Publ. RIMS, Kyoto Univ.
22 (1986), 119-141

Reflection Positivity for the Complementary Series of $SL(2n, C)$

By

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Abstract

We apply the concept of reflection positivity in euclidean quantum field theory to the complementary series of $SL(2n, C)$ as given by Gelfand and Neumark for $n=1$ and by Stein for $n>1$. The result is a virtual representation in the sense of Fröhlich. Osterwalder and Seiler or equivalently a strongly continuous representation of a closed subsemigroup by contractions on a new Hilbert space. Analytic continuation gives a unitary representation of a certain dual group of $SL(2n, C)$. The possible relation to the theory of noncommuting monodromy matrices appearing in the theory of integrable quantum systems is briefly discussed.

§ 1. Introduction

The concept of reflection positivity originates in relativistic quantum field theory [OS]. It allows to recover the Green's functions (Wightman functions) of a relativistic quantum field theory from its values at the euclidean points (Schwinger functions) by providing the scalar product for the Hilbert space of the relativistic theory.

On the other hand, the euclidean theory itself carries a scalar product, referred to as Symanzik positivity [Sy] and an associated unitary representation of the euclidean group. When combined with reflection positivity this unitary representation leads, via an analytic continuation process, to a unitary representation of the Poincaré group, the symmetry group of special relativity. In particular, the euclidean time translations first give a contraction semigroup, called a transfer matrix because of the interpretation of euclidean quantum field theory as a statistical theory. Its analytic continuation is the one-parameter unitary group describing the time evolution of the relativistic quantum theory (for a detailed account see e. g. [GJ]).

The insight obtained from the concept of reflection positivity was soon applied to other groups. In particular the idea of analytic continuation led to a non-commutative version of the Hille-Yosida theorem [LM] and the notion of a

Communicated by H. Araki, August 7, 1985.
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virtual representation [FOS] sometimes also called a local representation (see e. g. [KL1], [KL2], [J1], [J2], [Se]). The aim of this article is to extend this discussion and in particular to apply it to the complementary series of $SL(2n, \mathbf{C})$ as given by Gelfand and Neumark for $n=1$ [GN] (see also [N]) and by Stein for $n>1$ [St].

In Section 2 we present a general discussion of reflection positivity in the context of unitary representations of groups. Although well known in special examples, we believe some of our material to be new. It starts with a unitary representation π of a group G in a Hilbert space $\mathscr{H}$, viewing its scalar product as a version of Symanzik positivity. Reflection positivity is then defined in terms of a closed subspace $\mathscr{H}^+$ and a unitary involution θ leading to a new scalar product in $\mathscr{H}^+$. In addition we require the existence of an involutive automorphism ϑ of the group which in a specified sense is compatible with θ. This leads to a representation π_θ of a subsemigroup G^+ in G. The elements in G^+ fixed under ϑ are still represented by unitaries. However, some elements in G^+ are represented as selfadjoint contractions. We interpret them as transfer matrices. In general they do not commute. In case G is a Lie group, the noncommutative Hille-Yosida theorem then gives an analytic continuation of π_θ to G^*, which is a dual group of G and is obtained from the involutive automorphism ϑ. In Section 3 we apply these concepts to the complementary series of $SL(2n, \mathbf{C})$. Actually we will obtain reflection positivity also for a larger range of the parameter describing the representation and for which Symanzik positivity does not hold.

In Section 4 we show that π_θ also defines a local representation in the sense of [J1], [J2] (called a virtual representation in [FOS]). The proof is obtained by establishing a result known as the Reeh-Schlieder theorem in the context of relativistic quantum field theory [RS] (see also e. g. [SW]). Application of the main result in [J2] will provide an alternative proof that π_θ extends to a unitary representation of G^*. Now transfer matrices appear also in another important context, namely in the theory of completely integrable quantum systems, where they are called monodromy matrices (see e. g. [F] for an account). They obey certain relations called Yang-Baxter equations [Y], [B]. In the last years considerable efforts have been undertaken, in particular by Soviet mathematicians and physicists (see e. g. [Sem] for references), to arrive at a group theoretical understanding of the Yang-Baxter equations. The author's interest in this subject and the starting point for the present investigation arouse from a lucid talk given by Semanov-Tyan-Shanskii in Kyoto in the fall of 1984, where he proposed a double Lie algebra structure to describe the so-called classical Yang-Baxter equations [Sem]. Recall that a Lie algebra defines a symplectic structure on the dual of the Lie algebra, first discovered by Lie himself. Hence for given Hamilton function one has two ways to obtain classical equations of motion. The present author was struck by the close similarity to the quantum case discussed above, where one also deals with one 'Hamiltonian', i.e. the infinitesimal generator of time translations, and two scalar products, namely Symanzik positivity and reflection positivity.

In the second part of Section 2 we try to point out some structural similarities of our approach with the quantum mechanical Yang-Baxter relations. In this context we find the following observation worth mentioning. The reason that reflection positivity holds for the complementary series of $SL(2n, \mathbf{C})$ is due to a remarkable property of the intertwining operator appearing in this context. Now as was also first noted by Semenov-Tyan-Shanskii the Yang-Baxter equations exhibit a structural behaviour intrigueingly similar to ones encountered in the theory of intertwining operators, an understanding of which was missing (see the quotation in [KRS]).

It is the hope of the authors that the present investigation will stimulate further investigations in this direction.

Acknowledgements

This work was begun when the author was visiting RIMS (Kyoto) on a sholarship provided by the Japan Society for the Promotion of Sciences. He would like to thank H. Araki and H. Hironaka for their hospitality. Discussions with H. Araki, T. Hirai, M. Karowski and W. Müller and a correspondence with P. E. T. Jorgensen have been very stimulating.

§2. Reflection Positivity for Group Representations

In this section we give a general discussion of reflection positivity adapted to the theory of unitary representations of groups.

Definition 2.1. Given a Hilbert space $\mathcal{H}$ with scalar product $(\,,)$, a closed subspace $\mathcal{H}^+$ and a unitary involution θ, the triple $(\mathcal{H}, \mathcal{H}^+, \theta)$ is called reflection positive, if the quadratic form

$$\langle f, f' \rangle_\theta = (f, \theta f') \tag{2.1}$$

is positive semidefinite on $\mathcal{H}^+$.

Let $(\mathcal{H}, \mathcal{H}^+, \theta)$ be reflection positive and consider the subset $\mathcal{N}_\theta$ of $\mathcal{H}^+$ consisting of those elements f for which $\langle f, f \rangle_\theta = 0$. By Schwarz' inequality for $\langle \cdot, \cdot \rangle_\theta$, $\mathcal{N}_\theta$ is equal to the set of all f's in $\mathcal{H}^+$ with $\langle f', f \rangle_\theta = 0$ for all f' in $\mathcal{H}^+$. On $\mathcal{H}^+/\mathcal{N}_\theta$, $\langle \cdot, \cdot \rangle_\theta$ induces a positive definite scalar product, also denoted by $\langle \cdot, \cdot \rangle_\theta$. This makes $\mathcal{H}^+/\mathcal{N}_\theta$ a pre Hilbert space and we denote by $\mathcal{H}_\theta$ its completion. Let $[\;]_\theta : \mathcal{H}^+ \to \mathcal{H}_\theta$ be the induced map obtained from the canonical projection $\mathcal{H}^+ \to \mathcal{H}^+/\mathcal{N}_\theta$ and the canonical injection $\mathcal{H}^+/\mathcal{N}_\theta \to \mathcal{H}_\theta$. Since

$$0 \leqq \langle f, f \rangle_\theta \leqq (f, f) \tag{2.2}$$

for all $f \in \mathcal{H}^+$, the map $[\ ]_\theta$ is a contraction

$$\| [f]_\theta \|_\theta \leqq \| f \| , \qquad f \in \mathcal{H}^+. \tag{2.3}$$

Note that those $f \in \mathcal{H}^+$ with $\theta f = f$ are mapped isometrically into $\mathcal{H}_\theta$. Now let in addition π be a unitary representation of a group G in $\mathcal{H}$. Denote by G^+ the subsemigroup of G consisting of those $g \in G$ for which

$$\pi(g)\mathcal{H}^+ \subseteqq \mathcal{H}^+. \tag{2.4}$$

Furthermore assume there is an involutive automorphism ϑ on G compatible with θ in the sense that

$$\theta \pi(g) = \pi(\vartheta g)\theta , \qquad g \in G . \tag{2.5}$$

Finally assume that

$$\vartheta g^{-1} = (\vartheta g)^{-1} \in G^+ \quad \text{whenever} \quad g \in G^+. \tag{2.6}$$

With these assumptions we have the first result. Its proof is an adaption of arguments used in [OS].

Theorem 2.2. *Let the triple* $(\mathcal{H}, \mathcal{H}^+, \theta)$ *be reflection positive and* ϑ *an involution on G satisfying* (2.5) *and* (2.6). *Then the representation* π *defines a representation* π_θ *of the semigroup* G^+ *of G into the contractions of* $\mathcal{H}_\theta$ *such that*

$$[\pi(g)f]_\theta = \pi_\theta(g)[f]_\theta \tag{2.7}$$

holds for $f \in \mathcal{H}^+$ *and* $g \in G^+$. *Furthermore, for* $f, f' \in \mathcal{H}_\theta$ *and* $g \in G^+$

$$\langle \pi_\theta(g) f, f' \rangle_\theta = \langle f, \pi_\theta(\vartheta g^{-1}) f' \rangle_\theta . \tag{2.8}$$

If in addition G is a topological group and π *a strongly continuous representation (such that* G^+ *is closed in G), then* π_θ *is strongly continuous* G^+.

Proof. First we note that the null space $\mathcal{N}_\theta$ in $\mathcal{H}^+$ is left invariant by any $\pi(g)$ $(g \in G^+)$. Indeed by (2.5) for any $f, f' \in \mathcal{H}^+$ and $g \in G^+$ we have

$$\begin{aligned} \langle \pi(g) f, f' \rangle_\theta &= (\pi(g) f, \theta f') \\ &= (f, \pi(g^{-1}) \theta f') \\ &= (f, \theta \pi(\vartheta g^{-1}) f') \\ &= \langle f, \pi(\vartheta g^{-1}) f' \rangle_\theta . \end{aligned} \tag{2.9}$$

Here we have used assumption (2.6), relation (2.4) and the unitarity of π. In particular if $f \in \mathcal{N}_\theta$ the last expression in (2.9) is zero, showing that $\pi(g) f \in \mathcal{N}_\theta$. Thus $\pi(g)$ $(g \in G^+)$ induces a map of $\mathcal{H}^+/\mathcal{N}_\theta$ into itself. We will show that this map is a contraction w.r.t. the norm $\| \ \|_\theta$ on $\mathcal{H}^+/\mathcal{N}_\theta$. By definition $\mathcal{H}^+/\mathcal{N}_\theta$ is dense in $\mathscr{H}_\theta$. This map therefore extends to a contraction of $\mathscr{H}_\theta$ into itself yielding the desired π_θ. Relation (2.8) is then a direct consequence of relation (2.9). To establish the desired contractivity, let $f \in \mathscr{H}^+$ and $g \in G^+$. We have

$$
\begin{aligned}
(2.10) \qquad \langle \pi(g)f,\, \pi(g)f \rangle_\theta &= (\pi(g)f,\, \theta\pi(g)f) \\
&= (f,\, \theta\pi((\vartheta g^{-1})\cdot g)f) \\
&\leqq \langle f,\, f\rangle_\theta^{1/2} \langle \pi(\vartheta g^{-1})\cdot g)f,\, \pi((\vartheta g^{-1})\cdot g)f\rangle_\theta^{1/2}.
\end{aligned}
$$

Again we have used the unitarity of π, relation (2.5) and Schwarz inequality for $\langle\cdot,\, \cdot\rangle_\theta$. We now iterate estimate (2.10). This leads to

$$
(2.11) \qquad \langle \pi(g)f,\, \pi(g)f\rangle_\theta \leqq \langle f,\, f\rangle_\theta^{1-2^{-n}} \langle \pi(((\vartheta g^{-1})\cdot g)^{2^{n-1}})f,\, \pi(((\vartheta g^{-1})\cdot g)^{2^{n-1}})f\rangle_\theta^{2^{-n}}
$$

for arbitrary integers $n \geqq 1$.

Now by (2.2), the second factor on the right hand side of (2.10) is bounded by $\|f\|^{2^{-n+1}}$. Here we may let n tend to infinity and obtain

$$
(2.12) \qquad \langle \pi(g)f,\, \pi(g)f\rangle_\theta \leqq \langle f,\, f\rangle_\theta
$$

which is the desired contraction property. The continuity of π_θ, given the strong continuity of π, follows from (2.7) and the contractivity of the map $[\ ]_\theta$ (see (2.3)). This concludes the proof of the theorem.

Note that if $g \in G^+$ is such that $g^{-1} \in G^+$ then $\pi_\theta(g)$ is an invertible contraction with inverse $\pi_\theta(g^{-1})$ and hence a unitary operator. Now such g's form a subgroup K^+ of G contained in G^+. K^+ is closed if G is a topological group and π is continuous. In particular our construction leads to a unitary representation of K^+.

Next if $g \in G^+$ is such that $\vartheta g^{-1} = g$, then $\pi_\theta(g)$ is a self adjoint contraction. If in addition g is in K^+, then $\pi_\theta(g)$ is a unitary and selfadjoint contraction, hence a selfadjoint involution. Let $G_+ = \{g \in G \mid \vartheta g = g\}$ be the subgroup of G on which ϑ is the identity. By assumption (2.6) K^+ contains the subgroup $G^+ \cap G_+$:

$$
(2.13) \qquad K^+ \supseteqq G^+ \cap G_+ .
$$

From now on, we will assume G to be a Lie group, ϑ a C^∞ automorphism and $\pi(\cdot)$ to be continuous. Then the triple (G, G_+, ϑ) is a symmetric space [H], [KN]. We may now give a discussion on the Lie algebra level. Let $\mathfrak{G}$ denote the Lie algebra of G. We define $\mathfrak{G}^+$, the tangent space of G^+ at $g=e$, to be the set of all tangent vectors at $g(0)=e$ of C^∞ maps $g: [0, \varepsilon] \to G^+$ ($\varepsilon > 0$ arbitrary). If $g(\cdot)$ is such a map, then also $g_\lambda(\cdot)$ with $g_\lambda(t) = g(\lambda t)$ ($\lambda > 0$). Also if $g_1(\cdot)$ and $g_2(\cdot)$ are such maps then $g_1 \circ g_2$ with $g_1 \circ g_2(t) = g_1(t) \cdot g_2(t)$. Hence $\mathfrak{G}^+$ is a convex cone in $\mathfrak{G}$, which is obviously closed and invariant under Ad_{K^+}. Note that $\exp(ta) \in G^+$ if $a \in \mathfrak{G}^+$. In fact let a be tangent to $g(t)$ at the origin $g(0)=e$.

Then $\exp(ta)=\lim_{n\to\infty} g\left(\frac{t}{n}\right)^n \in G^+$ for all $0\leqq t<\infty$.

Let now $d\vartheta$ denote the derived map of ϑ. $d\vartheta$ is an involutive Lie algebra automorphism of $\mathfrak{G}$. The expression

$$a=\frac{1}{2}(a+d\vartheta a)+\frac{1}{2}(a-d\vartheta a) \tag{2.14}$$

for an element $a\in\mathfrak{G}$ describes the decomposition of $\mathfrak{G}$ into eigenspaces of $d\vartheta$

$$\mathfrak{G}=\mathfrak{G}_+\oplus\mathfrak{G}_- . \tag{2.15}$$

$\mathfrak{G}_+$ is of course the Lie algebra of G_+. Since (G, G_+, ϑ) is a symmetric space, we have the familiar relations

$$\begin{cases} [\mathfrak{G}_+, \mathfrak{G}_+]\subseteqq\mathfrak{G}_+ \\ [\mathfrak{G}_+, \mathfrak{G}_-]\subseteqq\mathfrak{G}_- \\ [\mathfrak{G}_-, \mathfrak{G}_-]\subseteqq\mathfrak{G}_+ . \end{cases} \tag{2.16}$$

On the Lie algebra level, the condition (2.6) takes the form

$$-d\vartheta a\in\mathfrak{G}^+ \quad \text{whenever} \quad a\in\mathfrak{G}^+. \tag{2.6'}$$

Lemma 2.3. *Assume that G^+ contains G_+ such that $\mathfrak{G}^+$ contains $\mathfrak{G}_+$. Then (2.6') holds and $\mathfrak{G}^+$ has the the decomposition*

$$\mathfrak{G}^+=\mathfrak{G}_+\oplus(\mathfrak{G}^+\cap\mathfrak{G}_-) . \tag{2.17}$$

Proof. Since (2.6') is an easy consequence of (2.17), it suffices to prove this last relation. Now let $g(\cdot):[0, \varepsilon]\to G^+$ be C^∞ with tangent vector $a\in\mathfrak{G}^+$ at the identity. By decreasing ε if necessary, we may assume $g(\cdot)$ to be of the form $g(t)=\exp a_1(t) \exp a_2(t)$, where $a_1(t)\in\mathfrak{G}_+$ $a_2(t)\in\mathfrak{G}_-$ depend on t in a C^∞-way. By the Campell-Hausdorff formula we have

$$\begin{cases} a_1(t)=\frac{t}{2}(a+d\vartheta a)+0(t^2) , \\ a_2(t)=\frac{t}{2}(a-d\vartheta a)+0(t^2) . \end{cases} \tag{2.18}$$

Now $\vartheta g(t)^{-1}=\exp a_2(t) \exp(-a_1(t))$. By assumption $\exp a_1(t)\in G_+\subseteqq G^+$. Hence $\exp a_2(t)\in G^+$ for all $0\leqq t<\varepsilon$. Therefore $\frac{1}{2}(a-d\vartheta a)\in\mathfrak{G}^+$, proving (2.17).

We want to discuss the consequences of the assumption made in Lemma 2.3 for the representation theory.

Define

$$d\pi_\theta(a)=\frac{d}{dt}\pi_\theta(\exp ta)|_{t=0} \tag{2.19}$$

to be the infinitesimal generator of the strongly continuous contraction semigroup $\pi_\theta(\exp ta)$ $(a \in \mathfrak{G}^+,\ t \geqq 0)$. The elements $d\pi_\theta(a)$ with $a \in \mathfrak{G}_+$ are antiselfadjoint. Moreover, this is generally true for any $a \in \mathfrak{G}^+$ with $-a \in \mathfrak{G}^+$. The elements $d\pi_\theta(a)$ with $a \in \mathfrak{G}^+ \cap \mathfrak{G}_-$ form a cone of nonpositive selfadjoint operators. Two operators $d\pi_\theta(a)$ and $d\pi_\theta(a')$ with $a,\ a' \in \mathfrak{G}^+ \cap \mathfrak{G}_-$ on the same orbit under Ad_{G_+} have the same spectrum. Also any a in the maximal linear subspace of $\mathfrak{G}^+ \cap \mathfrak{G}_-$ is mapped onto zero under $d\pi_\theta$. Note that this subspace is invariant under Ad_{G_+}.

Now consider the real Lie algebra $\mathfrak{G}^* = \mathfrak{G}_+ \oplus j\mathfrak{G}_-$ contained in the complexification $\mathfrak{G}_c = \mathfrak{G} \oplus j\mathfrak{G}$ $(j^2 = -1)$. Since $\mathfrak{G}$ will already be a complex vector space in the cases we will consider, we use j to describe the additional complexification. $\mathfrak{G}^*$ is called the dual of $\mathfrak{G}$ obtained from the symmetric space $(G,\ G_+,\ \vartheta)$. Under the additional condition that the cone $\mathfrak{G}^+ \cap \mathfrak{G}_-$ has nonempty interior in $\mathfrak{G}_-$, and hence spans $\mathfrak{G}_-$, we may apply the following noncommutative version of the Hille-Yosida theorem [LM].

Theorem 2.4. *Assume $G_+ \subseteqq G^+$ and let $\mathfrak{G}^+ \cap \mathfrak{G}_-$ have nonempty interior in $\mathfrak{G}_-$. Then π_θ has an analytic continuation to a unitary representation π_θ^* of the simply connected Lie group G^* whose Lie algebra is $\mathfrak{G}^*$.*

In our applications to the complementary series of $SL(2n, \mathbb{C})$, we will be able to establish the conditions of Theorem 2.4.

We turn to a brief discussion of possible physical applications. In the quantum version of the Yang-Baxter equations one encounters noncommuting monodromy matrices and therefore is interested in their behaviour if multiplied in different order [F]. In our context, we may even interpret the selfadjoint contraction semigroups $\pi_\theta(\exp ta)$ $(t \geqq 0,\ a \in \mathfrak{G}^+ \cap \mathfrak{G}_-)$ as so-called transfer matrices. In fact, in the examples we shall encounter, we shall see that they are positivity preserving in the sense that they leave a certain cone in $\mathcal{H}_\theta$ invariant. Hence each of these one-parameter semigroups may be used as a transition function in the construction of a Markov process.

Let $T: G \to G$ denote the map $g \mapsto \vartheta g^{-1}$ and introduce the set $G_- = \{g \mid g = \vartheta g^{-1}\}$. The tangent space to G_- at $g = e$ is obviously $\mathfrak{G}_-$ and G_- is just the fixed point set of the antiautomorphism T. Furthermore

$$T(g_1 \cdot g_2) = g_2 \cdot g_1 \tag{2.20}$$

wherever $g_1,\ g_2 \in G_-$. Thus T interchanges the order of multiplication on G_- and is therefore a way of measuring how much $g_1 \cdot g_2$ fails to be an element in G_-.

Next introduce the set $\mathcal{G} = G_+ \cdot G_-$. Note that $G_+ \cap G_-$ consists of involutive elements and is not necessarily a subgroup of G_+. Since G is a Lie group, $\mathcal{G}$ contains a neighborhood of the identity. More generally we have the following analogue of the polar decomposition theorem

Lemma 2.5. *Every element $g \in G$ such that $(\vartheta g^{-1}) \cdot g$ is a square in G_-, is contained in $\mathcal{G}$. In particular, if any element in G_- is a square, $\mathcal{G}=G$.*

Proof. For given g, let $h=(\vartheta g^{-1}) \cdot g$. We have $\vartheta h^{-1}=h$, such that $h \in G_-$. Let $g_- \in G_-$ be such that $g_-^{\,2}=h$. Set $g_+=g \cdot g_-^{\,-1}$. Then $\vartheta g_+=\vartheta g \cdot \vartheta g_-^{\,-1}= (\vartheta g) \cdot g_-=g \cdot (g^{-1} \cdot \vartheta g) \cdot g_-=g \cdot g_-^{\,-2} \cdot g_-=g \cdot g_-^{\,-1}=g_+$ and $g=g_+ \cdot g_-$ is the decomposition.

Now let $g=g_+ \cdot g_-$. We have

$$T(g)=T(g_+ \cdot g_-)=\vartheta(g_+ \cdot g_-)^{-1} = g_-^{\,-1} \cdot g_+^{\,-1}=g_- \cdot g_+^{\,-1}=g_+^{\,-1} \cdot g \cdot g_+^{\,-1}. \tag{2.21}$$

Applying (2.20) and (2.21) we have proved

Theorem 2.6. *For all sufficiently small $a, b \in \mathfrak{G}^+ \cap \mathfrak{G}_-$ there is an element $g_+(a, b) \in G_+$ such that*

$$\exp a \cdot \exp b=g_+(a, b)^{-1} \cdot \exp b \cdot \exp a \cdot g_+(a, b)^{-1}. \tag{2.22}$$

If every element in G_- is a square in G_-, such a relation holds for all $a, b \in \mathfrak{G}^+ \cap \mathfrak{G}_-$.

We now apply π_θ to this relation. Then $\pi_\theta(\exp b \cdot \exp a)$ is the adjoint of $\pi_\theta(\exp a \cdot \exp b)$ and $\pi_\theta(g_+(a, b))$ is unitary. Recall that by the polar decomposition theorem any bounded operator A in the Hilbert space with zero kernel and dense range is related to its adjoint via $A^*=U^{-1} \cdot A \cdot U^{-1}$ for a suitable unitary U. Hence the upshot of relation (2.22) is that for the choice $A=\pi_\theta(\exp a \cdot \exp b)$ the resulting U is in the image of G_+ under π_θ. Next we note a certain similarity to the quantum mechanical Yang-Baxter relations, which are typically of the form (see e.g. [F]).

$$L_2 \cdot L_1=R \cdot L_1 \cdot L_2 \cdot R^{-1} \tag{2.23}$$

for monodromy matrices L_1 and L_2 and a unitary R-matrix. The difference to our situation is that the R-matrix appears in the form of conjugation. In this context it would be interesting to see whether our approach is related to the appearance of anticommutation relations in the discussion of the Yang-Baxter relations as given in [Sk1], [Sk2]. On the other hand there are striking similarities. First we note that in analogy to the R-matrix our unitary $\pi_\theta(g_+(a, b))$ acts on a "smaller space" in the following sense. The unitary representation G_+ given by π_θ is in general not irreducible such that $\mathcal{H}_\theta$ decomposes and only operators $\pi_\theta(g)$ $(g \in G^+ \setminus G_+)$ may interpolate between different components. Secondly in relation (2.22) we may look at the behaviour of $g_+(a, b)$ as a and b vary in (different) orbits under Ad_{G_+}. More specifically choose $a=\mathrm{Ad}_{h_1} a_0$, $b=\mathrm{Ad}_{h_2} b_0$ with $a_0, b_0 \in \mathfrak{G}^+ \cap \mathfrak{G}_-$ and $h_1, h_2 \in G_+$. Note that

$$\text{(2.24)} \qquad spectrum\ d\pi_\theta(a) = spectrum\ d\pi_\theta(a_0).$$

Writing $g_+(a, b)$ as $g_+(a_0, b_0, h_1, h_2)$, by construction we have $g_+(a_0, a_0, h_1, h_1) = \mathbb{1}$ for all $h_1 \in G_+$. More generally we have the covariance property

$$\text{(2.25)} \qquad g_+(a_0, b_0, h_1, h_2) = h_1 \cdot g_+(a_0, b_0, e, h_1^{-1}h_2) \cdot h_1^{-1}.$$

Hence we may interpret h_1 and h_2 as abstract spectral parameters. This analogy may even be pushed further, although at the moment in only a speculative way. For integrable systems the inverse scattering problem is used to express the relevant objects in terms of conserved quantities. To achieve this one is led to consider a Riemann-Hilbert problem for the transfer matrix, viewed as a function of the spectral parameter (see e.g. [F] and references quoted there). In our context introduce $\rho \in \mathfrak{G}_+$ via $h = \exp \rho$ to describe the spectral parameter and set

$$\text{(2.26)} \qquad \begin{aligned} M(a, \rho) &= \exp \mathrm{Ad}_{\exp \rho} a \\ &= \exp \rho \cdot \exp a \cdot \exp -\rho \end{aligned}$$

with $a \in \mathfrak{G}^+ \cap \mathfrak{G}_-$ such that $\pi_\theta(M(a, \rho))$ is a selfadjoint contraction for all $\rho \in \mathfrak{G}_+$. Next let $G^- = (G^+)^{-1}$ with tangent space $\mathfrak{G}^- = -\mathfrak{G}^+$. Assume in addition G to be a complex analytic Lie group such that the interior $\mathrm{int}\,\mathfrak{G}^+$ of $\mathfrak{G}^+$ is an analytic domain (as will be the case for the examples we will consider). The Riemann-Hilbert problem may now be formulated as that of finding $L^+(a, \rho) \in G^+$ and $L^-(a, \rho) \in G^-$ analytic in ρ in $\mathrm{int}\,\mathfrak{G}^+$ and $\mathrm{int}\,\mathfrak{G}^-$ respectively such that the boundary values on $\mathfrak{G}_+ \subseteq \mathfrak{G}^+ \cap \mathfrak{G}^-$ exist and satisfy

$$\text{(2.27)} \qquad L^+(a, \rho) = L^-(a, \rho) \cdot M(a, \rho), \qquad \rho \in \mathfrak{G}_+.$$

Similarly one might look at the equation

$$\text{(2.28)} \qquad R^+(a, \rho) = M(a, \rho) \cdot R^-(a, \rho), \qquad \rho \in \mathfrak{G}_+$$

with $R^\pm(a, \rho) \in G^\pm$ analytic for ρ in $\mathrm{int}\,\mathfrak{G}^\pm$. If ϑ is holomorphic or antiholomorphic, then

$$\text{(2.29)} \qquad \begin{aligned} R^\pm(a, \rho) &= T(L^\pm(a, d\vartheta\rho)) \\ &= \vartheta L^\pm(a, d\vartheta\rho)^{-1} \end{aligned}$$

is a solution to (2.28) if $L^\pm$ is a solution to (2.27) and vice versa. Hence it suffices to consider equation (2.29). Now any solution to (2.27) is highly non-unique, for example $\exp \rho \cdot L^\pm(a, \rho)$ is a solution wherever $L^\pm(a, \rho)$ is a solution. Also a particular solution to (2.29) is given by $L^\pm(a, \rho) = \exp \pm \frac{a}{2} \exp \rho$. To make the problem nontrivial, we therefore introduce a normalization condition by requiring that

$$\text{(2.30L)} \qquad L^\pm(0, \rho) = e$$

and similarly

$$R^{\pm}(0, \rho)=e \tag{2.30R}$$

for all ρ. If the solution to (2.29), (2.30L) is unique and ϑ is holomorphic or antiholomorphic, it is easy to verify that

$$L^{-}(a, \rho)=\vartheta L^{+}(a, d\vartheta\rho), \qquad \rho\in \mathrm{int}\, \mathfrak{G}^{-}. \tag{2.31}$$

Similarly the covariance property

$$M(\mathrm{Ad}_{g'}a, \mathrm{Ad}_{g'}\rho)=g'\cdot M(a, \rho)\cdot g'^{-1}, \qquad g'\in G_{+} \tag{2.32}$$

gives

$$L^{+}(\mathrm{Ad}_{g'}a, \mathrm{Ad}_{g'}\rho)=g'\cdot L^{+}(a, \rho)\cdot g'^{-1}, \qquad g'\in G_{+}. \tag{2.33}$$

Provided a solution to (2.27), (2.30L) exists, then motivated by the discussion in [GW], we may formulate the following abstract scattering problem. Consider the limit (if it exists)

$$L^{+}(a)=\lim_{t\to\infty} L^{+}(a, ta)\in G^{+}, \qquad L^{-}(a)=\lim_{t\to-\infty} L^{-}(a, ta)\in G^{-}. \tag{2.34}$$

In case $L^{\pm}(a)\in\mathrm{ran(exp)}$, let $\phi^{\pm}(a)\in\mathfrak{G}^{+}$ be given by

$$L^{\pm}(a)=\exp \pm\phi^{\pm}(a). \tag{2.35}$$

By (2.30L)

$$\phi^{\pm}(0)=0 \tag{2.36}$$

and

$$\phi^{\pm}(\mathrm{Ad}_{g'}a)=\mathrm{Ad}_{g'}\phi^{\pm}(a). \tag{2.37}$$

The scattering transformation is the map $\phi^{-}(a)\mapsto\phi^{+}(a)$ from $\mathrm{Im}\,\phi^{-}$ to $\mathrm{Im}\,\phi^{+}$ provided the map $a\mapsto\phi^{-}(a)$ is injective. Note that since $\phi^{\pm}(a)\in\mathfrak{G}^{+}$ by construction, the accretive operators $-d\pi_{\theta}(\phi^{\pm}(a))$ are well defined. At the moment we do not know whether the above approach has a solution in the concrete context of our examples of $SL(2n, \boldsymbol{C})$ to be discussed in the next section.

§ 3. The Complementary Series of $SL(2n, \boldsymbol{C})$

In this section we will prove reflection positivity for the representations of $SL(2n, \boldsymbol{C})$ as given by Stein [St] and which reduce to the complementary series of Gelfand Neumark for $SL(2, \boldsymbol{C})$ when $n=1$.

Actually we shall show that one obtains a Hilbert space $\mathcal{H}_{\theta}$ and a representation π_{θ} of G^{+} even for a larger range of the parameter describing the representation π than the one which is known to give unitary representations of $SL(2n, \boldsymbol{C})$. To deal with these extra cases, we will extend the general arguments of Section 2. From the start we shall work with arbitrary n although specialization to $n=1$ would sometimes simplify the calculations.

Let $M_n=M_n(\boldsymbol{C})$ denote the linear space of all complex $n\times n$ matrices $z=(z_{ij})_{1\leq i,j\leq n}$ $(z_{ij}\in\boldsymbol{C})$. Let $dz=\prod_{i,j} d\,\mathrm{Re}(z_{ij})d\,\mathrm{Im}(z_{ij})$ be the canonical Lebesgue measure on M_n. Let $z^*=\bar{z}^t$ be the hermitean adjoint of z and let

$$z=x+iy \tag{3.1}$$

$$\mathrm{Re}\,z=x=\frac{1}{2}(z+z^*)\,, \quad \mathrm{Im}\,z=y=\frac{1}{2i}(z-z^*)$$

be the resulting expression for z in terms of hermitean x and y. For $n=1$, the star operation is simply complex conjugation such that $\mathrm{Re}\,z$ and $\mathrm{Im}\,z$ are real numbers. We will use the notation $z>0$ to describe positive definite $n\times n$ matrices z. Let $M_n^{\pm}$ be the open subspace of M_n consisting of elements $z=x+iy$ with $\pm y>0$. M_n^+ and M_n^- are spaces of the first Cartan type [C]. They are isometric to $SU(n, n)/S(U_n\times U_n)$ and are denoted as type AIII in [H] (see also the discussion at the end of this section). Some authors (see e.g. [K1]) refer to M_n^+ as a Siegel domain of the first genus. Each of the maps $z\mapsto -z$, $z\mapsto z^*$ and $z\mapsto z^{-1}$ maps M_n^+ onto-to-one onto M_n^- in a C^∞-way. Note that $\det z\neq 0$ on $M_n^+\cup M_n^-$. For $n=1$, M_n^+ and M_n^- are simply the upper and lower complex half plane in $\boldsymbol{C}$. Next write any element $g\in SL(2n, \boldsymbol{C})$ in the form

$$g=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{3.2}$$

where each g_{ik} $(1\leqq i, k\leqq n)$ is a complex $n\times n$ matrix. $SL(2n, \boldsymbol{C})$ acts as a transformation group on M_n by

$$\pi(g)z=(g_{22}z-g_{21})\cdot(-g_{12}z+g_{11})^{-1} \tag{3.3}$$

with inverse

$$\pi(g^{-1})z=(zg_{12}+g_{22})^{-1}(zg_{11}+g_{21})\,. \tag{3.4}$$

Here zg_{12} etc. denotes matrix multiplication in M_n. Note that $\pi(g)z$ is not defined for those pairs (g, z) for which $\det(-g_{12}z+g_{11})=0$. However, they form a lower dimensional closed set in $SL(2n, \boldsymbol{C})\times M_n$. In what follows, it is always understood that this set is excluded. With this convention $(g, z)\mapsto\pi(g)z$ is a C^∞ map from $SL(2n, \boldsymbol{C})\times M_n$ into M_n. We now use the procedure of Gelfand and Neumark, by which the representation properties of π follow easily if we identify an element $z\in M_n$ with an element in $SL(2, \boldsymbol{C})$ of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$. These elements form (modulo a set of measure zero) the coset space of the subgroup of $SL(2n, \boldsymbol{C})$ consisting of elements g with $g_{21}=0$. The action of

$SL(2n, \boldsymbol{C})$ on this coset space in terms of these representatives is now given exactly in terms of (3.4), i. e.

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}=\begin{pmatrix} g_{11}-g_{12}\pi(g^{-1})z & g_{12} \\ 0 & zg_{12}+g_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pi(g^{-1})z & 1 \end{pmatrix}. \tag{3.5}$$

We introduce the symplectic matrix

$$J=\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{3.6}$$

and define the involutive automorphism ϑ on $SL(2n, \boldsymbol{C})$ by

$$\vartheta g=Jg^{*-1}J\,. \tag{3.7}$$

For $n=1$, ϑ is simply complex conjugation. On the Lie algebra level, the derived map $d\vartheta$ of ϑ takes the form

$$d\vartheta a=-Ja^*J\,, \quad a\in sl(2n, \boldsymbol{C})\,. \tag{3.8}$$

The group $G_+=\{g\in SL(2n, \boldsymbol{C}) \mid \vartheta g=g\}$ is the subgroup of $SL(2n, \boldsymbol{C})$ consisting of those elements $g\in SL(2n, \boldsymbol{C})$ leaving the symplectic form J invariant

$$G_+=\{g\in SL(2n, \boldsymbol{C}) \mid g^*Jg=J\}. \tag{3.9}$$

For $n=1$ this group is just $SL(2, \boldsymbol{R})$. By making use of (3.5) one easily derives

$$(\pi(g)z)^*=\pi(\vartheta g)z^*. \tag{3.10}$$

Also a routine calculation shows that under π, G_+ maps M_n^+ one-to-one onto itself. By (3.10) the same is true when M_n^+ is replaced by M_n^-. We now define G^+ to be the subsemigroup of $SL(2n, \boldsymbol{C})$ consisting of elements mapping M_n^+ into itself under π. In particular we have $G_+\subseteqq G^+$. We claim G^+ is closed. In fact if the sequence $g_n\in G^+$ tends to g, then by continuity g maps M_n^+ into the closure of M_n^+. Since g has an inverse in $SL(2n, \boldsymbol{C})$, $\pi(g)M_n^+$ is an open set and hence contained in M_n^+, since M_n^+ is the interior of its closure.

Now with

$$\gamma=\begin{pmatrix} 0 & 0 \\ -i\mathbf{1} & 0 \end{pmatrix} \tag{3.11}$$

$\exp t\gamma\in G^+$ $(t>0)$, such that $\gamma\in\mathfrak{G}^+\cap\mathfrak{G}_-$. In fact, we have $\pi(\exp t\gamma)z=z+it\mathbf{1}$. Since $J\in G_+$, this also gives

$$\gamma^*=J\gamma J^{-1}\in\mathfrak{G}^+\cap\mathfrak{G}_-\,. \tag{3.12}$$

Therefore $iJ=\gamma+\gamma^*\in\mathfrak{G}^+\cap\mathfrak{G}_-$. We claim iJ is an interior point of $\mathfrak{G}^+\cap\mathfrak{G}_-$ in $\mathfrak{G}_-$. To see this write the elements $a\in\mathfrak{G}$ in a neighborhood of iJ as

$$a=\begin{pmatrix} -i\alpha & i(\mathbf{1}+\gamma) \\ -i(\mathbf{1}+\beta) & i\alpha^* \end{pmatrix} \tag{3.13}$$

with β, γ hermitean. We have to show that for α, β, γ sufficiently small, $g(t)=\exp ta$ $(t\geqq 0)$ is in G^+. To see this, we consider the flow generated by $g(t)$ on M_n. Thus it suffices to show that $\frac{d}{dt}\pi(g(t))z|_{t=0}\in M_n{}^+$ for any $z\in M_n$. Now

$$\frac{d}{dt}\pi(g(t))z|_{t=0}=i\alpha^*z+i(\mathbf{1}+\beta)+zi(\mathbf{1}+\gamma)z+zi\alpha\,, \tag{3.14}$$

such that

$$\begin{aligned}\operatorname{Im}\frac{d}{dt}\pi(g(t))z|_{t=0}=&\frac{1}{2}z(\mathbf{1}+\gamma)z+\frac{1}{2}z^*(\mathbf{1}+\gamma)z^*+\frac{1}{2}(z-z^*)\alpha\\&+\frac{1}{2}\alpha^*(z-z^*)+(\mathbf{1}+\beta)\,.\end{aligned} \tag{3.15}$$

For $(1+\gamma)>0$ we may complete the square. With the choice $\rho=2(1+\gamma)^{-1}\alpha$ we obtain

$$\begin{aligned}\operatorname{Im}\frac{d}{dt}\pi(g(t))z|_{t=0}=&\frac{1}{4}((1+\gamma)^{1/2}(z+z^*))^*((1+\gamma)^{1/2}(z+z^*))\\&+\frac{1}{4}((1+\gamma)^{1/2}(z-z^*+\rho))^*((1+\gamma)^{1/2}(z-z^*+\rho))\\&-\alpha^*(1+\gamma)^{-1}\alpha+\mathbf{1}+\beta\,.\end{aligned} \tag{3.16}$$

Hence all α, β, γ with

$$\begin{cases}1+\gamma>0\\ \alpha^*(1+\gamma)^{-1}\alpha<\mathbf{1}+\beta\end{cases} \tag{3.17}$$

give

$$\operatorname{Im}\frac{d}{dt}\pi(g(t))z|_{t=0}>0\,. \tag{3.18}$$

Since the set of such α, β, γ via (3.13) define an open neighborhood of iJ in $\mathfrak{G}_-$, the claim that iJ is an interior point of the cone is proved.

An explicit calculation for $n=1$ shows that $\mathfrak{G}^+\cap\mathfrak{G}_-$ is the convex cone in $\mathfrak{G}_-$ spanned by the elements

$$\begin{pmatrix}0 & i\\ 0 & 0\end{pmatrix},\ \begin{pmatrix}0 & 0\\ -i & 0\end{pmatrix},\ \begin{pmatrix}i & i\\ -i & -i\end{pmatrix}\ \text{and}\ \begin{pmatrix}-i & i\\ -i & -i\end{pmatrix}.$$

Note that the orbits in $\mathfrak{G}^+\cap\mathfrak{G}_-$ under $\mathrm{Ad}_{SL(2,\boldsymbol{R})}$ are 2-dimensional. In general Ad_{G_+} does not act transitively on $\mathfrak{G}^+$. We collect our result in

Lemma 3.1. *The cone $\mathfrak{G}^+\cap\mathfrak{G}_-$ has nonempty interior in $\mathfrak{G}_-$.*

Next we claim that the dual $\mathfrak{G}^*=\mathfrak{G}_+\oplus j\mathfrak{G}_-$ $(j^2=-1)$ is isomorphic to $\mathfrak{G}_+\oplus\mathfrak{G}_+$. For $n=1$ this follows by inspection. For the general case consider the $\boldsymbol{R}$-linear map I on $\mathfrak{G}$ given by $I(a)=ia$. Then $d\vartheta I(a)=-I(d\vartheta a)$ such that I

maps $\mathfrak{G}_+$ into $\mathfrak{G}_-$ and vice versa. Since $I^4=id$, these maps are actually onto, giving an easy proof that $\dim \mathfrak{G}_+=\dim \mathfrak{G}_-=1/2 \dim \mathfrak{G}$. Now define two $\boldsymbol{R}$-linear maps S and D from $\mathfrak{G}_+$ into $\mathfrak{G}^*$ by

$$\begin{cases} S(a)=\dfrac{1}{\sqrt{2}}(a+jI(a)) \\ D(a)=\dfrac{1}{\sqrt{2}}(a-jI(a)) . \end{cases} \tag{3.19}$$

A routine calculation shows that S and D are Lie algebra homomorphisms with trivial kernel and that $\mathfrak{G}^*=\mathrm{Im}\, S \oplus \mathrm{Im}\, D$ as a direct sum of Lie algebras. This proves

Lemma 3.2. *The dual $\mathfrak{G}^*$ of $sl(2n, \boldsymbol{C})$ with respect to the involution given by* (3.9) *is isomorphic to $\mathfrak{G}_+ \oplus \mathfrak{G}_+$.*

With these preparations we turn to a discussion of the complementary series. First the Hilbert-space for these representations of $SL(2n, \boldsymbol{C})$ as used in [GN] and [St] is obtained from the scalar product for measurable functions on M_n of the form

$$\int_{z, z' \in M_n} \overline{f(z)} |\det(z-z')|^{-2n+2} f'(z') dz dz' \tag{3.20}$$

with the restriction $-1<\lambda<1$ for $n>1$ and $0<\lambda \leqq 1$ for $n=1$. In the resulting Hilbert space let $\mathcal{H}^+$ denote the closed subspace formed by functions with support in the closure of $M_n{}^+$. Also let θ be given by

$$(\theta f)(z)=f(z^*) . \tag{3.21}$$

To establish reflection positivity in the form discussed in Section 2, we will actually discuss a more general set-up. Namely on $C_0^\infty(M_n{}^+)$ and for fixed λ, $\mu<1$ consider the quadratic form on $\mathcal{H}^+$ given as

$$\begin{aligned} \langle f, f' \rangle_\theta^{\lambda, \mu} &= \int_{z, z' \in M_n^+} \overline{f(z)} (\det(-i(z-z'^*)))^{\lambda-n} (\det(+i(z^*-z')))^{\mu-n} f'(z') dz dz' \\ &= \int_{z, z' \in M_n} \overline{f(z)} (\det(-i(z-z')))^{\lambda-n} (\det(i(z^*-z'^*))^{\mu-n} (\theta f')(z') dz dz'. \end{aligned} \tag{3.22}$$

The reflection positivity for the scalar product (3.20) is now a special case of

Theorem 3.3. *The quadratic form* (3.22) *on $C_0^\infty(M_n{}^+)$ is positive semidefinite for all λ, $\mu<1$.*

Proof. We will apply the theory of Laplace transforms on the symmetric space K_n consisting of all positive definite complex $n \times n$ matrices. We recall there is a measure dk on K_n which is invariant under the map

(3.23) $$k \mapsto zkz^*(k \in K_n,\ z \in M_n,\ \det z \neq 0).$$

dk is unique up to a constant. With the choice

(3.24) $$dk = (\det k)^{-n} \prod_i dk_{ii} \prod_{i<j} d\,\mathrm{Re}\,k_{ij}\, d\,\mathrm{Im}\,k_{ij}$$

the resulting Γ function $\Gamma(s, K_n)$ on K_n

(3.25) $$\Gamma(s, K_n) = \int e^{-\mathrm{trace}\,k} (\det k)^s dk$$

may be calculated explicitly and equals

(3.26) $$\Gamma(s, K_n) = \pi^{n/2(n-1)} \Gamma(s)\Gamma(s-1) \cdots \Gamma(s-n+1)$$

The proof of this dates back to Siegel ([Si], Hilfssatz 37) and Selberg [Se], (see e. g. [M]).

From (3.25) one deduces

(3.27) $$(\det y)^{-s} = \Gamma(s, K_n)^{-1} \int e^{-\mathrm{trace}(yk)} (\det k)^s dk$$

for any positive definite $n \times n$ matrix y and $s > n-1$. By analytic continuation this gives

(3.28) $$\det(-iz)^{-s} = \Gamma(s, K_n)^{-1} \int e^{i\,\mathrm{trace}(kz)} (\det k)^s dk,$$

for any $z \in M_n^+$ and $s > n-1$.

We now represent $(\det(-i(z-z'^*)))^{\lambda-n}$ and $(\det i(z^*-z'))^{\mu-n}$ using (3.28). Inserting this into (3.22) and interchanging the order of integration proves the claim.

Remark 3.4. Let $(v, v') = \sum v_i v_i'$ denote the canonical scalar product and $d^n v$ the canonical Lebesgue measure on $\boldsymbol{R}^n$. Using the representation

(3.29) $$\det(-iz)^{-1/2} = (2\pi)^{-n/2} \int_{v \in \boldsymbol{R}^n} e^{i(v, zv)} d^n v$$

for $z \in M_n^+$, by a similar argument one shows that $\langle \cdot, \cdot \rangle_\theta^{\lambda, \mu}$ is positive semidefinite whenever both $-\lambda+n$ and $-\mu+n$ are nonnegative halfintegers. We do not know whether Theorem 3.3 continues to hold for arbitrary $-\lambda+n$ and $-\mu+n$ in the interval $[0, n-1)$.

Next define by

(3.30) $$(\pi^{\lambda,\mu}(g)f)(z) = \det(zg_{12}+g_{22})^{-n-\lambda} \det(zg_{12}+g_{22})^{-n-\mu} f(\pi(g^{-1})z)$$

a representation of $SL(2n, \boldsymbol{C})$ on the space of measurable functions f on M_n. The representation properties follow from the relation

(3.31) $$\det(z(gg')_{12}+(gg')_{22})=\det(zg_{12}+g_{22})\det(\pi^{-1}(g)zg'_{12}+g'_{22})$$

which may easily be read off (3.5). For any $g\in G_+$, $\pi^{\lambda,\mu}(g)$ leaves $C_0^\infty(M_n^+)$ invariant.

The relevance of the scalar product $\langle\cdot,\cdot\rangle_\theta^{\lambda,\mu}$ for this representation is due to an important intertwining property of $\det(z-z')$ ($z, z'\in M_n$). The following result and its proof are essentially contained in [St]. For the convenience of the reader, we repeat the arguments leading to

Proposition 3.5. *For almost all $g\in SL(2n, \mathbf{C})$ and $z, z'\in M_n$*

(3.32) $$\det(zg_{12}+g_{22})\det(\pi(g^{-1})z-\pi(g^{-1})z')\det(z'g_{12}+g_{22})=\det(z-z')\,.$$

Proof. If we introduce the quantity

(3.33) $$F(g, z, z')=\frac{\det(zg_{12}+g_{22})\det(\pi(g^{-1})z-\pi(g^{-1})z')\det(z'g_{12}+g_{22})}{\det(z-z')},$$

the aim is to show that $F(g, z, z')$ is independent of g and hence equals 1. First relation (3.31) directly gives

(3.34) $$F(gg', z, z')=F(g, z, z')F(g', \pi(g^{-1})z, \pi(g^{-1})z')\,.$$

Now any $g\in SL(2, \mathbf{C})$ may be written as $g_1g_2\cdots g_k$ ($k\leqq N(n)=(2n)\,!$) where each g_i is of the form

(3.35) $$\begin{cases} \text{a)} \begin{pmatrix} \mathbf{1} & 0 \\ g_{21} & \mathbf{1}\end{pmatrix} \\ \text{b)} \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22}\end{pmatrix} \\ \text{c)}\ J\,. \end{cases}$$

By (3.34), to show that $F(g, z, z')$ does not depend on g, it therefore suffices to check it on elements of the form (3.35). For g of the form (3.35a) $F(g, z, z')=1$ trivially holds since $\pi(g^{-1})z=z+g_{21}$. For g of the form (3.35b) we have $\pi(g^{-1})z=g_{22}{}^{-1}zg_{11}$ such that $\det(\pi(g^{-1})z-\pi(g^{-1})z')=(\det g_{22})^{-1}\det(z-z')(\det g_{11})$. Since $\det g_{11}=(\det g_{22})^{-1}$ we have $F(g, z, z')=1$ also in that case. Finally if $g=J$, then

$$\begin{aligned}\det(\pi(g^{-1})z-\pi(g^{-1})z')&=\det(-z^{-1}+z'^{-1})=\det(z^{-1}(z-z')z'^{-1})\\ &=(\det z)^{-1}\det(z-z')(\det z')^{-1}.\end{aligned}$$

Also $\det(zg_{12}+g_{22})=\det z$, which again gives $F(J, z, z')=1$. This concludes the proof of Proposition 3.5.

Finally we will need the relation

(3.36) $$\left|\frac{\partial\pi(g^{-1})z}{\partial z}\right|=\det(zg_{12}+g_{22})^{-2n}$$

for the complex Jacobian.

Let now $\pi_\theta^{\lambda,\mu}$ denote the restriction of the representation $\pi^{\lambda,\mu}$ of G^+ to $C_0^\infty(M_n{}^+)$. If we combine (3.22), (3.30), (3.32), and (3.36) we obtain the analog of (2.8) for the present situation.

Theorem 3.6. *The representations $\pi_\theta^{\lambda,\mu}(\lambda, \mu<1$ or $-\lambda+n$ and $-\mu+n$ half-integers) satisfy the relation*

$$\langle \pi_\theta^{\mu,\lambda}(g)f, f'\rangle_\theta^{\lambda,\mu} = \langle f, \pi_\theta^{\lambda,\mu}(\vartheta g^{-1})f'\rangle_\theta . \tag{3.37}$$

Hence the interesting cases arise when we make the additional restriction $\lambda=\mu$. Write $\langle\cdot,\cdot\rangle_\theta^\lambda$ and π_θ^λ for $\langle\cdot,\cdot\rangle_\theta^{\lambda,\lambda}$ and $\pi_\theta^{\lambda,\lambda}$ respectively. Since our range for the parameter λ is larger than the one given for the unitary representations of $SL(2n, \boldsymbol{C})$, we will also give a direct proof of the contractive property of π_θ^λ on G^+. First let $\mathcal{H}_\theta^\lambda$ denote the Hilbert space obtained from the semidefinite quadratic form $\langle\cdot,\cdot\rangle_\theta$ on $C_0^\infty(M_n{}^+)$.

By arguments similar to those used in Section 2, π_θ^λ extends to a continuous representation of G^+ on $\mathcal{H}_\theta^\lambda$ and satisfies property (3.37) for $f, f'\in\mathcal{H}_\theta^\lambda$. In particular, by (3.37) the operators $\pi_\theta^\lambda(g)$ are unitary for g in G_+. To establish the contractivity in the general case, we first note that estimate (2.10) continues to hold in our present case.

Now let $g(t)=\exp ta$ with $a\in\mathfrak{G}^+\cap\mathfrak{G}_-$. It suffices to show that

$$\langle f, \pi_\theta^\lambda(g(t))f\rangle_\theta^\lambda \tag{3.38}$$

is bounded in t for $t\geqq 0$ and $f\in C_0^\infty(M_n{}^+)$, since then we may repeat the arguments which led to (2.12). Since $C_0^\infty(M_n{}^+)$ is dense in this will prove the contractivity of $\pi_\theta^\lambda(g(t))$. Now (3.38) may be written as

$$\int_{z,z'\in M_n^+} \overline{f(z)}\,|\det(z-(\pi(g(t))z')^*)|^{-2n+2\lambda} f(z')dzdz'. \tag{3.39}$$

We need the following elementary

Lemma 3.7. *Let A and B be hermitean $n\times n$ matrices. If $A>0$, then*

$$|\det(A+iB)|\geqq \det A . \tag{3.40}$$

If $A>B>0$ then

$$\det A>\det B . \tag{3.41}$$

Proof. If $A>0$, then

$$A+C=A^{1/2}(\mathbf{1}+A^{-1/2}CA^{-1/2})A^{1/2}.$$

Apply this first to $C=iB$, and (3.40) follows. To get (3.41), write $A=B+(A-B)$, and factor $B^{1/2}$ out on the left, and the right.

We apply this lemma as follows. Since $\pi(g)z' \in M_n{}^+$ we also have $-(\pi(g)z')^* \in M_n{}^+$ and therefore

$$|\det(z-(\pi(g)z')^*)| \geqq \det(\mathrm{Im}(z-(\pi(g)z')^*)) \geqq \det \mathrm{Im}\, z \tag{3.42}$$

and the right hand side is bounded below by some $c>0$ on the support of $f(z)$. Inserting this in (3.38) proves the boundedness in $t \geqq 0$. We collect our results in

Theorem 3.8. *The representation π_θ^λ of G^+ on $\mathcal{H}_\theta^\lambda$ is continuous and contractive and satisfies*

$$\langle \pi_\theta^\lambda(g)f, f' \rangle_\theta^\lambda = \langle f, \pi_\theta^\lambda(\vartheta g^{-1})f' \rangle_\theta^\lambda \tag{3.43}$$

for $f, f' \in \mathcal{H}_\theta^\lambda$. Here $\lambda<1$ or $-\lambda+n$ is a nonnegative halfinteger. Each $\pi_\theta^\lambda(g)$ $(g \in G^+)$ leaves the cone, obtained by forming the closure of positive functions in $C_0^\infty(M_n{}^+)$ invariant. The representation π_θ^λ has an analytic continuation to a unitary representation of the simply connected Lie group G^* with Lie algebra $\mathfrak{G}_+ \oplus \mathfrak{G}_+$, denoted by $\pi_\theta^{\lambda *}$.

We conclude this section by giving an alternative description of our construction in terms of functions living on the unit disc $D(M_n)$ in M_n

$$D(M_n) = \{z \in M_n \mid z^* z < \mathbf{1}\}. \tag{3.45}$$

In fact let $\rho = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -i\mathbf{1} \\ -i\mathbf{1} & \mathbf{1} \end{pmatrix}$ and define

$$\varphi(z) = (\pi^{\lambda,\mu}(\rho) f)(z). \tag{3.45}$$

Then if f lives on $M_n{}^+$, φ lives on $D(M_n)$ and under the pull back by $\pi^{\lambda,\mu}(\rho)$ the scalar product $\langle f, f' \rangle_\theta^{\lambda,\mu}$ takes the form

$$\int_{z, z' \in D(M_n)} \overline{\varphi(z)} \det(1-zz'^*)^{\lambda-n} \det(1-z^*z')^{\mu-n} \varphi'(z') dz dz'. \tag{3.46}$$

We note that the kernel $\det(1-zz'^*)^{-m}$ plays an important role in the theory of automorphic functions (see e.g. [Ka]).

For $\lambda=\mu$ the pullback of π_θ^λ gives a representation of the subsemigroup of $SL(2n, \boldsymbol{C})$, consisting of elements mapping $D(M_n)$ into itself. This subsemigroup is just the conjugate to G^+ under ρ. The subgroup of $SL(2n, \boldsymbol{C})$ which maps $D(M_n)$ one-to-one onto itself is then the conjugate under ρ to G_+. It is given as $SU(n, n)$, defined to be the set of elements $g \in SL(2n, \boldsymbol{C})$ with

$$g^* \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} g = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{3.47}$$

(see e.g. [H], Chap. IX, § 4, [KN], Chap. IX, § 6, Example 6.5, Chap. XI, § 10, Example 10.9). It is known that $SU(n, n)$ is connected but not simply connected.

In particular our group G_+ is connected but not simply connected. Moreover the representation $\pi_\theta^{\lambda *}$ of G^* does not reduce to a representation of $G_+ \times G_+$. Indeed, $h(m) = \exp_{G^*} \frac{2\pi m}{\sqrt{2}}(J + j \cdot I(J))$ $(m \in \boldsymbol{Z})$ is in the kernel of the covering homomorphism and therefore would have to be mapped into the identity operator on $\mathcal{H}_\theta^\lambda$ under $\pi_\theta^{\lambda *}$. Now $d\pi_\theta^\lambda(J)$ has discrete spectrum $\subseteqq 2\pi \boldsymbol{Z}$ and commutes with $d\pi_\theta^\lambda(iJ) = d\pi_\theta^\lambda(I(J))$. Also by definition of the analytic continuation $\pi_\theta^{\lambda *}(\exp tjI(J)) = \exp \text{id}\, \pi_\theta^\lambda(I(J))$. But $d\pi_\theta^\lambda(I(J))$ has nonpositive spectrum. Therefore $h(m)$ cannot be mapped into the identity under $\pi_\theta^{\lambda *}$.

§ 4. Construction of the Local Representation

The main result of this section is

Theorem 4.1. *For all $\lambda < 1$ and all λ such that $-\lambda + n$ is a nonnegative half-integer, π_θ^λ extends to a local representation of $SL(2n, \boldsymbol{C})$ on $\mathcal{H}_\theta^\lambda$ in the sense of* [J1], [J2].

Thus by a theorem in [J2] we obtain another proof of the second part of Theorem 3.8. This theorem in [J2] relies on the exponentiation theory for Lie algebras of unbounded operators, see for example [FSSS], [GoJ], [JM], [P], [Si].

We recall the defining properties of a local representation as given in [J2]. We have to find a dense domain $\mathcal{D}$ in $\mathcal{H}_\theta^\lambda$ and a neighborhood U of the identity in $SL(2n, \boldsymbol{C})$ with the following properties.

(i) For all $g \in U$, $\mathcal{D}$ is in the domain $\mathcal{D}(\pi_\theta^\lambda(g))$ of $\pi_\theta^\lambda(g)$.

(ii) If g_1, g_2 and $g_1 \cdot g_2$ are in U and $f \in \mathcal{D}$, then

$$\pi_\theta^\lambda(g_2) f \in \mathcal{D}(\pi_\theta^\lambda(g_1))$$

and

$$\pi_\theta^\lambda(g_1)\pi_\theta^\lambda(g_2) f = \pi_\theta^\lambda(g_1 \cdot g_2) f .$$

(iii) If $a \in \mathfrak{G}_-$ is such that $\exp a \in U$ then $\exp ta \in U$ for $0 \leqq t \leqq 1$ and

$$s\text{-}\lim \pi_\theta^\lambda(\exp ta) f = f \qquad (0, 1) \ni t \to 0 \tag{4.1}$$

for all $f \in \mathcal{D}$.

(iv) $d\pi_\theta^\lambda(a)\, \mathcal{D} \subset \mathcal{D}$ for $a \in \mathfrak{G}_-$.

(v) For each $\varphi \in \mathcal{D}$, there is a neighborhood V_φ of e in G_+ such that $U \cdot V_\varphi \subset U^2$ and

$$\pi_\theta^\lambda(g) f \in \mathcal{D} , \qquad g \in V_\varphi . \tag{4.2}$$

Finally

(vi) For $a \in \mathfrak{G}_-$ and $f \in \mathcal{D}$ the function

$$g \mapsto \pi_\theta^\lambda(\exp \text{Ad}_g a) f \tag{4.3}$$

is locally integrable on the subset $\{g\in G_+ \mid \exp \mathrm{Ad}_g a\in U\}$.

To construct $\mathscr{D}$, let $\mathcal{O}$ be any open, precompact set in $M_n{}^+$. Let $U_{\mathcal{O}}$ be the interior of a compact neighborhood K of the identity such that $\pi(g)\mathcal{O}\subset M_n{}^+$ for any $g\in K$. Let $a_1, \cdots, a_{4n^2-2}$ be a basis in $sl(2n, \boldsymbol{C})$ such that $a_1, \cdots, a_{2n^2-1}\in\mathfrak{G}_+$ and $a_{2n^2}, \cdots, a_{4n^2-2}\in\mathfrak{G}_-$. By taking $\varepsilon>0$ sufficiently small, we will choose K to be of the form

$$K=\left\{\exp\sum_{i=1}^{4n^2-2}\lambda_i a_i \mid \lambda_i\in\boldsymbol{R}_1 \sum_{i=1}^{4n^2-2}\lambda_i{}^2\leqq\varepsilon\right\}.$$

With this choice $\{a\in\mathfrak{G}_- \mid \exp a\in U_{\mathcal{O}}\}$ is trivially star shaped and $U_{\mathcal{O}}$ is invariant under the map $g\mapsto\vartheta g^{-1}$.

Let $C_0^\infty(\mathcal{O})$ denote the space of C^∞ functions with compact support in $\mathcal{O}$. We extend π_θ^λ to $U_{\mathcal{O}}$ on $C_0^\infty(\mathcal{O})$ by

$$(\pi_\theta^\lambda(g)f)(z)=|\det(zg_{12}+g_{22})|^{-2n-2\lambda}f(\pi^{-1}(g)z)\,. \tag{4.4}$$

Thus relation (3.42) is still valid for all $g\in U_{\mathcal{O}}$, $f, f'\in C_0^\infty(\mathcal{O})$. For any $z_0\in M_n{}^+$, $r>0$ let

$$B_{z_0}(r)=\{z\in M_n{}^+ \mid (z-z_0)^*(z-z_0)<r\mathbf{1}\}. \tag{4.5}$$

We now take $\mathscr{D}=C_0^\infty\ (B_{i1}(1/2))$ and $U=U_{B_{i1}(1/2)}$.

Lemma 4.2. *$\mathscr{D}$ is dense in $\mathscr{H}_\theta^\lambda$ for any λ in the range given in Theorem* 4.1.

This result is of a type known as the Reeh-Schlieder theorem in quantum field theory [RS].

Assuming this lemma to be true, the conditions (i) and (ii) for a local representation are trivially valid. As for (iii), we now use the relations

$$\begin{cases}\lim\limits_{U_{\mathcal{O}}\ni g\to e}\langle f, \pi_\theta^\lambda(g)f'\rangle_\theta^\lambda=\langle f, f'\rangle_\theta^\lambda\\[2ex] \lim\limits_{U_{\mathcal{O}}\ni g\to e}\langle\pi_\eta^\lambda(g)f, \pi_\theta^\lambda(g)f'\rangle=\langle f, f'\rangle_\theta^\lambda\end{cases} \tag{4.6}$$

valid for any $f, f'\in C_0^\infty(\mathcal{O})$.

Relations (4.6) are an easy consequence of Lebesgue dominated convergence theorem. Relation (4.1) in turn is also an easy consequence of relation (4.6). Condition (iv) is even satisfied for all $a\in\mathfrak{G}$. Next, if $f\in\mathscr{D}$ then for all g in a neighborhood of e, depending on the support of f, $\pi_\theta^\lambda(g)f\in\mathscr{D}$, proving (v). (vi) again is trivially satisfied.

It remains to prove Lemma 4.2.

Let $\varphi\in\mathscr{D}^\perp$, the orthogonal complement of $\mathscr{D}$ in $\mathscr{H}_\theta^\lambda$. We will show that

$$\langle\varphi, f\rangle_\theta^\lambda=0 \tag{4.5}$$

for all $f\in C_0^\infty(M_n{}^+)$. Since $C_0^\infty(M_n{}^+)$ is dense in $\mathscr{H}_\theta^\lambda$ by construction of $\mathscr{H}_\theta^\lambda$, this implies that $\varphi=0$ and $\mathscr{D}$ is dense. Now to prove (4.5), fix $f\in C_0^\infty(M_n{}^+)$. Take

$g(1)=\begin{pmatrix} s\mathbf{1} & 0 \\ 0 & s^{-1}\mathbf{1} \end{pmatrix}\in G_+$. $s>0$ depends on the support of f and is chosen so large that for all $g\in G_+$ in a small neighborhood of $g(1)$, $\pi_\theta^\lambda(g)f$ has support in $M_n^+\cap\{z \mid z^*z<1/4\}$. This is possible due to the relation

$$(\pi_\theta^\lambda(g(1))f)(z)=s^{n(2n+2\lambda)}f(s^2z)\,. \tag{4.6}$$

Next choose (see (3.11))

$$h(\tau)=\begin{pmatrix} \mathbf{1} & 0 \\ -is^2\tau\mathbf{1} & \mathbf{1} \end{pmatrix}=\exp\tau s^2\gamma\,. \tag{4.7}$$

Then

$$(\pi_\theta^\lambda(h(\tau)g(1))f)(z)=s^{n(2n+2\lambda)}f(s^2(z-i\tau\mathbf{1}))\,. \tag{4.8}$$

Therefore $\pi_\theta^\lambda(h(\tau)g)f$ has support in $B_{i1}(1/2)$ for τ in an open interval containing 1 and all g in a small neighborhood $W\subset G_+$ of $g(1)$. Therefore

$$\langle\varphi,\ \pi_\theta^\lambda(h(\tau)g)f\rangle_\theta^\lambda=0 \tag{4.9}$$

for τ in an open interval containing 1 and all $g\in W$. Now by our discussion in Section 3, $\pi_\theta^\lambda(h(\tau))$ defines a selfadjoint contraction semigroup for $\tau\geqq 0$. By the spectral theorem, we obtain an analytic contraction semigroup for complex τ with $\mathrm{Re}\,\tau>0$.

For fixed $g\in W$, the left hand side of (4.9) extends to an analytic function in $\mathrm{Re}\,\tau>0$. Since it vanishes for τ in a real open interval, it is actually the zero function in τ for all $g\in W$. Hence, if we let $\tau\to 0$, by continuity we obtain

$$\langle\varphi,\ \pi_\theta^\lambda(g)f\rangle_\theta^\lambda=0 \tag{4.10}$$

for $g\in W$.

Now for arbitrary $g\in G_+$, we may write

$$\langle\varphi,\ \pi_\theta^\lambda(g)f\rangle_\theta^\lambda \tag{4.10}$$
$$=\int_{z,z'\in M_n^+}\overline{\varphi(z)}\cdot\det(z-\pi(g)z'^*)^{-n-\lambda}\cdot\det(z-\pi(g)z')^{-n-\lambda}f(z')dzdz'.$$

We have used relation (3.10). Now we consider G_+ as a real analytic Lie group. Then the right hand side of (4.10) is real analytic in g. In particular by Heine-Borel it extends to an analytic function in a complex neighborhood V in $SL(2n, \boldsymbol{C})$ of the set $\{g(t)\}_{0\leq t\leq 1}$, where

$$g(t)=\begin{pmatrix} (st+(1-t))\mathbf{1} & 0 \\ 0 & (st+(1-t))^{-1}\mathbf{1} \end{pmatrix}. \tag{4.11}$$

Since this function vanishes on $V\cap W$, a real open set in V, it must vanish identically on V. In particular it is zero at $g(0)=e$. This proves (4.5), concluding the proof of Lemma 4.2.

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