\phi+\psi)=X(\phi)+X(\psi)$. Therefore $X(\phi)$ is linear in ϕ. By the identity $\|X(\phi)\|^2=\rho(\phi * \tilde{\bar{\phi}})$ we obtain the continuity of X. Thus our theorem is completely proved.

Q. E. D.

Next we shall discuss the case of real stationary Fourier hyperprocesses. By Theorem 2.3 we see that $\rho=\bar{\rho}$ in this case. But we have

$$\bar{\rho}(\phi)=\overline{\rho(\bar{\phi})}=\int\bar{\hat{\bar{\phi}}}(\lambda)d\mu(\lambda)=\int\hat{\phi}(-\lambda)d\mu(\lambda)=\int\hat{\phi}(\lambda)d\check{\mu}(\lambda)$$

$$(\check{\mu}(E)=\mu(-E),\ -E=\{t\,;\,-t\in E\}).$$

By the uniqueness of the spectral measure we will obtain the following

Theorem 3.4. *In the case of a real stationary Fourier hyperprocess, the spectral measure μ is symmetric with respect to 0, viz. $\mu(E)=\mu(-E)$.*

Conversely we have

Theorem 3.5. *Any Fourier hyperfunction of the form (*) with a symmetric measure μ is the covariance Fourier hyperfunction of a certain stationary Fourier hyperprocess which is real normal.*

The proof is similar to that of Theorem 3.3; we use the existence theorem of real normal systems instead of complex normal ones.

Example. B' is a real stationary Fourier hyperprocess whose spectral measure is the ordinary Lebesgue measure, because we have

$$\delta(\phi)=\phi(0)=\int\hat{\phi}(\lambda)d\lambda.$$

§4. Spectral Decomposition of Stationary Fourier Hyperprocesses

We will first introduce a random hypomeasure. Let μ be a nonnegative measure defined for all Borel sets in $\boldsymbol{R}$ and $\boldsymbol{B}^*$ denote the system of all Borel sets with finite μ-measure.

Definition 4.1. An H-valued function $M(E)$ defined for $E\in\boldsymbol{B}^*$ is called a random hypomeasure with respect to μ if

$$(M(E_1),\ M(E_2))=\mu(E_1\cap E_2), \qquad E_1,\ E_2\in\boldsymbol{B}^*$$

holds.

We get, by the definition,

Theorem 4.2. *Let $M(E)$ be a random hypomeasure with respect to μ. Then we have*

(1) $\|M(E)\|^2=\mu(E)$,

(2) $M(E_1)\perp M(E_2)$ *if* $E_1\cap E_2=\emptyset$,

(3) *If* $E_1,\ E_2,\ \cdots$ *are disjoint to each other and belong to* $\boldsymbol{B}^*$ *with their sum* $E=\sum_{n=1}^{\infty}E_n$, $M(E)=\sum_{n=1}^{\infty}M(E_n)$, *(in H).*

We can easily define the integral with respect to the random hypomeasure (Doob [1], IX, §2):

$$M(f)=\int f(\lambda)dM(\lambda)$$

for $f\in L^2(\boldsymbol{R},\ \mu)$.

Then we have the following

Theorem 4.3. *Let $M(f)$ be as above. Then we have, for $f_1, f_2\in L^2(\boldsymbol{R},\ \mu)$ and $c_1,\ c_2\in\boldsymbol{C}$,*

(1) $(M(f_1),\ M(f_2))=(f_1,\ f_2)\ \left(\equiv\int f_1(\lambda)\overline{f_2(\lambda)}d\mu(\lambda)\right)$,

(2) $M(c_1f_1+c_2f_2)=c_1M(f_1)+c_2M(f_2)$.

Theorem 4.4. *Let X be any stationary Fourier hyperprocess with the spectral measure μ. Then $X(\phi)$ will be expressible in the form*

$$X(\phi)=\int\hat{\phi}(\lambda)dM(\lambda)=M(\hat{\phi}) \tag{**}$$

in one and only one way, M being a random hypomeasure with respect to μ. Conversely, any Fourier hyperprocess of such form is a stationary Fourier hyperprocess.

Definition 4.5. We will call the expression (**) the spectral decomposition of X and M the spectral hypomeasure of X.

Proof of Theorem 4.4. We will first remark that $\underset{\sim}{A}$ is dense in $L^2 \equiv L^2(R, \mu)$. Then the Fourier transformation is a topological isomorphism from $\underset{\sim}{A}$ onto itself. Thus the uniqueness of the expression is clear.

In order to prove the possibility of the expression, we will put

$$T(\psi)=X(\phi) \qquad \text{for} \quad \psi=\hat{\phi}\,.$$

Then T will be a mapping from $\underset{\sim}{A}$ $(\subset L^2)$ into H, which is clearly linear and isometric on account of the identity:

$$\begin{aligned}\|T(\psi)\|^2&=(X(\phi),\ X(\phi))=\rho(\phi * \tilde{\phi})\\ &=\int |\psi(\lambda)|^2 d\mu(\lambda)=\|\psi\|^2,\end{aligned}$$

since $(\phi * \tilde{\phi})^{\wedge}=|\hat{\phi}|^2$. $\underset{\sim}{A}$ being dense in L^2, we can extend $T(\psi)$ to a linear isometric mapping from L^2 into H. As the characteristic function $\chi_E(\lambda)$ of a set $E \in \boldsymbol{B}^*$ belongs to L^2, we may define $M(E)$ as follows:

$$M(E)=T(\chi_E)\,.$$

Then we have

$$(M(E_1),\ M(E_2))=\int \chi_{E_1}(\lambda)\overline{\chi_{E_2}(\lambda)}\,d\mu(\lambda)=\mu(E_1 \cap E_2)\,,$$

since T is isometric. In addition to this, we will have

$$(***) \qquad M(f)=T(f) \qquad \text{for} \quad f \in L^2,$$

for this is evidently true for any simple function f in L^2 by the definition and we will easily see that it is also true for any $f \in L^2$, by taking into account the fact that both sides of (***) are isometric in f and any $f \in L^2$ is expressed as the L^2-limit of a sequence of simple functions. If we put $f=\hat{\phi}$ in (***), we obtain (**) at once. The last part of the theorem is clear by the definitions.

Q. E. D.

Making use of this theorem we can characterize the class of slowly increasing stationary processes.

Theorem 4.6. *A slowly increasing stationary process X is a stationary Fourier hyperprocess with the spectral measure such that*

$$\int d\mu(\lambda) < \infty .$$

Proof. A slowly increasing stationary process X is a stationary continuous random process. By Khinchine's Theorem (see Khinchine [15]), it spectral measure μ satisfies the assumption of the theorem and, by virtue of Doob [1], XI, § 4, we have the expression

$$X(\phi) = \int \hat{\phi}(\lambda) dM(\lambda) = M(\hat{\phi}) .$$

Thus, by Theorem 4.4, X induces a stationary Fourier hyperprocess with the spectral measure μ.

Conversely, let X be a stationary Fourier hyperprocess with the spectral measure μ which satisfies the assumption of the theorem. Then we have

$$X(\phi) = \int \hat{\phi}(\lambda) dM(\lambda) , \qquad (M(E_1), M(E_2)) = \mu(E_1 \cap E_2) ,$$

where

$$\int d\mu(\lambda) < \infty .$$

Put

$$Y(t) = \int e^{-i2\pi\lambda t} dM(\lambda) ,$$

which may be defined, since the λ-function $e^{-i2\pi\lambda t}$ belongs to L^2 by virtue of the assumption on μ, $Y(t)$ proves to be a stationary continuous random process and, what is more, it becomes a slowly increasing stationary continuous random process. Therefore, we have, for $\phi \in \underset{\sim}{A}$,

$$\int Y(t)\phi(t) dt = \int \phi(t) \int e^{-i2\pi\lambda t} dM(\lambda) dt$$

$$= \int \hat{\phi}(\lambda) dM(\lambda) = X(\phi) ,$$

which implies that $X(\phi)$ is induced by a slowly increasing stationary process Y.

Q. E. D.

In the proof of the above theorem, we have the following

Corollary. *A slowly increasing stationary continuous random process and a stationary continuous random process are identical.*

By the same way as in Theorem 3.4, we will obtain

Theorem 4.7. *In the case of a real stationary Fourier hyperprocess the spectral random hypomeasure M is hermitian symmetric, i.e. $M(E) = \overline{M(-E)}$.*

§ 5. Derivatives of Stationary Fourier Hyperprocesses

Any Fourier hyperprocess has derivatives of any order, which are also Fourier hyperprocesses.

Theorem 5.1. *Let X be a stationary Fourier hyperprocess with the spectral measure μ and the spectral random hypomeasure M. Then $X^{(k)}$ $(=D^k X)$ is also a stationary Fourier hyperprocess whose spectral measure μ_k and spectral random hypomeasure M_k are given by*

$$d\mu_k(\lambda)=(2\pi\lambda)^{2k}d\mu(\lambda), \quad dM_k(\lambda)=(i2\pi\lambda)^k dM(\lambda).$$

Proof. We have, by definition,

$$\begin{aligned} X^{(k)}(\phi)&=(-1)^k X(\phi^{(k)})=(-1)^k\int\hat{\phi}^{(k)}(\lambda)dM(\lambda) \\ &=\int(i2\pi\lambda)^k\phi(\lambda)dM(\lambda), \end{aligned}$$

since we have, for $\phi\in\underset{\sim}{A}$,

$$\hat{\phi}^{(k)}(\lambda)=(-1)^k(i2\pi\lambda)^k\phi(\lambda).$$

Thus $X^{(k)}$ proves to be a stationary Fourier hyperprocess satisfying the above conditions. Q. E. D.

By Theorem 4.6 we have the following

Theorem 5.2. *In order that $X^{(k)}$ is a stationary continuous process, it is necessary and sufficient that the spectral measure μ of X satisfies*

$$\int\lambda^{2k}d\mu(\lambda)<\infty.$$

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