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On the Stable Homotopy of the Real Projective Space of Even Low Dimension

Dedicated to Professor Itiro Tamura on his 60th birthday

By

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§ 0. Introduction

We denote by $\{X, Y\}$ the group of stable homotopy classes of mappings from X to Y. We denote by P^n the real n-dimensional projective space. The purpose of this note is to determine the group structure of $\{P^{2n}, P^{2n}\}$ for $2\leqq n\leqq 4$ (Theorems 2.4, 3.4 and 5.6). As an application, the stable group of self-homotopy equivalences of P^{2n} will be determined in our case (Corollaries to the above theorems).

We denote by $\gamma_n: S^n\to P^n$ the projection. Let $\bar{\gamma}_{2n}: E^{2n-1}P^2\to P^{2n}$ be a stable extension of γ_{2n} such that P^{2n+2} is the mapping cone of $\bar{\gamma}_{2n}$. Then our method is to use the cofibre sequence starting with $\bar{\gamma}_{2n}$ and to use the following: The order of the identity class of P^{2n} [11], the order of the Kahn-Priddy map [6] and the ring structure for $k\leqq 8$ of the stable homotopy ring of spheres $\pi_*=\sum \pi_k(S^0)$ [10]. The *EHP*-sequence is used to show that the generator σ of the 2-component of $\pi_7(S^0)$ survives in $\{P^8, P^4\}$ (Lemma 5.2).

§ 1. Main Results Used in the Computations

Throughout this note we work in the stable category, unless otherwise stated. First we shall give a remark about the stable secondary compositions. The last part of Chap. III of [10] deals with them in the only case of the stable homotopy groups of spheres. But the definition of the stable secondary composition is still valid in the case of the stable homotopy groups between finite CW-complexes. The properties (3.5), (3.6), (3.7) and (3.8) of [10] are valid in our case. For example, we have the following:

$$\alpha\circ\langle\beta, \gamma, \delta\rangle\subset(-1)^{|\alpha|}\langle\alpha\circ\beta, \gamma, \delta\rangle$$

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and

$$\langle\alpha, \beta, \gamma\rangle\circ\delta=(-1)^{|\alpha|+1}\alpha\circ\langle\beta, \gamma, \delta\rangle,$$

where $|\alpha|=\dim Y-\dim Z$ for $\alpha\in\{Y, Z\}$.

These properties of the stable secondary compositions will be freely used in the subsequent arguments.

We denote by $s(n)$ the number of i such that $0<i\leqq n$ and $i\equiv 0, 1, 2$ or 4 mod 8. By Theorem 6.2 of [1], P^{n-1} is reducible if and only if $2^{s(n-1)}$ is a divisor of n. So we have the following

Theorem 1.1. *γ_{2n} is trivial if and only if $n=1$ or 3.*

It is well known that $2\gamma_{2n}=0$.

We denote by ι the identity class of S^0, by $i:S^1\rightarrow P^2$ and $p:P^2\rightarrow S^2$ the canonical maps. Then we have a cofire sequence

$$\text{(1.1)} \qquad S^1 \xrightarrow{2\iota} S^1 \xrightarrow{i} P^2 \xrightarrow{p} S^2 \longrightarrow \cdots.$$

We take $\bar{\gamma}_{2n}\in\langle\gamma_{2n}, 2\iota, p\rangle$ such that P^{2n+2} is its mapping cone. Exactly we have a cofibre sequence

$$\text{(1.2)}^{(n)} \qquad E^{2n-1}P^2 \xrightarrow{\bar{\gamma}_{2n}} P^{2n} \xrightarrow{i^{(n)}} P^{2n+2} \xrightarrow{p^{(n)}} E^{2n}P^2 \longrightarrow \cdots,$$

where $i^{(n)}$ and $p^{(n)}$ are the canonical maps.

Let η be the generator of $\pi_1(S^0)\approx Z/2$ and ι'_n the identity class of P^n Then, by [11], we have the following

Theorem 1.2. i) *ι'_{2n} is of order $2^{s(2n)}$.*

ii) *$2\iota'_2=i\eta p$.*

A mapping $\phi:P^{2n}\rightarrow S^0$ is called a Kahn-Priddy map if the restriction $\phi|S^1=\eta$. We denote by $\xi(X)$ the stable group of self-homotopy equivalences of X. Then, by Theorems 1.1 and 3.1 of [6], we have the following

Theorem 1.3. *Let $\phi_{2n}:P^{2n}\rightarrow S^0$ be a Kahn-Priddy map. Then*

i) *ϕ_{2n} is of order $2^{s(2n)}$.*

ii) *There exists an element $\varepsilon'_{2n}\in\xi(P^{2n})$ such that $\langle\phi_{2n}\circ\varepsilon'_{2n}, \bar{\gamma}_{2n}, p^{(n)}\rangle$ contains a Kahn-Priddy map from P^{2n+2} to S^0.*

This theorem will be used putting $\varepsilon'_{2n}=\iota'_{2n}$ in our arguments.

We shall use the following [10]

Theorem 1.4. i) *$\pi_k(S^0)$ for $0\leqq k\leqq 8$ (the 2-component for $k=3$ or 7) is isomorphic to the corresponding group in the following table:*

k	0	1	2	3	4, 5	6	7	8
$\pi_k(S^0)$	(∞)	(2)	(2)	(8)	0	(2)	(16)	$(2)^2$
gen.	ι	η	η^2	ν		ν^2	σ	$\eta\sigma,\ \varepsilon$

Here (n) *means* Z/n *and* $(2)^2=(2)\oplus(2)$ *(direct summand).*

ii) *There exist the following relations*:

$$\eta^2=\langle 2\iota, \eta, 2\iota\rangle,\quad \eta^3=4\nu,\quad \eta\nu=\nu\eta=0,\quad 2\nu\in\langle\eta, 2\iota, \eta\rangle \bmod 4\nu,$$

$$\nu^2=\langle\eta, \nu, \eta\rangle,\quad \eta\sigma=\sigma\eta,\quad \varepsilon\in\langle\eta, 2\iota, \nu^2\rangle=\langle\eta, \nu, 2\nu\rangle \bmod \eta\sigma,$$

$$\eta\sigma+\varepsilon=\langle\nu, \eta, \nu\rangle.$$

By use of (1.1) and Theorem 1.4, we have the following (Theorems 3.1 and 3.2 of [4] and §2 of [5])

Theorem 1.5. i) $G_k=\pi^0(E^{k-2}P^2)$ *and* $G_k^*=\pi_{k+1}(P^2)$ *are for* $0\leqq k\leqq 8$ *isomorphic to the corresponding group in the following table*:

k	0	1	2	3	4	5	6	7	8
$G_k\approx G_k^*$	(2)	(2)	(4)	$(2)^2$	(2)	0	(2)	$(2)^2$	$(2)^3$
gen. of G_k	p	ηp	$\bar\eta$	$\eta\bar\eta,\ \nu p$	$\eta^2\bar\eta$		$\nu^2 p$	$\overline{\nu^2},\ \sigma p$	$\overline{8\sigma},\ \eta\sigma p,\ \varepsilon p$
gen. of G_k^*	i	$i\eta$	$\tilde\eta$	$\tilde\eta\eta,\ i\nu$	$\tilde\eta\eta^2$		$i\nu^2$	$\widetilde{\nu^2},\ i\sigma$	$\widetilde{8\sigma},\ i\eta\sigma,\ i\varepsilon$

ii) *There exist the following relations*:

$$pi=0,\quad \iota\in\langle p, i, 2\iota\rangle=\langle 2\iota, p, i\rangle \bmod 2\iota,\quad \bar\eta\in\langle\eta, 2\iota, p\rangle,\quad \tilde\eta\in\langle i, 2\iota, \eta\rangle,$$

$$2\bar\eta=\eta^2 p,\quad 2\tilde\eta=i\eta^2,\quad \bar\eta\tilde\eta=\pm 2\nu,\quad \nu\bar\eta=0,\quad \tilde\eta\nu=0,\quad \overline{\nu^2}\in\langle\nu^2, 2\iota, p\rangle,$$

$$\widetilde{\nu^2}\in\langle i, 2\iota, \nu^2\rangle,\quad \eta\overline{\nu^2}=\varepsilon p,\quad \widetilde{\nu^2}\eta=i\varepsilon,\quad \overline{8\sigma}\in\langle 8\sigma, 2\iota, p\rangle,\quad \widetilde{8\sigma}\in\langle i, 2\iota, 8\sigma\rangle,$$

$$\bar\eta\widetilde{\nu^2}=\overline{\nu^2}\tilde\eta=\varepsilon.$$

By Theorem 3.3 of [4] and by Proposition 2.1 of [5], we have the following

Theorem 1.6. i) $H_k=\{E^k P^2, P^2\}$ *for* $-1\leqq k\leqq 6$ *is isomorphic to the corresponding group in the following table*:

k	-1	0	1	2	3	4	5	6
H_k	(2)	(4)	$(2)^2$	$(2)^3$	$(4)\oplus(2)$	(2)	(2)	$(2)^3$
gen.	ip	ι_2'	$i\bar\eta,\ \tilde\eta p$	$i\eta\bar\eta,\ \tilde\eta\eta p,\ i\nu p$	$\tilde\eta\bar\eta,\ \overline{i\nu}$	$\tilde\eta\eta\bar\eta$	$i\nu^2 p$	$i\overline{\nu^2},\ \widetilde{\nu^2}p,\ i\sigma p$

ii) *There exist the following relations*:

$$2\iota_2'=i\eta p,\quad (i\bar{\eta})^2=i\eta\bar{\eta},\quad (\tilde{\eta}p)^2=\tilde{\eta}\eta p,\quad \overline{i\nu}\in\langle i\nu, 2\iota, p\rangle=\langle i, 2\iota, \nu p\rangle$$
$$\ni\widetilde{\nu p}\bmod 2\tilde{\eta}\bar{\eta},\quad \tilde{\eta}p(\tilde{\eta}\eta\bar{\eta})=(\tilde{\eta}\eta\bar{\eta})(i\bar{\eta})=\tilde{\eta}\eta^2\bar{\eta}=0.$$

We shall give a proof of the last relation (cf. the proof of Proposition 2.1. vi) of [5]):

$$\tilde{\eta}\eta^2\bar{\eta}=\tilde{\eta}\langle 2\iota, \eta, 2\iota\rangle\bar{\eta}\subset\langle 2\tilde{\eta}, \eta, 2\bar{\eta}\rangle=\langle i\eta^2, \eta, \eta^2 p\rangle\subset\langle i\eta, \eta^3, \eta p\rangle$$
$$=\langle i\eta, 4\nu, \eta p\rangle.$$

On the other hand,

$$\langle i\eta, 4\nu, \eta p\rangle\supset\langle i\eta\nu, 4\iota, \eta p\rangle=\langle 0, 4\iota, \eta p\rangle\ni 0\bmod (i\eta)\pi^0(E^3P^2)+\pi_6(P^2)(\eta p)=0.$$

This completes the proof.

The theorems in this section will be often used without any references.

§2. Determination of $\{P^4, P^4\}$

Hereafter $Z/2$ is taken as the coefficients group of the cohomology. Since $Sq^2:\tilde{H}^2(P^4)\to\tilde{H}^4(P^4)$ is nontrivial, we have

$$\bar{\gamma}_2=\tilde{\eta}p. \tag{2.1}$$

By Theorem 1.3 and (2.1), we have a Kahn-Priddy map $\bar{\eta}'\in\pi^0(P^4)$ of order 8 satisfying $\bar{\eta}'i'=\bar{\eta}$, i.e.,

$$\bar{\eta}'\in\langle\bar{\eta}, \tilde{\eta}p, p'\rangle. \tag{2.2}$$

Here $i'=i^{(1)}$ and $p'=p^{(1)}$ in (1.2)′. So, by use of the exact sequence induced from (1.2)′, we have $4\bar{\eta}'=\eta^2\tilde{\eta}p'$ and $\pi^0(P^4)=\{\bar{\eta}'\}\approx Z/8$.

We put $p_{2n}=E^{2n-2}p\circ p^{(n-1)}:P^{2n}\to S^{2n}$. Similarly as above, by use of (1.2)′, we have the following

Proposition 2.1. *$\pi^k(P^4)$ for $0\leqq k\leqq 4$ is isomorphic to the corresponding group in the following table:*

k	0	1	2	3	4
$\pi^k(P^4)$	(8)	$(2)^2$	(2)	0	(2)
gen.	$\bar{\eta}'$	$\eta\tilde{\eta}p'$, νp_4	$\tilde{\eta}p'$		p_4

Here $4\bar{\eta}'=\eta^2\tilde{\eta}p'$.

By use of (1.2)′, we have a short exact sequence

$$0 \longrightarrow \{\tilde{\eta}\} \xrightarrow{i'_*} \pi_3(P^4) \xrightarrow{p'_*} \{i\} \longrightarrow 0 .$$
$$\{\tilde{\eta}\} \approx Z/4, \quad \{i\} \approx Z/2$$

We define an element $\tilde{i} \in \pi_3(P^4)$ by $p'\tilde{i}=i$, i.e.,

(2.3) $$\tilde{i} \in \langle i', \tilde{\eta} p, i \rangle .$$

Then $2\tilde{i} \in \langle \iota', \tilde{\eta} p, i \rangle \circ 2\iota = -i' \langle \tilde{\eta} p, i, 2\iota \rangle \supset i' \tilde{\eta} \langle p, i, 2\iota \rangle \ni i' \tilde{\eta} \bmod 2i' \pi_3(P^2) = \{2i' \tilde{\eta}\}$. So we have $2\tilde{i} = \pm i' \tilde{\eta}$ and $\pi_3(P^4) = \{\tilde{i}\} \approx Z/8$.

By the similar arguments to the above, we have the following

Proposition 2.2.

k	0	1	2	3
$\pi_{k+3}(P^4)$	(8)	$(2)^2$	(2)	(2)
gen.	$\tilde{i}$	$\tilde{i}\eta,\ i'i\nu$	$\tilde{i}\eta^2$	$\tilde{i}\nu$

Here $2\tilde{i} = \pm i' \tilde{\eta}$.

Hereafter the inclusions $i, i', \cdots$ (resp. the projections $p, p', \cdots$) are often used to denote the compositions of the inclusions (resp. the projections), unless any confusion occurs.

By use of (1.1) and Proposition 2.2, we have the following

Proposition 2.3.

k	0	1	2	3	4
$\{E^k P^2, P^4\}$	(4)	$(2)^2$	$(2)^3$	$(4)\oplus(2)$	$(2)^2$
gen.	i'	$i\tilde{\eta},\ \tilde{i}p$	$i\eta\tilde{\eta},\ \tilde{i}\eta p,\ i\nu p$	$\tilde{i}\tilde{\eta},\ i'\overline{i\nu}$	$\tilde{i}\eta\tilde{\eta},\ \tilde{i}\nu p$

Theorem 2.4. i) $4\iota'_4 \equiv i\eta\tilde{\eta}p' \bmod i\nu p_4$.
ii) $\{P^4, P^4\} = \{\iota'_4, i\nu p_4\} \approx Z/8 \oplus Z/2$.

Proof. Consider the following exact sequence induced from (1.2)':

$$\{P^2, P^4\} \xleftarrow{i'^*} \{P^4, P^4\} \xleftarrow{p'^*} \{E^2P^2, P^4\} \xleftarrow{(\tilde{\eta} p)^*} \{EP^2, P^4\}.$$

Then, $(\tilde{\eta} p)^*(i\tilde{\eta})=0$ and by Proposition 2.3, $(\tilde{\eta} p)^*(\tilde{i}p) = \tilde{i}\eta p \neq 0$. So we have a short exact sequence

$$0 \longleftarrow \{i'\} \xleftarrow{i'^*} \{P^4, P^4\} \xleftarrow{p'^*} \{i\eta\tilde{\eta}, i\nu p\} \longleftarrow 0 .$$
$$\{i'\} \approx Z/4, \quad \{i\eta\tilde{\eta}, i\nu p\} \approx (Z/2)^2$$

By (2.2) and Proposition 2.1, $\bar{\eta}'(i\eta\bar{\eta}p')=\eta^2\bar{\eta}p'=4\bar{\eta}'$ and $\bar{\eta}'(i\nu p)=\eta\nu p=0$. This completes the proof.

We denote by $(Z/n)^*$ the multiplicative group of Z/n and by $G\times H$ the direct product of groups G and H. In the above theorem, $(i\nu p_4)^2=0$. So we have the following

Corollary. $$\xi(P^4)\approx(Z/8)^*\times Z/2\,.$$

By (2.2) and (2.3), $2\bar{\eta}'\tilde{i}\in\langle\bar{\eta},\ \bar{\eta}p,\ i\rangle\circ 2\iota=-\bar{\eta}\langle\bar{\eta}p,\ i,\ 2\iota\rangle\supset\bar{\eta}\bar{\eta}\langle p,\ i,\ 2\iota\rangle\ni\pm 2\nu$ $\bmod\,(2\bar{\eta})\pi_3(P^2)=\{4\nu\}$. So we have

$$\bar{\eta}'\tilde{i}\equiv\nu \bmod 2\nu\,. \tag{2.4}$$

§3. Determination of $\{P^6,\ P^6\}$

Since $Sq^4:\tilde{H}^1(P^5)\to\tilde{H}^5(P^5)$ is trivial, $\gamma_4=\tilde{i}\eta$ by Proposition 2.2. So, by Proposition 2.3, we can take

$$\bar{\gamma}_4=\tilde{i}\bar{\eta}\,. \tag{3.1}$$

By (2.4), $\bar{\eta}'(\tilde{i}\bar{\eta})=\nu\bar{\eta}=0$. So, by Theorem 1.3, we have a Kahn-Priddy map $\bar{\bar{\eta}}\in\pi^0(P^6)$ of order 8 satisfying $\bar{\bar{\eta}}i''=\bar{\eta}'$, i.e.,

$$\bar{\bar{\eta}}\in\langle\bar{\eta}',\ \tilde{i}\bar{\eta},\ p''\rangle\,. \tag{3.2}$$

By use of (1.2)″ and (3.2), we have a split exact sequence

$$\begin{array}{ccccccccc} 0 & \xleftarrow{} & \{\bar{\eta}'\} & \xleftarrow{i''^*} & \pi^0(P^6) & \xleftarrow{p''^*} & \{\nu^2 p\} & \xleftarrow{} & 0\,. \\ & & \wr\wr & & & & \wr\wr & & \\ & & Z/8 & & & & Z/2 & & \end{array}$$

Therefore $\pi^0(P^6)=\{\bar{\bar{\eta}},\ \nu^2p_6\}\approx Z/8\oplus Z/2$.

By (2.3), $p_4\tilde{i}=pi=0$. So we can define an element $\bar{p}_4\in\pi^4(P^6)$ by $\bar{p}_4i''=p_4$, i.e.,

$$\bar{p}_4\in\langle p_4,\ \tilde{i}\bar{\eta},\ p''\rangle\,. \tag{3.3}$$

Then, $2\bar{p}_4\in 2\iota\circ\langle p_4,\ \tilde{i}\bar{\eta},\ p''\rangle=-\langle 2\iota,\ p_4,\ \tilde{i}\bar{\eta}\rangle p''\supset-\langle 2\iota,\ p,\ p'\tilde{i}\bar{\eta}\rangle p''\supset-\langle 2\iota,\ p,\ i\rangle\bar{\eta}p''\ni$ $-\bar{\eta}p''$ $\bmod 2\pi^0(P^2)p''+\pi^4(EP^4)(\tilde{i}\bar{\eta}p'')=\{2\bar{\eta}p''\}$ by Proposition 2.1. So we have $2\bar{p}_4=\pm\bar{\eta}p''$. Therefore (1.2)″ and Proposition 2.1 lead us to the following

Proposition 3.1. *$\pi^k(P^6)$ for $0\leqq k\leqq 6$ is isomorphic to the corresponding group in the following table:*

k	0	1	2	3	4	5	6
$\pi^k(P^6)$	$(8)\oplus(2)$	(2)	0	(2)	(8)	(2)	(2)
gen.	$\bar{\bar{\eta}},\ \nu^2 p_6$	$\nu\bar{p}_4$		νp_6	$\bar{p}_4$	ηp_6	p_6

Here $2\bar{p}_4=\pm\bar{\eta}p''$.

By use of (1.2)″ and Proposition 2.2, we have the following

Proposition 3.2.

k	0	1	2
$\pi_{k+4}(P^6)$	(2)	0	(2)
gen.	$i\nu$		$i''\tilde{i\nu}$

By use of (1.2)″ and Proposition 2.3, we have the following

Proposition 3.3.

k	2	3	4
$\{E^kP^2, P^6\}$	$(2)^2$	(2)	(2)
gen.	$i\eta\bar{\eta},\ i\nu p$	$i'\overline{i\nu}$	$i''\tilde{i\nu}p$

Consider the following exact sequence:

$$\pi^4(P^4)\xrightarrow{\gamma_{4*}}\{P^4, P^4\}\xrightarrow{i''_*}\{P^4, P^5\}\longrightarrow 0\,.$$

Then, by (1.2)′, $\gamma_4 p_4=\tilde{i}\eta pp'=p'^*(\bar{\eta}p)^*(\tilde{i}p)=0$. So, by Theorem 2.4, we have $\{P^4, P^6\}=\{i'', i\nu p_4\}\approx Z/8\oplus Z/2$.

Theorem 3.4. $\{P^6, P^6\}=\{\iota_6', i\nu\bar{p}_4, i''\tilde{i\nu}p_6\}\approx Z/8\oplus(Z/2)^2$.

Proof. Consider the following exact sequence induced from (1.2)″:

$$\{E^3P^2, P^6\}\xleftarrow{(\tilde{i}\bar{\eta})^*}\{P^4, P^6\}\xleftarrow{i''^*}\{P^6, P^6\}\xleftarrow{p''^*}\{E^4P^2, P^6\}\xleftarrow{(\tilde{i}\bar{\eta})^*}\{EP^4, P^6\}\,.$$

Then $(\tilde{i}\bar{\eta})^*(i\nu p_4)=0$ and by Proposition 3.2, $(\tilde{i}\bar{\eta})^*\{EP^4, P^6\}\subset\pi_4(P^6)\bar{\eta}=0$. So we have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \{i'', i\nu p_4\} & \xleftarrow{i''^*} & \{P^6, P^6\} & \xleftarrow{p''^*} & \{i''\tilde{i\nu}p\} & \longleftarrow & 0\,. \\ & & \wr\wr & & & & \wr\wr & & \\ & & Z/8\oplus Z/2 & & & & Z/2 & & \end{array}$$

By (3.3), there exists an element $i\nu\bar{p}_4$ of order 2 in $\{P^6, P^6\}$. Since ι'_6 is of order 8, the above sequence is split. This completes the proof.

We put $\alpha=i\nu\bar{p}_4$ and $\beta=i''\tilde{i}\nu p_6$. Then $\alpha^2=\beta^2=0$ and $\alpha\beta=\beta\alpha=0$. This leads us to the following

Corollary. $\xi(P^6)\approx(Z/8)^*\times(Z/2)^2$.

By use of (1.1) and Proposition 3.1, we have the following

Proposition 3.5. *$\{P^6, E^kP^2\}$ for $-1\leqq k\leqq 4$ is isomorphic to the corresponding group in the following table:*

k	-1	0	1	2	3	4
$\{P^6, E^kP^2\}$	$(2)^3$	(2)	(2)	$(2)^2$	$(2)^2$	(4)
gen.	$\widetilde{\nu\bar{p}_4}, i\bar{\eta}, i\nu^2p_6$	$i\nu\bar{p}_4$	$\widetilde{\nu}pp''$	$\bar{\eta}\eta p_6, i\nu p_6$	$\bar{\eta}p_6, i\bar{p}_4$	p''

Here $\widetilde{\nu\bar{p}_4}\in\langle i, 2\iota, \nu\bar{p}_4\rangle$ and $2p''=i\eta p_6$.

By use of (1.2)′, (1.2)″ and Proposition 3.5, we have the following

Proposition 3.6. i) $\{P^6, P^4\}=\{\widetilde{\bar{\eta}\eta p_6}, \tilde{i}\nu p_6, i\nu\bar{p}_4\}\approx(Z/2)^3$, *where* $\widetilde{\bar{\eta}\eta p_6}\in\langle i', \bar{\eta}p, \bar{\eta}\eta p_6\rangle$.

ii) $4\iota'_6\equiv i''\widetilde{\bar{\eta}\eta p_6}$ mod $\{i''\tilde{i}\nu p_6, i\nu\bar{p}_4\}$.

§4. Determination of Generators of $\pi^k(P^8)$ for $0\leqq k\leqq 8$

By use of (1.2)′, we have a short exact sequence

$$0\longrightarrow\{i\nu^2\}\xrightarrow{i'_*}\pi_7(P^4)\xrightarrow{p'_*}\{\bar{\eta}\eta^2\}\longrightarrow 0\,.$$

We define an element $\widetilde{\bar{\eta}\eta^2}\in\pi_7(P^4)$ by $p'\widetilde{\bar{\eta}\eta^2}=\bar{\eta}\eta^2$, i.e.,

$$\widetilde{\bar{\eta}\eta^2}\in\langle i', \bar{\eta}p, \bar{\eta}\eta^2\rangle\,. \tag{4.1}$$

By use of (1.2)″, we have a short exact sequence

$$0\longrightarrow\pi_7(P^4)\xrightarrow{i''_*}\pi_7(P^6)\xrightarrow{p''_*}\{\bar{\eta}\}\longrightarrow 0\,.$$

We define an element $\bar{\eta}'\in\pi_7(P^6)$ by $p''\bar{\eta}'=\bar{\eta}$, i.e.,

$$\bar{\eta}'\in\langle i'', \tilde{i}\bar{\eta}, \bar{\eta}\rangle\,. \tag{4.2}$$

By (3.3) and (4.2), $\bar{p}_4\bar{\eta}'\in\langle p_4, \tilde{i}\bar{\eta}, \bar{\eta}\rangle\subset\langle p, i\bar{\eta}, \bar{\eta}\rangle\supset\langle p, i, \pm 2\nu\rangle\supset\langle p, i, 2\iota\rangle\circ(\pm\nu)\ni\pm\nu$

mod $p\pi_5(P^2)+\pi^0(P^2)\tilde{\eta}=\{2\nu\}$. So we have

$$\bar{p}_4\tilde{\eta}'\equiv\nu \bmod 2\nu\,. \tag{4.3}$$

Proposition 4.1. i) *$\tilde{\eta}'$ is of order* 8.

ii) $4\tilde{\eta}'\equiv i''\widetilde{\tilde{\eta}\eta^2} \bmod i\nu^2$.

iii) $\pi_7(P^4)=\{\widetilde{\tilde{\eta}\eta^2},\ i\nu^2\}\approx(Z/2)^2$.

iv) $\pi_7(P^6)=\{\tilde{\eta}',\ i\nu^2\}\approx Z/8\oplus Z/2$.

Proof. i) follows from (4.3). By (4.2), $4\tilde{\eta}'\in\langle i'',\ \tilde{i\eta},\ \tilde{\eta}\rangle\circ 4\iota=-i''\langle\tilde{i\eta},\ \tilde{\eta},\ 4\iota\rangle$. On the other hand, $p'\langle\tilde{i\eta},\ \tilde{\eta},\ 4\iota\rangle\subset\langle i\tilde{\eta},\ \tilde{\eta},\ 4\iota\rangle\subset\langle i,\ \pm 2\nu,\ 4\iota\rangle\supset\langle i,\ 2\iota,\ 4\nu\rangle\ni\tilde{\eta}\eta^2$ mod $i\pi_4(S^0)+4\pi_5(P^2)=0$. So we have $\langle\tilde{i\eta},\ \tilde{\eta},\ 4\iota\rangle\ni\widetilde{\tilde{\eta}\eta^2}$ mod $i\nu^2$. This leads us to ii). iii) and iv) follow from i) and ii). This completes the proof.

By Proposition 4.1, $\bar{\gamma}_6\in\langle\gamma_6,\ 2\iota,\ p\rangle=\langle 0,\ 2\iota,\ p\rangle=\pi_7(P^6)p=\{\tilde{\eta}'p,\ i\nu^2p\}$. By (4.2), Propositions 3.3 and 4.1, $\tilde{\eta}'\in\langle i'',\ \tilde{i\eta},\ \tilde{\eta}\rangle$ mod $i''\pi_7(P^4)+\{E^4P^2,\ P^6\}\tilde{\eta}=\{4\tilde{\eta}',\ i\nu^2\}$. Since $Sq^2:\tilde{H}^6(P^8)\to\tilde{H}^8(P^8)$ is nontrivial, we can take

$$\bar{\gamma}_6=\tilde{\eta}'p \quad \textit{for a suitable representative} \quad \tilde{\eta}'\in\langle i'',\ \tilde{i\eta},\ \tilde{\eta}\rangle\,. \tag{4.4}$$

By use of (1.2)″, we have a short exact sequence

$$0\longrightarrow\pi_7(P^6)\xrightarrow{i'''_*}\pi_7(P^8)\xrightarrow{p'''_*}\{i\}\longrightarrow 0\,.$$

We define an element $\tilde{i}'\in\pi_7(P^8)$ by $p'''\tilde{i}'=i$, i.e.,

$$\tilde{i}'\in\langle i''',\ \tilde{\eta}'p,\ i\rangle\,. \tag{4.5}$$

Then $2\tilde{i}'\in\langle i''',\ \tilde{\eta}'p,\ i\rangle\circ 2\iota=-i'''\langle\tilde{\eta}'p,\ i,\ 2\iota\rangle\ni i'''\tilde{\eta}'$ mod $2i'''_*\pi_7(P^6)=\{2i'''\tilde{\eta}'\}$ by Proposition 4.1. This leads us to the following

Proposition 4.2. $2\tilde{i}'\equiv i'''\tilde{\eta}'$ mod $2i'''\tilde{\eta}'$ *and* $\pi_7(P^8)=\{\tilde{i}',\ i\nu^2\}\approx Z/16\oplus Z/2$.

By use of $(1.2)^{(n)}$ for $n=1$, 2 and 3, we have the following

Proposition 4.3. *$\pi_8(P^{2k})$ for $2\leq k\leq 4$ is isomorphic to the corresponding group in the following table:*

k	2	3	4
$\pi_8(P^{2k})$	$(2)^2$	$(2)^4$	$(2)^4$
gen.	$i'\widetilde{\nu^2},\ i\sigma$	$\tilde{\eta}'\eta,\ \widetilde{i\nu},\ i'\widetilde{\nu^2},\ i\sigma$	$\tilde{i}'\eta,\ i'''\widetilde{i\nu},\ i'\widetilde{\nu^2},\ i\sigma$

Here $\widetilde{i\nu}\in\langle i'',\ \tilde{i\eta},\ i\nu\rangle$.

Since $Sq^2:\tilde{H}^7(P^9)\to\tilde{H}^9(P^9)$ is nontrivial and $Sq^8:\tilde{H}^1(P^9)\to\tilde{H}^9(P^9)$ is trivial,

$\gamma_8 \equiv \tilde{i}'\eta \bmod \{i'''\widetilde{i\nu}, i'\widetilde{\nu^2}\}$ by Proposition 4.3. So we have the following

Proposition 4.4. i) $\gamma_8 \equiv \tilde{i}'\eta \bmod \{i'''\widetilde{i\nu}, i'\widetilde{\nu^2}\}$.
ii) $\pi_8(P^9)=\{i'''\widetilde{i\nu}, i'\widetilde{\nu^2}, i\sigma\} \approx (Z/2)^3$.

Remark 1. By Proposition 3.2, $\pi_6(P^8)=\{i''\tilde{i}\nu\} \approx Z/2$. So Propositions 2.2, 3.2, 4.2 and 4.4 overlap with Theorem 2.6 of [2] and Table 4.1 of [3].

Next we shall determine generators of $\pi^k(P^8)$ for $0 \leqq k \leqq 8$. We define an element $\overline{\nu p} \in \pi^0(EP^4)$ by $\overline{\nu p}i'=\nu p$, i.e., $\overline{\nu p} \in \langle \nu p, \bar{\eta}p, p' \rangle$. Then, by use of (1.2)′, we have $\pi^0(EP^4)=\{\eta\bar{\eta}', \overline{\nu p}\} \approx (Z/2)^2$.

Consider the following exact sequence induced from (1.2)″:

$$\pi^0(E^4P^2) \xleftarrow{(\tilde{i}\bar{\eta})^*} \pi^0(EP^4) \xleftarrow{i''^*} \pi^0(EP^6) \xleftarrow{p''^*} \pi^0(E^5P^2) \xleftarrow{(\tilde{i}\bar{\eta})^*} \pi^0(E^2P^4).$$

Then $(\tilde{i}\bar{\eta})^*\pi^0(E^kP^4) \subset \pi_{k+3}(S^0)\bar{\eta}=0$ for $k=1$ or 2, and so we have a short exact sequence

$$0 \longleftarrow \underset{\substack{\wr\wr \\ (Z/2)^2}}{\{\eta\bar{\eta}', \overline{\nu p}\}} \xleftarrow{i''^*} \pi^0(EP^6) \xleftarrow{p''^*} \underset{\substack{\wr\wr \\ (Z/2)^2}}{\{\overline{\nu^2}, \sigma p\}} \longleftarrow 0.$$

We define an element $\overline{\nu p}' \in \pi^0(EP^6)$ by $\overline{\nu p}'i''=\overline{\nu p}$, i.e., $\overline{\nu p}' \in \langle \overline{\nu p}, \tilde{i}\bar{\eta}, p'' \rangle$. Then $2\overline{\nu p}' \in 2\iota \circ \langle \overline{\nu p}, \tilde{i}\bar{\eta}, p'' \rangle = -\langle 2\iota, \overline{\nu p}, \tilde{i}\bar{\eta} \rangle p'' \subset -\langle 2\iota, 0, \bar{\eta} \rangle p'' = 2\pi^0(E^5P^2)p'' + \pi_5(S^0)\bar{\eta}p'' =0$. So, by (3.2), we have the following

Proposition 4.5. $\pi^0(EP^6)=\{\eta\bar{\bar{\eta}}, \overline{\nu p}', \overline{\nu^2}p'', \sigma p_6\} \approx (Z/2)^4$.

Consider the following exact sequence induced from (1.2)‴:

$$\pi^0(E^5P^2) \xleftarrow{(\bar{\eta}'p)^*} \pi^0(P^6) \xleftarrow{i'''^*} \pi^0(P^8) \xleftarrow{p'''^*} \pi^0(E^6P^2) \xleftarrow{(\bar{\eta}'p)^*} \pi^0(EP^6).$$

Then, by Proposition 3.1, $(\bar{\eta}'p)^*(\nu^2p_6)=0$ and $(\bar{\eta}'p)^*\bar{\bar{\eta}}=\bar{\bar{\eta}}\bar{\eta}'p$. Since $\bar{\bar{\eta}}$ is of order 8, $\bar{\bar{\eta}}\bar{\eta}'=2a\sigma$ for some integer a. So the first $(\bar{\eta}'p)^*$ is trivial. By (4.2) and Proposition 4.5, $(\bar{\eta}'p)^*(\eta\bar{\bar{\eta}})=0$, $(\bar{\eta}'p)^*(\overline{\nu^2}p'')=\varepsilon p$, $(\bar{\eta}'p)^*(\sigma p_6)=\sigma\eta p$ and $(\bar{\eta}'p)^*\overline{\nu p}' \in \pi_8(S^0)p$. Therefore we have

$$\pi_8(S^0)p_8=0 \tag{4.6}$$

and a short exact sequence

$$0 \longleftarrow \underset{\substack{\wr\wr \\ Z/8\oplus Z/2}}{\{\bar{\bar{\eta}}, \nu^2p_6\}} \xleftarrow{i'''^*} \pi^0(P^8) \xleftarrow{p'''^*} \underset{\substack{\wr\wr \\ Z/2}}{\{\overline{8\sigma}\}} \longleftarrow 0.$$

We define an element $\overline{\overline{\nu p}}_6 \in \pi^3(P^8)$ by $\overline{\overline{\nu p}}_6 i'''=\nu p_6$, i.e.,

$$\overline{\overline{\nu p_6}} \in \langle \nu p_6, \tilde{\eta}' p, p''' \rangle . \tag{4.7}$$

By (4.6) and (4.7), $2\nu\overline{\overline{\nu p_6}} \in 2\nu \circ \langle \nu p_6, \tilde{\eta}' p, p''' \rangle = \langle 2\nu, \nu p_6, \tilde{\eta}' p \rangle p''' \supset \langle 2\nu, \nu, \eta p \rangle p''' \supset \langle 2\nu, \nu, \eta \rangle p p''' = 0 \bmod 2\nu\pi^0(E^3P^2) = 0$. So we have $2\nu\overline{\overline{\nu p_6}} = 0$. By Theorem 1.3, we have a Kahn-Priddy map $\bar{\bar{\eta}}' \in \pi^0(P^8)$ of order 16 satisfying $\bar{\bar{\eta}}' i''' = \bar{\bar{\eta}}$, i.e.,

$$\bar{\bar{\eta}}' \in \langle \bar{\bar{\eta}}, \tilde{\eta}' p, p''' \rangle . \tag{4.8}$$

Therefore we have the following

Theorem 4.6. $8\bar{\bar{\eta}}' = \overline{8\sigma} p'''$ *and* $\pi^0(P^8) = \{\bar{\bar{\eta}}', \nu\overline{\overline{\nu p_6}}\} \approx Z/16 \oplus Z/2$.

By (4.6) and (4.8), $8\bar{\bar{\eta}}' \in 8\iota \circ \langle \bar{\bar{\eta}}, \tilde{\eta}' p, p''' \rangle = -\langle 8\iota, \bar{\bar{\eta}}, \tilde{\eta}' p \rangle p''' \subset -\langle 8\iota, 2a\sigma, p \rangle p''' \supset -a\langle 8\sigma, 2\iota, p \rangle p''' \ni a\overline{8\sigma} p''' \bmod 8\pi^0(E^6P^2)p''' + \pi_8(S^0)p_8 = 0$. So, by Theorem 4.6, a must be odd and we have

$$\bar{\bar{\eta}}\bar{\eta}' \equiv 2\sigma \bmod 4\sigma . \tag{4.9}$$

Consider the following exact sequence induced from (1.2)''':

$$\pi^0(EP^2) \xleftarrow{(\tilde{\eta}'p)^*} \pi^4(P^6) \xleftarrow{i'''^*} \pi^4(P^8) \xleftarrow{p'''^*} \pi^0(E^2P^2) \xleftarrow{(\tilde{\eta}'p)^*} \pi^3(P^6) .$$

Then, by Proposition 3.1 and (4.3), $(\tilde{\eta}'p)^*(\nu p_6) = 0$, $(\tilde{\eta}'p)^*\bar{p}_4 = \nu p$ and $\operatorname{Im} i'''^* = \{\bar{\eta} p''\} \approx Z/4$. So we have a short exact sequence

$$0 \longleftarrow \underset{\substack{\wr\wr \\ Z/4}}{\{\bar{\eta} p''\}} \xleftarrow{i'''^*} \pi^4(P^8) \xleftarrow{p'''^*} \underset{\substack{\wr\wr \\ Z/2}}{\{\eta^2 \bar{\eta}\}} \longleftarrow 0 .$$

We define an element $\overline{\bar{\eta} p''} \in \pi^4(P^8)$ by $\overline{\bar{\eta} p''} i''' = \bar{\eta} p''$, i.e., $\overline{\bar{\eta} p''} \in \langle \bar{\eta} p'', \tilde{\eta}' p, p''' \rangle$. Then $4\overline{\bar{\eta} p''} \in 4\iota \circ \langle \bar{\eta} p'', \tilde{\eta}' p, p''' \rangle = -\langle 4\iota, \bar{\eta} p'', \tilde{\eta}' p \rangle p''' \subset -\langle 4\iota, \pm 2\nu, p \rangle p''' \supset -\langle 4\nu, 2\iota, p \rangle p''' \ni \eta^2 \bar{\eta} p''' \bmod 4\pi^0(E^2P^2)p''' = 0$. So we have $4\overline{\bar{\eta} p''} = \eta^2 \bar{\eta} p'''$ and $\pi^4(P^8) = \{\overline{\bar{\eta} p''}\} \approx Z/8$.

By the similar arguments to the above, we have the following

Proposition 4.7. *$\pi^k(P^8)$ for $1 \leq k \leq 8$ is isomorphic to the corresponding group in the following table:*

k	1	2	3	4	5	6	7	8
$\pi^k(P^8)$	$(2)^2$	0	(2)	(8)	(2)	(2)	0	(2)
gen.	$\overline{\nu^2 p'''}$, σp_8		$\overline{\overline{\nu p_6}}$	$\overline{\bar{\eta} p''}$	$\eta \bar{\eta} p'''$	$\bar{\eta} p'''$		p_8

Here $\overline{\bar{\eta} p''} \in \langle \bar{\eta} p'', \tilde{\eta}' p, p''' \rangle$ *and* $4\overline{\bar{\eta} p''} = \eta^2 \bar{\eta} p'''$.

Remark 2. Hideaki Ōshima pointed out the following:

Let $V_{n,k}$ denote the Stiefel manifold of k-frames in R^n. Then, according to [9], $\pi^{n-m}(P^n) \approx \pi_{N+m-n-1}(V_{N-1,n})$, where $N=j2^{s(n-1)}$ for a large integer j. So the group structures of $\pi^{2n-m}(P^{2n})$ for $n \leqq 4$ are also obtained from the well-known works of G. F. Paechter and C. S. Hoo.

By (4.5), (4.8) and (4.9), $\bar{\bar{\eta}}'\tilde{i}' \in \langle \bar{\bar{\eta}}, \tilde{\eta}'p, i \rangle \supset \langle 2\sigma, p, i \rangle \ni -\sigma \bmod \bar{\bar{\eta}}\pi_7(P^6) + \pi^0(E^6P^2)i = \{2\sigma\}$. So we have

$$(4.10) \qquad \bar{\bar{\eta}}'\tilde{i}' \equiv \sigma \bmod 2\sigma .$$

§ 5. Determination of $\{P^8, P^8\}$

By use of (1.1) and Proposition 4.7, we have the following

Proposition 5.1. *$\{P^8, E^kP^2\}$ for $0 \leqq k \leqq 6$ is isomorphic to the corresponding group in the following table:*

k	0	1	2	3	4	5	6
$\{P^8, E^kP^2\}$	$(2)^2$	(2)	$(2)^2$	$(2)^2$	(4)	(2)	(2)
gen.	$i\overline{\nu^2}p''', i\sigma p_8$	$\overset{\sim}{\bar{\bar{\nu p}}}_6$	$\tilde{\eta}\eta\bar{\eta}p''', i\bar{\bar{\nu p}}_6$	$\tilde{\eta}\bar{\eta}p''', i\overline{\bar{\eta}p''}$	$\widetilde{\bar{\eta}p'''}$	$i\bar{\eta}p'''$	p'''

Here $\overset{\sim}{\bar{\bar{\nu p}}}_6 \in \langle i, 2\iota, \bar{\bar{\nu p}}_6 \rangle$, $\widetilde{\bar{\eta}p'''} \in \langle i, 2\iota, \bar{\eta}p''' \rangle$ *and* $2\widetilde{\bar{\eta}p'''} = i\eta\bar{\eta}p'''$.

Consider the following exact sequence induced from (1.2)''':

$$\{E^5P^2, P^2\} \xleftarrow{(\tilde{\eta}'p)^*} \{P^6, P^2\} \xleftarrow{i'''^*} \{P^8, P^2\} \xleftarrow{p'''^*} \{E^6P^2, P^2\} \xleftarrow{(\tilde{\eta}'p)^*} \{EP^6, P^2\} .$$

Then, by Proposition 3.5, (4.3) and (4.9), $(\tilde{\eta}'p)^*(i\nu\bar{p}_4) = i\nu^2p$ and $(\tilde{\eta}'p)^*(i\eta) = (\tilde{\eta}'p)^*(i\nu^2p_6) = 0$. We have also $(\tilde{\eta}'p)^*\widetilde{\nu\bar{p}}_4 \in \langle i, 2\iota, \nu\bar{p}_4 \rangle \circ \tilde{\eta}'p \subset \langle i, 2\iota, \nu^2 \rangle p \ni \widetilde{\nu^2}p$ mod $i\sigma p$. So we have

$$(5.1) \qquad \widetilde{\nu^2}p_8 \equiv 0 \bmod i\sigma p_8 .$$

Remark 1. By the same arguments as the ones in the proof of Lemma 5.2, we have $\widetilde{\nu^2}p_8 = 0$.

Consider the following exact sequence induced from (1.2)':

$$\{P^8, EP^2\} \xrightarrow{(\tilde{\eta}p)_*} \{P^8, P^2\} \xrightarrow{i'_*} \{P^8, P^4\} \xrightarrow{p'_*} \{P^8, E^2P^2\} \xrightarrow{(\tilde{\eta}p)_*} \{P^8, EP^2\} .$$

Then, by Proposition 5.1 and (4.7), $(\tilde{\eta}p)_*\overset{\sim}{\bar{\bar{\nu p}}}_6 = \tilde{\eta}\bar{\bar{\nu p}}_6 \in \tilde{\eta}\langle \nu p_6, \tilde{\eta}'p, p''' \rangle = \langle \tilde{\eta}, \nu p_6, \tilde{\eta}'p \rangle p''' \supset \langle \tilde{\eta}, \nu, \eta p \rangle p''' \supset \langle \tilde{\eta}, \nu, \eta \rangle p_8$ mod $\tilde{\eta}\pi^0(E^3P^2)p''' = 0$. Since $p\langle \tilde{\eta}, \nu, \eta \rangle \subset \langle \eta, \nu, \eta \rangle = \nu^2$, $\langle \tilde{\eta}, \nu, \eta \rangle \ni \widetilde{\nu^2}$ mod $i\sigma$. So, by (5.1), $\tilde{\eta}\bar{\bar{\nu p}}_6 = 0$ or $i\sigma p_8$.

Lemma 5.2. $\bar{\eta}\bar{\bar{\nu}}p_6=0$ *in* $\{P^8, P^2\}$ *and* $i\sigma p_8\neq 0$ *in* $\{P^8, P^4\}$.

Proof. It suffices to prove $i\sigma p_8\neq 0$ in $\{P^8, P^4\}$. Consider the following EHP-sequence:

$$\begin{array}{ccccc} [E^9P^8, E(E^7P^4\wedge E^7P^4)] & \xrightarrow{\Delta} & [E^7P^8, E^7P^4] & \xrightarrow{E} & [E^8P^8, E^8P^4] \longrightarrow 0. \\ \wr\wr & & \Big\downarrow H & & \wr\wr \\ [E^9P^8, E^{16}P^2] & & & & \{P^8, P^4\} \\ \wr\wr & & & & \\ \{P^8, E^7P^2\} & & [E^7P^8, E(E^6P^4\wedge E^6P^4)] & & \\ & & \wr\wr & & \\ & & \{P^8, E^7P^2\} & & \end{array}$$

Here we have used the following: $P^2\wedge P^2=EP^2\cup_{2\iota'_2}C(EP^2)$ and so the 3-skeleton of $P^2\wedge P^2$ is stably equivalent to $EP^2\vee S^3$. Then, by use of (1.2)‴, $\{P^8, E^7P^2\}=\{ip_8\}\approx Z/2$. By inspecting Proposition 2.2 of [10], $H(E^7(i'i)\circ\sigma\circ E^7p_8)=H(E^7(i'i)\circ\sigma)\circ E^7p_8=E^{14}i\circ E^7p_8\neq 0$ since $H(\sigma)=\iota_{15}$. Here ι_n denotes the identity class of S^n. By Proposition 3.3 of [6], (5.16) of [10] and Lemma 3.1 of [8], $\Delta(E^{16}i\circ E^9p_8)=\Delta(E^{16}i)\circ E^7p_8=E^7i\circ\Delta(\iota_{17})\circ E^7p_8=E^7i\circ(2\sigma-E\sigma')\circ E^7p_8=E^7i\circ E\sigma'\circ E^7p_8$, where σ' denotes the generator of the 2-component of $[S^{14}, S^7]$. So we have $H(\Delta(E^{16}i\circ E^9p_8))=0$. Therefore $E^7(i'i)\circ\sigma\circ E^7p_8$ is not in the image of Δ. This completes the proof.

By Proposition 5.1 and Lemma 5.2, we have a short exact sequence

$$0\longrightarrow \underset{\underset{(Z/2)^2}{\wr\wr}}{\{i\overline{\nu^2}p''', i\sigma p_8\}} \xrightarrow{i'_*} \{P^8, P^4\} \xrightarrow{p'_*} \underset{\underset{(Z/2)^2}{\wr\wr}}{\{\bar{\eta}\eta\bar{\eta}p''', i\bar{\bar{\nu}}p_6\}} \longrightarrow 0.$$

We define an element $\widetilde{\bar{\eta}\eta\bar{\eta}}\in\{E^6P^2, P^4\}$ by $p'\widetilde{\bar{\eta}\eta\bar{\eta}}=\bar{\eta}\eta\bar{\eta}$, i.e.,

$$\widetilde{\bar{\eta}\eta\bar{\eta}}\in\langle i', \bar{\eta}p, \bar{\eta}\eta\bar{\eta}\rangle. \tag{5.2}$$

By (2.3) and Proposition 4.7, there exists an element $\widetilde{i\bar{\bar{\nu}}}p_6\in\{P^8, P^4\}$ and by Proposition 2.2 and Lemma 5.2, $2\widetilde{i\bar{\bar{\nu}}}p_6=i'\bar{\eta}\bar{\bar{\nu}}p_6=0$. This leads us to the following

Proposition 5.3. $\{P^8, P^4\}=\{\widetilde{\bar{\eta}\eta\bar{\eta}}p''', \widetilde{i\bar{\bar{\nu}}}p_6, i\overline{\nu^2}p''', i\sigma p_8\}\approx(Z/2)^4$.

Lemma 5.4. i) $\eta\overline{\bar{\eta}p''}=0$. ii) $\bar{\eta}\overline{\bar{\eta}p''}=0$. iii) $\widetilde{\eta\bar{\eta}p'''}=\pm 2\overline{\bar{\eta}p''}$.

Proof. By Proposition 3.1, $\eta\bar{\eta}p''=\eta\circ 2\bar{p}_4=0$. By Proposition 4.7, $\eta\overline{\bar{\eta}p''}\in\eta\langle\bar{\eta}p'', \bar{\eta}'p, p'''\rangle=\langle\eta, \bar{\eta}p'', \bar{\eta}'p\rangle p'''\subset\pi^0(E^3P^2)p'''=0$. Therefore $\bar{\eta}\widetilde{\overline{\eta p''}}\in\langle i, 2\iota, \eta\rangle\overline{\bar{\eta}p''}=i\langle 2\iota, \eta, \overline{\bar{\eta}p''}\rangle\subset i_*\pi^2(P^8)=0$. By Propositions 4.7 and 5.1, $2\widetilde{\eta\bar{\eta}p'''}=\eta^2\bar{\eta}p'''=4\overline{\bar{\eta}p''}\in\pi^4(P^8)=\{\overline{\bar{\eta}p''}\}\approx Z/8$. This completes the proof.

Remark 2. i) By Theorem 4.6, Proposition 4.7, (4.6) and Lemma 5.4. i),

$\langle\eta, \nu, p_8\rangle \ni 0 \mod \eta\pi^4(P^8)=0$. So, by Proposition 4.3, $\widetilde{i\nu}p_8 \in \langle i'', \tilde{i}\bar{\eta}, i\nu\rangle p_8 = -i''\langle\tilde{i}\bar{\eta}, i\nu, p_8\rangle \supset i''\tilde{i}\langle\eta, \nu, p_8\rangle \ni 0 \mod i''\pi_8(P^4)p_8=\{i'\widetilde{\nu^2}p_8, i\sigma p_8\}$. Therefore, by (5.1), $\widetilde{i\nu}p_8 \equiv 0 \mod i\sigma p_8$. Since $\widetilde{i\nu}p_8$ can desuspend on E^6P^8, we have $\widetilde{i\nu}p_8=0$.

ii) By Proposition 4.7, $\tilde{i}'\eta p_8=0$. So, by i), Proposition 4.4 and Remark 1, $\gamma_8 p_8=0$.

Conjecture. *$\gamma_{2n}p_{2n}=0$ for all n.*

Consider the following exact sequence induced from (1.2)″:

$$\{P^8, E^3P^2\} \xrightarrow{(\tilde{i}\bar{\eta})_*} \{P^8, P^4\} \xrightarrow{i''_*} \{P^8, P^6\} \xrightarrow{p''_*} \{P^8, E^4P^2\} \xrightarrow{(\tilde{i}\bar{\eta})_*} \{P^8, EP^4\}\,.$$

Then, by Propositions 2.5, 5.1 and Lemma 5.4, $(\tilde{i}\bar{\eta})(\bar{\eta}\bar{\eta}p''')=0$, $(\tilde{i}\bar{\eta})(\overline{i\bar{\eta}p''})=\tilde{i}\eta\overline{\bar{\eta}p''}=0$ and $(\tilde{i}\bar{\eta})\widetilde{\bar{\eta}p'''}=\pm 2\tilde{i}\overline{\bar{\eta}p''}=i'\bar{\eta}\overline{\bar{\eta}p''}=0$. So we have a short exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & \{\widetilde{\bar{\eta}\eta\bar{\eta}p'''}, \tilde{\bar{\bar{i\nu}}}p_6, i\overline{\nu^2}p''', i\sigma p_8\} & \xrightarrow{i''_*} & \{P^8, P^6\} & \xrightarrow{p''_*} & \{\widetilde{\bar{\eta}p'''}\} & \longrightarrow 0\,. \\ & \wr\wr & & & & \wr\wr & \\ & (Z/2)^4 & & & & Z/4 & \end{array}$$

We define an element $\widetilde{\bar{\eta}p'''}' \in \{P^8, P^6\}$ by $p''\widetilde{\bar{\eta}p'''}'=\widetilde{\bar{\eta}p'''}$, i.e.,

$$\widetilde{\bar{\eta}p'''}' \in \langle i'', \tilde{i}\bar{\eta}, \widetilde{\bar{\eta}p'''}\rangle\,. \tag{5.3}$$

By use of (1.2)‴ and Proposition 5.1, we have a short exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \{P^8, P^6\} & \xrightarrow{i'''_*} & \{P^8, P^8\} & \xrightarrow{p'''_*} & \{p'''\} & \longrightarrow 0\,. \\ & & & & \wr\wr & \\ & & & & Z/2 & \end{array}$$

Since ι_8' is of order 16, there exists an element of order 8 in $\{P^8, P^6\}$ which is mapped onto $2\iota_8'$ by i'''_*. So this element must be $\widetilde{\bar{\eta}p'''}'$ modulo elements of order 2. Therefore we have $2\iota_8' \equiv i'''\widetilde{\bar{\eta}p'''}' \mod \{\textit{some elements of order } 2\}$. By Theorem 4.6, $\bar{\bar{\eta}}(4\widetilde{\bar{\eta}p'''}')=8\bar{\bar{\eta}}'=\overline{8\sigma}p'''$. This leads us to

$$4\bar{\bar{\eta}}\widetilde{\bar{\eta}p'''}'=\overline{8\sigma}p'''. \tag{5.4}$$

Lemma 5.5. $\bar{\eta}'\widetilde{\bar{\eta}\eta\bar{\eta}} \equiv \overline{8\sigma} \mod \pi_8(S^0)p\,.$

Proof. By (2.2) and (5.2), $\bar{\eta}'\widetilde{\bar{\eta}\eta\bar{\eta}} \in \langle\bar{\eta}, \bar{\eta}p, \bar{\eta}\eta\bar{\eta}\rangle \subset \langle\bar{\eta}, \bar{\eta}\eta, \eta\bar{\eta}\rangle \mod \bar{\eta}\{E^6P^2, P^2\}+\pi_5(S^0)(\eta\bar{\eta})=\pi_8(S^0)p$. By (4.9) and Proposition 4.1. ii), $8\sigma=4\bar{\bar{\eta}}\bar{\eta}' \equiv \bar{\eta}'\widetilde{\bar{\eta}\eta^2} \mod \bar{\bar{\eta}}(i\nu^2)=0$. So, by (2.2) and (4.1), $8\sigma=\bar{\eta}'\widetilde{\bar{\eta}\eta^2} \in \langle\bar{\eta}, \bar{\eta}p, \bar{\eta}\eta^2\rangle \subset \langle\bar{\eta}, \bar{\eta}\eta, \eta^2\rangle \mod \bar{\eta}\pi_7(P^2)+\pi_5(S^0)\eta^2=0$. By (2.2), (4.1) and (5.2), $\bar{\eta}'\widetilde{\bar{\eta}\eta\bar{\eta}}i \in \langle\bar{\eta}, \bar{\eta}p, \bar{\eta}\eta^2\rangle \ni \bar{\eta}'\widetilde{\bar{\eta}\eta^2} \mod \bar{\eta}\pi_7(P^2)+\pi^0(E^2P^2)\bar{\eta}\eta^2=0$. Therefore $\bar{\eta}'\widetilde{\bar{\eta}\eta\bar{\eta}}i=\bar{\eta}'\widetilde{\bar{\eta}\eta^2}=8\sigma$. This completes the proof.

Remark 3. According to [7], the equality $\langle\bar{\eta}, \tilde{\eta}\eta, \eta^2\rangle=8\sigma$ holds on S^5.

By Lemma 5.5 and (4.6), $\bar{\bar{\eta}}(\widetilde{i''\tilde{\eta}\eta\bar{\eta}p'''})=\overline{8\sigma}p'''$. By (2.4) and Theorem 4.6, $\bar{\bar{\eta}}(i''\tilde{i}\overline{\overline{\nu p}}_6)=\nu\overline{\overline{\nu p}}_6\neq 0$. So, by (5.4), we have

(5.5) $$4\widetilde{\bar{\eta}p'''}'\equiv i''\widetilde{\tilde{\eta}\eta\bar{\eta}p'''} \bmod \{i\overline{\nu^2}p''', i\sigma p_8\}$$

and

(5.6) $$2\iota_8'\equiv i'''\widetilde{\bar{\eta}p'''}' \bmod \{i''\tilde{i}\overline{\overline{\nu p}}_6, i\overline{\nu^2}p''', i\sigma p_8\}\ .$$

Therefore (5.5) and (5.6) lead us to the following

Theorem 5.6.

i) $\{P^8, P^6\}=\{\widetilde{\bar{\eta}p'''}', i''\tilde{i}\overline{\overline{\nu p}}_6, i\overline{\nu^2}p''', i\sigma p_8\}\approx Z/8\oplus(Z/2)^3$.

ii) $\{P^8, P^8\}=\{\iota_8', i''\tilde{i}\overline{\overline{\nu p}}_6, i\overline{\nu^2}p''', i\sigma p_8\}\approx Z/16\oplus(Z/2)^3$.

We put $\alpha=i''\tilde{i}\overline{\overline{\nu p}}_6$, $\beta=i\overline{\nu^2}p'''$ and $\gamma=i\sigma p_8$. Then $\alpha^2=\beta^2=\gamma^2=0$, $\alpha\beta=\beta\alpha=0$, $\beta\gamma=\gamma\beta=0$ and $\alpha\gamma=\gamma\alpha=0$. So we have the following

Corollary. $\xi(P^8)\approx(Z/16)^*\times(Z/2)^3$.

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