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Forgetful Homomorphisms in Equivariant K-Theory

Dedicated to Professor Nobuo Shimada on his sixtieth birthday

By

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Introduction

The purpose of this note is to analyse the surjectivity of the forgetful homomorphism $f(G, X): K_G(X) \to K(X)$, which gives some useful information about lifting group actions in stable vector bundles. Here G is a compact connected Lie group and X is a compact G-space such that $K_G^*(X)$ is finitely generated over $R(G)$. Moreover let T denote a maximal torus of G throughout this paper. It is known that if $\pi_1(G)$ is torsion free, then the homomorphism $\alpha(G, T): R(T) \to K(G/T)$ which is interpreted as $f(G, G/T)$ via the isomorphism $K_G(G/T) \approx R(T)$ is surjective (cf. [5], [6]). We shall use a theorem which Pittie [6] presented to prove this fact.

In Section 1 we shall give a sufficient condition for the surjectivity of $f(G, X)$ for G a torus (Theorem 1) and further we shall prove that if $\pi_1(G)$ is torsion free and $f(T, X)$ is surjective, then $f(G, X)$ is also surjective (Theorem 2). Section 2 consists of applications of the preceding theorems to actions on homotopy complex projective spaces, pseudo-linear G-spheres and complex quadrics. In Section 3 we shall give a generalized form of Theorem 2 for the case when $\mathrm{Tor}\, \pi_1(G) \neq 0$ (Theorem 5) and using some results due to Hodgkin we shall obtain examples of actions of these groups. In the last section we shall prove that if $\alpha(G, T)$ is surjective, then $\pi_1(G)$ is torsion free (Theorem 6).

§ 1. Some Criterions for the Surjectivity

First we provide a criterion for the case when G is a torus. Let G be the n-dimensional torus $S_1^1 \times S_2^1 \cdots \times S_n^1$ where S_j^1 is the circle subgroup, and let $T(i) = e \times \cdots \times e \times S_{n-i+1}^1 \times \cdots \times S_n^1$ for $1 \leqq i \leqq n$ where e is the trivial subgroup.

Theorem 1. *Suppose that* $K_{T(i)}^1(X) = 0$ *for* $1 \leqq i \leqq n$. *Then* $f(G, X)$ *is sur-*

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jective.

Proof. Let $S^1=S^1_1$, $T^{n-1}=S^1_2\times\cdots\times S^1_n$ and denote by X' the $S^1\times T^{n-1}$-action on X induced by $S^1\times T^{n-1}\xrightarrow{p_2}T^{n-1}\xrightarrow{i}S^1\times T^{n-1}$. Denote by C the complex one dimensional $S^1\times T^{n-1}$-module on which T^{n-1} acts trivially and S^1 acts as complex multiplication, and denote by $D(C)$ the unit disc with boundary $S(C)$. There is an $S^1\times T^{n-1}$-homeomorphism $S(C)\times X'\to S(C)\times X$ given by $(z, x)\to(z, zx)$ (cf. the proof of Theorem 1.1 in [4]). Consider the exact sequence associated with the Puppe sequence, $S(C)\times X\overset{i}{\subset}D(C)\times X\xrightarrow{j}D(C)\times X/S(C)\times X$,

$$\begin{array}{l}
K^*_{S^1\times T^{n-1}}(X)\overset{\phi}{\approx}K^*_{S^1\times T^{n-1}}(C\times X)\xrightarrow{j^*}K^*_{S^1\times T^{n-1}}(D(C)\times X)\approx K^*_{S^1\times T^{n-1}}(X)\\
\quad\nwarrow\delta \qquad\qquad i^*\swarrow\ \swarrow\pi^* \qquad \swarrow f\\
\qquad K^*_{S^1\times T^{n-1}}(S(C)\times X)\\
\qquad\approx K^*_{S^1\times T^{n-1}}(S(C)\times X')\\
\qquad\approx K^*_{T^{n-1}}(X')\approx K^*_{T^{(n-1)}}(X)\,,
\end{array}$$

where ϕ denotes the Thom isomorphism and $\pi: S(C)\times X\to X$ is the projection, and f is a forgetful homomorphism. By the assumption $K^1_{S^1\times T^{n-1}}(X)=0$, then f is surjective. By an obvious induction we obtain the theorem.

Suppose that the fundamental group $\pi_1(G)$ is torsion free. Then we have

Theorem 2. *If $f(T, X)$ is surjective, then $f(G, X)$ is so.*

Proof. Since it follows from [5] that Hypothesis 3.2 in [7] is true, we obtain by [7] an isomorphism

$$K_G(G/T\times X)\approx R(T)\otimes_{R(G)}K_G(X)\,,$$

therefore

$$K_T(X)\approx R(T)\otimes_{R(G)}K_G(X)\,.$$

Since $R(T)$ is a free $R(G)$-module (Theorem 1 in [6]), we have a decomposition as an $R(G)$-module,

$$R(T)=R(G)\oplus u_1R(G)\oplus\cdots\oplus u_{s-1}R(G)\,,$$

where $u_i\in R(T)$ for $1\leqq i\leqq s-1$ and s is the order of the Weyl group of G. Hence we have an isomorphism

$$(1)\qquad K_T(X)\approx K_G(X)\oplus u_1K_G(X)\oplus\cdots\oplus u_{s-1}K_G(X)$$

which we take to be an equality. By the assumption, for any $x\in K(X)$, there exists $y\in K_T(X)$ such that $f(T, X)(y)=x$. We see by (1) that y can be written in the form

$$y=x_0+u_1x_1+\cdots+u_{s-1}x_{s-1} \quad \text{for some } x_i\in K_G(X),\ 0\leqq i\leqq s-1.$$

Then

$$x=f(G, X)(x_0)+\varepsilon(u_1)f(G, X)(x_1)+\cdots+\varepsilon(u_{s-1})f(G, X)(x_{s-1})$$

where $\varepsilon: R(T)\to Z$ is the augmentation. Thus $f(G, X)(x_0+\varepsilon(u_1)x_1+\cdots+\varepsilon(u_{s-1})x_{s-1})=x$, and the proof is completed.

§2. Applications of the Preceding Theorems

First we consider compact connected Lie group actions on homotopy complex projective spaces, and we have

Proposition 3. *Let G satisfy the relation $H^3(BG, Z)=0$. Let X be a homotopy complex projective G-space. Then $f(G, X)$ is surjective.*

Proof. Let $h: X\to CP^m$ be a homotopy equivalence, where CP^m denotes the m-dimensional complex projective space. Consider the principal bundle $S^1\to\Sigma\to X$ induced from the principal bundle $S^1\to S^{2m+1}\to CP^m$. Then Σ is a homotopy sphere. We denote by H the associated complex line bundle $\Sigma\times_{S^1}C$. We have $K(X)=Z[H]/(1-H)^{m+1}$. Since G is connected, the Chern class $c_1(\Sigma)$ is invariant under the action of G. Consider the cohomology spectral sequence associated with the fibering $X\to X\times_G EG\to BG$, then we have

$$E_2^{p,1}=H^p(BG, H^1(X, Z))=0,$$

$$d_2: H^2(X, Z)\to E_2^{2,1}=H^2(BG, H^1(X, Z))=0,$$

$$d_3: E_3^{0,2}=H^2(X, Z)\to E_3^{3,0}=\text{a quotient group of } E_2^{3,0}$$
$$=\text{a quotient group of } H^3(BG, Z)=0.$$

Thus by Corollary 1.3 in [2], we obtain the proposition.

Now suppose that $\pi_1(G)$ is torsion free, and let X be a *smooth* homotopy complex projective G-space. Then we have

Corollary to Proposition 3. *$f(G, X)$ is surjective.*

Proof. Since $H^3(BT, Z)=0$, we see by Proposition 3, $f(T, X)$ is surjective, and hence by Theorem 2 that so is $f(G, X)$.

Next let Σ be a pseudo-linear G sphere of even dimension [4] where $\pi_1(G)$ is torsion free then we have

Corollary to Theorems 1 and 2. *$f(G, \Sigma)$ is surjective.*

Proof. Let $T(n)$ be a maximal torus of G. By Proposition 2.3 in [4],

$K^1_{T(i)}(\Sigma)=0$ for $1\leqq i\leqq n$ where $T(i)$ is as in §1. Then by Theorem 1, $f(T(n), \Sigma)$ is surjective, hence by Theorem 2 we obtain the corollary.

Now let us consider the complex quadric W^{2m} defined by an equation $z_0^2+z_1^2+\cdots+z_{2m+1}^2=0$ in CP^{2m+1}. We have an inclusion map $U(m)\subset SO(2m)$ which is given by the realification $A+\sqrt{-1}B\to\begin{pmatrix}A & -B\\ B & A\end{pmatrix}$. For any $M\in SO(2m)$ we have

$$M\begin{pmatrix}\rho z_2\\ \vdots\\ \rho z_{2m+1}\end{pmatrix}=\rho M\begin{pmatrix}z_2\\ \vdots\\ z_{2m+1}\end{pmatrix}\qquad \text{for}\quad \rho\in S^1$$

then we obtain actions of $SO(2m)$ and $U(m)$ on W^{2m}. Then we have

Proposition 4. *$f(U(m), W^{2m})$ is surjective.*

Proof. Let $T(m)$ be the standard maximal torus of $U(m)$ and $T(i)$ be as in §1. By Theorems 1, 2, it is sufficient to prove that $K^1_{T(i)}(W^{2m})=0$ for $1\leqq i\leqq m$. By 3.6 Theorem in [1], $K^1(W^{2m})=K^1(SO(2m+2)/SO(2m)\times SO(2))=0$, then by Theorem 1.1 and Lemma 1.6 in [4], $S^{-1}K^1_{T(i)}(W^{2m})=0$ for $1\leqq i\leqq m$. Here let $I_{T(i)}$ be the kernel of the augmentation $R(T(i))\to Z$ and $S=1+I_{T(i)}$. Let $\mathcal{P}\subset R(T(i))$ be a prime ideal associated to the trivial group e, then $(1+I_{T(i)})\cap\mathcal{P}=\emptyset$, therefore

$$(1)\qquad K^1_{T(i)}(W^{2m})_e=0 \qquad \text{for}\quad 1\leqq i\leqq m\,.$$

Now we consider the case $m=1$. For any non trivial subgroup H of $T(1)=S^1$,

$$K^1_{S^1}(W^2)_H\approx K^1_{S^1}((W^2)^H)_H\approx(K^1_{S^1/H}(\text{two points})\otimes_{R(S^1/H)}R(S^1))_H=0\,,$$

because of $(W^2)^H=\{[1, i, 0, 0], [1, -i, 0, 0]\}$, therefore by (1) $K^1_{S^1}(W^2)_{\mathcal{P}}=0$ for any prime ideal $\mathcal{P}$ of $R(S^1)$, thus $K^1_{S^1}(W^2)=0$. Next let us suppose that $K^1_{T(n')}(W^{2n})=0$ for $n'\leqq n<m$. For $m'\leqq m$ and a subgroup $H=e\times\cdots\times e\times H_1\times\cdots\times H_k$ of $T(m')$ where $H_i\neq e$ for $1\leqq i\leqq k$, we have

$$T(m')/H=T(m'-k)\times T(k)/H'$$

where $H'=H_1\times\cdots\times H_k$ and let $T(m'-k)$, $T(k)$ be viewed as $S^1_{k+1}\times\cdots\times S^1_{m'}$, $S^1_1\times\cdots\times S^1_k$ respectively. Then

$$K^1_{T(m')/H}((W^{2m})^H)\approx K^1_{T(m')/H}(W^{2(m-k)})\approx K^1_{T(m'-k)}(W^{2(m-k)})\otimes R(T(k)/H')=0\,,$$

therefore

$$K^1_{T(m')}(W^{2m})_H\approx K^1_{T(m')}((W^{2m})^H)_H\approx(K^1_{T(m')/H}(W^{2m})^H\otimes_{R(T(m')/H)}R(T(m'))_H=0\,,$$

hence by (1), $K^1_{T(m')}(W^{2m})_{\mathcal{P}}=0$ for any prime ideal $\mathcal{P}$ of $R(T(m'))$. Thus $K^1_{T(m')}(W^{2m})=0$ for $m'\leqq m$.

§3. A Generalization of Theorem 2

In this section we consider the case $\mathrm{Tor}\,\pi_1(G)\neq 0$. It is known that G is isomorphic to a quotient group of a compact connected Lie group $\tilde{G}$ with $\mathrm{Tor}\,\pi_1(\tilde{G})=0$ by a finite subgroup F of its center ($\tilde{G}$ is uniquely determined up to an isomorphism). So we write $G=\tilde{G}/F$. Here we assume that F is cyclic of order d and there exist complex representations $W_{i,1},\cdots,W_{i,l_i}$ of $\tilde{G}$ for $1\leqq i\leqq d-1$ such that

$$(1)\qquad W_{i,k}|F=m(i,k)V^{\otimes i}\qquad(1\leqq k\leqq l_i),$$

where $W|F$ denotes the restriction of W on F, $m(i,k)$ the degree of $W_{i,k}$ and V a non trivial canonical 1-dimensional complex representation of F.

In fact, if $\tilde{G}$ is simple and simply connected then we have such a system of representations. Because $\tilde{G}$ admits at least one faithful irreducible representation W and so we may consider that $W|F=mV$ which implies that the i-fold exterior power of W is of the form $\binom{m}{i}V^{\otimes i}$ $(1\leqq i\leqq d-1)$.

Let m_i be the greatest common divisor (G. C. D.) of $m(i,1),\cdots,m(i,l_i)$ for $1\leqq i\leqq d-1$ and m be the least common multiple of $m_1,\cdots,m_{d-1}$ (L. C. M.). Then we have

Theorem 5. *If $f(T,X)$ is surjective, then* Image $f(G,X)\supset mK(X)$.

Proof. Choose a maximal torus $\tilde{T}$ of $\tilde{G}$ such that $T=\tilde{T}/F$. Since we can view a T-vector bundle over X as a $\tilde{T}$-vector bundle over X in a natural way, we see by the assumption that $f(\tilde{T},X)$ is surjective, and so by Theorem 2 that $f(\tilde{G},X)$ must also be. Let $E\to X$ be a $\tilde{G}$-vector bundle. Then we have a decomposition of a F-vector bundle

$$E\approx E^F\oplus\textstyle\sum_{i=1}^{d-1}\underline{V}^{\otimes i}\otimes\mathrm{Hom}_F(\underline{V}^{\otimes i},E),$$

where E^F is the invariant subbundle of E and $\underline{A}=A\times X$ a product vector bundle. Therefore we have an equality

$$(2)\qquad f(\tilde{G},X)[E]=[E^F]+\textstyle\sum_{i=1}^{d-1}[\mathrm{Hom}_F(\underline{V}^{\otimes i},E)],$$

in $K(X)$. Since E^F becomes a G-vector bundle, $[E^F]\in\mathrm{Image}\,f(G,X)$. Now by (1)

$$(\underline{W}_{s,t}\otimes E)^F\approx m(s,t)(\underline{V}^{\otimes s}\otimes E)^F,$$

as usual vector bundles. Therefore

$$m(s,t)[\mathrm{Hom}_F(\underline{V}^{\otimes(d-s)},E)]=[(\underline{W}_{s,t}\otimes E)^F]\in\mathrm{Image}\,f(G,X),$$

because of $(\underline{V}^{\otimes s}\otimes E)^F\approx\mathrm{Hom}_F(\underline{V}^{\otimes(d-s)},E)$. This implies that $m[\mathrm{Hom}_F(\underline{V}^{\otimes(d-s)},E)]\in\mathrm{Image}\,f(G,X)$ for $1\leqq s\leqq d-1$. Hence by (2) we see that $mf(\tilde{G},X)[E]\in\mathrm{Image}\,f(G,X)$, and the proof is completed.

From the facts mentioned in § 12 of [3] we have the following examples of m (we use the notations of [3]).

1. $\tilde{G}=SU(l)$, $F=Z_l$ the center of $\tilde{G}$.
   $m=$L. C. M. of $\binom{l}{i}$ $(1\leqq i\leqq l-1)$, for $\lambda_i|F=\binom{l}{i}V^{\otimes i}$.

2. $\tilde{G}=\mathrm{Spin}(2l+1)$, $F=Z_2(-1)$.
   $m=2^l$, for $\Delta|F=2^lV$.

3. $\tilde{G}=\mathrm{Sp}(l)$, $F=Z_2(-1)$.
   $m=$G. C. D. of $\binom{2l}{2i+1}$ $(1\leqq 2i+1\leqq l)$, for $\lambda_i|F=\binom{2l}{i}V^{\otimes i}$,

4. $\tilde{G}=\mathrm{Spin}(2l)$, $F=Z_2(-1)$.
   $m=2^{l-1}$, for $\Delta^+|F=\Delta^-|F=2^{l-1}V$.

5. $\tilde{G}=\mathrm{Spin}(4l+2)$, $F=Z_4(e_1\cdots e_{4l+2})$.
   $m=$L. C. M. $\left\{2^{2l},\ \text{G. C. D.}\binom{4l+2}{2i+1},\ 1\leqq 2i+1\leqq 2l-1)\right\}$, for $\Delta^+|F=2^{2l}V$, $\lambda_1|F=(4l+2)V^{\otimes 2}$, $\Delta^-|F=2^{2l}V^{\otimes 3}$ and $\lambda_i|F=\binom{4l+2}{i}V^{\otimes 2i}$.

6. $\tilde{G}=\mathrm{Spin}(4l)$, $F=Z_2(-e_1\cdots e_{4l})$.
   $m=$G. C. D. $\left\{2^{2l-1},\ \binom{4l}{2i-1},\ 1\leqq 2i-1<2l-2\right\}$, for $\lambda_i|F=\binom{4l}{i}V^{\otimes i}$, $\Delta^+|F=2^{2l-1}\cdot 1$ and $\Delta^-|F=2^{2l-1}V$.

7. $\tilde{G}=E_6$, $F=Z_3$ the center of $\tilde{G}$.
   $m=27$, for $\rho_1|F=27V$, $\rho_5|F=13\cdot 27V$, $\rho_6|F=27V^{\otimes 2}$ and $\rho_3|F=13\cdot 27V^{\otimes 2}$.

8. $\tilde{G}=E_7$, $F=Z_2$ the center of $\tilde{G}$.
   $m=8$, for $\rho_2|F=8\cdot 95V$, $\rho_5|F=16\cdot 5187V$ and $\rho_7|F=8\cdot 7V$.

§ 4. On the Atiyah-Hirzebruch Map

In this section we shall make a remark about the map $\alpha(G, T)$.

Theorem 6. *If the map $\alpha(G, T)$ is surjective, then the group $\pi_1(G)$ is torsion free.*

Proof. We suppose that the group $\pi_1(G)$ has a p-torsion subgroup, where p is a prime number. As remarked in § 3 if G is semisimple, then there are a simply connected Lie group $\tilde{G}$ and a non trivial subgroup F of the center $Z(\tilde{G})$ and we can write $G=\tilde{G}/F$. Let Γ be a cyclic subgroup of F and of order p. We have $R(\Gamma)=Z[V]/(V^{\otimes p}-1)$, where V is a canonical non trivial one dimensional representation. By § 12 in [3],

$$(1)\qquad Z[\alpha(V)]/(\alpha(V^{\otimes p})-1,\ p^k(\alpha(V)-1))\subset K(\tilde{G}/\Gamma),$$

where $\alpha: R(\Gamma)\to K(\tilde{G}/\Gamma)$ is given by the map $U\to\tilde{G}\times_{\Gamma}U$ for a Γ-module U, and $k\geqq 1$. Denote by $\pi:\tilde{G}\to G$ the projection map, and let $\tilde{T}$ be a maximal torus of $\tilde{G}$. Then $\Gamma\subset\tilde{T}$ and $T=\pi(\tilde{T})$ is a maximal torus of G. Let $\bar{\alpha}: R(\Gamma)\otimes_{R(G)}Z\to K(\tilde{G}/\Gamma)$ be the map induced from the map α. By Theorem 3 in [6], we have an isomorphism $\bar{\alpha}(\tilde{G},\tilde{T}): R(\tilde{T})\otimes_{R(\tilde{G})}Z\to K(\tilde{G}/\tilde{T})$. Then we have a commutative diagram

$$\begin{array}{ccc}
R(T) & \xrightarrow{\alpha(G,\,T)} & K(G/T) \\
\pi^*\downarrow & & \approx\downarrow \\
R(\tilde{T}) & \xrightarrow{\alpha(\tilde{G},\,\tilde{T})} & K(\tilde{G}/\tilde{T}) \\
q\downarrow & \nearrow{\scriptstyle \bar{\alpha}(\tilde{G},\,\tilde{T})}\ \approx & \Big\downarrow \\
R(\tilde{T})\otimes_{R(\tilde{G})}Z & & \Big\downarrow \\
i^*\otimes 1\downarrow & & \Big\downarrow \\
R(\Gamma)\otimes_{R(\tilde{G})}Z & \xrightarrow{\bar{\alpha}} & K(\tilde{G}/\Gamma),
\end{array}$$

where q denotes the natural projection. Suppose that $\alpha(G,T)$ is surjective, then $q\circ\pi^*$ is surjective, therefore $(i^*\otimes 1)\circ q\circ\pi^*$ is so. Thus $V\otimes 1$ in $R(\Gamma)\otimes_{R(\tilde{G})}Z$ is contained in the image of $(i^*\otimes 1)\circ q\circ\pi^*$. Since the image of π^* are trivial on Γ, $\bar{\alpha}(V\otimes 1-1)=0$. On the other hand, by (1) $\bar{\alpha}(V\otimes 1-1)=(\alpha(V)\otimes 1-1)\neq 0$, which is a contradiction. Hence $\alpha(G,T)$ is not surjective.

Now we consider the case where G is not semisimple. We have a compact simply connected semisimple Lie group G_0, a torus S and a finite subgroup F of the center of $G_0\times S$ such that $F\cap(1\times S)=e$ and $G=(G_0\times S)/F$. Then we have an exact sequence,

$$0\longrightarrow\pi_1(G_0\times S)\longrightarrow\pi_1(G)\longrightarrow\pi_0(F)\longrightarrow 0,$$

and isomorphisms

$$\pi_1(G_0\times S)\approx\pi_1(S)\approx\overset{l}{\oplus}Z \quad\text{and}\quad \pi_0(F)=F,$$

where l is the dimension of S. Hence $\pi_1(G)\approx\overset{l}{\oplus}Z\oplus T$ for some torsion group T. Denote by $\pi: G_0\times S\to G$ the projection map. Consider the exact sequence

$$e\longrightarrow F\cap(G_0\times e)\longrightarrow G_0\times e\longrightarrow\pi(G_0\times e)\longrightarrow e.$$

Suppose that $F\cap(G_0\times e)=e$, then $G_0\times e\approx\pi(G_0\times e)$, therefore from the fibration $G_0\approx\pi(G_0\times e)\to G\to G/\pi(G_0\times e)=a$ torus of rank l, we have $\pi_1(G)\approx\pi_1(G/\pi(G_0\times e))\approx\overset{l}{\oplus}Z$, hence $T=0$. Thus if $T\neq 0$, then $F\cap(G_0\times e)\neq e$, and we have an element $g=(g_0, 1)$ in $F\cap(G_0\times e)$ where 1 denotes the unit element, such that the order of g is a prime number p. Let Γ_0 be the cyclic group generated by g_0, and T_0 be a maximal torus of G_0. Now we have a commutative diagram

$$
\begin{array}{ccc}
R(T) & \xrightarrow{\alpha(G,\, T)} & K(G/T) \\
\pi^{*}\downarrow & & \approx\downarrow \\
R(T_0\times S) & \xrightarrow{\alpha(G_0\times S,\, T_0\times S)} & K((G_0\times S)/(T_0\times S)) \\
q\ \downarrow & \nearrow_{\approx}\ \bar{\alpha}(G_0\times S,\, T_0\times S) & \approx\downarrow \\
R(T_0\times S)\otimes_{R(G_0\times S)}Z & & \\
\approx\downarrow & & \\
R(T_0)\otimes_{R(G_0)}Z & \xrightarrow{\bar{\alpha}(G_0,\, T_0)} & K(G_0/T_0) \\
i^{*}\otimes 1\downarrow & & \downarrow \\
R(\Gamma_0)\otimes_{R(G_0)}Z & \xrightarrow{\bar{\alpha}} & K(G_0/\Gamma_0)\,,
\end{array}
$$

then by the same argument as in the case of semisimple, we can prove that $\alpha(G, T)$ is not surjective.

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