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On the Spaces of Self Homotopy Equivalences for Fibre Spaces II

By

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Introduction

Let X be a connected CW complex with non-degenerate base point x_0. And let $G_0(X)$ be the space of self homotopy equivalences of (X, x_0).

The purpose of this paper is to study $G_0(E)$ when E is a fibre space of a fibration with fibre $K(G, n)$ $(n>1)$:

$$K(G, n) \xrightarrow{i} E \xrightarrow{p} B.$$

If a base space B is simply connected, we had some results on $G_0(E)$ in the previous papers [16, 17, 18, 19]. Here we treat $G_0(E)$ for the case of a non-simply connected base space B.

Let G be an abelian group and let $\mathrm{Aut}(G)$ be its group of automorphisms. Denote by $L(G, n+1)$ the classifying space for fibrations with fibre $K(G, n)$ and by W an Eilenberg-MacLane complex $K(\mathrm{Aut}(G), 1)$. Then we have the fibration:

$$K(G, n+1) \xrightarrow{i_0} L(G, n+1) \xrightarrow{p_0} W.$$

Under these notations our main results (Theorem 3.3, 4.4 and 4.7) are stated as follows.

Theorem 3.3. *Let X be a CW complex, k be a fixed map of (X, x_0) to $(L(G, n+1), l_0)$ and $p_0\circ k=k':(X, x_0)\to(W, w_0)$ be a space over (W, w_0). Then the space $\mathrm{map}_0(X, L(G, n+1))_W$ of maps over (W, w_0) has the same weak homotopy type as*

$$H^{n+1}(X, x_0; G)\times \prod_{i=1}^{n} K(H^{n+1-i}(X, x_0; G), i)$$

where the cohomology is taken with local coefficients classified by the map $k':X\to W=K(\mathrm{Aut}(G), 1)$.

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Denote by $\mathcal{G}\ (E \bmod F)$ the space of self fibre homotopy equivalences of E leaving a fibre F fixed in a fibration: $F \xrightarrow{i} E \xrightarrow{p} B$. We denote by $X \underset{w}{\simeq} Y$ when X has the same weak homotopy type as Y. Then, by using the result proved in [18, 19] we have

Theorem 4.4. *Let $p: E \to B$ be a fibration with fibre $F=K(G, n)$ $(n>1)$ such that B is a CW complex. Then if we denote by $k:(B, b_0) \to (L(G, n+1), l_0)$ a corresponding map to the fibration: $F \xrightarrow{i} E \xrightarrow{p} B$, we have*

$$\mathcal{G}\,(E \bmod F) \underset{w}{\simeq} \mathrm{map}_0(B,\ L(G,\ n))_W\,.$$

Let $\varepsilon(X)$ denote the group $\pi_0(G_0(X))$ for a CW complex X. Then we have the following theorem which is a generalization of Theorem 10 in [18].

Theorem 4.7. *For a given $1 \leqq m < n$, let*

$$F=K(G,\ n) \xrightarrow{i} E \xrightarrow{p} K(\pi,\ m)=B$$

be a fibration with a corresponding map $k:(B, b_0) \to (L(G, n+1), l_0)$. Then we have

$$G_0(E) \underset{w}{\simeq} R \times H^n(B,\ G) \times \prod_{i=1}^{n-1} K(H^{n-i}(B,\ b_0\,;\ G),\ i)$$

where R is the subgroup of $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G)=\varepsilon(B) \times \varepsilon(F)$ consisting of $([g], [h])$ with

$$[\chi_\infty(h)] \circ [k]=[k] \circ [g]\,,$$

and the cohomology is taken with local coefficients classified by the map $p_0 \circ k$: $B \to K(\mathrm{Aut}(G), 1)=W$.

Thus as a corollary of Theorem 4.7 we have the following theorem [9, 11, 15].

Theorem 4.8. *Under the same hypothesis of Theorem 4.7 there exists the following exact sequence*

$$1 \longrightarrow H^n(B,\ G) \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1\,,$$

where R is the same group as the group stated in Theorem 4.7 and the cohomology is taken with local coefficients classified by the map $p_0 \circ k: B \to K(\mathrm{Aut}(G), 1)$.

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§ 1. Fibrations

Throughout this paper, we shall work within the category of compactly generated Hausdorff spaces [13] and by a base point we mean a non-degenerate base point.

Let X and Y be spaces with base points x_0 and y_0 respectively. The space of maps of X to Y will be denoted by $\mathrm{map}(X, Y)$ and $\mathrm{map}_0(X, Y)$ will be the subspace of $\mathrm{map}(X, Y)$ of maps of (X, x_0) to (Y, y_0). Moreover, when k is a map of X to Y, we denote by $\mathrm{map}(X, Y; k)$ the path component of k in $\mathrm{map}(X, Y)$, and $\mathrm{map}_0(X, Y; k)$ is defined similarly.

Furthermore, throughout this paper a CW complex means a connected CW complex with base point, unless otherwise stated.

Let $k:(X, x_0)\to(B, b_0)$ and $k':(Y, y_0)\to(B, b_0)$ be spaces over (B, b_0), then we denote by $\mathrm{map}_0(X, Y)_B$ the subspace of $\mathrm{map}_0(X, Y)$ of maps over (B, b_0) of k to k'. That is, each element f of $\mathrm{map}_0(X, Y)_B$ satisfies $k'\circ f=k$,

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{\ f\ } & (Y, y_0) \\ & {\scriptstyle k}\searrow \quad \swarrow{\scriptstyle k'} & \\ & (B, b_0) & \end{array}$$

Let $p:E\to B$ be a space over B and let X be a space. We denote by E_B^X the space of maps each of which is a map of X to E such that its composition with p is a constant map of X to B. Then the following diagram is commutative:

$$\begin{array}{ccc} E_B^X & \xrightarrow{\ i\ } & \mathrm{map}\,(X, E) \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p_\#} \\ B & \xrightarrow{\ c\ } & \mathrm{map}\,(X, B) \end{array}$$

where $p_\#:\mathrm{map}(X, E)\to\mathrm{map}(X, B)$ is the map induced by p, $p':E_B^X\to B$ is defined by $p'(f)=p\circ f(x)$, c is a map defined by

$$c(b)(x)=b \qquad (b\in B,\ x\in X)$$

and i is the inclusion map.

Let $p:E\to B$ be a map and let X be a space in the category. Then we say that p is a fibration, if and only if it has the homotopy lifting property with respect to every X. Thus our fibration $p:E\to B$ is not necessary surjective.

Note that $p':E_B^X\to B$ is a fibration if $p:E\to B$ is a fibration. Let X be a space with base point x_0. Define a map $\omega:\mathrm{map}(X, E)\to E$ by $\omega(f)=f(x_0)$. The restriction of ω on E_B^X will be denoted by the same ω, then we have the following

Proposition 1.1. *With the above notations,* $\omega: E_B^X \to E$ *is a fibration and the following diagram is commutative*:

$$\begin{array}{ccc} E_B^X & \xrightarrow{\omega} & E \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

Proof. Define a map $\bar{p}: \mathrm{map}(X, E) \to E \times \mathrm{map}(X, B)$ by

$$\bar{p}(f) = (\omega(f),\ p_{\#}(f)).$$

Then we can easily see that the following diagram is commutative:

$$\begin{array}{ccc} E_B^X & \xrightarrow{\ i\ } & \mathrm{map}\,(X, E) \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \bar{p}} \\ E & \xrightarrow{\ \bar{c}\ } & E \times \mathrm{map}\,(X, B) \end{array}$$

where $\bar{c}: E \to E \times \mathrm{map}(X, B)$ is defined by

$$\bar{c}(e) = (e,\ c_{p(e)}),$$
$$c_{p(e)}(x) = p(e) \qquad (e \in E,\ x \in X).$$

Since $\bar{p}: \mathrm{map}(X, E) \to E \times \mathrm{map}(X, B)$ is a fibration (see Theorem 10 in [14]) and $\bar{c}$ is injective, we see that $\omega: E_B^X \to E$ is a fibration.

The equality $p \circ \omega = p'$ follows immediately from the definition ω, p and p'.

Remark 1.2. In Proposition 1.1, when a fibration $p: E \to B$ has a cross-section $s: B \to E$, the fibration $p_{\#}: \mathrm{map}(X, E) \to \mathrm{map}(X, B)$ has also a cross-section $s_{\#}: \mathrm{map}(X, B) \to \mathrm{map}(X, E)$. Thus, since $p': E_B^X \to B$ is a pullback of the fibration $p_{\#}$, the fibration p' has a cross-section $s': B \to E_B^X$ defined by

$$s'(b)(x) = s(b) \qquad (b \in B,\ x \in X),$$

and the following diagram is commutative:

$$\begin{array}{ccc} & B & \\ & {\scriptstyle s'}\swarrow \quad \searrow{\scriptstyle s} & \\ E_B^X & \xrightarrow{\omega} & E \end{array}$$

We need the following

Proposition 1.3. *Let* $p: (E', e_0') \to (E, e_0)$ *and* $k: (E, e_0) \to (B, b_0)$ *be fibrations. Put* $k' = k \circ p$. *Moreover, let* $s: (B, b_0) \to (E, e_0)$ *and* $s': (B, b_0) \to (E', e_0')$ *be cross-sections with* $p \circ s' = s$ *for the fibrations* k *and* k' *respectively. When* k'' *is a map of* (X, x_0) *to* (B, b_0), *we have the following fibration*

$$p_{\#}: \mathrm{map}_0(X, E')_B \to \mathrm{map}_0(X, E)_B .$$

Proof can be done easily, so it is omitted.

§2. Fibrations with Fibre $K(G, n)$

Let $p: E \to B$ be a fibration with fibre $K(G, n)$ (G is an abelian group) over a CW complex B. In the following, we denote by B_∞ the classifying space for fibrations with fibre $K(G, n)$ and we shall investigate the loop space $\Omega\,\mathrm{map}_0(B, B_\infty; k)$ [18] of $\mathrm{map}_0(B, B_\infty; k)$, where k is the classifying map of the above fibration.

For this purpose we prove the following

Theorem 2.1. *Let X be a CW complex and let π be an arbitrary group, then every path component of* $\mathrm{map}_0(X, K(\pi, 1))$ *is weakly contractible.*

Proof. First we shall show $\pi_i(\mathrm{map}_0(X, K(\pi, 1); k))=0$ for $i \geqq 2$, where k is a map of (X, x_0) to $(K(\pi, 1), y_0)$. Let $\bar{f}$ be a map of $(S^i, *)$ to $(\mathrm{map}(X, K(\pi, 1); k), k)$. Then we have its associated map $f: S^i \times X \to K(\pi, 1)$ with $f | * \times X = k$. The map $f: (S^i \times X, * \times x_0) \to (K(\pi, 1), y_0)$ induces the homomorphism f_*:

$$\pi_1(S^i \times X) \cong \pi_1(X) \to \pi_1(K(\pi, 1)) \cong \pi$$

which is the same as the homomorphism $k_*: \pi_1(X) \to \pi$ induced by the map k.

Let c be a map of $S^i \times X$ to $K(\pi, 1)$ defined by

$$c(y, x) = k(x) \qquad (x \in X,\ y \in S^i) .$$

Then obviously c induces the homomorphism $c_*: \pi_1(S^i \times X) \cong \pi_1(X) \to \pi$ which may be regarded as the homomorphism $k_*: \pi_1(X) \to \pi$. Therefore f and c are homotopic relative to $(*, x_0)$ [20]. This means that every map of S^i to $\mathrm{map}(X, K(\pi, 1); k)$ is freely homotopic to the constant map $\bar{c}$ defined by $\bar{c}(y) = k$ for all $y \in S^i$. Therefore we have $\pi_i(\mathrm{map}(X, K(\pi, 1); k)) = 0$ for $i \geqq 2$.

Now, let ω be a map of $\mathrm{map}(X, K(\pi, 1); k)$ to $K(\pi, 1)$ defined by

$$\omega(f) = f(x_0) \qquad (f \in \mathrm{map}(X, K(\pi, 1); k)) .$$

We get the following fibration:

$$F \xrightarrow{j} \mathrm{map}(X, K(\pi, 1); k) \xrightarrow{\omega} K(\pi, 1) ,$$

where F is the fibre over y_0 which contains $\mathrm{map}_0(X, K(\pi, 1); k)$. Since $K(\pi, 1)$ is aspherical, it holds that

$$\pi_i(\mathrm{map}_0(X, K(\pi, 1); k) \cong \pi_i(\mathrm{map}(X, K(\pi, 1); k))$$

for $i \geqq 2$. Consequently we have

$$\pi_i(\mathrm{map}_0(X, K(\pi, 1); k))=0$$

for $i \geqq 2$.

Next we note that the following lemma holds.

Lemma 2.2. *$\pi_1(\mathrm{map}_0(X, K(\pi, 1); k)$ is trivial.*

A proof of this lemma is similarly performed to the proof of Lemma 3 in [3], so it is omitted.

Thus our proof of Theorem 2.1 is completed.

On the homotopy sequence of the fibration:

$$F \xrightarrow{j} \mathrm{map}(X, K(\pi, 1); k) \xrightarrow{\omega} K(\pi, 1),$$

we have the following

Corollary 2.3. *Let $k_*: \pi_1(X) \to \pi$ be the homomorphism induced by the map k and denote by C_k the centralizer of $k_*(\pi_1(X))$ in π. Then we have the following homotopy sequence of the above fibration*

$$1 \xrightarrow{j_*} C_k \xrightarrow{\omega_*} \pi \xrightarrow{\partial} R \longrightarrow 1$$

where C_k is isomorphic to $\pi_1(\mathrm{map}(X, K(\pi, 1); k)$ and R is the subset of $\mathrm{Hom}(\pi_1(X), \pi)$ consisting of elements $\alpha^{-1}k_*\alpha$ $(\alpha \in \pi)$.

Proof. We shall show in the following that the boundary $\partial: \pi_1(K(\pi, 1)) \cong \pi \to \pi_0(F)$ is just given by

$$\partial(\alpha)=\alpha^{-1}k_*\alpha,$$

where $\pi_0(F)$ may be regarded as R. For a given element α of $\pi_1(K(\pi, 1))$, let $f:(I, \partial I) \to (K(\pi, 1), y_0)$ be a map representing α and let $\bar{F}:(I, 0) \to (\mathrm{map}(X, K(\pi, 1); k), k)$ be a map such that $\omega \circ \bar{F}=f$. Then there exists the map $F: X \times I \to K(\pi, 1)$ associated with $\bar{F}$ such that

$$F(x, 0)=k(x)$$
$$F(x_0, t)=f(t) \qquad (x \in X,\ t \in I).$$

If we put $F(x, 1)=k'(x)$ for $x \in X$, we easily see

$$k'_*=\alpha^{-1}k_*\alpha.$$

Namely, for every λ of $\pi_1(X)$ it holds that

$$k'_*(\lambda)=\alpha^{-1}k_*(\lambda)\alpha.$$

Thus we see $\partial(\alpha)=\alpha^{-1}k_*\alpha$.

Furthermore we can easily see that

$$\partial(\alpha\beta)=\beta^{-1}\partial(\alpha)\beta.$$

By using this equality we find that $\partial^{-1}(k_*)$ is a subgroup of $\pi_1(K(\pi, 1))$ which is the centralizer of $k_*(\pi_1(X))$ in $\pi_1(K(\pi, 1))$. By Theorem 2.1 $\mathrm{map}_0(X, K(\pi, 1); k)$ is weakly contractible. Therefore we have

$$\pi_1(\mathrm{map}(X, K(\pi, 1)); k) \cong C_k .$$

Note that this is known in Lemma 2 of Gottlieb [4].

Now, for the classifying space B_∞ we have the following fibration:

$$K(G, n+1) \xrightarrow{i_0} B_\infty \xrightarrow{p_0} K(\mathrm{Aut}(G), 1) .$$

Let X be a CW complex, then we have the following fibration:

$$F \longrightarrow \mathrm{map}_0(X, B_\infty; k) \xrightarrow{p_{0\#}} \mathrm{map}_0(X, K(\mathrm{Aut}(G), 1); p_0 \circ k) ,$$

where F is the fibre over $p_0 \circ k$.

Proposition 2.4. *With the above notations, we have*

$$\Omega\, \mathrm{map}_0(X, B_\infty; k) \underset{w}{\simeq} \Omega F ,$$

where $Y \underset{w}{\simeq} Z$ *means that* Y *has the same weak homotopy type as* Z.

Proof. By Theorem 2.1 $\mathrm{map}_0(X, K(\mathrm{Aut}(G), 1); p_0 \circ k)$ is weakly contractible. Therefore F is weakly homotopy equivalent to $\mathrm{map}_0(X, B_\infty; k)$. Thus we have

$$\Omega F \underset{w}{\simeq} \Omega\, \mathrm{map}_0(X, B_\infty; k) .$$

§ 3. The Classifying Space for Fibration with Fibre $K(G, n)$

Let G be an abelian group and $\mathrm{Aut}(G)$ its group of automorphisms. Then there exists an Eilenberg-MacLane complex $K(G, n+1)$ $(n \geqq 0)$ which is a topological abelian group [7] and on which $\mathrm{Aut}(G)$ acts on the left by base point preserving cellular homeomorphisms ((5.2.5) Lemma in [1], 1.2. Lemma in [12]), here the base point is the identity element of $K(G, n+1)$. Let W be a complex $K(\mathrm{Aut}(G), 1)$ and $\widetilde{W}$ its universal covering complex. Then $\mathrm{Aut}(G)$ acts on $\widetilde{W}$ freely and cellularly on the left. Thus $\mathrm{Aut}(G)$ acts on $\widetilde{W} \times K(G, n+1)$ diagonally. We denote by $L(G, n+1)$ the quotient

$$(\widetilde{W} \times K(G, n+1))/\mathrm{Aut}(G) .$$

The projection of $\widetilde{W} \times K(G, n+1)$ onto $\widetilde{W}$ induces a map $p_0 : L(G, n+1) \to W = K(\mathrm{Aut}(G), 1)$ which is a fibre bundle with fibre $K(G, n+1)$ and with structure group $\mathrm{Aut}(G)$. Each fibre of this fibration is a topological abelian group isomorphic to $K(G, n+1)$ and there exists a canonical cross-section $s_0 : W \to L(G, n+1)$. It is well known that $L(G, n+1)$ is a classifying space B_∞ for fibrations with fibre $K(G, n)$ [5, 10].

Let X be a CW complex and let $k:(X, x_0)\to(W, w_0)=(K(\mathrm{Aut}(G), 1), w_0)$ be a space over (W, w_0). Then we define a multiplication in the space $\mathrm{map}_0(X, L(G, n+1))_W$, where the fibration $p_0:(L(G, n+1), l_0)\to(W, w_0)$ is the space over (W, w_0) and the canonical cross-section $s_0:(W, w_0)\to(L(G, n+1), l_0)$ is equipped.

Let f and g be any elements of $\mathrm{map}_0(X, L(G, n+1))_W$, then we define multiplication $f\cdot g$ of f and g by

$$(f\cdot g)(x)=f(x)g(x) \qquad (x\in X),$$

because both $f(x)$ and $g(x)$ are contained in the fibre $p^{-1}(k(x))$ over $k(x)$. We can easily see that

$$f\cdot g\in \mathrm{map}_0(X, L(G, n+1))_W .$$

Thus we obtain the following

Proposition 3.1. *Let X be a CW complex and $k:(X, x_0)\to(W, w_0)=(K(\mathrm{Aut}(G), 1), w_0)$ be a space over (W, w_0). Then, with respect to the multiplication defined above* $\mathrm{map}_0(X, L(G, n+1))_W$ *is a topological abelian group.*

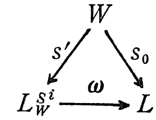
Proof is easily done, so it is omitted.

We observed that for $n\geqq 0$ there exists the following fibre bundle with structure group $\mathrm{Aut}(G)$:

$$K(G, n+1) \xrightarrow{i_0} L(G, n+1) \xrightarrow{p_0} K(\mathrm{Aut}(G), 1)=W.$$

In the following we abbreviate this fibration by the fibration $p_0: L\to W$. So we have the space $p_0:(L, l_0)\to(W, w_0)$ over (W, w_0) and the canonical cross-section $s_0:(W, w_0)\to(L, l_0)$.

In Remark 1.2, if we replace (X, x_0) by $(S^i, *)$ $(i\geqq 0)$ and replace a fibration $p: E\to B$ by the fibration $p_0: L\to W$, then we have the fibration $\omega: L_W^{S^i}\to L$, and the cross-section $s':(W, w_0)\to(L_W^{S^i}, c_{l_0})$ for the fibration $p_0\circ\omega: L_W^{S^i}\to W$, where c_{l_0} denotes the constant map of S^i to l_0. Also we have the following commutative diagram



Let us denote $\omega^{-1}(s_0(W))$ by $\bar{\Omega}^i L$ for $n+1>i\geqq 0$, then we have a fibration ((2.5) in [1], the proof of Lemma 1.2 in [2], [6]):

$$K(G, n+1-i) \longrightarrow \bar{\Omega}^i L \xrightarrow{p_0} W,$$

such that $\bar{\Omega}^i L$ may be regarded as a space $L(G, n+1-i)$.

With these notations we have the following

Lemma 3.2. *Let X be a CW complex, k be a fixed map of $(X, x_0)\to(L, l_0)$ and $p_0\circ k=k':(X, x_0)\to(W, w_0)$ be a space over (W, w_0). We have the following isomorphisms*:

$$\pi_i(\mathrm{map}_0(X, L)_W, s_0\circ k')\cong[X, \bar{\Omega}^i L]_W^0\cong H^{n+1-i}(X, x_0; G),$$

where $[X, \bar{\Omega}^i L]_W^0$ denotes the pointed homotopy classes over W of maps from (X, x_0) to $(\bar{\Omega}^i L, c_{l_0})$ and the cohomology is taken with local coefficients classified by the map $k':X\to W$.

Proof. Let $\bar{f}$ be a map of $(S^i, *)$ to $(\mathrm{map}_0(X, L)_W, s_0\circ k')$. Then we have its associated map $f:S^i\times(X, x_0)\to(L, l_0)$ with $f|*\times X=s_0\circ k'$ such that the following diagram is commutative

$$\begin{array}{ccc} S^i\times(X, x_0) & \xrightarrow{\quad f \quad} & (L, l_0) \\ & {\scriptstyle k'\circ p_2}\searrow \quad \swarrow{\scriptstyle p_0} & \\ & (W, w_0) & \end{array}$$

where p_2 is the projection of $S^i\times(X, x_0)$ onto (X, x_0).

Furthermore f corresponds to the map $\tilde{f}$ of (X, x_0) to $(L_W^{S^i}, c_{l_0})$ defined by

$$\tilde{f}(x)(y)=f(y, x) \qquad (x\in X,\ y\in S^i).$$

In fact, since for $y\in S^i$ we have

$$p_0(\tilde{f}(x)(y))=p_0\circ f(y, x)=k'(x) \qquad (x\in X),$$

we see that $\tilde{f}(x)$ is an element of $L_W^{S^i}$ for $x\in X$. By using

$$\omega\circ\tilde{f}(x)=\tilde{f}(x)(*)=s_0\circ k'(x) \qquad (x\in X),$$

we can easily see that $\tilde{f}$ is a map of (X, x_0) to $(\bar{\Omega}^i L, c_{l_0})$ and the following diagram is commutative:

$$\begin{array}{ccccc} (X, x_0) & \xrightarrow{\ \tilde{f}\ } & (L_W^{S^i}, c_{l_0}) & \xrightarrow{\ \omega\ } & (L, l_0) \\ & {\scriptstyle k'}\searrow & \downarrow{\scriptstyle p'} & \swarrow{\scriptstyle p_0} & \\ & & (W, w_0) & & \end{array}$$

where p' denotes $p_0\circ\omega$.

Similarly a homotopy $\bar{H}:(S^i, *)\times I\to(\mathrm{map}_0(X, L)_W, s_0\circ k')$ corresponds to a homotopy $\tilde{H}:(X, x_0)\times I\to(\bar{\Omega}^i L, c_{l_0})$. We can easily see that this correspondence induces the bijection

$$\pi_i(\mathrm{map}_0(X, L)_W, s_0\circ k')\cong[X, \bar{\Omega}^i L]_W^0.$$

Since $\bar{\Omega}^i L$ is a $L(G, n+1-i)$, we have the following isomorphisms ((5.2.4)

Theorem in [1], 3.1 Definition in [10], (6.13) Theorem in Chapter VI of [20])

$$[X, \bar{\Omega}^i L]^0_W \cong [X, L(G, n+1-i)]^0_W \cong H^{n+1-i}(X, x_0; G),$$

where $H^{n+1-i}(X, x_0; G)$ is the cohomology group with local coefficients classified by the map k'.

From this lemma we get the following

Theorem 3.3. *Let X be a CW complex, k be a fixed map of (X, x_0) to (L, l_0) and $p_0 \circ k = k' : (X, x_0) \to (W, w_0)$ be a space over (W, w_0). Then* $\mathrm{map}_0(X, L)_W$ *has the same weak homotopy type as*

$$H^{n+1}(X, x_0; G) \times \prod_{i=1}^{n} K(H^{n+1-i}(X, x_0; G), i),$$

where the cohomology is taken with local coefficients classified by the map $k' : X \to W$.

Proof. Notice that $[X, L]^0_W$ is isomorphic to the group $H^{n+1}(X, x_0; G)$. Thus by using Proposition 3.1 and the theorem of J. C. Moore [8] we have

$$\mathrm{map}_0(X, L)_W \underset{w}{\simeq} H^{n+1}(X, x_0; G) \times \prod_{i=1}^{n} K(H^{n+1-i}(X, x_0; G), i).$$

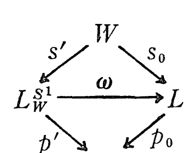
§ 4. Main Results

Recall the fibration:

$$F \longrightarrow \mathrm{map}_0(X, B_\infty; k) \xrightarrow{p_{0\#}} \mathrm{map}_0(X, K(\mathrm{Aut}(G), 1); p_0 \circ k),$$

where F is the fibre over $p_0 \circ k = k'$. In the following we shall investigate the loop space ΩF of F.

Since the fibration $p_0 : L \to W$ has a canonical cross-section $s_0 : (W, w_0) \to (L, l_0)$, by Proposition 1.1 and Remark 1.2 we have the following commutative diagram:



where $p' = p_0 \circ \omega$ and $s_0 = \omega \circ s'$. Let k be a map of a given CW complex (X, x_0) to (L, l_0). If we put $p_0 \circ k = k'$, by Proposition 1.3 we have the following fibration $\omega_\#$:

$$\mathrm{map}_0(X, L^{S^1}_W)_W \longrightarrow \mathrm{map}_0(X, L)_W.$$

On a relation between this fibration and ΩF, we have the following

Lemma 4.1. *ΩF is homeomorphic to the fibre $\omega_\#^{-1}(k)$ over k.*

Proof. Let $\tilde{f}$ be a map of $(S^1, *)$ to (F, k). Then we have its associated map $f: S^1\times(X, x_0)\to(L, l_0)$ such that

$$f(*, x)=k(x),$$

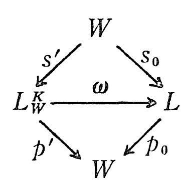
$$p_0\circ f(t, x)=k'(x) \quad (=p_0\circ k(x)) \quad (t\in S^1,\ x\in X).$$

We may define a map $\tilde{f}$ of (X, x_0) to $(L_W^{S^1}, c_{l_0})$ by $\tilde{f}(x)(t)=f(t, x)$ for $t\in S^1$ and $x\in X$, because

$$\begin{aligned} p_0\circ\omega\circ\tilde{f}(x) &= p_0(\tilde{f}(x)(*))=p_0\circ f(*, x) \\ &= p_0\circ k(x)=k'(x) \qquad (x\in X). \end{aligned}$$

Thus we see that $\tilde{f}:(X, x_0)\to(L_W^{S^1}, c_{l_0})$ is a map over (W, w_0). Also we see that $\omega_\#(\tilde{f})=k$, that is, $\tilde{f}$ is an element of a fibre $\omega_\#^{-1}(k)$ of the fibration $\omega_\#$: $\mathrm{map}_0(X, L_W^{S^1})_W\to\mathrm{map}_0(X, L)_W$. As easily seen, this correspondence gives rise to a homeomorphism of ΩF onto the fibre $\omega_\#^{-1}(k)$ over k.

Now, for any CW complex K we have the following commutative diagram:



We denote by $L_W^K\times_W L_W^K$ the fibred product of the fibration $p': L_W^K\to W$ and itself. It should be noted that there exists a map μ of $L_W^K\times_W L_W^K$ to L_W^K defined by

$$\mu(f, g)(y)=f(y)g(y) \qquad (y\in K),$$

because $f(y)$ and $g(y)$ for every $y\in K$ are contained in the same fibre $p_0^{-1}(p'(f))=p_0^{-1}(p'(g))$. Especially we have a map μ of $L_W^{S^1}\times_W L_W^{S^1}$ to $L_W^{S^1}$. By using this multiplication of $L_W^{S^1}$ we have the following

Proposition 4.2. *Let X be a CW complex, k be a fixed map of (X, x_0) to (L, l_0) and $p_0\circ k=k':(X, x_0)\to(W, w_0)$ be a space over (W, w_0). Then* $\mathrm{map}_0(X, L_W^{S^1})_W$ *is a topological abelian group and the projection $\omega_\#:\mathrm{map}_0(X, L_W^{S^1})_W\to\mathrm{map}_0(X, L)_W$ is an epimorphism of topological groups.*

Proof. For any elements $\tilde{f}$ and $\tilde{g}$ of $\mathrm{map}_0(X, L_W^{S^1})_W$ the multiplication $\tilde{f}\cdot\tilde{g}$ is defined by

$$(\tilde{f}\cdot\tilde{g})(x)=\tilde{f}(x)\cdot\tilde{g}(x) \qquad (x\in X),$$

where $\tilde{f}(x)\cdot\tilde{g}(x)$ means $\mu(\tilde{f}(x), \tilde{g}(x))$. Note that $\mu(\tilde{f}(x), \tilde{g}(x))$ is well defined, because $(\tilde{f}(x), \tilde{g}(x))$ is an element of $L_W^{S^1}\times_W L_W^{S^1}$ by

$$p'(\tilde{f}(x))=p'(\tilde{g}(x))=k'(x) \qquad (x\in X).$$

Since $p'(\tilde{f}\circ\tilde{g})(x)=p'\circ\tilde{f}(x)=k'(x)$, we see that $\tilde{f}\cdot\tilde{g}$ is an element of $\text{map}_0(X, L_W^{S^1})_W$.

We can easily prove that with respect to this multiplication $\text{map}_0(X, L_W^{S^1})_W$ is a topological abelian group with the identity element $s'\circ k'$.

Next we shall show that $\omega_\#$ is a homomorphism of $\text{map}_0(X, L_W^{S^1})_W$ onto the group $\text{map}_0(X, L)_W$. Let $\tilde{f}$ and $\tilde{g}$ be any elements of $\text{map}_0(X, L_W^{S^1})_W$. Then we have for $x\in X$

$$\begin{aligned}\omega_\#(\tilde{f}\cdot\tilde{g})(x)&=(\tilde{f}\cdot\tilde{g})(x)(*)=(\tilde{f}(x)\cdot\tilde{g}(x))(*)\\&=(\tilde{f}(x)(*))(\tilde{g}(x)(*))\\&=(\omega_\#(\tilde{f})(x))(\omega_\#(\tilde{g})(x))\\&=(\omega_\#(f)\cdot\omega_\#(g))(x)\,.\end{aligned}$$

Thus we have $\omega_\#(\tilde{f}\cdot\tilde{g})=\omega_\#(\tilde{f})\cdot\omega_\#(\tilde{g})$.

Define a map $\sigma: L\to L_W^{S^1}$ as follows:

$$\sigma(l)(t)=l \qquad (l\in L,\ t\in S^1)\,.$$

Then we see easily that σ is a map over (W, w_0) and a cross-section for the fibration $\omega: L_W^{S^1}\to L$. So σ induces the map $\sigma_\#: \text{map}_0(X, L)_W\to\text{map}_0(X, L_W^{S^1})_W$ which is a cross-section for the fibration $\omega_\#: \text{map}_0(X, L_W^{S^1})_W\to\text{map}_0(X, L)_W$. Therefore we see that $\omega_\#$ is surjective.

We can easily see $\omega_\#^{-1}(s_0\circ k')=\text{map}_0(X, \bar{\Omega}L)_W$. Hence the following is an immediate consequence of Proposition 4.2.

Corollary 4.3. $\text{Ker}\,\omega_\#=\omega_\#^{-1}(s_0\circ k')$ *is homeomorphic to the fibre* $\omega_\#^{-1}(k)$ *over* k. *In other words,*

$$\omega_\#^{-1}(k)\cong\text{map}_0(X, \bar{\Omega}L)_W=\text{map}_0(X, L(G, n))_W\,.$$

Now, let $p: E\to B$ be a fibration with fibre $F=K(G, n)$ $(n>1)$ over a CW complex B. In [18, 19] the following result is shown for a simply connected B:

$$\begin{aligned}\mathcal{G}(E \bmod F) &\underset{w}{\simeq} \text{map}_0(B, K(G, n))\\&\underset{w}{\simeq} H^n(B, G)\times\prod_{i=1}^{n-1}K(H^{n-i}(B, G), i)\,,\end{aligned}$$

where $\mathcal{G}(E \bmod F)$ is the space of self fibre homotopy equivalences of E leaving a fibre $p^{-1}(b_0)=F$ fixed. Here without assuming that B is simply connected, we have a following generalization of this result which follows from our previous Proposition 2.4, Lemma 4.1, Corollary 4.3 and Theorem 3.3.

Theorem 4.4. *Let* $p: E\to B$ *be a fibration with fibre* $F=K(G, n)$ $(n>1)$ *such that* B *is a CW complex. Then if we denote by* $k:(B, b_0)\to(L(G, n+1), l_0)$ *a corresponding map to the fibration*: $F\xrightarrow{i}E\xrightarrow{p}B$, *we have*

$$\mathcal{G}(E \bmod F) \underset{w}{\simeq} \mathrm{map}_0(B, L(G, n))_W$$
$$\underset{w}{\simeq} H^n(B, G) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, b_0; G), i)$$

where the cohomology is taken with local coefficients classified by the map $p_0 \circ k : B \to K(\mathrm{Aut}(G), 1)$ (p_0 *is the projection of* $L(G, n+1)$ *to* $K(\mathrm{Aut}(G), 1)$).

Now, we quote the following two theorems [18, 19], where $G_0(X)$ denotes the space of self homotopy equivalences of a CW complex (X, x_0).

Theorem 4.5. *Let* E *and* B *be CW complexes and let* $p: E \to B$ *be a fibration with fibre* F. *For a given* $n>1$ *if* F *is* $(n-1)$*-connected and* $\pi_i(B)=0$ *for every* $i \geqq n$, *then we have the following fibration*:

$$\mathcal{G}(E \bmod F) \longrightarrow G_0(E) \xrightarrow{\rho} G_0(B) \times G_0(F).$$

Theorem 4.6. *Under the hypothesis of Theorem* 4.5, *the image of* $\rho : G_0(E) \to G_0(B) \times G_0(F)$ *is just the union of the path components in* $G_0(B) \times G_0(F)$ *each of which contains* (g, h) *satisfying*

$$[\chi_\infty(h)] \circ [k] = [k] \circ [g],$$

where $\chi_\infty(h)$ *is a self map of* (B_∞, b_∞) *and* $k : (B, b_0) \to (B_\infty, b_\infty)$ *is a corresponding map to the fibration*: $F \xrightarrow{i} E \xrightarrow{p} B$.

Let $\varepsilon(X)$ denote the group $\pi_0(G_0(X))$ for a CW complex X. Since $G_0(K(\pi, n)) \underset{w}{\simeq} \mathrm{Aut}(\pi)$ for $n \geqq 1$, by Theorem 4.4, 4.5 and 4.6 we have the following theorem which is a generalization of Theorem 10 in [18].

Theorem 4.7. *For given* $1 \leqq m < n$, *let*

$$F = K(G, n) \xrightarrow{i} E \xrightarrow{p} K(\pi, m) = B$$

be a fibration with a corresponding map $k : (B, b_0) \to (L(G, n+1), l_0)$. *Then we have*

$$G_0(E) \underset{w}{\simeq} R \times H^n(B, G) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, b_0; G), i),$$

where R *is the subgroup of* $\mathrm{Aut}(\pi) \times \mathrm{Aut}(G) = \varepsilon(B) \times \varepsilon(F)$ *consisting of* $([g], [h])$ *with*

$$[\chi_\infty(h)] \circ [k] = [k] \circ [g],$$

and the cohomology is taken with local coefficients classified by the map $p_0 \circ k : B \to K(\mathrm{Aut}(G), 1) = W$ (p_0 *is the projection of* $L(G, n+1)$ *to* $K(\mathrm{Aut}(G), 1)$.

Note that the map $\rho : G_0(E) \to G_0(B) \times G_0(F)$ induces the homomorphism ρ_* of $\varepsilon(E)$ into $\varepsilon(B) \times \varepsilon(F)$, that the image of ρ_* is just R in Theorem 4.7 and that the kernel of ρ_* may be regarded as $H^n(B, G)$. Thus, as a corollary of Theorem 4.7 we have the following theorem [9, 11, 15].

Theorem 4.8. *Under the same hypothesis of Theorem* 4.7 *there exists the following exact sequence*

$$1 \longrightarrow H^n(B, G) \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1,$$

where R is the same group as the group stated in Theorem 4.7 *and the cohomology is taken with local coefficients given by the map $p_0 \circ k : B \to K(\mathrm{Aut}(G), 1)$.*

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