Publ. RIMS, Kyoto Univ.
22 (1986), 103-117

A Dichotomy for Derivations on $\mathcal{O}_n$

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Abstract

Let $\mathcal{O}_n$ be the Cuntz algebra generated by $s_1, \cdots, s_n$, and let $\mathcal{P}(\mathcal{O}_n)$ be the *-subalgebra of *-polynomials in the generators. We show that if δ is a gauge-invariant derivation mapping $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{P}(\mathcal{O}_n)$, and δ is approximately inner, then δ is inner.

§ 1. Introduction

The Cuntz algebra $\mathcal{O}_n$ is uniquely defined as the C*-algebra generated by $n=2, 3, \cdots$ isometries $s_1, \cdots, s_n$ satisfying

$$s_i^* s_j = \delta_{ij} 1 , \quad \sum_{j=1}^{n} s_i s_i^* = 1 ,$$

[7]. There is a canonical representation of the n-dimensional unitary group $U(n)$ in the automorphism group of $\mathcal{O}_n$ defined by

$$\alpha_g(s_i) = \sum_{k=1}^{n} g_{ki} s_k$$

for $g = [g_{ij}]_{i,j=1}^{n} \in U(n)$. In [4, Theorem 2.4] it was proved that if δ is a *-derivation defined on the $U(n)$-finite elements

$$\mathcal{O}_{nF}^{\alpha} = \{x \in \mathcal{O}_n \mid C\alpha_{U(n)}(x) \text{ is finite dimensional}\}$$

for this action, then δ has a unique decomposition

$$\delta = \delta_0 + \tilde{\delta} ,$$

where δ_0 is the generator of a one-parameter subgroup of the action α, and $\tilde{\delta}$ is bounded. Now, none of the generators δ_0 are approximately inner on the

Communicated by H. Araki, July 30, 1985.
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polynomial *-algebra $\mathcal{P}(\mathcal{O}_n)$ generated by $s_1, \cdots, s_n$, except for $\delta_0=0$, and hence this theorem has the remarkable consequence that if $\delta: \mathcal{O}_{nF}^{\alpha} \to \mathcal{O}_n$ is any derivation which is approximately inner on $\mathcal{P}(\mathcal{O}_n)$, then δ is actually inner, [4, Remark 2 to Theorem 2.4] (See also the end of § 2). This paper grew out of a desire to understand this fact more algebraically, and hence pave the ground for an understanding of the Lie algebra of all derivations mapping $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{P}(\mathcal{O}_n)$. It is already known that all these derivations are pregenerators, i. e. they are closable and the closures are infinitesmal generators of one-parameter groups of *-automorphisms, [3, Corollary 2.6]. Also, $\mathcal{P}(\mathcal{O}_n)$ consists of analytic elements for the derivations in Der$(\mathcal{P}(\mathcal{O}_n), \mathcal{P}(\mathcal{O}_n))$, [3], and hence it seems plausible that the exponential map defines a representation of the covering group of Der$(\mathcal{P}(\mathcal{O}_n), \mathcal{P}(\mathcal{O}_n))$, see [13]. Here we will take up the more restricted problem whether all approximately inner derivations in Der$(\mathcal{P}(\mathcal{O}_n), \mathcal{P}(\mathcal{O}_n))$ are inner, and our main result, Theorem 4.1, is that this is indeed true for gauge-invariant derivations, i. e. derivations commuting with the restriction of α to the centre $\boldsymbol{T}$ of $U(n)$. We expect this also to be true for derivations which are not gauge invariant, but we do not have a proof for the moment.

As a byproduct of these considerations we will in § 2 give an alternative construction of the action of the symplectic group $U(n, 1)$ on $\mathcal{O}_n$ defined in [16] and studied further in [6]; our construction is based on infinitesmal analysis. We will also give an alternative introduction to the Cuntz states from that of [8], [6], and use these states to show that none of the non-zero generators of the $U(n, 1)$ action are approximately inner.

In section 5 we will give examples showing that if $\delta \in$ Der$(\mathcal{P}(\mathcal{O}_n), \mathcal{O}_n)$, then δ is not necessarily a pregenerator, although $\pm\delta$ are dissipative by [3, Proposition 3.5], and also that δ need not be inner if it is approximately inner, even when δ is gauge invariant.

§ 2. Preliminaries

First we recall some facts about Cuntz and Toeplitz algebras from [6], [7], [8], [11], [12], [14], [15], [16].

Let $\mathcal{H}_n$ be a n-dimensional complex Hilbert space, where $2 \leq n < \infty$, with complete orthonormal basis $\{\xi_i: i=1, 2, \cdots, n\}$. The Toeplitz algebra $\mathcal{T}_n$ is the unique unital C*-algebra generated by the range of a linear map l defined on $\mathcal{H}_n$ such that

$$l(\psi)^* l(\xi) = \langle \psi, \xi \rangle 1, \qquad \psi, \xi \in \mathcal{H}_n,$$

and

$$\sum_{i=1}^{n} l(\xi_i) l(\xi_i)^* < 1.$$

The Cuntz algebra $\mathcal{O}_n$ is the unique unital C*-algebra generated by the range of a linear map s defined on $\mathcal{H}_n$ satisfying

$$s(\psi)^*s(\xi)=\langle\psi, \xi\rangle 1, \qquad \psi, \xi\in\mathcal{H}_n,$$

and

$$\sum_{i=1}^{n} s(\xi_i)s(\xi_i)^*=1.$$

We write l_i for $l(\xi_i)$ and s_i for $s(\xi_i)$. Then the Toeplitz algebra $\mathcal{T}_n$ can be regarded as a C*-subalgebra of the Cuntz algebra $\mathcal{O}_n$, by identifying l_i in $\mathcal{T}_n$, with s_i in $\mathcal{O}_{n+1}$ for $1\leqq i\leqq n$. Also $\mathcal{T}_n$ is an extension of $\mathcal{O}_{n+1}$ by the compacts. More precisely, let $\mathcal{F}_n=\mathcal{F}(\mathcal{H}_n)$ denote the full Fock space

$$\bigoplus_{m=0}^{\infty}(\otimes^m\mathcal{H}_n),$$

where $\otimes^0\mathcal{H}_n$ denotes a one-dimensional Hilbert space spanned by a unit vector Ω called the vacuum. Then the projection

$$p=1-\sum_{i=1}^{n} l_i l_i^*$$

generates a closed two sided ideal $\mathcal{K}_n$ in $\mathcal{T}_n$, which is isomorphic to the compact operators on $\mathcal{F}_n$, and contains p as a minimal projection. Moreover, $\mathcal{K}_n$ is generated by matrix units

$$l_{i_1}\cdots l_{i_r}pl_{j_m}^*\cdots l_{j_1}^*$$

which can be identified with the rank one operators

$$[\xi_{i_1}\otimes\cdots\otimes\xi_{i_r}]\otimes[\xi_{j_1}\otimes\cdots\otimes\xi_{j_m}]^-$$

on $\mathcal{F}_n$, where $\xi_{i_1}\otimes\cdots\otimes\xi_{i_r}=\Omega$ if $r=0$, and $\eta\otimes\bar{\phi}$ denotes the rank one operator $\psi\to\langle\phi, \psi\rangle\eta$ on $\mathcal{F}_n$, ψ, ϕ, $\eta\in\mathcal{F}_n$. Then if ϕ denotes the quotient map from $\mathcal{T}_n$ onto $\mathcal{T}_n/\mathcal{K}_n$, $\mathcal{O}_n$ is isomorphic to $\mathcal{T}_n/\mathcal{K}_n$, if we identify s_i with $\phi(l_i)$, $i=1, \cdots, n$.

The Fock or regular representation of $\mathcal{T}_n$ on $\mathcal{F}_n$ is constructed as follows. Define bounded operators $l(\psi)$ on $\mathcal{F}_n$, for $\psi\in\mathcal{H}_n$, by

$$l(\psi)\eta=\psi\otimes\eta \quad \eta\in\otimes^m\mathcal{H}_n, \qquad m\geqq 1,$$
$$l(\psi)\Omega=\psi.$$

If $u\in U(n)=U(\mathcal{H}_n)$, the group of unitaries on $\mathcal{H}_n$, let $\Gamma(u)$ denote the unitary

$$\bigoplus_{m=0}^{\infty}(\otimes^m u)$$

on $\mathcal{F}_n$. Then

$$\Gamma(u)l(\psi)\Gamma(u)^*=l(u\psi), \qquad \psi\in\mathcal{H}_n.$$

There is an automorphism $\beta_u=\mathrm{Ad}\,\Gamma(u)|_{\mathcal{T}_n}$ on $\mathcal{T}_n$ leaving $\mathcal{K}_n$ invariant defined by

$$\beta_u(l(\psi))=l(u\psi), \qquad \psi\in\mathcal{H}_n,$$

and an induced automorphism α_u on $\mathcal{O}_n=\mathcal{T}_n/\mathcal{K}_n$ defined by

$$\alpha_u s(\psi)=s(u\psi), \qquad \psi\in\mathcal{H}_n.$$

In particular, if $\gamma=\alpha|_T$, then the fixed point algebra $\mathcal{A}=\mathcal{A}(\mathcal{H}_n)=\mathcal{O}_n^\gamma$ is a UHF algebra, isomorphic to $\bigotimes_1^\infty M_n$, where we identify

$$\{s_{i_1}\cdots s_{i_r}s_{j_r}^*\cdots s_{j_1}^*:\ 1<i_1,\ \cdots,\ i_r,\ j_1,\ \cdots,\ j_r\leqq n\}$$

in $\mathcal{A}$ with canonical matrix units

$$e_{i_1j_1}\otimes\cdots\otimes e_{i_rj_r}$$

in $\bigotimes_1^r M_n\subset\bigotimes_1^\infty M_n$, if $\{e_{ij}:1\leqq i,\ j\leqq n\}$ are canonical matrix units in M_n, the algebra of $n\times n$ complex matrices.

We let $\mathcal{P}(\mathcal{O}_n)$ denote the *-algebra generated by $s_1,\cdots,s_n$ and $\mathcal{P}(\mathcal{A})=\mathcal{P}(\mathcal{O}_n)\cap\mathcal{A}$. Recall, [3], that there is a bijection between derivations δ from the polynomial algebra $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{O}_n$ and skew adjoint operators L in $\mathcal{O}_n$, given by

$$\delta_L(s_i)=Ls_i$$

$$L_\delta=\sum_{i=1}^n \delta(s_i)s_i^*.$$

Then δ is gauge invariant (i.e. $\delta\gamma(t)=\gamma(t)\delta$ on $\mathcal{P}(\mathcal{O}_n)$, or $\delta(\mathcal{A})\subset\mathcal{A}$) if and only if $L_\delta\in\mathcal{A}$. If $\delta=\mathrm{ad}\,H|_{\mathcal{P}(\mathcal{O}_n)}$, where $H\in\mathcal{O}_n$, then $L_\delta=H-\sigma(H)$, if σ denotes the shift $\sum_{i=1}^n s_i(\cdot)s_i^*$, (Note that $\sigma|_{\mathcal{A}}$ is the one-sided shift on $\bigotimes_1^\infty M_n$). In this case δ is gauge-invariant if and only if H is so. Thus an arbitrary δ on $\mathcal{P}(\mathcal{O}_n)$ is inner (respectively approximately inner) if and only if $L_\delta\in(1-\sigma)(\mathcal{O}_n)$ (respectively $L_\delta\in\overline{(1-\sigma)(\mathcal{O}_n)}$). Also a derivation δ leaves $\mathcal{P}(\mathcal{O}_n)$ (respectively $\mathcal{P}(\mathcal{A})$) globably invariant if and only if $L_\delta\in\mathcal{P}(\mathcal{O}_n)$ (respectively $L_\delta\in\mathcal{P}(\mathcal{A})$).

As an example of the use of the correspondence between L and δ we give an infinitesimal construction of the action of $U(n, 1)$ on $\mathcal{O}_n$ defined by Voiculescu [16] (see also [6]). We take $U(n, 1)$ to be the group of $(n+1)\times(n+1)$ invertible matrices A with

$$AJA^*=J,$$

where $J=\begin{pmatrix}-1 & 0\\ 0 & 1_n\end{pmatrix}$, and 1_n is the identity $n\times n$ matrix. We will write

$$A=\begin{pmatrix}a_0 & \langle\xi_1,\ \cdot\rangle\\ \xi_2 & A_1\end{pmatrix},$$

where $a_0\in\boldsymbol{C}$, A_1 is an $n\times n$ matrix, and ξ_1, ξ_2 are vectors in $\mathcal{H}_n$. The Lie algebra $u(n, 1)$ of $U(n, 1)$ consists of $(n+1)\times(n+1)$ matrices of the form

$$X=\begin{pmatrix}x_0 & \langle\xi,\ \cdot\rangle\\ \xi & X_1\end{pmatrix},$$

where $x_0\in i\boldsymbol{R}$, $X_1^*=-X_1\in M_n$ and $\xi\in\mathcal{H}_n$. Define $sXs^*=\sum_{ij}X_{ij}s_is_j^*$ if $X=[X_{ij}]\in M_n$. We can then define for each $X\in u(n, 1)$ a skew adjoint operator

L_X in $\mathcal{P}(\mathcal{O}_n)$ by

$$L_X = x_0 1 + s(\xi) - s(\xi)^* + sX_1 s^*.$$

We let δ_X denote the corresponding derivation of $\mathcal{P}(\mathcal{O}_n)$. Then straightforward computations show that $X \to \delta_X$ is a Lie algebra homomorphism from $u(n, 1)$ into $\mathrm{Der}(\mathcal{P}(\mathcal{O}_n), \mathcal{P}(\mathcal{O}_n))$. This amounts to showing

$$L_{[X,Y]} = [L_Y, L_X] + \delta_X(L_Y) - \delta_Y(L_X)$$

for all $X, Y \in u(n, 1)$. By [3, Corollary 2.6] and its proof, it follows that $\mathcal{P}(\mathcal{O}_n)$ consists of analytic elements for each δ_X, δ_X is closable and its closure $\bar{\delta}_X$ generates a one-parameter group of *-automorphisms of $\mathcal{O}_n$. By [13, Theorem 3.1], we can thus integrate $X \to \delta_X$ to get an action α of $U(n, 1)$ on $\mathcal{O}_n$ such that

$$\alpha_{\exp tX} = \exp t\bar{\delta}_X\,, \qquad t \in \boldsymbol{R}\,, \quad x \in u(n, 1)\,.$$

The exponentiated action of the simply connected covering group $\tilde{U}(n, 1)$ can be seen, by a direct calculation, to be trivial on the kernel of the covering map, $\tilde{U}(n, 1) \to U(n, 1)$. The corresponding action β of $u(n, 1)$ on $\mathcal{F}_n$ is unitarily implemented by an action u on $\mathcal{F}_n$, [17]. In fact

$$du(X) = d\Gamma(X_1 - x_0) - x_0 1 - a(\xi) + a^*(\xi)$$

where $a^*(\xi)$, $a(\xi)$ are the unbounded 'creation' and 'annihilation' operators:

$$a^*(\xi)(\eta_1 \otimes \cdots \otimes \eta_m) = \sum_{i=0}^{m} \eta_1 \otimes \cdots \otimes \eta_i \otimes \xi \otimes \eta_{i+1} \otimes \cdots \otimes \eta_m$$

$$a(\xi)(\eta_1 \otimes \cdots \otimes \eta_n) = \sum_{i=1}^{m} \langle \xi, \eta_i \rangle \eta_1 \otimes \cdots \otimes \eta_{i-1} \otimes \eta_{i+1} \otimes \cdots \otimes \eta_n\,.$$

Then $d\beta(X)(Y) = \mathrm{ad}(du(X))(Y)$ for $Y \in \mathcal{P}(\mathcal{F}_n)$ (acting on $\mathcal{P}(\mathcal{F}_n)\Omega$).

In considering the range of $1-\sigma$, it is useful to have available a large class of shift invariant states. A family of shift invariant states was constructed by Cuntz [8], and appeared in [6] as the weak limits of $\alpha_{\exp tX}(t \to \pm\infty)$, for hyperbolic elements $X \in u(n, 1)$. Here we give an alternative construction of these states based on the following general considerations about completely positive maps.

There is a well known correspondence between endomorphisms α of $\mathcal{O}_n$ and unitaries u in $\mathcal{O}_n$ [8], (and, as we just explained, between derivations on $\mathcal{P}(\mathcal{O}_n)$ and skew adjoint operators in $\mathcal{O}_n$, [3]), given by $\alpha(s_i) = us_i$ and $u = \sum_j \alpha(s_j)s_j^*$. Now let ϕ be a completely positive map $\mathcal{O}_n$ into itself. Then

$$x = \sum_{i=1}^{n} \phi(s_i)s_i^*$$

is a contraction since $x = [(\phi \otimes 1)(S)]S^*$, where $S = \begin{pmatrix} s_1 & \cdots & s_n \\ & 0 & \end{pmatrix}$ is a partial isometry in $M_n(\mathcal{O}_n)$. Also, $\phi(s_i) = xs_i$. Conversely:

Proposition 2.1. *Let x be a contraction in $\mathcal{O}_n$. Then there exists a completely positive unital linear map ϕ on $\mathcal{O}_n$, such that*

$$\phi(s_i)=xs_i\,.$$

If x is a co-isometry, then ϕ is unique and given by

$$(*)\qquad \phi(s_{i_1}\cdots s_{i_r}s_{j_m}^*\cdots s_{j_1}^*)=(xs_{i_1})(xs_{i_2})\cdots(xs_{i_r})(xs_{j_m})^*\cdots(xs_{j_1})^*.$$

Proof. Define a morphism $\pi:\mathcal{O}_n\to M_2(\mathcal{O}_n)$ by

$$\pi(s_i)=u\begin{pmatrix} s_i & 0\\ 0 & s_i\end{pmatrix},$$

where $u=\begin{pmatrix} x & -(1-xx^*)^{1/2}\\ (1-x^*x)^{1/2} & x^*\end{pmatrix}$ is a unitary dilation of x. If $V=(1,0)$, define

$$\phi(a)=V\pi(a)V^*,\qquad a\in\mathcal{O}_n\,.$$

If $xx^*=1$, it is clear that (*) holds. In this case let θ be any completely positive unital linear map such that $\theta(s_i)=xs_i$. Then

$$(\theta\otimes 1)(S)(\theta\otimes 1)(S^*)=(\theta\otimes 1)(SS^*)\,.$$

Then by the Cauchy-Schwarz inequality (see the proof of [10, Theorem 31])

$$(\theta\otimes 1)(SA)=(\theta\otimes 1)(S)(\theta\otimes 1)(A)$$

for all $A\in M_n(\mathcal{O}_n)$. In particular

$$\theta(s_ia)=\theta(s_i)\theta(a)\,,\qquad \text{for all}\quad a\in\mathcal{O}_n\,,$$

and so (*) follows for θ.

In particular take $x=s(\xi)^*$ where ξ is a unit vector in $\mathcal{H}_n$. Then there is a unique completely positive unital map ϕ_ξ on $\mathcal{O}_n$ such that

$$\phi_\xi(s(\psi))=\langle\xi,\psi\rangle\,,$$

and ϕ_ξ is the Cuntz state:

$$\phi_\xi(s(\psi_1)\cdots s(\psi_r)s(\eta_s)^*\cdots s(\eta_1)^*)=\prod_{i=1}^{r}\langle\xi,\psi_i\rangle\prod_{j=1}^{s}\langle\eta,\xi_i\rangle\,,$$

c.f. [8], [6].

If $\xi\in\mathcal{H}_n$, $\|\xi\|=1$, the Cuntz state ϕ_ξ is clearly σ-invariant. If

$$L_X=x_0 1+s(\eta)-s(\eta)^*+sX_1s^*$$

is the skew-adjoint operator defining a typical generator of a one-parameter subgroup of the action of $U(n,1)$ on $\mathcal{O}_n$, we have

$$\phi_\xi(L_X)=x_0+\langle\xi,\eta\rangle-\langle\eta,\xi\rangle+\langle\xi,X_1^T\xi\rangle\,,$$

where X_1^T is the transpose of X_1. Thus, if $\phi_\xi(L_X)=0$ for all ξ, then $X=0$. This proves that none of the nonzero generators of the $U(n,1)$ action are approximately inner.

We end this section by mentioning that $L=s_1(s_1s_1^*-\sigma(s_1s_1^*))$ is annihilated by all the Cuntz states, but nevertheless $L\notin(1-\sigma)(\mathcal{O}_n)$.

§ 3. The One-Sided Shift on a UHF Algebra

In this section, let $\mathcal{A}$ be the C*-tensor product of infinitely many copies of the full $n\times n$ matrix algebra M_n, i. e. $\mathcal{A}=\bigotimes_1^\infty M_n$, and let σ be the one-sided shift on $\mathcal{A}$ defined on monomials by:

$$\sigma(x_1\otimes x_2\otimes\cdots\otimes x_M\otimes 1\otimes 1\otimes\cdots)=1\otimes x_1\otimes x_2\otimes\cdots\otimes x_M\otimes 1\otimes 1\otimes\cdots$$

for $x_i\in M_n$, $i=1,\cdots,M$. The map σ extends to an injective morphism from $\mathcal{A}$ into $\mathcal{A}$. As noted in section 2, $\mathcal{A}$ is the fixed point algebra in $\mathcal{O}_n$ for the gauge action of $\boldsymbol{T}$, and σ is nothing but the restriction to $\mathcal{A}$ of the shift $\sigma(\cdot)=\sum_{i=1}^n s_i\cdot s_i^*$ on $\mathcal{O}_n$.

If $M\in\boldsymbol{N}$, define $\mathcal{A}_M=\bigotimes_1^M M_n=$the tensor product of the M first factors M_n in $\mathcal{A}$, and define the polynomial algebra of $\mathcal{A}$ as $\mathcal{P}(\mathcal{A})=\bigcup^M\mathcal{A}_M$, without closure. The reason for this terminology is of course that $\mathcal{P}(\mathcal{A})=\mathcal{P}(\mathcal{O}_n)\cap\mathcal{A}$. Use $\overline{(1-\sigma)(\mathcal{A})}$ to denote the norm closure of $(1-\sigma)(\mathcal{A})$.

Theorem 3.1. $\overline{(1-\sigma)(\mathcal{A})}\cap\mathcal{A}_M=(1-\sigma)(\mathcal{A}_{M-1})$ *for* $M=1,2,\cdots$, *with the convention that* $\mathcal{A}_0=\{0\}$.

Remark 3.2. Before proving Theorem 3.1, it is interesting to remark that the corresponding result is not true for the unilateral shift on $\boldsymbol{N}$, i. e. the morphism σ defined on the C*-algebra $\mathcal{A}=c_0=$all sequences converging to 0, by

$$\sigma(x)_i=\begin{cases}0 & \text{if } i=1\\ x_{i-1} & \text{if } i\geqq 2\,.\end{cases}$$

If one defines $\mathcal{A}_M$ as the set of sequences $x=\{x_i\}$ such that $x_i=0$ for $i>M$, then $x\in\mathcal{A}_M$ is in $(1-\sigma)(\mathcal{A})$ if and only if $\sum_i x_i=0$, but it is easy to check that $\overline{(1-\sigma)(\mathcal{A})}=\mathcal{A}$.

We prove Theorem 3.1 via two lemmas.

Lemma 3.3. *If* $L\in\overline{(1-\sigma)(\mathcal{A})}\cap\mathcal{A}_M$ *then*

$$(\phi\otimes\phi)((1+\sigma+\cdots+\sigma^{M-1})(L))=0$$

for all $\phi\in\mathcal{A}_M^*$, *where we have made the obvious identification* $\mathcal{A}_{2M}=\mathcal{A}_M\otimes\mathcal{A}_M$. *and*

$\mathcal{A}_M^*$ is the dual of $\mathcal{A}_M$.

Proof. Assume first that ϕ is a state. We have the identification

$$\mathcal{A}=\mathcal{A}_M\otimes\mathcal{A}_M\otimes\mathcal{A}_M\otimes\cdots,$$

and ϕ defines a state ω on $\mathcal{A}$ by

$$\omega=\phi\otimes\phi\otimes\phi\otimes\cdots.$$

But as $\phi(1)=1$, we have

$$\omega\circ\sigma^M=\omega,$$

and thus

$$\omega\circ(1+\sigma+\cdots+\sigma^{M-1})$$

is a σ-invariant functional on $\mathcal{A}$. But as $L\in\overline{(1-\sigma)(\mathcal{A})}$ it follows that

$$\omega((1+\sigma+\cdots+\sigma^{M-1})(L))=0,$$

and since $\sigma(\mathcal{A}_N)\subseteq\mathcal{A}_{N+1}$ for all N, we have

$$(1+\sigma+\cdots+\sigma^{M-1})(L)\subseteq\mathcal{A}_{2M-1}\subseteq\mathcal{A}_{2M}=\mathcal{A}_M\otimes\mathcal{A}_M,$$

and thus

$$\omega((1+\sigma+\cdots+\sigma^{M-1})(L))=\phi\otimes\phi((1+\sigma+\cdots+\sigma^{M-1})(L)).$$

This establishes that

$$\phi\otimes\phi((1+\sigma+\cdots+\sigma^{M-1})(L))=0$$

if ϕ is a positive functional, and thus by polarization (use $\phi=\phi_1+\phi_2$):

$$(\phi_1\otimes\phi_2+\phi_2\otimes\phi_1)((1+\sigma+\cdots\sigma^{M-1})(L))=0$$

if ϕ_1 and ϕ_2 are positive functionals. As any functional on $\mathcal{A}_M$ is a linear combination of four positive functionals, this identity is valid for general $\phi_1, \phi_2\in\mathcal{A}_M^*$ by linearity. This establishes the lemma.

Define the cyclic shift σ_N on $\mathcal{A}_N$ by

$$\sigma_N(x_1\otimes x_2\otimes\cdots\otimes x_N)=x_N\otimes x_1\otimes\cdots\otimes x_{N-1},$$

and define the flip β_{2M} on $\mathcal{A}_{2M}=\mathcal{A}_M\otimes\mathcal{A}_M$ by

$$\beta_{2M}(x_1\otimes x_2\otimes\cdots\otimes x_M\otimes x_{M+1}\otimes\cdots\otimes x_{2M})$$
$$=(x_{M+1}\otimes\cdots\otimes x_{2M}\otimes x_1\otimes\cdots\otimes x_M).$$

With these definitions, we prove:

Lemma 3.4. *If $L\in\mathcal{A}_M$, the following conditions are equivalent:*

1. $(\phi\otimes\phi)((1+\sigma+\cdots+\sigma^{M-1})(L))=0$ *for all* $\phi\in\mathcal{A}_M^*$.

2. $(1+\sigma+\cdots+\sigma^{M-1})(L)$ *is antisymmetric under the flip on* $\mathcal{A}_{2M}=\mathcal{A}_M\otimes\mathcal{A}_M$: $\beta_{2M}((1+\sigma+\cdots+\sigma^{M-1})(L))=-(1+\sigma+\cdots+\sigma^{M-1})(L)$.
3. $(1+\sigma_{2M}+\sigma_{2M}^2+\cdots+\sigma_{2M}^{2M-1})(L)=0$.
4. $L\in(1-\sigma_{2M})(\mathcal{A}_{2M})$.

Proof. Put $L_\sigma\equiv(1+\sigma+\cdots+\sigma^{M-1})(L)$.

$1\Rightarrow2$: The condition 1 implies by polarization that

$$(\psi\otimes\phi+\phi\otimes\psi)(L_\sigma)=0$$

for all ψ, $\phi\in\mathcal{A}_M^*$. But as

$$\psi\otimes\phi+\phi\otimes\psi=\psi\otimes\phi\circ(1+\beta_{2M}),$$

it follows that

$$(1+\beta_{2M})(L_\sigma)=0,$$

which is 2.

$2\Rightarrow3$: Using 2, it suffices to show that

$$\begin{aligned}&(1+\sigma+\cdots+\sigma^{M-1})|_{\mathcal{A}_M}+\beta_{2M}(1+\sigma+\cdots+\sigma^{M-1})|_{\mathcal{A}_M}\\&=(1+\sigma_{2M}+\cdots+\sigma_{2M}^{2M-1})|_{\mathcal{A}_M}.\end{aligned}$$

But

$$\begin{aligned}&((1+\sigma+\cdots+\sigma^{M-1})+\beta_{2M}(1+\sigma+\cdots+\sigma^{M-1}))\\&\qquad\qquad\times(x_1\otimes x_2\otimes\cdots\otimes x_M\otimes1\otimes\cdots\otimes1)\\&=x_1\otimes x_2\otimes\cdots\otimes x_M\otimes1\otimes\cdots\otimes1\\&\quad+1\otimes x_1\otimes\cdots\otimes x_{M-1}\otimes x_M\otimes1\otimes\cdots\otimes1\\&\quad+\cdots\\&\quad+1\otimes1\otimes\cdots\otimes x_1\otimes x_2\otimes\cdots\otimes x_M\otimes1\\&\quad+1\otimes1\otimes\cdots\otimes1\otimes x_1\otimes\cdots\otimes x_M\\&\quad+x_M\otimes1\otimes\cdots\otimes1\otimes1\otimes x_1\otimes\cdots\otimes x_{M-1}\\&\quad+\cdots\\&\quad+x_2\otimes x_3\otimes\cdots\otimes x_M\otimes1\otimes1\otimes\cdots\otimes1\otimes x_1\\&=(1+\sigma_{2M}+\cdots+\sigma_{2M}^{2M-1})(x_1\otimes\cdots\otimes x_M\otimes1\otimes\cdots\otimes1).\end{aligned}$$

$3\Rightarrow4$: σ_{2M} defines a representation of the cyclic group $\mathbf{Z}_{2M}$ of order $2M$ on $\mathcal{A}_{2M}$, and if $\omega=e^{2\pi i/2M}$, then L has a Fourier decomposition

$$L=\sum_{k=0}^{2M-1}L_k$$

with respect to this representation. Here

$$L_k = \frac{1}{2M} \sum_{m=0}^{2M-1} \bar{\omega}^{km} \sigma_{2M}^m(L)$$

is the Fourier component such that

$$\sigma_{2M}(L_k) = \omega^k L_k .$$

But condition 3 just says that

$$L_0 = 0 ,$$

so putting

$$H = \sum_{k=1}^{2M-1} \frac{L_k}{1-\omega^k} ,$$

we have

$$L = (1-\sigma_{2M})(H) .$$

The implication 4⇒3 is trivial, and the implications 3⇒2 and 2⇒1 follows by reversing the arguments in 2⇒3 and 1⇒2.

Proof of Theorem 3.1. Let $L \in \overline{(1-\sigma)(\mathcal{A})} \cap \mathcal{A}_M$. Since then $L \in \mathcal{A}_{KM}$ for all $K \in N$, it follows from Lemma 3.3 and Lemma 3.4 that

$$(1+\sigma_{2KM}+ \cdots +\sigma_{2KM}^{2KM-1})(L) = 0$$

for $K=1, 2, 3, \cdots$. But as $L \in \mathcal{A}_M$ we have that

$$\sigma_{2KM}^m(L) = \sigma^m(L)$$

for $m=0, 1 \cdots, 2KM-M$, and thus

$$(1+\sigma+\sigma^2+ \cdots +\sigma^{2KM-M})(L) = -(\sigma_{2KM}^{2KM-M+1}+ \cdots +\sigma_{2KM}^{2KM-1})(L) .$$

From this we deduce two facts:

$$\|(1+\sigma+\sigma^2+ \cdots +\sigma^{2KM-M})(L)\| \leqq (M-1)\|L\| ,$$

i.e. the sequence $(1+\sigma+ \cdots +\sigma^m)(L)$ is uniformly bounded in m, and

$$\begin{aligned}&(1+\sigma+\sigma^2+ \cdots +\sigma^{2KM-M})(L) \\ &\subseteqq \Big(\bigotimes_1^{M-1} M_n\Big) \otimes \Big(\bigotimes_M^{2KM-M} 1\Big) \otimes \Big(\bigotimes_{2KM-M+1}^{2KM} M_n\Big) \otimes \Big(\bigotimes_{2KM+1}^{\infty} 1\Big)\end{aligned}$$

for $K=1, 2, \cdots$. From the first fact we deduce that the sequence $H_K = (1+\sigma+\sigma^2+ \cdots +\sigma^{2KM-M})(L)$ has a weak limit point H as $K \to \infty$ in the trace representation of $\mathcal{A}$, and from the second fact it follows that this limit point H must commute with all factors in the decomposition $\bigotimes_1^{\infty} M_n$ except for the $M-1$ first ones. But the relative commutant of these factors in the trace representation is just the finite dimensional algebra $\mathcal{A}_{M-1}$, and thus $H \in \mathcal{A}_{M-1}$. Furthermore, as

$$H_K-\sigma(H_K)=L-\sigma^{2KM-M+1}(L)\,,$$

$K\mapsto\sigma^{2KM-M+1}(L)$ is a central sequence in $\mathcal{A}$, and the trace representation is a factor representation, it follows that

$$H-\sigma(H)=L-\lambda 1$$

where λ is a scalar. But as the trace state τ on $\mathcal{A}$ is σ-invariant and $L\in\overline{(1-\sigma)(\mathcal{A})}$ it follows that $\tau(L)=0$, and it follows by applying the trace to the relation above that $\lambda=0$. Thus

$$L=H-\sigma(H)$$

where $H\in\mathcal{A}_{M-1}$, and the theorem is proved.

§ 4. The Dichotomy

Theorem 4.1. *Let δ be a derivation mapping the polynomial *-subalgebra $\mathcal{P}(\mathcal{O}_n)$ of the Cuntz's algebra $\mathcal{O}_n$ into itself, and assume there exists a sequence $H_m\in\mathcal{O}_n$ such that*

$$\lim_{m\to\infty}\|\delta(x)-[H_m,\ x]\|=0$$

for $x\in\mathcal{P}(\mathcal{O}_n)$. Assume that $\delta\gamma_t=\gamma_t\delta$ for all $t\in\mathbf{T}$, where γ is the gauge action on $\mathcal{O}_n$. It follows that there exists a $H\in\mathcal{O}_n^{\gamma}\cap\mathcal{P}(\mathcal{O}_n)$ such that

$$\delta(x)=[H,\ x]$$

for all $x\in\mathcal{P}(\mathcal{O}_n)$.

Proof. Without loss of generality we may assume that δ is a *-derivation and $H_m=-H_m^*$. As $\delta\gamma_t=\gamma_t\delta$ we may also replace H_m by $\int_{\mathbf{T}}dt\,\gamma_t(H_m)$, and hence we may assume that $H_m\in\mathcal{O}_n^{\gamma}=\mathcal{A}$. But if $L=\sum_i\delta(s_i)s_i^*$ is the skew adjoint operator defining δ, we have that

$$L=\lim_m(H_m-\sigma(H_m))\,,$$

where σ identifies with the one-sided shift on $\mathcal{A}=\bigotimes_1^{\infty}M_n$. As $\delta(\mathcal{P}(\mathcal{O}_n))\subseteq\mathcal{P}(\mathcal{O}_n)$, we have $L\in\mathcal{P}(\mathcal{A})=\mathcal{A}\cap\mathcal{P}(\mathcal{O}_n)$, and it now follows from Theorem 3.1 that there exists an $H=-H^*\in\mathcal{P}(\mathcal{A})$ such that

$$L=H-\sigma(H)\,.$$

But this means that

$$\delta(x)=[H,\ x]$$

for $x=s_i$, $i=1,\cdots,n$, and thus for all $x\in\mathcal{P}(\mathcal{O}_n)$. This ends the proof of Theorem 4.1.

§ 5. Some Counterexamples

We now know that if δ is a *-derivation such that $D(\delta)=\mathcal{P}(\mathcal{O}_n)$ and $\delta(\mathcal{P}(\mathcal{O}_n))\subseteqq\mathcal{P}(\mathcal{O}_n)$, then δ is a pregenerator, [3, Corollary 2.6] and if δ in addition is gauge-invariant and approximately inner, then δ is inner, Theorem 4.1. We now exhibit two examples showing that both these statements are no longer true if the condition $\delta(\mathcal{P}(\mathcal{O}_n))\subseteqq\mathcal{P}(\mathcal{O}_n)$ is removed.

Example 5.1. *We first show that a gauge-invariant derivation δ from $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{O}_n$ which is approximately inner is not necessarily inner.*

Assume ad absurdum that all approximately inner gauge-invariant derivations were inner. This would mean that the range $\mathcal{R}$ of the operator $1-\sigma$ on $\mathcal{A}=\bigotimes_1^\infty M_n$ were closed. The kernel of $1-\sigma$ is $\boldsymbol{C}1$ (since σ is asymptotically abelian and $\mathcal{A}$ is simple), and thus $1-\sigma$ induces a continuous bijection $\mathcal{A}/_{\boldsymbol{C}1}\mapsto\mathcal{R}$. But as $\mathcal{R}$ is closed, the inverse of this injection is bounded, i. e.

$$\|x+\boldsymbol{C}1\|\leqq C\|x-\sigma(x)\|$$

for some $C>0$, and all $x\in\mathcal{A}$, where

$$\|x+\boldsymbol{C}1\|=\inf\{\|x+\lambda 1\| \mid \lambda\in\boldsymbol{C}\}.$$

But if $h\in\mathcal{A}$, define

$$h_m=h+\sigma(h)+\cdots+\sigma^{m-1}(h)$$

for $m=1, 2, \cdots$, and put $x=h_m$ in the above relation. Then

$$\|h_m+\boldsymbol{C}1\|\leqq C\|h-\sigma^m(h)\|\leqq 2C\|h\|\,.$$

If h has the form

$$h=p\otimes 1\otimes 1\otimes\cdots,$$

where p is a nontrivial orthogonal projection in M_n, then

$$\mathit{Spectrum}(h_m)=\{0, 1, 2, \cdots, m\},$$

and hence

$$\|h_m+\boldsymbol{C}1\|=m/2\,.$$

But this contradicts the uniform boundedness of $\|h_m+\boldsymbol{C}1\|$, and this contradiction establishes that $\mathcal{R}$ is not closed, and hence there exist gauge-invariant derivations from $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{O}_n$ which are approximately inner, but not inner.

Example 5.2. *We will now exhibit a derivation δ from $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{O}_n$ which is not a pre-generator.*

The shift algebra $\mathcal{T}_1=C^*(s_1)$ generated by s_1 contains the compact operators $\mathcal{K}$ as the ideal generated by the projection $\mathbf{1}-s_1s_1^*$, and $C^*(s_1)/\mathcal{K}=C(\boldsymbol{T})$ where

$\boldsymbol{T}$ is the circle, [9].

If $f\in C(\boldsymbol{T})$, let $M_f: L^2(\boldsymbol{T})\mapsto L^2(\boldsymbol{T})$ be the operator of multiplication by f, and consider the Toeplitz operator $T_f=PM_f: L^2(\boldsymbol{T})\mapsto L^2(\boldsymbol{T})$, where P is the orthogonal projection on $L^2(\boldsymbol{T})$ defined by

$$P\Big(\sum_{m=-\infty}^{\infty} a_m e^{imt}\Big)=\sum_{m=0}^{\infty} a_m e^{imt}.$$

The C^*-algebra $C^*(T_f \mid f\in C(\boldsymbol{T}))$ generated by the bounded operators T_f on $L^2(\boldsymbol{T})$ is canonically isomorphic to the shift algebra $C^*(s_1)$, the isomorphism is determined by $T_{id}\to s_1$ where $\mathrm{id}(z)=z$ for all $z\in\boldsymbol{T}$. Also, if $f, g\in C(\boldsymbol{T})$ then $T_fT_g-T_{fg}\in\mathcal{K}$, and hence if $\phi: C^*(s_1)\to C^*(s_1)/\mathcal{K}=C(\boldsymbol{T})$ is the quotient map, $f\mapsto\phi(T_f)$ is a morphism, and thus

$$\phi(T_f)=f\,.$$

In particular, if $f(\boldsymbol{T})\subseteq i\boldsymbol{R}$, then

$$\phi\Big(\frac{1}{2}(T_f-T_f^*)\Big)=\frac{1}{2}(f-\bar{f})=f\,,$$

and thus T_f is skew-adjoint modulo compacts.

Now, let $f\in C(\boldsymbol{T})$ be a function such that $f(\boldsymbol{T})\subseteq i\boldsymbol{R}$ and $f(e^{it})=i\sqrt{|t|}$ when $|t|<\frac{\pi}{2}$. Let

$$L=\frac{1}{2}(T_f-T_f^*)\,,$$

and let δ be the *-derivation from $\mathcal{P}(\mathcal{O}_n)$ into $\mathcal{O}_n$ defined by

$$\delta(s_i)=Ls_i\,,\qquad i=1,\cdots,n\,.$$

We will argue that δ is not a pregenerator by using an ad absurdum argument: If $e^{t\delta}$ exists, then

$$e^{t\delta}(C^*(s_1))\subseteq C^*(s_1)$$

since if S_t is the strongly continuous one-parameter family of morphisms from $\mathcal{O}_n$ into $\mathcal{O}_n$ determined by

$$S_ts_i=e^{tL}s_i\,,\qquad i=1,\cdots,n\,,$$

then

$$e^{t\delta}(s_1)=\lim_{n\to\infty}(S_{t/n})^n(s_1)\in C^*(s_1)\,,$$

where the first equality follows from [5, Theorem 3.1.30], and the last inclusion from $e^{tL}\in C^*(s_1)$. But then $e^{t\delta}$ map the canonical ideal $\mathcal{K}$ in $C^*(s_1)$ onto itself, and using the quotient map $\phi: C^*(s_1)\mapsto C(\boldsymbol{T})$, $e^{t\delta}$ defines a one-parameter group of automorphisms of $C(\boldsymbol{T})$. But as

$$\begin{aligned}\phi(\delta(s_1))&=\phi(Ls_1)\\&=\phi(T_f T_z)\\&=\phi(T_{fz})=f(e^{it})e^{it},\end{aligned}$$

we see that the generator of the latter group is an extension of

$$-if(e^{it})\frac{d}{dt}$$

defined on the polynomials in z and $\bar{z}$. But since $1/f$ is integrable near the zero at $t=0$, this derivation has no generator extensions [1], [2]. This contradiction establishes that δ is not a pregenerator.

Acknowledgements

This collaboration started when the four authors visited the University of Toronto with support from NSERC, continued during a visit by O. B. to the University of Iowa and completed while O. B. and D. E. E. were visiting Kyoto University. We are indepted to George A. Elliott, Paul Muhly and Huzihiro Araki for making these visits possible, and also to Elliott and Muhly for useful discussions of the problems under consideration here. O. B. was partly supported by NAVF, Norway, D. E. E. by SERC, United Kingdom and F. M. G. and P. E. T. J. by NSF, USA, during this research.

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