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On Foliations Associated with Differential Equations of Conformal Type

By

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Introduction

The differentiability is of class C^∞ through this paper unless otherwise stated. Let $\mathcal{F}$ be a foliation on a manifold M of codimension q with trivial normal bundle defined by 1-forms $\omega_1, \cdots, \omega_q$. Assume that $d\omega_i = \sum_{j,k} c_{ijk}\omega_j \wedge \omega_k$ where c_{ijk} are functions on M. Then $\mathcal{F}$ is called a generalized Lie foliation.

Let E^n be the n-dimensional Euclidean space and consider a differential equation whose solutions are local mappings of R^m to E^n. In this paper we shall deal with differential equations $\mathcal{E}$ whose automorphism pseudogroups $\mathcal{A}(\mathcal{E})$ equal to the pseudogroup $\mathcal{P}$ of local isometries of E^n and construct and study foliations $\mathcal{F}(\mathcal{E})$ which are invariants of these differential equations $\mathcal{E}$. These foliations admit transversally conformal structures and if codim $\mathcal{F}(\mathcal{E}) \geq 3$, they are prolonged to generalized Lie foliations which are also invariants of $\mathcal{E}$.

In §1, we study a prolongation scheme of a principal bundle on M associated to a foliation $\mathcal{F}$ on M. The method of the prolongation is a generalization of that of Singer-Sternberg ([5]). Principal bundles considered are subbundles of the frame bundle associated to the normal bundle of $\mathcal{F}$ satisfying a few invariant conditions with respect to $\mathcal{F}$. Theorem 4.3 or Corollary 4.4 is the main theorem in §1.

§2 is devoted to the presentation of a differential equation $\mathcal{E}$ with $\mathcal{A}(\mathcal{E}) = \mathcal{P}$.

In §3, we construct a foliation $\mathcal{F}(\mathcal{E})$ on a manifold $M(\mathcal{E})$ for a given differential equation $\mathcal{E}$ with $\mathcal{A}(\mathcal{E}) = \mathcal{P}$ satisfying a regularity condition. This foliation $(\mathcal{F}(\mathcal{E}), M(\mathcal{E}))$ is Riemannian and is an invariant of $\mathcal{E}$ i.e. if two such differ-

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ential equations $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally isomorphic at a point $z_0 \in E^n$, then $(\mathcal{F}(\mathcal{E}_1), M(\mathcal{E}_1))$ is conformally isomorphic to $(\mathcal{F}(\mathcal{E}_2), M(\mathcal{E}_2))$. Applying the result of §1, we see that if $\operatorname{codim}\mathcal{F}(\mathcal{E}_i) \geq 3$, $\mathcal{F}(\mathcal{E}_i)$ is prolonged to a generalized Lie foliation $\mathcal{F}'(\mathcal{E}_i)$ and any local isomorphism ϕ of $\mathcal{E}_1$ to $\mathcal{E}_2$ with $\phi(z_0)=z_0$ induces an isomorphism ϕ' of $\mathcal{F}'(\mathcal{E}_1)$ to $\mathcal{F}'(\mathcal{E}_2)$ as a generalized Lie foliation. Theorem 12.2 is the main theorem of this paper.

Theorem 8.1 and 12.2 are modified in forms of Corollaries 14.1 and 14.2 where foliations on subvarieties are introduced. These foliations are naturally associated with analytic involutive differential equations.

§1. Prolongations of Foliations

1. Let $\mathcal{F}$ be a foliation of codimension q on a manifold M and let π': $Q \to M$ be the normal bundle of $\mathcal{F}$. Let $\bar{\pi}$: $F(Q) \to M$ be the frame bundle associated to the vector bundle π': $Q \to M$. This frame bundle is called the normal frame bundle of $\mathcal{F}$. Denote by E the tangent bundle to $\mathcal{F}$ and by $T(M)$ the tangent bundle of M. We have then $Q = T(M)/E$. The following definition is suggested by H. Imanishi. Let G be a Lie subgroup of $GL(q, R)$.

Definition 1.1. Let P be a G-subbundle of $F(Q)$ i.e. a G-bundle$\subset F(Q)$ with projection $\pi = \bar{\pi}|P$. If there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of M with $\pi^{-1}(U_\alpha) \approx U_\alpha \times G$ such that, for any leaf L of $\mathcal{F}$, each transition function $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to G$ is constant on each connected component of $U_\alpha \cap U_\beta \cap L$, P is called a $(G, \mathcal{F})$-subbundle of $F(Q)$.

Example 1.1. Assume that a foliation $\mathcal{F}$ is given by a collection $\{U_\alpha, f_\alpha\}_{\alpha \in A}$ where $\{U_\alpha\}_{\alpha \in A}$ is an open covering of M and f_α: $U_\alpha \to R^q$ is a submersion such that $f_\alpha(x) = \varphi_{\alpha\beta}(x) \circ f_\beta(x)$ for $x \in U_\alpha \cap U_\beta$ and for a local diffeomorphism $\varphi_{\alpha\beta}(x)$ where $\varphi_{\alpha\beta}$ is constant on each component in $U_\alpha \cap U_\beta$. Let P be any G-subbundle of $F(Q)$ such that $\pi^{-1}(U_\alpha) \approx U_\alpha \times G$ for any $\alpha \in A$ and transition functions are given by $g_{\alpha\beta}(x) = (\varphi_{\alpha\beta}(x))_*$ for $x \in U_\alpha \cap U_\beta$. Then $g_{\alpha\beta}$ is constant on each component in $U_\alpha \cap U_\beta \cap L$. Therefore P is a $(G, \mathcal{F})$-subbundle of $F(Q)$.

Example 1.2. Let ρ: $M \to N$ be a G-structure on N. Then ρ defines a foliation $\mathcal{F}$ on M as the pull back of the foliation on N with point-leaves. Denote by P the induced bundle on M of the G-bundle M on N by ρ. Then P is a $(G, \mathcal{F})$-subbundle of $F(Q)$.

The following lemma has been obtained by a discussion with N. Shimada.

Lemma 1.1. *If P is a $(G, \mathcal{F})$-subbundle, $\mathcal{F}$ is naturally lifted to a foliation*

$\tilde{\mathcal{F}}$ on P such that $\dim\tilde{\mathcal{F}}=\dim\mathcal{F}$ *and each leaf of $\tilde{\mathcal{F}}$ is mapped on a leaf of $\mathcal{F}$ by the projection π: $P\to M$.*

Proof. Let V be a cubic neighbourhood of the unit element of G. Then $U_\alpha\times V\subset\pi^{-1}(U_\alpha)$ and $\{U_\alpha\times\sigma\circ V\}_{\sigma\in G}$ is an open covering of $\pi^{-1}(U_\alpha)$. Therefore $\{U_\alpha\times\sigma\circ V\}_{\alpha\in A,\sigma\in G}$ is an open covering of P. We may consider that $V\subset R^r$, $r=\dim G$. Define a map $f_{\alpha\sigma}$: $U_{\alpha\sigma}=U_\alpha\times\sigma\circ V\to R^{q+r}=R^q\times R^r$ by $U_\alpha\times\sigma\circ V\ni(x,\sigma\circ g)\to(f_\alpha(x),g)\in R^q\times R^r$. If $U_{\alpha\sigma}\cap U_{\beta\tau}\neq\phi$, we get the relation

$$f_{\alpha\sigma}=\xi_{\alpha\beta\sigma\tau}\circ f_{\beta\tau}\quad\text{on}\quad U_{\alpha\sigma}\cap U_{\beta\tau}\,,$$

where $\xi_{\alpha\beta\sigma\tau}(p)=(\varphi_{\alpha\beta}(\pi(p)),\ \sigma^{-1}\circ g_{\alpha\beta}(\pi(p))\circ\tau$. Since P is a $(G,\mathcal{F})$-subbundle, $g_{\alpha\beta}$ is constant on each component in $U_\alpha\cap U_\beta\cap L$ for any leaf L of $\mathcal{F}$. This proves that the cocycle $\{U_{\alpha\sigma},f_{\alpha\sigma}\}_{\alpha\in A,\sigma\in G}$ defines a foliation on P. The construction of $\tilde{\mathcal{F}}$ asserts that $\tilde{\mathcal{F}}$ is a foliation of codimension $q+r$ and that each leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ is a covering space of some leaf L of $\mathcal{F}$. This completes the proof of Lemma 1.1.

Definition 1.2. The foliation $\tilde{\mathcal{F}}$ on a $(G,\mathcal{F})$-subbundle P is called the lift of $\mathcal{F}$ to P.

2. In this section we shall make preparations for defining the structure function C: $P\to\mathrm{Hom}(V\wedge V,V)/\partial\,\mathrm{Hom}(V,\mathcal{G})$ for a $(G,\mathcal{F})$-subbundle satisfying some invariant condition, which will be a generalization of the structure function for a G-structure. The statements will be parallel to Singer-Sternberg ([5]).

Let $\mathcal{F}$ be a foliation of codimension q on M and let P be any G-bundle $\subset F(Q)$, $G\subset GL(q,R)$, with projection $\pi=\bar{\pi}|F$. Let ρ be the projection of the tangent bundle $T(M)$ onto the normal bundle $Q=T(M)/E$. Denote by V a q-dimensional real vector space. Then there exists a V-valued linear differential form ω on P defined as follows:

$$\omega(X)=p^{-1}\rho\pi_*(X)\qquad\text{for any}\quad X\in T_p(P)\,. \tag{2.1}$$

Since p is considered as an isomorphism of V with $Q_{\pi(p)}=T_{\pi(p)}(M)/E_{\pi(p)}$, the above equality makes sense. ω is called the basic form of P.

Let $a\in G$ and $X\in T_p(P)$. Since $\pi=\pi\circ R_a$, we have $\pi_*R_{a^*}(X)=\pi_*(X)$. Thus $\omega(R_{a^*}(X))=(pa)^{-1}\rho\pi_*R_{a^*}(X)=(pa)^{-1}\rho\pi_*(X)=a^{-1}p^{-1}\rho\pi_*(X)=a^{-1}\omega(X)$. Therefore $R_{a^*}\omega=a^{-1}\omega$.

Let P be a $(G,\mathcal{F})$-subbundle of $F(Q)$. Denote by $\tilde{E}$ the tangent bundle to $\tilde{\mathcal{F}}$ on P and by $\tilde{\rho}$ the projection of $T(P)$ onto $T(P)/\tilde{E}$. Let $p\in P$ and let

H_p be a q-dimensional subspace of $T_p(P)$. H_p is called a transversally horizontal subspace at p if $\rho\pi_*(H_p)=Q_{\pi(p)}$. Note that ω_p restricted to H_p gives an isomorphism of H_p with V. Let $\mathcal{H}_p$ be the set of transversally horizontal subspaces at p and introduce an equivalence relation $\sim$ in $\mathcal{H}_p$ by $H_p \sim H'_p$ if and only if $\tilde{\rho}(H_p)=\tilde{\rho}(H'_p)$. We set $\tilde{\mathcal{H}}_p=\mathcal{H}_p/\sim$. An element of $\tilde{\mathcal{H}}_p$ is called a normally horizontal subspace at p.

If $H_p \sim H'_p$, then we have $\omega_p(X)=\omega_p(X')$ for $X \in H_p$, $X' \in H'_p$ with $\tilde{\rho}(X)=\tilde{\rho}(X')$.

Let $\tilde{H}^1_p$ and $\tilde{H}^2_p$ be two normally horizontal subspaces and let H^i_p be any representative of $\tilde{H}^i_p$. Then for $X_1 \in H^1_p$ and $X_2 \in H^2_p$ with $\omega(X_1)=\omega(X_2)=v$, $\tilde{\rho}(X_1-X_2)$ is independent of a choice of a representative H^i_p. Therefore we can get a linear map S $(=S\tilde{H}^1_p\tilde{H}^2_p)$ of V into the Lie algebra $\mathcal{G}$ of G defined by

$$\tilde{\rho}(X_1-X_2) = \tilde{\rho}(S(v)^*_p) \tag{2.2}$$

where $S(v)^*$ is the fundamental vector field on P corresponding to $S(v) \in \mathcal{G}$. Thus for $v \in V$, there corresponds a unique element $S(v) \in \mathcal{G}$. By this correspondence, $S \in \mathrm{Hom}(V, \mathcal{G})$ is defined for normally horizontal subspaces $\tilde{H}^1_p$ and $\tilde{H}^2_p$.

Let $\{e_1, \cdots, e_q\}$ be a basis of V and set $\omega=\sum_{i=1}^{q} \omega_i e_i$.

Lemma 2.1. *Let P be a $(G, \mathcal{F})$-subbundle. Then the basic form ω of P is locally expressed in terms of a basis for the integrals of $\tilde{\mathcal{F}}$, more precisely, for any point $p \in P$ there exist a neighbourhood U of p and C^∞-function α_{ij} $(i=1, \cdots, q$ and $j=1, \cdots, \tilde{q})$ with q variables such that $\omega_i \mid U=\sum_{j=1}^{q} \alpha_{ij}(y_1, \cdots, y_{\tilde{q}})dy_j$ $(i=1, \cdots, q)$ where $\{y_1, \cdots, y_{\tilde{q}}\}$ is a fundamental system of 1-st integrals of $\tilde{\mathcal{F}}$ on U.*

Proof. Since P is a G-subbundle of the normal frame bundle $F(Q)$ of $\mathcal{F}$, it follows easily that there exists a neighbourhood U_α of $\pi(p) \in M$ such that $U_\alpha/\mathcal{F}$ is a manifold and such that, if we set $P_\alpha=\pi^{-1}(U_\alpha)$, $P_\alpha/\tilde{\mathcal{F}}$ is a G-subbundle of the frame bundle over $U_\alpha/\mathcal{F}$ and P_α is the induced bundle by the projection $\bar{\mu}_\alpha$ of U_α onto $U_\alpha/\mathcal{F}$. Let $\bar{\pi}_\alpha$ be the projection of $P_\alpha/\tilde{\mathcal{F}}$ onto $U_\alpha/\mathcal{F}$ and denote by μ_α the bundle map of P_α to $P_\alpha/\tilde{\mathcal{F}}$. Then we have $\bar{\pi}_{\alpha*}\circ\tilde{\rho}=\rho\circ\pi_*$ on P_α and the basic form $\bar{\omega}_\alpha$ of the G-subbundle $P_\alpha/\tilde{\mathcal{F}}$ is defined by

$$\bar{\omega}_\alpha(X) = \bar{p}^{-1}\bar{\pi}_{\alpha*}(X),\ X \in T_{\bar{p}}(P_\alpha/\tilde{\mathcal{F}})\,.$$

For $Z\in T_p(P)$, we have

$$\omega(Z)=p^{-1}\rho\pi_*(Z)=p^{-1}\bar{\pi}_{\alpha*}\bar{\rho}(Z)$$
$$=\mu_\alpha(p)^{-1}\bar{\pi}_{\alpha*}\mu_{\alpha*}(Z)=(\mu_\alpha{}^*\bar{\omega}_\alpha)(Z).$$

This means that ω is locally expressed in terms of a basis for the integrals of $\tilde{\mathcal{F}}$.

Let P be a $(G,\mathcal{F})$-subbundle of $F(Q)$ with the basic form ω. The form $d\omega$ is an exterior 2-form with values in V. Let $H_p\in\mathcal{H}_p$ and let H_p be a representative of $\tilde{H}_p$. The restriction of $d\omega_p$ to $H_p\wedge H_p$ gives a map of $H_p\wedge H_p$ into V. By Lemma 2.1, we can easily see that $d\omega_p|H_p\wedge H_p$ is independent of a choice of a representative H_p. Therefore, via the identification of H_p with V by ω_p, a map $C\tilde{H}_p$ of $V\wedge V$ into V is defined by $C\tilde{H}_p(u\wedge v)=d\omega(X\wedge Y)$ where X, $Y\in H_p$, $\omega(X)=u$ and $\omega(Y)=v$.

Let $\tilde{H}_p^1$ and $\tilde{H}_p^2$ be two normally horizontal subspaces at $p\in P$. Then analogically to [5] (p. 42), we can prove that

$$C\tilde{H}_p^1(u\wedge v)-C\tilde{H}_p^2(u\wedge v)=S(v)u-S(u)v \tag{2.3}$$

where $S=S\tilde{H}_p^1\tilde{H}_p^2$. So as to prove this equality, we have only to note that there exists a representative H_p^i of $\tilde{H}_p^i$ such that (2.2) implies the equality $X_1-X_2=S(v)_p^*$. This is proved as follows. Let F_p be any complement of $\tilde{E}_p$ in $T_p(P)$ and choose a representative H_p^i of $\tilde{H}_p^i$ such that $H_p^i\subset F_p$. Then both X_1-X_2 and $S(v)_p^*$ are elements in F_p and so (2.2) implies $X_1-X_2=S(v)_p^*$.

We define the map ∂: $\mathrm{Hom}(V,\mathcal{G})\to\mathrm{Hom}(V\wedge V, V)$ by $\partial S(u\wedge v)=S(u)v-S(v)u$ for $S\in\mathrm{Hom}(V,\mathcal{G})$. Then (2.3) is written by $C\tilde{H}_p^2-C\tilde{H}_p^1=\partial S$.

3. Now we define the structure function C of the $(G,\mathcal{F})$-subbundle P and state the prolongation of P. For the sake of later convenience (cf. Lemma 4.2), it will be defined as a $\mathrm{Hom}(V\wedge V, V)$-valued function such that $C(p)$, $p\in P$, belongs to the chosen complement $\mathcal{C}(p)$ to $\partial\mathrm{Hom}(V,\mathcal{G})$.

Let V and W be finite dimensional vector spaces and let $\mathcal{G}$ be a subspace of $\mathrm{Hom}(V, W)$.

Denote by $\mathcal{G}^{(1)}$ the set of all $T\in\mathrm{Hom}(V,\mathcal{G})$ which satisfy $T(u)v=T(v)u$ for all u, $v\in V$. $\mathcal{G}^{(1)}$ is called the first prolongation of $\mathcal{G}$.

The first prolongation of the space $\mathcal{G}^{(k-1)}\subset\mathrm{Hom}(V,\mathcal{G}^{(k-2)})$, $k\geq 2$, is called the k-th prolongation of $\mathcal{G}$ and is denoted by $\mathcal{G}^{(k)}$.

Let $\mathcal{F}$ be a foliation of codimension q and let P be any $(G,\mathcal{F})$-subbundle of the normal frame bundle of $\mathcal{F}$. We denote by $\mathcal{G}$ the Lie algebra of G and by V a q-dimensional real vector space. We choose a complement $\mathcal{C}$ to $\partial\mathrm{Hom}(V, \mathcal{G})$ in $\mathrm{Hom}(V\wedge V, V)$.

Lemma 3.1. *For any $p\in P$, there exists a normally horizontal subspace $\tilde{H}_p$ at p with $C\tilde{H}_p\in\mathcal{C}$.*

Proof. Let $\tilde{H}'_p$ be any normally horizontal subspace at p. Then we have a unique decomposition $C\tilde{H}'_p=C+\partial S$ where $C\in\mathcal{C}$ and $\partial S\in\partial\mathrm{Hom}(V, \mathcal{G})$. For $X'\in H'_p$, we set $u=\omega_p(X')$ and $X=X'-S(u)^*_p$. Then $H_p=\{X(=X'-S(u)^*_p); X'\in H'_p\}$ is a transversally horizontal subspace at p. Clearly we have $\tilde{\rho}(X'-X)=\tilde{\rho}(S(u)^*_p)$ and as is stated before, we get $C\tilde{H}'_p-C\tilde{H}_p=\partial S$. Therefore $C\tilde{H}_p=C$ which proves Lemma 3.1.

Let $Gr_i(\mathrm{Hom}(V\wedge V, V))$ be the Grassmann manifold of i-planes in the vector space $\mathrm{Hom}(V\wedge V, V)$. We set $i=\dim\mathrm{Hom}(V\wedge V, V)-\dim\partial\mathrm{Hom}(V, \mathcal{G})$. Denote also by $\mathcal{C}$ a smooth map of a $(G, \mathcal{F})$-subbundle P to $Gr_i(\mathrm{Hom}(V\wedge V, V))$ such that each image $\mathcal{C}(p)$ of $p\in P$ is a complement to $\partial\mathrm{Hom}(V, \mathcal{G})$ in $\mathrm{Hom}(V\wedge V, V)$.

Let $\tilde{H}_p$ be a normally horizontal subspace at $p\in P$ with $C\tilde{H}_p\in\mathcal{C}(p)$. Then by (2.3), $C\tilde{H}_p$ is independent of a choice of $\tilde{H}_p$ and depends only on $p\in P$. Therefore we can set $C(p)=C\tilde{H}_p$, $C\tilde{H}_p\in\mathcal{C}(p)$.

Definition 3.1. This $\mathrm{Hom}(V\wedge V, V)$-valued function C is called the structure function with respect to $\mathcal{C}$ of a $(G, \mathcal{F})$-subbundle P of the normal frame bundle of $\mathcal{F}$.

A normally horizontal subspace $\tilde{H}_p$ at $p\in P$ defines a linear isomorphism of $V+\mathcal{G}$ with $\tilde{Q}_p=T_p(P)/\tilde{E}_p$ by $V\ni v\to\tilde{\rho}\omega_p^{-1}(v)\in\tilde{Q}_p$ and $\mathcal{G}\ni A\to\tilde{\rho}(A^*_p)\in\tilde{Q}_p$ where ω_p^{-1} is the isomorphism of V with a representative H_p of $\tilde{H}_p$ and this definition is independent of a choice of H_p.

Let $\tilde{H}^1_p$ and $\tilde{H}^2_p$ be two normally horizontal subspaces at $p\in P$ with $C\tilde{H}^1_p$, $C\tilde{H}^2_p\in\mathcal{C}(p)$ and let η_1 and η_2 be the corresponding isomorphisms of $V+\mathcal{G}$ with $T_p(P)/\tilde{E}_p$. Then for $S\in\mathcal{G}^{(1)}$ if we define $T_S\in\mathrm{Hom}(V+\mathcal{G}, V+\mathcal{G})$ by $T_S(A)=A$ for $A\in\mathcal{G}$ and $T_S(v)=v+S(v)$ for $v\in V$, we get $\eta_1=\eta_2\circ T_S$.

We set $G^{(1)}=\{T_S;\ S\in\mathcal{G}^{(1)}\}$. Then $G^{(1)}$ is a Lie subgroup of $GL(V+\mathcal{G})$. Denote by $P^{(1)}_{\mathcal{C}}$ the set of isomorphisms η of $V+\mathcal{G}$ with $\tilde{Q}_p$ defined by normally horizontal subspaces $\tilde{H}_p$ with $C\tilde{H}_p\in\mathcal{C}(p)$, $p\in P$.

Lemma 3.2. *$P^{(1)}_{\mathcal{C}}$ is a $G^{(1)}$-subbundle of the normal frame bundle of $\tilde{\mathcal{F}}$.*

Proof. The foliation $\tilde{\mathcal{F}}$ gives rise to a collection $\{V_\alpha, f_\alpha\}_{\alpha\in A}$ where $\{V_\alpha\}_{\alpha\in A}$ is an open covering of P and $f_\alpha\colon V_\alpha\to R^q$ is a submersion being constant along the connected components of any leaf of $\tilde{\mathcal{F}}$ in V_α. Let $\mathcal{W}_p$ be the set of normally horizontal subspaces at $p\in P$ and set $\mathcal{W}=\bigcup_{p\in P}\mathcal{W}_p$. Then $(\mathcal{W}, P, \lambda)$ is a fibred manifold where λ is the projection of $\mathcal{W}$ onto P. Then for each V_α we may assume that there exists a cross-section φ_α of V_α to $\mathcal{W}$. We set $\tilde{H}_p=\varphi_\alpha(p)$ for $p\in V_\alpha$. Then we have a unique decomposition $C\tilde{H}_p=C_p+\partial S_p$ where $C_p\in\mathcal{C}(p)$ and $\partial S_p\in\partial\mathrm{Hom}(V, \mathcal{G})$. As was proved in Lemma 3.1, if we set $H'_p=\{X';\ X'=X-S_p(u)^*_p,\ X\in H_p,\ \omega_p(X)=u\}$, then $C\tilde{H}'_p=C_p$. Setting $\varphi'_\alpha(p)=\tilde{H}'_p$, φ'_α is a cross-section of V_α to $\mathcal{W}$ with $C_{\varphi'_\alpha(p)}\in\mathcal{C}(p)$. This map φ'_α gives rise to a cross-section on V_α of the normal frame bundle $F(\tilde{Q})$ of $\tilde{\mathcal{F}}$ such that the image is contained in $P^{(1)}_C$. This shows that $P^{(1)}_C$ is a $G^{(1)}$-subbundle of $F(\tilde{Q})$.

Definition 3.2. $P^{(1)}_C$ is called the first prolongation of a $(G, \mathcal{F})$-subbundle P.

The group $G^{(1)}$ is also called the first prolongation of G. It is clear that the Lie algebra of $G^{(1)}$ is $\mathcal{G}^{(1)}$.

Remark 3.1. Let $\mathcal{F}$ be a foliation on M of codimension q and let P be a $(G, \mathcal{F})$-subbundle of $F(Q)$. Assume that $G^{(1)}$ consists of only the unit element. Then $P^{(1)}_C$ is clearly a $(G^{(1)}, \tilde{\mathcal{F}})$-subbundle. Therefore the lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ is lifted to a foliation $\mathcal{F}^{(1)}$ on $P^{(1)}_C$. Later we will discuss the possibility of lifting $\tilde{\mathcal{F}}$ to a foliation $\mathcal{F}^{(1)}$ on $P^{(1)}_C$ more generally (Theorem 4.4).

4. In the beginning of this section, we will explain the natural parallelism along the leaves of a foliation due to R. Hermann [3].

Let $\mathcal{F}$ be a foliation of codimension q on M given by a collection $\{U_\alpha, f_\alpha\}_{\alpha\in A}$ where $\{U_\alpha\}_{\alpha\in A}$ is an open covering of M and $f_\alpha\colon U_\alpha\to R^q$ is a submersion such that $f_\alpha(x)=\varphi_{\alpha\beta}(x)\circ f_\beta(x)$ for $x\in U_\alpha\cap U_\beta$ where $\varphi_{\alpha\beta}(x)$ is a local diffeomorphism of R^q.

Let L be a leaf of $\mathcal{F}$ and let $\gamma\colon[0, 1]\to L$ be a curve. Then there exists a parallel translation of vectors $v\in Q_{\gamma(0)}$ along γ, $[0, 1]\ni t\to v(t)\in Q_{\gamma(t)}$, such that

(1) $v(0)=v$,

(2) for t_1 and t_2 sufficiently near such that $\gamma(t)$, $t_1\le t\le t_2$, lies in a connected component of $L\cap U_\alpha$ for some $\alpha\in A$, we have $\bar{\bar{f}}_{\alpha\gamma(t)}(v(t))=\bar{\bar{f}}_{\alpha\gamma(t_i)}(v(t_i))$

where $\bar{\bar{f}}_{\alpha\gamma(t)}$ is the linear isomorphism of $T_{\gamma(t)}(M)/E_{\gamma(t)}$ onto $T_{f_\alpha(\gamma(t))}(R^q)$ induced from f_α.

It is easy to see that $v(1)$ obtained by this way depends only on γ and v. The mapping $v \rightarrow v(1)$ is denoted by k_γ. This parallel translation of normal vectors to $\mathcal{F}$ along curves lying in a leaf of $\mathcal{F}$ is called the natural parallelism along the leaves. It is known that two homotopic curves γ_1 and γ_2 with the same end points satisfy $k_{\gamma_1}=k_{\gamma_2}$. This natural parallelism is also defined for normal frames $(v_1, \cdots, v_q)\in F(Q)$. The set of all k_γ for all loops based at $x\in L$ forms a group which is called the holonomy group of the leaf L. It is clear that for any x and $y\in L$, the holonomy groups are isomorphic to each other.

Let P be a $(G, \mathcal{F})$-subbundle and let $\tilde{L}$ be a leaf of $\tilde{\mathcal{F}}$. For any curve $\tilde{\gamma}$: $[0, 1]\rightarrow\tilde{L}$, we set $\gamma=\pi\circ\tilde{\gamma}$. Then γ is a curve of $[0, 1]$ to the leaf $L=\pi(\tilde{L})$ of $\mathcal{F}$.

Since π maps any leaf of $\tilde{\mathcal{F}}$ to a leaf of $\mathcal{F}$, π induces the projection $\bar{\pi}_*$ of $\tilde{Q}=T(P)/\tilde{E}$ onto $Q=T(M)/E$. For any normal vector $\tilde{X}$ (resp. X) to $\tilde{\mathcal{F}}$ at $\tilde{\gamma}(0)$ (resp. to $\mathcal{F}$ at $\gamma(0)$), denote by $\tilde{\gamma}(t)\tilde{X}$ (resp. $\gamma(t)X$) the normal vector obtained by the parallel translation of $\tilde{X}$ along $\tilde{\gamma}$ (resp. X along γ).

Lemma 4.1. $\bar{\pi}_*(\tilde{\gamma}(t)\tilde{X})=\gamma(t)\bar{\pi}_*X.$

Proof. By taking a finite covering of the curve $\tilde{\gamma}$, it is sufficient to prove this equality in the case that $\tilde{\gamma}$ is contained in some $U_{\alpha\sigma}$ where $\{U_{\alpha\sigma}, f_{\alpha\sigma}\}_{\alpha\in A,\sigma\in G}$ is the cocycle of $\tilde{\mathcal{F}}$ given in Lemma 1.1. Then for $p=(x,\sigma\cdot g)\in U_{\alpha\sigma}=U_\alpha\times\sigma\cdot V$, $f_{\alpha\sigma}(x, \sigma\cdot g)=(f_\alpha(x), g)$. We have then $\bar{\pi}_*\circ(\bar{\bar{f}}_{\alpha\sigma\tilde{\gamma}(t)})^{-1}\circ\bar{\bar{f}}_{\alpha\sigma\tilde{\gamma}(0)}=(\bar{\bar{f}}_{\alpha\gamma(t)})^{-1}\circ\bar{\bar{f}}_{\alpha\gamma(0)}\circ\bar{\pi}_*$. This implies that $\bar{\pi}_*(\tilde{\gamma}(t)\tilde{X})=\gamma(t)\bar{\pi}_*\tilde{X}$ for any normal vector $\tilde{X}$ at $\tilde{\gamma}(0)$. The proof is completed.

Let P be a $(G, \mathcal{F})$-subbundle and let $\tilde{H}_p$ be a normally horizontal subspace at $p\in P$. $\tilde{H}_p$ is considered as a subspace of the fiber $\tilde{Q}_p$ over p of the normal bundle $\tilde{Q}$ of $\tilde{\mathcal{F}}$ on P. Let $\tilde{L}$ be the leaf through p and let $\tilde{\gamma}$: $[0, 1]\rightarrow\tilde{L}$ be any curve with $\tilde{\gamma}(0)=p$, $\tilde{\gamma}(1)=p'$. Choose a basis $\{v_1(p), \cdots, v_m(p)\}$ of $\tilde{H}_p$ and denote by $v_i(p')$ the normal vector at p' obtained by the parallel translation of $v_i(p)$ along $\tilde{\gamma}$. The subspace $\tilde{H}_{p'}\subset\tilde{Q}_{p'}$ generated by $\{v_1(p'), \cdots, v_m(p')\}$ is then a normally horizontal subspace at p'. It is clear that $\tilde{H}_{p'}$ is independent of a choice of a basis of $\tilde{H}_p$. Therefore we can say that $\tilde{H}_{p'}$ is obtained by the parallel translation of $\tilde{H}_p$ along $\tilde{\gamma}$ and so we can also say that $C\tilde{H}_{p'}$ is obtained by the parallel translation of $C\tilde{H}_p$ along the curve $\tilde{\gamma}$.

Lemma 4.2. *Let P be a $(G, \mathcal{F})$-subbundle. Then there exists a map C*

of P to $Gr_i(\mathrm{Hom}(V\wedge V, V))$ satisfying the following conditions:

(1) $\mathcal{C}(p)\oplus\partial\mathrm{Hom}(V, \mathcal{G})=\mathrm{Hom}(V\wedge V, V)$.

(2) *The structure function C with respect to $\mathcal{C}$ is invariant under the natural parallelism along the leaves of $\tilde{\mathcal{F}}$.*

Proof. Let $p\in P$ and let U be a neighbourhood of $\pi(p)$ such that $\pi^{-1}(U)=U\times G$. We set $\mathcal{K}_p=\{H_p\in\mathcal{H}_p;\ H_p\subset T_{\pi(p)}(U)\times\{0\}\}$. Then $\mathcal{K}_p$ is independent of a choice of a local triviality of $\pi^{-1}(U)$. Let $\tilde{L}$ be the leaf of $\tilde{\mathcal{F}}$ through p and let $\tilde{\gamma}$ be any curve in $\tilde{L}$ connecting p with p' and denote by $\tilde{H}_{p'}$ be the normally horizontal subspace at p' obtained by the parallel translation of $\tilde{H}_p$, $H_p\in\mathcal{K}_p$, along $\tilde{\gamma}$. Let v be any vector in $\tilde{H}_p$ and denote by $v'\in\tilde{H}_{p'}$ the normal vector at p' obtained by the parallel translation of v along $\tilde{\gamma}$. By the parallel translation of v along the leaves of $\tilde{\mathcal{F}}$, the property of the vertical component of v being zero is invariant. Therefore the vertical component of v' is also zero and so there exists a repersentative $H_{p'}$ of $\tilde{H}_{p'}$ such that $H_{p'}\subset T_{p'}(U')\times\{0\}$ where U' is a neighbourhood of $\pi(p')$ with $\pi^{-1}(U')=U'\times G$. Thus the family $\{\mathcal{K}_p\}_{p\in P}$ is invariant under the parallel translation along the leaves of $\tilde{\mathcal{F}}$.

If H_p and $H'_p\in\mathcal{K}_p$, then $H_p-H'_p\subset T_{\pi(p)}(U)\times\{0\}$. On the other hand $\tilde{\rho}(X-X')=\tilde{\rho}(S(u)^*_p)$ where $X\in H_p$, $X'\in H'_p$ with $\omega(X)=\omega(X')=u$. Therefore $S\ (=S\tilde{H}_p\tilde{H}'_p)=0$ and we have $C\tilde{H}_p=C\tilde{H}'_p$. Thus $C\tilde{H}_p$ is independent of the choice of $H_p\in\mathcal{K}_p$ and the family $\{C\tilde{H}_p\}_{p\in P, H_p\in\mathcal{K}_p}$ is invariant under the natural parallelism along the leaves of $\tilde{\mathcal{F}}$.

We can choose a map $\mathcal{C}\colon P\to Gr_i(\mathrm{Hom}(V\wedge V, V))$ such that $\mathcal{C}(p)\ni C\tilde{H}_p$, $H_p\in\mathcal{K}_p$, for any $p\in P$ and such that $\mathcal{C}(p)\oplus\partial\mathrm{Hom}(V, \mathcal{G})=\mathrm{Hom}(V\wedge V, V)$. Then the structure function C with respect to $\mathcal{C}$ is given by $C(p)=C\tilde{H}_p$, $H_p\in\mathcal{K}_p$, and this map $\mathcal{C}$ satisfies (1) and (2). This completes the proof.

Definition 4.1. A $(G, \mathcal{F})$-subbundle P is called a $\langle G, \mathcal{F}\rangle$-subbundle if P is invariant under the natural parallelism along the leaves of $\mathcal{F}$.

Remark 4.1. If P is the associated transverse G-structure of a G-foliation $\mathcal{F}$ in the sense of L. Conlon [1], then P is a $\langle G, \mathcal{F}\rangle$-subbundle.

Theorem 4.3. *Let P be a $\langle G, \mathcal{F}\rangle$-subbundle. Then there exists a map $\mathcal{C}\colon P\to Gr_i(\mathrm{Hom}(V\wedge V, V))$ such that $P^{(1)}_{\mathcal{C}}$ is a $\langle G^{(1)}, \tilde{\mathcal{F}}\rangle$-subbundle.*

Proof. Let $\mathcal{C}$ be a map obtained in Lemma 4.2 and let $\psi_\alpha\colon V_\alpha\to P^{(1)}_{\mathcal{C}}$ be a local cross-section which gives rise to a triviality of $(\pi^{(1)})^{-1}(V_\alpha)$ and such that we have a submersion $\tilde{f}_\alpha\colon V_\alpha \to \tilde{R}^q$ associated to $\tilde{\mathcal{F}}$. Since $V_\alpha/\tilde{\mathcal{F}}$ has a manifold structure, there exists also a cross-section s_α of $V_\alpha/\tilde{\mathcal{F}} \to V_\alpha$. We set $\psi'_\alpha = \psi_\alpha \circ s_\alpha$. Since any point in $P_C^{(1)}$ is identified with a normally horizontal subspace, we can write $\psi'_\alpha(p') = \tilde{H}_p$ where $p = s_\alpha(p')$ and $C\tilde{H}_p \in \mathcal{C}(p)$. Let $\tilde{L}$ be the leaf through p. Then for any $\tilde{p}$ in $\tilde{L} \cap V_\alpha$, by the parallel translation of $\tilde{H}_p$ along any curve in $\tilde{L} \cap V_\alpha$ connecting p and $\tilde{p}$, we have a unique normally horizontal subspace $\tilde{H}_{\tilde{p}}$ with $C\tilde{H}_{\tilde{p}} \in \mathcal{C}(\tilde{p})$ because there is a submersion $\tilde{f}_\alpha\colon V_\alpha \to \tilde{R}^q$. By this way we get a new cross-section $\bar{\psi}_\alpha$ of $V_\alpha \to P_C^{(1)}$.

Let $(V_\alpha, \bar{\psi}_\alpha)$ and $(V_\beta, \bar{\psi}_\beta)$ be two such pairs and assume that $V_\alpha \cap V_\beta \neq \phi$. For any point $p \in V_\alpha \cap V_\beta$, we can set $\bar{\psi}_\alpha(p) = \tilde{H}_p^\alpha$ and $\bar{\psi}_\beta(p) = \tilde{H}_p^\beta$ with $C\tilde{H}_p^\alpha$, $C\tilde{H}_p^\beta \in \mathcal{C}(p)$. Then we have a unique $S_p = S\tilde{H}_p^\alpha \tilde{H}_p^\beta \in \mathrm{Hom}(V, \mathcal{G})$ with $\partial S_p = 0$.

Let $\tilde{p}$ be any point in $V_\alpha \cap V_\beta \cap \tilde{L}$. We shall show that $S_{\tilde{p}}$ is independent of $\tilde{p}$ on each component of $V_\alpha \cap V_\beta \cap \tilde{L}$.

Let $X^\alpha \in H_p^\alpha$ and $X^\beta \in H_p^\beta$ such that $\omega(X^\alpha) = \omega(X^\beta) = u$. Then we have $\tilde{\rho}(X^\alpha) = \tilde{\rho}(X^\beta + S_p(u)_p^*)$. Let $\tilde{\rho}(Y^\alpha)$ (resp. $\tilde{\rho}(Y^\beta)$) be the vector in $\tilde{H}_{\tilde{p}}^\alpha$ (resp. $\tilde{H}_{\tilde{p}}^\beta$) obtained by the parallel translation of $\tilde{\rho}(X^\alpha) \in \tilde{H}_p^\alpha$ (resp. $\tilde{\rho}(X^\beta) \in \tilde{H}_p^\beta$) along a curve $\tilde{\gamma}\colon [0, 1] \to V_\alpha \cap V_\beta \cap \tilde{L}$. We shall prove that $\tilde{\rho}(Y^\alpha) = \tilde{\rho}(Y^\beta + S_{\tilde{p}}(u)_{\tilde{p}}^*)$. Then since $\tilde{\rho}(S_p(u)_p^*)$ is translated to $\tilde{\rho}(S_p(u)_{\tilde{p}}^*)$ by the parallel translation along $\tilde{\gamma}$ and so $\tilde{\rho}(Y^\alpha) = \tilde{\rho}(Y^\beta + S_p(u)_{\tilde{p}}^*)$, we get $S_{\tilde{p}}(u)_{\tilde{p}}^* = S_p(u)_{\tilde{p}}^*$ and so $S_{\tilde{p}} = S_p$ for any $\tilde{p}$ and p in the same component of $V_\alpha \cap V_\beta \cap \tilde{L}$.

Since $\dim \tilde{\mathcal{F}} = \dim \mathcal{F}$, there exists a neighbourhood U of $x = \pi(p)$ such that, by the projection $\pi\colon P \to M$, any component of $\tilde{U} \cap \tilde{L}$ is diffeomorphic to some component of $U \cap L$ where $\tilde{U} = \pi^{-1}(U)$. We may assume that $\tilde{U} \supset V_\alpha \cap V_\beta$. Let $\tilde{p}$ be any point in the component of $\tilde{U} \cap \tilde{L}$ containing p and set $\tilde{x} = \pi(\tilde{p})$. Let γ be a curve of $[0, 1] \to U \cap L$ with $\gamma(0) = x$ and $\gamma(1) = \tilde{x}$ and let $\tilde{\gamma}\colon [0, 1] \to F(Q)$ be the curve with $\tilde{\gamma}(0) = p$ obtained by the parallel translation of p along γ. Since P is invariant under the natural parallelism along the leaves of $\mathcal{F}$, $\tilde{\gamma}\colon [0, 1] \to \tilde{L} \subset P$. Since $\tilde{\gamma}$ is continuous and $\gamma\colon [0, 1] \to U \cap L$, we get $\tilde{\gamma}\colon [0, 1] \to \tilde{U} \cap \tilde{L}$. Therefore we have $\tilde{\gamma}(1) = \tilde{p}$. This means that any $\tilde{p}$ in the component of $\tilde{U} \cap \tilde{L}$ containing p is obtained by the parallel translation of p along some curve γ in $U \cap L$.

If we set $p = (v_1, \cdots, v_q)$ and $\tilde{p} = (\tilde{v}_1, \cdots, \tilde{v}_q)$, then $\tilde{v}_j$ is the normal vector to $\mathcal{F}$ obtained by the parallel translation of v_j along γ. The vector $\bar{\pi}_* \tilde{\rho}(X^\alpha)$ (resp. $\bar{\pi}_* \tilde{\rho}(Y^\alpha)$) is expressed as a linear combination $\sum_{i=1}^q c_i v_i$ (resp. $\sum_{i=1}^q \tilde{c}_i \tilde{v}_i$).

By Lemma 4.1, $\bar{\pi}_* \tilde{\rho}(Y^\alpha)$ is obtained by the parallel translation of $\bar{\pi}_* \tilde{\rho}(X^\alpha)$ along the curve γ.

On the other hand, the normal vector $\sum_{i=1}^{q} c_i\tilde{v}_i$ is obtained by the parallel translation of the normal vector $\sum_{i=1}^{q} c_i v_i$ along γ. Therefore we get $\sum_{i=1}^{q} \tilde{c}_i\tilde{v}_i = \sum_{i=1}^{q} c_i\tilde{v}_i$, that is, $\tilde{c}_i = c_i$ $(i=1, \cdots, q)$. Then by choosing a canonical basis $\{e_1, \cdots, e_q\}$ of V, $\omega(X^\alpha) = p^{-1}\rho\pi_*(X^\alpha) = p^{-1}\bar{\pi}_*\tilde{\rho}(X^\alpha) = p^{-1}(\sum_{i=1}^{q} c_i v_i) = \sum_{i=1}^{q} c_i e_i = \tilde{p}^{-1}(\sum_{i=1}^{q} c_i\tilde{v}_i) = \tilde{p}^{-1}(\bar{\pi}_*\tilde{\rho}(Y^\alpha)) = \omega(Y^\alpha)$.

Furthermore since $\omega(X^\beta) = \omega(X^\alpha)$, we have $\bar{\pi}_*\tilde{\rho}(X^\beta) = \bar{\pi}_*\tilde{\rho}(X^\alpha) = \sum_{i=1}^{q} c_i v_i$. Since $\tilde{\rho}(Y^\beta)$ is obtained by the parallel translation of $\tilde{\rho}(X^\beta)$ along $\tilde{\gamma}$, again by Lemma 4.1 $\bar{\pi}_*\tilde{\rho}(Y^\beta)$ is obtained by the parallel translation of $\bar{\pi}_*\tilde{\rho}(X^\beta)$ along γ. Therefore we get $\bar{\pi}_*\tilde{\rho}(Y^\beta) = \sum_{i=1}^{q} c_i\tilde{v}_i$. This implies that $\omega(Y^\beta) = \omega(Y^\alpha) = u$ and we get $\tilde{\rho}(Y^\alpha) = \tilde{\rho}(Y^\beta + S_{\tilde{p}}(u)_{\tilde{p}}^{*})$, $S_{\tilde{p}} = S\bar{H}_p^\alpha\bar{H}_p^\beta$.

The correspondence $\tilde{p} \to S_{\tilde{p}}$ is constant on each component of $V_\alpha \cap V_\beta \cap \tilde{L}$. Since the transition function for $(V_\alpha, \bar{\Psi}_\alpha)$ and $(V_\beta, \bar{\Psi}_\beta)$ is given by

$$g_{\alpha\beta}(p) = \begin{bmatrix} I & S_p \\ 0 & I \end{bmatrix}$$

and is constant on each component of $V_\alpha \cap V_\beta \cap \tilde{L}$, $P_C^{(1)}$ is a $(G^{(1)}, \tilde{\mathcal{F}})$-subbundle and so $\tilde{\mathcal{F}}$ is naturally lifted to a foliation $\tilde{\mathcal{F}}^{(1)}$ on $P_C^{(1)}$ by Lemma 1.1.

Furthermore $P_C^{(1)}$ is invariant under the natural parallelism along the leaves of $\tilde{\mathcal{F}}$ because C is chosen so as to satisfy the conditions in Lemma 4.2. Therefore $P_C^{(1)}$ is a $\langle G^{(1)}, \tilde{\mathcal{F}}\rangle$-subbundle and the proof is completed.

Definition 4.2. The foliation $\mathcal{F}^{(1)}$ is called the first prolongation of $\tilde{\mathcal{F}}$.

Corollary 4.4. *Let P be a $\langle G, \mathcal{F}\rangle$-subbundle. Then we have a sequence of foliated manifolds $\{(P^{(k)}, \mathcal{F}^{(k)})\}_{k\geq 0}$ satisfying the following conditions*:

(1) *$P^{(0)} = P$ and $\mathcal{F}^{(0)} = \tilde{\mathcal{F}}$.*

(2) *$P^{(k)}$ is a $\langle G^{(k)}, \mathcal{F}^{(k-1)}\rangle$-subbundle on $P^{(k-1)}$ for $k \geq 1$ and $P^{(k)}$ is the first prolongation of $P^{(k-1)}$.*

(3) *$\mathcal{F}^{(k)}$ is the first prolongation of $\mathcal{F}^{(k-1)}$ for $k \geq 1$.*

Definition 4.3. $(P^{(k)}, \mathcal{F}^{(k)})$ is called the k-th prolongation of $(P, \tilde{\mathcal{F}})$. The basic form on $P^{(k)}$ is denoted by $\omega^{(k)}$.

5. For a foliation $\mathcal{F}_i$ of codimension q on M_i $(i=1, 2)$, assume that there exists an isomorphism ϕ of $\mathcal{F}_1$ with $\mathcal{F}_2$. Then ϕ induces the bundle isomorphism ϕ' (resp. $\tilde{\phi}$) of the normal bundle Q_1 of $\mathcal{F}_1$ with the normal bundle Q_2 of $\mathcal{F}_2$ (resp. of the normal frame bundle $F(Q_1)$ of $\mathcal{F}_1$ with the normal frame

bundle $F(Q_2)$ of $\mathcal{F}_2$). Let G be a closed subgroup of $GL(q, R)$.

Definition 5.1. A $(G, \mathcal{F}_1)$-subbundle P_1 on M_1 is said to be isomorphic to a $(G, \mathcal{F}_2)$-subbundle P_2 on M_2 if there exists a foliation isomorphism ϕ of $\mathcal{F}_1$ to $\mathcal{F}_2$ with $\tilde{\phi}(P_1)=P_2$. ϕ is called an isomorphism of P_1 to P_2. In particular, if $M_1=M_2$ and $\mathcal{F}_1=\mathcal{F}_2$, ϕ is called an automorphism.

Lemma 5.1. *Let P_i be a $(G, \mathcal{F}_i)$-subbundle on M_i, $i=1, 2$, and let ω_i be basic forms of P_i. Assume that G is connected. If $\hat{\phi}$ is a foliation isomorphism of $\tilde{\mathcal{F}}_1$ on P_1 with $\tilde{\mathcal{F}}_2$ on P_2 such that $\hat{\phi}^*\omega_2=a\omega_1$ for some $a\in G$, then there exists a foliation isomorphism φ of $\mathcal{F}_1$ with $\mathcal{F}_2$ such that $\tilde{\varphi}\,|\,P_1=\hat{\phi}\circ R_a$.*

Proof. Let z_t be a vertical curve in P_1, that is, a transversal curve with $\omega_1(\dot{z}_t)=0$. By the assumption that $\hat{\phi}^*\omega_2=a\omega_1$ and $\hat{\phi}$ is an isomorphism of $\tilde{\mathcal{F}}_1$ with $\tilde{\mathcal{F}}_2$, we know that $\hat{\phi}(z_t)$ is also a transverse vertical curve in P_2. Therefore $\hat{\phi}$ is fiber-preserving and induces a diffeomorphism φ of M_1 with M_2. Then it is clear that φ is an isomorphism of $\mathcal{F}_1$ with $\mathcal{F}_2$. Now we shall prove that $\tilde{\varphi}\,|\,P_1=\hat{\phi}\circ R_a$. If we set $F(Q_1)\supset P_3=\tilde{\varphi}^{-1}(P_2)$, then P_3 is a $(G, \mathcal{F}_1)$-subbundle. $J=\tilde{\varphi}^{-1}\circ\hat{\phi}$ is a fiber-preserving map of P_1 to P_3 and since $\tilde{\varphi}$ is a bundle isomorphism of $F(Q_1)$ to $F(Q_2)$, we have $\tilde{\varphi}^*\bar{\omega}_2=\bar{\omega}_1$ where $\bar{\omega}_i$ is the basic form of $F(Q_i)$. Since the basic form ω_i on P_i is the restriction of $\bar{\omega}_i$ to P_i $(i=1, 2, 3)$ where $\bar{\omega}_3=\bar{\omega}_1$, we get $\tilde{\varphi}^{-1*}\omega_3=\omega_2$. Thus we get $J^*\omega_3=a\omega_1$. Furthermore J induces the identity transformation on M_1. Therefore we have $z^{-1}\rho_1\pi_*(Z)=\omega_1(Z)=a^{-1}\omega_3(J_*(Z))=a^{-1}(J(z))^{-1}\rho_1\pi_*(J_*(Z))=a^{-1}(J(z))^{-1}\rho_1\pi_*(Z)$ for any $z\in P_1$ and any $Z\in T_z(P_1)$. This proves $J(z)=za^{-1}$, that is, $\tilde{\varphi}\,|\,P_1=\hat{\phi}\circ R_a$. This completes the proof.

Lemma 5.2. *Let P_i be a $\langle G, \mathcal{F}_1\rangle$-subbundle on M_i $(i=1, 2)$. If ψ is an isomorphism of the foliation $\mathcal{F}_1^{(1)}$ on $P_1^{(1)}$ with the foliation $\mathcal{F}_2^{(1)}$ on $P_2^{(1)}$ such that $\psi^*\omega_2^{(1)}=\omega_1^{(1)}$, then there exists an isomorphism ϕ of the foliation $\tilde{\mathcal{F}}_1$ with the foliation $\tilde{\mathcal{F}}_2$ such that $\phi^*\omega_2=\omega_1$ and $\tilde{\phi}\,|\,P_1^{(1)}=\psi$.*

Proof. By Lemma 5.1, we know that ψ induces an isomorphism ϕ of $\tilde{\mathcal{F}}_1$ with $\tilde{\mathcal{F}}_2$ such that $\tilde{\phi}\,|\,P_1^{(1)}=\psi$. We have only to see that $\phi^*\omega_2=\omega_1$. But this relation is obtained from the relation $\psi^*\omega_2^{(1)}=\omega_1^{(1)}$ because, for $Z\in T_{\hat{p}}(P_i^{(1)})$ if we set $p=\pi_i^{(1)}(\hat{p})$, $X_p+A_p^*=(\pi_i^{(1)})_*Z$ and $v=\omega_i(X_p)$, then $\omega_i^{(1)}(Z)=\hat{p}^{-1}\rho_i(\pi_i^{(1)})_*Z=\hat{p}^{-1}\tilde{\rho}_i(X_p+A_p^*)=\hat{p}^{-1}\tilde{\rho}_i\omega_i^{-1}(v)+A=v+A=\omega_i(X_p)+A$. This completes the proof.

Proposition 5.3. *Let P_i be a $\langle G, \mathcal{F}_i\rangle$-subbundle on M_i $(i=1, 2)$ and let*

k be any positive integer. Then there exists an isomorphism ψ of the foliation $\mathcal{F}_1^{(k)}$ on $P_1^{(k)}$ with the foliation $\mathcal{F}_2^{(k)}$ on $P_2^{(k)}$ such that $\psi^*\omega_2^{(k)}=\omega_1^{(k)}$ if and only if there exists an isomorphism ϕ of the $\langle G, \mathcal{F}_1\rangle$-subbundle P_1 on M_1 with the $\langle G, \mathcal{F}_2\rangle$-subbundle P_2 on M_2.

This proposition is an immediate consequence of the construction of $P_i^{(k)}$ and Lemma 5.2.

Definition 5.2. Let $\mathcal{F}$ be a foliation of codimension q on M and E the tangent bundle to $\mathcal{F}$. If there exists 1-forms $\omega_1, \cdots, \omega_q$ on M satisfying the following conditions (1) and (2), then the pair $(\mathcal{F}, \omega)$ is called a generalized Lie foliation where $\omega=(\omega_1, \cdots, \omega_q)$:

(1) $E\ni X$ if and only if $\omega(X)=0$.

(2) $d\omega_i=\sum_{j<k} c_{ijk}\omega_j\wedge\omega_k$ where c_{ijk} is a function on M. In particular if all c_{ijk} are constants, $(\mathcal{F}, \omega)$ is called a Lie foliation.

Definition 5.3. Let $\mathcal{F}$ be a foliation of codimension q on M. If there exists an $\langle \{e\}, \mathcal{F}\rangle$-subbundle P where e is the unit element of $GL(q, R)$, then the pair $(\mathcal{F}, P)$ is called an $\{e\}$-foliation.

Proposition 5.4. *There exists a natural one to one correspondence between generalized Lie foliations on M and $\{e\}$-foliations on M.*

Proof. Given a generalized Lie foliation $(\mathcal{F}, \omega)$ on M, the tangent vectors $Y_x^1, \cdots, Y_x^q$ at $x\in M$ can be chosen so as to satisfy $\omega_i(Y_x^j)=\delta_{ij}$ $(i, j=1, \cdots, q)$. Y_x^j is determined up to the vectors belonging to E_x and so uniquely determined as a normal vector $Y_x^j\in Q_x=T(M)_x/E_x$. Then for any $x\in M$ there exists a neighbourhood U of x such that there are vector fields $Y^1, \cdots, Y^q$ on U satisfying $\mathcal{P}(Y_y^j)=Y_y^j$ where $\mathcal{P}$ is the canonical projection of $T(M)$ onto Q.

Let L_{Y^i} (resp. ι_{Y^i}) denote the Lie differentiation (resp. interior product) with respect to Y^i. Then

$$\begin{aligned} L_{Y^i}(\omega_j) &= (d\iota_{Y^i}+\iota_{Y^i}d)\omega_j \\ &= \iota_{Y^i}(d\omega_j) \\ &= \iota_{Y^i}(\sum_{k<h} c_{jkh}\omega_k\wedge\omega_h) \\ &= \sum_{k<h} (c_{jkh}(\iota_{Y^i}\omega_k)\omega_h - c_{jkh}(\iota_{Y^i}\omega_h)\omega_k) \\ &= \sum_k a_k\omega_k \end{aligned}$$

where a_k is a constant. Therefore for any local cross-section X of E,

$$\omega_j([Y^i, X]) = \iota_{[Y^i,X]}\omega_j$$

$$\begin{aligned}
&= [L_{Y^i}, \iota_X]\omega_j \\
&= L_{Y^i}(\iota_X\omega_j) - \iota_X(L_{Y^i}(\omega_j)) \\
&= -\iota_X(L_{Y^i}(\omega_j)) \\
&= -\iota_X(\sum_k a_k\omega_k) \\
&= 0
\end{aligned}$$

and this implies that $[Y^i, X]$ is also a local cross-section of E. Then $\bar{Y}^1, \cdots, \bar{Y}^q$ are invariant under the natural parallelism along the leaves of $\mathcal{F}$. This shows that if we set $P=\{(Y_x^1, \cdots, Y_x^q); x\in M\}$, the pair $(\mathcal{F}, P)$ gives an $\{e\}$-foliation on M.

Conversely given an $\{e\}$-foliation $(\mathcal{F}, P)$ on M. Then there are given vector fields $Y^1, \cdots, Y^q$ on M such that $(\mathcal{P}(Y_x^1), \cdots, \mathcal{P}(Y_x^q))\in P$ for any $x\in M$. These vector fields satisfy the property that, for any local cross-section X of E, the blacket $[Y^i, X]$ is also a local cross-section of E, $i=1, \cdots, q$. Define 1-forms $\omega^i, \cdots, \omega_q$ on M by

$$\omega_i | E = 0\,,$$
$$\omega_i(Y^j) = \delta_{ij} \quad (i, j = 1, \cdots, q)\,.$$

Then since $\omega_1, \cdots, \omega_q$ define a foliation $\mathcal{F}$, the system $\Omega=\{\omega_i;\ i=1, \cdots, q\}$ is a completely integrable differential system. Therefore we have locally

$$d\omega_i = \sum_{j<k} c_{ijk}\omega_j\wedge\omega_k + \sum_{h,l} \bar{c}_{ihl}\omega_h\wedge\bar{\omega}_{li} \tag{5.1}$$

where $\bar{\omega}_{li}$ is a local 1-form. Let X be any cross-section of E. Then because $[Y^j, X]$ is a local cross-section of E, we have

$$d\omega_i(Y^j, X) = Y^j\omega_i(X) - X\omega_i(Y^j) - \omega_i([Y^j, X]) = 0\,.$$

On the other hand by (5.1) we have

$$d\omega_i(Y^j, X) = \sum_l \bar{c}_{ijl}\omega_j(Y^j)\bar{\omega}_{li}(X)$$

and since X is any local cross-section of E we get $\bar{c}_{ijl}=0$. Thus we obtain the relation

$$d\omega_i = \sum_{j<k} c_{ijk}\omega_j\wedge\omega_k$$

and so by setting $\omega=(\omega_1, \cdots, \omega_q)$ $(\mathcal{F}, \omega)$ is a generalized Lie foliation on M. This completes the proof of Proposition 5.4.

Definition 5.4. A $\langle G, \mathcal{F}\rangle$-subbundle P on M is said to be of finite type if the following statement (S) holds:

(S) We set $\omega=\omega^{(0)}$ and $\tilde{\mathcal{F}}=\mathcal{F}^{(0)}$. Then there exists an integer $k\geq 0$ such that the pair $(\mathcal{F}^{(k)}, \omega^{(k)})$ is a generalized Lie foliation where $(P^{(k)}, \mathcal{F}^{(k)})$ is the k-th prolongation of $(P, \tilde{\mathcal{F}})$ and $\omega^{(k)}$ is the basic form on $P^{(k)}$.

The minimum integer such that $G^{(k)}=\{e\}$ is called the order of P.

§2. Differential Equations of Euclidean Type

6. Denote by E^n the Euclidean space of dimension n i.e. the real number space R^n of dimension n with the canonical inner product.

Let $J^k(R^m, E^n)$ be the space of k-jets of local maps of R^m to E^n and each element of $J^k(R^m, E^n)$ is denoted by $j_x^k(f)$ where $x\in R^m$ and f is a map of a neighbourhood of x to E^n. Then $J^k(R^m, E^n)$ admits the canonical product structure $R^m\times R^n\times R^{n(1)}\times \cdots \times R^{n(k)}$ where $n(l)=\dim J^l(R^m, E^n)-\dim J^{l-1}(R^m, E^n)$ $(1\leq l\leq k)$. Denote by $\{x_1, \cdots, x_m\}$ (resp. $\{u_1, \cdots, u_n\}$) the canonical coordinate system on R^m (resp. E^n) and set

$$p^i_{j_1\cdots j_l}=\frac{\partial^l u_i}{\partial x_{j_1}\cdots\partial x_{j_l}}.$$

Then $\{p^i_{j_1\cdots j_l};\ 1\leq i\leq n,\ 1\leq j_1\leq \cdots \leq j_l\leq m\}$ is the coordinate system on $R^{n(l)}$.

Denote by $\tilde{J}^k(R^m, E^n)$ the set of k-jets of local maps of R^m to E^n of maximal rank i.e. k-jets of local immersions if $m<n$ and k-jets of local submersions if $m\geq n$. Then it is clear that $\tilde{J}^k(R^m, E^n)$ is open and dense in $J^k(R^m, E^n)$.

Definition 6.1. A subset $\mathcal{E}$ of $J^k(R^m, E^n)$ is called a differential equation if, for any point $p\in\mathcal{E}$, there exist a neighbourhood U and functions $f_1, \cdots, f_{r_p}$ on U such that $\mathcal{E}\cap U=\{p'\in U;\ f_1(p')=\cdots=f_{r_p}(p')=0\}$. Any local map s: $R^m\supset\mathcal{U}\to E^n$ is called a solution of $\mathcal{E}$ if $\{j_x^k(s);\ x\in\mathcal{U}\}\subset\mathcal{E}$.

Let Γ be a pseudogroup on E^n and for any $\phi\in\Gamma$ and $j_x^k(f)\in J^k(R^m, E^n)$ we set $\phi^{(k)}(j_x^k(f))=j_x^k(\phi\circ f)$ if the composite $\phi\circ f$ is defined on a neighbourhood of x. Thus Γ can be regarded as a pseudogroup on $J^k(R^m, E^n)$ which is denoted by $\Gamma^{(k)}$. Note that $\tilde{J}^k(R^m, E^n)$ is preserved by the action of $\Gamma^{(k)}$.

Definition 6.2. A function F defined on a neighbourhood $\mathcal{U}^k$ of $j_x^k(f)$ $\in J^k(R^m, E^n)$ is called a Γ-differential invariant if $\phi^{(k)*}F=F$ for any $\phi^{(k)}\in$ $\Gamma^{(k)}|\mathcal{U}^k$ where $\Gamma^{(k)}|\mathcal{U}^k$ is the restriction of $\Gamma^{(k)}$ to $\mathcal{U}^k$.

Denote by $\mathcal{P}$ the pseudogroup of local isometries of E^n and consider the function $\rho_1=\sum_{i=1}^n(p^i_1)^2$ (resp. $\rho_2=\sum_{i=1}^n(p^i_{11})^2$) on $J^1(R^m, E^n)$ (resp. $J^2(R^m, E^n)$).

Then it is clear that ρ_1 and ρ_2 are $\mathcal{P}$-differential invariants.

Let $\mathcal{E}_i \subset J^k(R^m, E^n)$ be a differential equation and denote by $\mathcal{S}(\mathcal{E}_i)$ the set of solutions of $\mathcal{E}_i$ $(i=1, 2)$. For any open subset $\mathcal{U} \subset E^n$, we set $\mathcal{S}(\mathcal{E}_i)|\mathcal{U} = \{s \in \mathcal{S}(\mathcal{E}_i);\ s$ is a map into $\mathcal{U}\}$.

Definition 6.3. A local transformation $\varphi: \mathcal{U} \to \mathcal{V}$ of E^n is called a local isomorphism of $\mathcal{E}_1$ to $\mathcal{E}_2$ if $\varphi(\mathcal{S}(\mathcal{E}_1)|\mathcal{U}) = \mathcal{S}(\mathcal{E}_2)|\mathcal{V}$. In particular in the case $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$, a local isomorphism is called a local automorphism and the pseudogroup generated by all local automorphisms of $\mathcal{E}$ is called the automorphism pseudogroup of $\mathcal{E}$ and is denoted by $\mathcal{A}(\mathcal{E})$.

7. Let f be a map of R^m to E^n and set $\lambda_1^f(x) = \rho_1(j_x^1(f))$ and $\lambda_2^f(x) = \rho_2(j_x^2(f))$. Furthermore we define the vector-valued functions a_1^f and a_2^f by $a_1^f(x) = (p_1^1(j_x^1(f)), \cdots, p_1^n(j_x^1(f)))$ and $a_2^f(x) = (p_{11}^1(j_x^2(f)), \cdots, p_{11}^n(j_x^2(f)))$.

Proposition 7.1. *Consider the differential equation $\mathcal{E} \subset J^k(R^m, E^n)$ $(m, n \geq 2)$ given on a neighbourhood $\mathcal{U}^k$ of $j_{x_0}^k(f)$ by $\mathcal{P}$-differential invariants $F_1, \cdots, F_r$ such that $\mathcal{E} \cap \mathcal{U}^k = \{p \in \mathcal{U}^k;\ F_1(p) = \cdots = F_r(p) = 0\}$ where $F_1 = \rho_1 - \lambda_1^f$ and $F_2 = \rho_2 - \lambda_2^f$ and suppose that, if we restrict f to a neighbourhood of x_0, it is contained in $\mathcal{S}(\mathcal{E})$. Furthermore suppose that $\lambda_1^f(x_0) \neq 0$, $\lambda_2^f(x_0) \neq 0$, $a_1^f(x_0) \neq \alpha a_2^f(x_0)$ for any $\alpha \in R$ and the map $\lambda^f = (\lambda_1^f, \lambda_2^f)$ is a submersion on a neighbourhood U of x_0 to R^2. Then if f and $\mathcal{E}$ are analytic, we have $\mathcal{A}(\mathcal{E}) = \mathcal{P}$ on a neighbourhood of $f(x_0)$.*

Proof. Consider the linear orthogonal transformation group $O(n)$ on R^n and define the action of $O(n)$ on the product space $R^n \times R^n$ by $\sigma(a, b) = (\sigma a, \sigma b)$ where $\sigma \in O(n)$ and $(a, b) \in R^n \times R^n$. Let $O(n)_{(a,b)}$ (resp. $O(n)_a$) denote the isotropy group of $O(n)$ at $(a, b) \in R^n \times R^n$ (resp. at $a \in R^n$). Then $O(n)_{(a,b)} = O(n)_a \cap O(n)_b$ and if $a \neq 0$, $b \neq 0$ and $a \neq \alpha b$ for any $\alpha \in R$, we have $\dim O(n) - \dim O(n)_{(a,b)} = 2n-3$. Therefore if we set $M(a, b) = \{\sigma(a, b);\ \sigma \in O(n)\}$, then $M(a, b)$ is a $(2n-3)$-dimensional submanifold of $R^n \times R^n$. Let π_1 (resp. π_2) be the projection of $R^n \times R^n$ to the 1-st component (resp. 2-nd component). Then it is clear that $\pi_1(M(a, b)) = S_{|a|}^{n-1}$ and $\pi_2(M(a, b)) = S_{|b|}^{n-1}$ where S_r^{n-1} is the $(n-1)$-sphere of the radius r.

Since $\lambda_1^f(x_0) \neq 0$, $\lambda_2^f(x_0) \neq 0$ and $a_1^f(x_0) \neq \alpha a_2^f(x_0)$ for any $\alpha \in R$, we may assume that we have $\lambda_1^f(x) \neq 0$, $\lambda_2^f(x) \neq 0$ and $a_1^f(x) \neq \alpha a_2^f(x)$ for any $\alpha \in R$ and any $x \in U$, a neighbourhood of x_0. Then $M(a_1^f(x), a_2^f(x))$ is a $(2n-3)$-dimensional submanifold of $R^n \times R^n$. Denote by π the map of $R^n \times R^n$ to R^2 defined by $\pi(a, b) = (|a|^2, |b|^2) \in R^2$. Then $\pi(M(a_1^f(x), a_2^f(x))) = (|a_1^f(x)|^2,$

$|a_2^f(x)|^2)$. On the other hand if we denote by a^f the map $U \ni x \to (a_1^f(x), a_2^f(x)) \in R^n \times R^n$, then $\lambda^f = \pi \circ a^f$ and since λ^f is a submersion, the image of a^f contains a 2-dimensional submanifold $N(x_0) \subset R^n \times R^n$ through $a^f(x_0)$ such that $\pi | N(x_0)$ is a diffeomorphism of $N(x_0)$ onto a neighbourhood $V(x_0)$ of $\lambda^f(x_0)$. This shows that $\pi^{-1}(V(x_0)) \cap \{\bigcup_{x \in U} M(a^f(x))\}$ contains a $(2n-1)$-dimensional submanifold H of $R^n \times R^n$ through $a^f(x_0)$.

Now for any local transformation ϕ of E^n, we have

$$\phi^{(2)*}\rho_2 = \sum_{i=1}^{n} \left(\frac{\partial^2 \phi_i(u)}{\partial x_1^2} \right)^2$$

where $\phi_i(u) = u_i(\phi)$ and since

$$\frac{\partial^2 \phi_i(u)}{\partial x_1^2} = \sum_{h,l=1}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} \frac{\partial u_h}{\partial x_1} \frac{\partial u_l}{\partial x_1} + \sum_{h=1}^{n} \frac{\partial \phi_i(u)}{\partial u_h} \frac{\partial^2 u_h}{\partial x_1^2},$$

we have

$$\phi^{(2)*}\rho_2 = \sum_{i=1}^{n} \left(\sum_{h,l=1}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} \frac{\partial u_h}{\partial x_1} \frac{\partial u_l}{\partial x_1} + \sum_{h=1}^{n} \frac{\partial \phi_i(u)}{\partial u_h} \frac{\partial^2 u_h}{\partial x_1^2} \right)^2.$$

Let $\mathcal{V}^\alpha$ be a neighbourhood of $j_{x_0}^\alpha(f) \in J^\alpha(R^m, E^n)$ and set $S(\mathcal{E}, \mathcal{V}^\alpha) = \{j_x^\alpha(s);\ s\colon \mathcal{U} \to E^n \textit{ belongs to } \mathcal{S}(\mathcal{E}),\ x \in \mathcal{U}\} \cap \mathcal{V}^\alpha$. Then if $\phi \in \mathcal{A}(\mathcal{E})$, we have

$$(7.1) \qquad \sum_{i=1}^{n} \left(\sum_{h,l=1}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} \frac{\partial u_h}{\partial x_1} \frac{\partial u_l}{\partial x_1} + \sum_{h=1}^{n} \frac{\partial \phi_i(u)}{\partial u_h} \frac{\partial^2 u_h}{\partial x_1^2} \right)^2 = \sum_{h=1}^{n} \left(\frac{\partial^2 u_h}{\partial x_1^2} \right)^2 \quad \text{on } S(\mathcal{E}, \mathcal{V}^2).$$

The jet space $J^k(R^m, E^n)$ $(k \geq 2)$ admits the product structure $R^{2n} \times R^{q-2n}$ where $q = \dim J^k(R^m, E^n)$, R^{2n} is the space with the coordinate system $\{p_1^1, \cdots, p_1^n, p_{11}^1, \cdots, p_{11}^n\}$ and R^{q-2n} is the space with the coordinate system $\{x_1, \cdots, x_m, u_1, \cdots, u_n, \cdots, p_{j_1 \cdots j_l}^i, \cdots\}$ $(p_{j_1 \cdots j_l}^i \neq p_1^i, p_{11}^i)$. Let $\mathcal{P}$ be the projection of $R^{2n} \times R^{q-2n}$ onto R^{2n}. Then because $\mathcal{S}(\mathcal{E}) \ni f$ and $\mathcal{A}(\mathcal{E}) \supset O(n)$, $\mathcal{P}(S(\mathcal{E}, \mathcal{V}^k))$ contains $H \cap \mathcal{O}$ where $\mathcal{O}$ is a neighbourhood of $a^f(x_0) \in R^{2n}$.

Since $\pi_1(M(a^f(x))) = S^{n-1}_{|a_1^f(x)|}$ and $\pi_2(M(a^f(x))) = S^{n-1}_{|a_2^f(x)|}$, it is easy to know that $\dim \pi_1(H \cap \mathcal{O}) = \dim \pi_2(H \cap \mathcal{O}) = n$. Therefore we may suppose that $2n-1$ functions $p_1^2, \cdots, p_1^n, p_{11}^1, \cdots, p_{11}^n$ are independent on $H \cap \mathcal{O}$. Then p_1^1 is written by $p_1^1 = A(p_1^2, \cdots, p_1^n, p_{11}^1, \cdots, p_{11}^n)$ where A is an analytic function with $2n-1$ variables.

Denote by $A = c + \sum_{r \geq 1} A^r$ the Taylor's expansion of A at $(p_1^2(j_{x_0}^1(f)), \cdots, p_1^n(j_{x_0}^1(f)), p_{11}^1(j_{x_0}^2(f)), \cdots, p_{11}^n(j_{x_0}^2(f))) \in R^{2n-1}$ where c is the constant term and

A^r is the sum of the terms of degree r.

Firstly suppose that $c \neq 0$. Then (7.1) is written by

$$c^4 \sum_{i=1}^{n} \left(\frac{\partial \phi_i(u)}{\partial u_1^2} \right)^2 + 4c^2 \sum_{l=2}^{n} \sum_{i=1}^{n} \left(\frac{\partial^2 \phi_i(u)}{\partial u_1 \partial u_l} \right)^2 (p_1^l)^2 \tag{7.2}$$
$$+ B = \sum_{h=1}^{n} (p_{11}^h)^2 \qquad \text{on} \quad H \cap \tilde{\mathcal{O}}$$

where $\tilde{\mathcal{O}}$ is a sufficiently small neighbourhood of $a^f(x_0) \in R^{2n}$ and B is a polynomial with respect to p_1^i $(i \neq 1)$ and p_{11}^i which contains neither the constant term nor the terms of linear combinations with respect to $(p_1^2)^2, \cdots, (p_1^n)^2$. Since the relation (7.2) holds identically on $H \cap \tilde{\mathcal{O}}$, we get

$$\frac{\partial^2 \phi_i(u)}{\partial u_1 \partial u_h} = 0 \quad (h = 1, \cdots, n).$$

Therefore in this case (7.1) is reduced to the relation

$$\sum_{i=1}^{n} \left(\sum_{h,l=2}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} p_1^h p_1^l + \sum_{\alpha=1}^{n} \frac{\partial \phi_i(u)}{\partial u_\alpha} p_{11}^\alpha \right)^2 \tag{7.3}$$
$$= \sum_{\beta=1}^{n} (p_{11}^\beta)^2 \qquad \text{on } H \cap \tilde{\mathcal{O}}.$$

Then we get

$$\frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} = 0 \quad (h, l = 2, \cdots, n),$$
$$\sum_{i=1}^{n} \frac{\partial \phi_i(u)}{\partial u_\alpha} \frac{\partial \phi_i(u)}{\partial u_\beta} = \delta_{\alpha\beta}. \tag{7.4}$$

This shows that ϕ is a local isometry of E^n, that is, $\phi \in \mathcal{P}$.

Secondly we suppose that $c = 0$. In this case the term $\sum_{h,l=1}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} p_1^h p_1^l$ in (7.1) is written by

$$\frac{\partial^2 \phi_i(u)}{\partial u_1^2} (A)^2 + 2 \sum_{l=2}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_1 \partial u_l} A\, p_1^l + \sum_{h,l=2}^{n} \frac{\partial^2 \phi_i(u)}{\partial u_h \partial u_l} p_1^h p_1^l$$

and it does not contain terms of degree 1 with respect to the variables $\{p_1^2, \cdots, p_1^n, p_{11}^1, \cdots, p_{11}^n\}$. Therefore from (7.1) we get

$$\sum_{i=1}^{n} \frac{\partial \phi_i(u)}{\partial u_\alpha} \frac{\partial \phi_i(u)}{\partial u_\beta} = \delta_{\alpha\beta}.$$

This shows that ϕ is a local isometry of E^n.

In any case we obtain $\mathcal{A}(\mathcal{E})\subset\mathcal{P}$. Since the relation $\mathcal{P}\subset\mathcal{A}(\mathcal{E})$ clearly holds, the proof of Proposition 7.1 is thereby completed.

§3. Conformally Foliated Structures of Differential Equations of Euclidean Type

8. Let α^k (resp β^k) be the map of $J^k(R^m, E^n)$ to R^m (resp. E^n) defined by $\alpha^k(j_x^k(f))=x$ (resp. $\beta^k(j_x^k(f))=f(x)$) and set $J_{xz}^k(R^m, E^n)=\{p\in J^k(R^m, E^n);\ \alpha^k(p)=x,\ \beta^k(p)=z\}$. Then $J_{xz}^k(R^m, E^n)$ is diffeomorphic to the product space $R^{n(1)}\times\cdots\times R^{n(k)}$ and so admits the canonical Euclidean structure. In the following we fix the points $x_0\in R^m$ and $z_0\in E^n$ and denote the space $J_{x_0z_0}(R^m, E^n)$ simply by $J^k(E^n)$.

Let G be the transformation group of isometries of E^n. Then it is known that $G\ni\phi$ if and only if

$$\phi_i(u)=\sum_{i=1}^{n} a_{ij}(\phi)u_j+b_i(\phi)$$

where $\phi_i(u)=u_i(\phi)$, $(a_{ij}(\phi))\in O(n)$ and $b_i(\phi)\in R$. Denote by $\mathcal{P}$ the pseudogroup on E^n generated by G and by Θ the sheaf of vector fields which generate local 1-parameter groups of local transformations contained in $\mathcal{P}$. For any local transformation φ of E^n with $\varphi(z_0)=z_0$, $\varphi^{(k)}$ induces a local transformation on $J^k(E^n)$. Thus the isotropy $\mathcal{P}_{z_0}^0$ of $\mathcal{P}$ at z_0 induces a pseudogroup $\mathcal{P}_*^{(k)}$ on $J^k(E^n)$ and the isotropy $\Theta_{z_0}^0$ of the stalk Θ_{z_0} at z_0 induces a sheaf $\Theta_*^{(k)}$ on $J^k(E^n)$.

For $p\in J^k(E^n)$, we denote by $\Theta_{*p}^{(k)}$ the stalk of $\Theta_*^{(k)}$ at p and by $\Theta_{*p}^{(k),0}$ the isotropy of $\Theta_{*p}^{(k)}$. Then $\Theta_{*p}^{(k)}/\Theta_{*p}^{(k),0}$ is considered as a subspace of the tangent space $T(J^k(E^n))_p$ of $J^k(E^n)$ at p and we obtain the correspondence $D^{(k)}: J^k(E^n)\ni p\to D_p^{(k)}=\Theta_{*p}^{(k)}/\Theta_{*p}^{(k),0}\subset T(J^k(E^n))_p$.

Let J^k denote the set of points $p\in J^k(E^n)$ such that $\dim D^{(k)}$ is constant on a neighbourhood of p. Then J^k is open and dense in $J^k(E^n)$. Denote by J_r^k the union of connected components of J^k on which $\dim D^{(k)}=r$. Then $D^{(k)}$ induces a foliation $\mathcal{F}_r^k$ on J_r^k.

Before stating Theorem 8.1, we give here the definition of a Riemannian foliation on a manifold M.

Definition 8.1. Let $\mathcal{F}$ be a foliation of codimension q on M. $\mathcal{F}$ is called a Riemannian foliation if it admits a following transversally Riemannian structure

$$(\{(U_\alpha, f_\alpha)\},\ \{\gamma_{\alpha\beta}\},\ \{(R_\alpha^q, g_\alpha)\})$$

where

i) $\{U_\alpha\}$ is an open covering of M,

ii) $f_\alpha\colon U_\alpha \to R^q_\alpha$ is a submersion,

iii) g_α is a Riemannian metric on R^q_α,

iv) $f_\alpha = r_{\alpha\beta}\circ f_\beta$ on $U_\alpha \cap U_\beta$ where $r_{\alpha\beta}\colon (R^q_\alpha, g_\alpha)\to(R^q_\beta, g_\beta)$ are local isometries.

Now consider a differential equation $\mathcal{E}\subset J^k(R^m, E^n)$ and set $S^k(\mathcal{E})=\{j^k_{x_0}(s);$ $s\in\mathcal{S}(\mathcal{E})$ *with* $s(x_0)=z_0\}$ and $S^k_r(\mathcal{E})=S^k(\mathcal{E})\cap J^k_r$.

Theorem 8.1. *Assume that* $\mathcal{A}(\mathcal{E})=\mathcal{P}$. *If* $S^k_r(\mathcal{E})$ *is a regular submanifold of* J^k_r, *then* $D^{(k)}$ *induces a Riemannian foliation* $\mathcal{F}^k_r(\mathcal{E})$ *on* $S^k_r(\mathcal{E})$ *such that each leaf of* $\mathcal{F}^k_r(\mathcal{E})$ *is an orbit of* $D^{(k)}$.

The proof of this theorem will be given in the subsequent sections.

9. For a function φ on a neighbourhood $\mathcal{U}^k$ of $j^k_{x_0}(f)\in J^k_{x_0}(E^n)=\{p\in J^k(R^m, E^n);\ \alpha^k(p)=x_0\}$, we can define the function $\partial^{\#}_j\varphi$ by

$$\partial^{\#}_j\varphi = \sum_{i=1}^n p^i_j\frac{\partial\varphi}{\partial u_i}+\sum_{i=1}^n\sum_{j_1=1}^m p^i_{jj_1}\frac{\partial\varphi}{\partial p^i_{j_1}}+\cdots +\sum_{i=1}^n\sum_{j_1,\cdots,j_k=1}^m p^i_{jj_1,\cdots,j_k}\frac{\partial\varphi}{\partial p^i_{j_1,\cdots,j_k}}.$$

Then $\partial^{\#}_j\varphi$ is defined on a neighbourhood $\mathcal{U}^{k+1}=(\pi^{k+1}_k)^{-1}(\mathcal{U}^k)$ of $j^{k+1}_{x_0}(f)\in J^{k+1}_{x_0}(E^n)$ where π^k_l $(k>l)$ is the canonical projection of $J^k_{x_0}(E^n)$ onto $J^l_{x_0}(E^n)$.

Let ϕ be a local transformation on E^n such that $\phi^{(k)}$ maps an open subset $\mathcal{U}^k\subset J^k_{x_0}(E^n)$ onto another open subset $\mathcal{V}^k$. Then $\phi^{(k+1)}$ maps $\mathcal{U}^{k+1}=(\pi^{k+1}_k)^{-1}(\mathcal{U}^k)$ onto $\mathcal{V}^{k+1}=(\pi^{k+1}_k)^{-1}(\mathcal{V}^k)$. Let F be a function defined on $\mathcal{V}^k$.

Lemma 9.1. $\partial^{\#}_j(\phi^{(k)*}F)=\phi^{(k+1)*}(\partial^{\#}_jF)$ *on* $\mathcal{U}^{k+1}$ *for* $k\geq 0$.

Proof. We have

$$\frac{\partial(F\circ\phi^{(k)})}{\partial p^\lambda_{j_1,\cdots,j_l}} = \sum_\mu\frac{\partial F}{\partial u_\mu}(\phi^{(k)})\frac{\partial u_\mu(\phi^{(k)})}{\partial p^\lambda_{j_1,\cdots,j_l}} +\sum_{\mu,j}\frac{\partial F}{\partial p^\mu_j}(\phi^{(k)})\frac{\partial p^\mu_j(\phi^{(k)})}{\partial p^\lambda_{j_1,\cdots,j_l}}+\cdots +\sum_\mu\sum_{j_1,\cdots,j_k}\frac{\partial F}{\partial p^\mu_{j_1,\cdots,j_k}}(\phi^{(k)})\frac{\partial p^\mu_{j_1,\cdots,j_k}(\phi^{(k)})}{\partial p^\lambda_{j_1,\cdots,j_l}}.$$

Then

$$\partial^{\#}_i(\phi^{(k)*}F) = \partial^{\#}_i(F\circ\phi^{(k)})$$

$$
\begin{aligned}
&= \sum_\mu p^\mu_j \frac{\partial(F\circ\phi^{(k)})}{\partial u_\mu} + \sum_\mu \sum_{j_1} p^\mu_{ij_1} \frac{\partial(F\circ\phi^{(k)})}{\partial p^\mu_{j_1}} \\
&\quad + \cdots + \sum_\mu \sum_{j_1,\cdots,j_k} p^\mu_{ij_1,\cdots,j_k} \frac{\partial(F\circ\phi^{(k)})}{\partial p^\mu_{j_1,\cdots,j_k}} \\
&= \sum_\mu \Big(\sum_\lambda p^\lambda_i \frac{\partial u_\mu(\phi^{(k)})}{\partial u_\lambda} + \sum_\lambda \sum_{h_1} p^\lambda_{ih_1} \frac{\partial u_\mu(\phi^{(k)})}{\partial p^\lambda_{h_1}} \\
&\quad + \cdots + \sum_\lambda \sum_{h_1,\cdots,h_k} p^\lambda_{ih_1,\cdots,h_k} \frac{\partial u_\mu(\phi^{(k)})}{\partial p^\lambda_{h_1,\cdots,h_k}}\Big) \frac{\partial F}{\partial u_\mu}(\phi^{(k)}) \\
&\quad + \sum_\mu \sum_{j_1} \Big(\sum_\lambda p^\lambda_i \frac{\partial p^\mu_{j_1}(\phi^{(k)})}{\partial u_\lambda} + \sum_\lambda \sum_{h_1} p^\lambda_{ih_1} \frac{\partial p^\mu_{j_1}(\phi^{(k)})}{\partial p^\lambda_{h_1}} \\
&\quad + \cdots + \sum_\lambda \sum_{h_1,\cdots,h_k} p^\lambda_{ih_1,\cdots,h_k} \frac{\partial p^\mu_{j_1}(\phi^{(k)})}{\partial p^\lambda_{h_1,\cdots,h_k}}\Big) \frac{\partial F}{\partial p^\mu_{j_1}}(\phi^{(k)}) \\
&\quad + \cdots \\
&\quad + \sum_\mu \sum_{j_1,\cdots,j_k} \Big(\sum_\lambda p^\lambda_i \frac{\partial p^\mu_{j_1,\cdots,j_k}(\phi^{(k)})}{\partial u_\lambda} + \sum_\lambda \sum_{h_1} p^\lambda_{ih_1} \frac{\partial p^\mu_{j_1,\cdots,j_k}(\phi^{(k)})}{\partial p^\lambda_{h_1}} \\
&\quad + \cdots + \sum_\lambda \sum_{h_1,\cdots,h_k} p^\lambda_{ih_1,\cdots,h_k} \frac{\partial p^\mu_{j_1,\cdots,j_k}(\phi^{(k)})}{\partial p^\lambda_{h_1,\cdots,h_k}}\Big) \frac{\partial F}{\partial p^\mu_{j_1,\cdots,j_k}}(\phi^{(k)}) \\
&= \sum_\mu p^\mu_i(\phi^{(k+1)}) \frac{\partial F}{\partial u_\mu}(\phi^{(k+1)}) + \sum_\mu \sum_{j_1} p^\mu_{ij_1}(\phi^{(k+1)}) \frac{\partial F}{\partial p^\mu_{j_1}}(\phi^{(k+1)}) \\
&\quad + \cdots + \sum_\mu \sum_{j_1,\cdots,j_k} p^\mu_{ij_1,\cdots,j_k}(\phi^{(k+1)}) \frac{\partial F}{\partial p^\mu_{j_1,\cdots,j_k}}(\phi^{(k+1)}) \\
&= (\partial^\sharp_i F)\circ\phi^{(k+1)} \equiv \phi^{(k+1)*}(\partial^\sharp_i F)\,.
\end{aligned}
$$

Any local vector field X on E^n is naturally lifted to a local vector field $X^{(k)}$ on $J^k_{x_0}(E^n)$ via the lift of the local 1-parameter group of local transformations generated by X. Then as the infinitesimal version of Lemma 9.1 we have $\partial^\sharp_i X^{(k)} = X^{(k+1)}\partial^\sharp_i$.

Lemma 9.2. *Let $\bar{X}$ be a local vector field on $J^k_{x_0}(E^n)$. Then there exists a local cross-section X of Θ with $\bar{X}=X^{(k)}$ if and only if*

$$
\begin{aligned}
\bar{X} &= \sum_{i,j=1}^n \alpha_{ij}\Big(u_j \frac{\partial}{\partial u_i} + \sum_{j_1=1}^m p^j_{j_1} \frac{\partial}{\partial p^i_{j_1}} + \cdots \\
&\quad + \sum_{j_1,\cdots,j_k=1}^m p^j_{j_1,\cdots,j_k} \frac{\partial}{\partial p^i_{j_1,\cdots,j_k}}\Big) + \sum_{i=1}^n \beta_i \frac{\partial}{\partial u_i}
\end{aligned}
$$

where (α_{ij}) is any skew-symmetric matrix and β_i is any constant.

Proof. It is known that X is a local cross-section of Θ if and only if X is of the form

$$X=\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\alpha_{ij}u_j+\beta_i\Big)\frac{\partial}{\partial u_i}$$

where (α_{ij}) is any skew-symmetric matrix and β_i is any constant. Then we have only to show that

$$X^{(k)}=\sum_{i,j=1}^{n}\alpha_{ij}\Big(u_j\frac{\partial}{\partial u_i}+\sum_{j_1=1}^{m}p^j_{j_1}\frac{\partial}{\partial p^i_{j_1}}+\cdots$$
$$+\sum_{j_1,\cdots,j_k=1}^{m}p^j_{j_1,\cdots,j_k}\frac{\partial}{\partial p^i_{j_1,\cdots,j_k}}\Big)+\sum_{i=1}^{n}\beta_i\frac{\partial}{\partial u_i}\,.$$

If we set $X^{(-1)}=0$ and $X^{(0)}=X$, we have $X^{(k)}=X^{(k-1)}+Y^{(k)}$ for $k\geq 1$ where $Y^{(k)}$ is of the form $\sum_{i=1}^{n}\sum_{j_1,\cdots,j_k=1}^{m}A^i_{j_1,\cdots,j_k}\frac{\partial}{\partial p^i_{j_1,\cdots,j_k}}$. Let us show that

$$A^i_{j_1,\cdots,j_k}=\sum_{j=1}^{n}\alpha_{ij}p^j_{j_1,\cdots,j_k}\,.$$

Since

$$X^{(0)}=\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\alpha_{ij}u_j+\beta_i\Big)\frac{\partial}{\partial u_i}\,,$$

we have

$$\partial^\sharp_h(X^{(0)}u_i)=\partial^\sharp_h\Big(\sum_j\alpha_{ij}u_j+\beta_i\Big)=\sum_j\alpha_{ij}p^j_h\,.$$

Then by Lemma 9.1

$$X^{(1)}(\partial^\sharp_h u_i)=\sum_j\alpha_{ij}p^j_h$$

and since

$$X^{(1)}(\partial^\sharp_h u_i)=X^{(1)}p^i_h=Y^{(1)}p^i_h=A^i_h\,,$$

we get

$$A^i_h=\sum_j\alpha_{ij}p^j_h\quad\text{i.e.}\quad X^{(1)}=X^{(0)}+\sum_i\sum_h\Big(\sum_j\alpha_{ij}p^j_h\Big)\frac{\partial}{\partial p^i_h}\,.$$

So assume that

$$Y^{(l-1)}=\sum_{i=1}^{n}\sum_{j_1,\cdots,j_{l-1}=1}^{m}\Big(\sum_{j=1}^{n}\alpha_{ij}p^j_{j_1,\cdots,j_{l-1}}\Big)\frac{\partial}{\partial p^i_{j_1,\cdots,j_{l-1}}}\,.$$

Then

$$\begin{aligned}X^{(l)}(\partial^\sharp_{j_l}p^i_{j_1,\cdots,j_{l-1}})&=\partial^\sharp_{j_l}(X^{(l-1)}p^i_{j_1,\cdots,j_{l-1}})\\&=\partial^\sharp_{j_l}(Y^{(l-1)}p^i_{j_1,\cdots,j_{l-1}})\\&=\partial^\sharp_{j_l}\Big(\sum_j\alpha_{ij}p^j_{j_1,\cdots,j_{l-1}}\Big)\\&=\sum_j\alpha_{ij}p^j_{j_1,\cdots,j_l}\,.\end{aligned}$$

Since

$$X^{(l)}(\partial^{\#}_{j_l} p^i_{j_1,\cdots,j_{l-1}}) = X^{(l)} p^i_{j_1,\cdots,j_l}$$

and

$$Y^{(l)} p^i_{j_1,\cdots,j_l} = X^{(l)} p^i_{j_1,\cdots,j_l},$$

we get

$$Y^{(l)} p^i_{j_1,\cdots,j_l} = \sum_j \alpha_{ij} p^j_{j_1,\cdots,j_l}.$$

Thus we obtain

$$Y^{(l)} = \sum_{i=1}^{n} \sum_{j_1,\cdots,j_l=1}^{n} (\sum_{j=1}^{n} \alpha_{ij} p^j_{j_1,\cdots,j_l}) \frac{\partial}{\partial p^i_{j_1,\cdots,j_l}}.$$

This proves that

$$X^{(k)} = \sum_{i,j=1}^{n} \alpha_{ij} (u_j \frac{\partial}{\partial u_i} + \sum_{j_1=1}^{m} p^j_{j_1} \frac{\partial}{\partial p^i_{j_1}} + \cdots + \sum_{j_1,\cdots,j_k=1}^{m} p^j_{j_1,\cdots,j_k} \frac{\partial}{\partial p^i_{j_1,\cdots,j_k}}) + \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial u_i}$$

and the proof of Lemma 9.2 is thereby completed.

10. We have seen that the involutive distribution $D^{(k)}$ is induced from the sheaf Θ of germs of Killing vector fields on E^n on the submanifold J^k_r of J^k. The foliation given by the orbits of $D^{(k)}$ on J^k_r has been denoted by $\mathcal{F}^k_r$.

Lemma 10.1. *The foliation $\mathcal{F}^k_r$ on J^k_r is Riemannian.*

Proof. Let G^0 be the connected component of the group of isometries on E^n and denote by $\mathcal{G}$ the Lie algebra of G^0. Then $G^0 \ni \phi$ if and only if

$$\phi_i(u) = \sum_{j=1}^{n} a_{ij}(\phi) u_j + b_i(\phi)$$

where $(a_{ij}(\phi)) \in SO(n)$ and $b_i(\phi)$ is any constant and $\mathcal{G}$ is identified with the Lie algebra of vector fields X on E^n of the form

$$X = \sum_{i=1}^{n} (\sum_{j=1}^{n} \alpha_{ij} u_j + \beta_i) \frac{\partial}{\partial u_i}$$

where (α_{ij}) is any skew-symmetric n-matrix and β_i is any constant. If we denote by $\mathcal{G}^{(k)}$ the Lie algebra of vector fields $X^{(k)}$ on $J^k_{x_0}(E^n)$ of the form

$$X^{(k)} = \sum_{i<j} \alpha_{ij} \{u_j \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial u_j} + \sum_{j_1} (p^j_{j_1} \frac{\partial}{\partial p^i_{j_1}} - p^i_{j_1} \frac{\partial}{\partial p^j_{j_1}}) + \cdots + \sum_{j_1,\cdots,j_k} (p^j_{j_1,\cdots,j_k} \frac{\partial}{\partial p^i_{j_1,\cdots,j_k}} - p^i_{j_1,\cdots,j_k} \frac{\partial}{\partial p^j_{j_1,\cdots,j_k}})\} + \sum_i \beta_i \frac{\partial}{\partial u_i},$$

then by Lemma 9.2 $\mathcal{G}$ is canonically isomorphic to $\mathcal{G}^{(k)}$. Let $\mathcal{G}_0$ be the isotropy algebra of $\mathcal{G}$ at $z_0 \in E^n$. Then there corresponds to $\mathcal{G}_0$ the Lie subalgebra $\mathcal{G}_0^{(k)}$ of $\mathcal{G}^{(k)}$ consisting of vector fields $Z^{(k)}$ on $J^k_{z_0}(E^n)$ of the form

$$\begin{aligned} Z^{(k)} = \sum_{i<j} \alpha_{ij} \{ & (u_j - u_j(z_0)) \frac{\partial}{\partial u_i} - (u_i - u_i(z_0)) \frac{\partial}{\partial u_j} \\ & + \sum_{j_1} (p^j_{j_1} \frac{\partial}{\partial p^i_{j_1}} - p^i_{j_1} \frac{\partial}{\partial p^j_{j_1}}) + \cdots \\ & + \sum_{j_1,\cdots,j_k} (p^j_{j_1,\cdots,j_k} \frac{\partial}{\partial p^i_{j_1,\cdots,j_k}} - p^i_{j_1,\cdots,j_k} \frac{\partial}{\partial p^j_{j_1,\cdots,j_k}}) \} . \end{aligned}$$

This Lie algebra $\mathcal{G}_0^{(k)}$ is isomorphic to the Lie algebra $\tilde{\mathcal{G}}_0^{(k)}$ of vector fields $\tilde{Z}^{(k)}$ on $J^k(E^n)$ $(=J^k_{x_0 z_0}(R^m, E^n))$ of the form

$$\begin{aligned} \tilde{Z}^{(k)} = \sum_{i<j} \alpha_{ij} \{ & \sum_{j_1} (p^j_{j_1} \frac{\partial}{\partial p^i_{j_1}} - p^i_{j_1} \frac{\partial}{\partial p^j_{j_1}}) + \cdots \\ & + \sum_{j_1,\cdots,j_k} (p^j_{j_1,\cdots,j_k} \frac{\partial}{\partial p^i_{j_1,\cdots,j_k}} - p^i_{j_1,\cdots,j_k} \frac{\partial}{\partial p^j_{j_1,\cdots,j_k}}) \} . \end{aligned}$$

The isotropy group G_0 of G^0 at z_0 is naturally lifted to a Lie transformation group $G_0^{(k)}$ on $J^k(E^n)$ and $\tilde{\mathcal{G}}_0^{(k)}$ is the Lie algebra of $G_0^{(k)}$. By Lemma 9.2, the distribution $D^{(k)}$ on J^k_r is just obtained from $\tilde{\mathcal{G}}_0^{(k)}$ i.e. for each $p \in J^k_r$, the space $D^{(k)}_p$ is just the space $\tilde{\mathcal{G}}^{(k)}_{0p} = \{Z^{(k)}_p$ (the value at p); $Z^{(k)} \in \tilde{\mathcal{G}}_0^{(k)}\}$ and the orbit of the component of $G_0^{(k)}$ through $p \in J^k_r$ is just the orbit of $D^{(k)}$ through p.

Since $J^k(E^n)$ admits the canonical Euclidean structure associated with the natural product structure $R^{n(1)} \times \cdots \times R^{n(k)}$, we see that $\tilde{Z}^{(k)}$ is an infinitesimal isometry on $J^k(E^n)$ with respect to the Euclidean structure and so that $\tilde{\mathcal{G}}_0^{(k)}$ is a Lie subalgebra of the Lie algebra of infinitesimal isometries on $J^k(E^n)$. Therefore the component of $G_0^{(k)}$ is a subgroup of the group of isometries on $J^k(E^n)$ i.e. acts on J^k_r as a group of isometries. By the theorem of Reinhart [5], the foliated manifold $(J^k_r, \mathcal{F}^k_r)$ is bundle-like with respect to the metric on J^k_r and $\mathcal{F}^k_r$ is a Riemannian foliation. This completes the proof.

11. Let M' be a regular submanifold of a manifold M and assume that there exists a Riemannian foliation $\mathcal{F}$ on M.

Lemma 11.1. *If M' consists of leaves of $\mathcal{F}$, $\mathcal{F}$ induces a Riemannian foliation on M'.*

Proof. Let ψ_U be a submersion of a neighbourhood U of $p \in M$ to R^q

where $q = \operatorname{codim} \mathcal{F}$ and denote by $\psi_{U'}$ the restriction of ψ_U to $U' = M' \cap U$. Then $\psi_{U'}$ is also a submersion of U' to $R^{q'}$ $(q' < q)$ because M' is a union of leaves of $\mathcal{F}$. Since M' is a regular submanifold of M, $\bar{U}' = \psi_{U'}(U')$ is also a regular submanifold of $\bar{U} = \psi_U(U)$.

Assume that (U, ψ_U) and (V, ψ_V) are two local submersions of $\mathcal{F}$ with $U \cap V \neq \phi$ and $\psi_U = g_{UV} \circ \psi_V$ on $U \cap V$ where g_{UV} is an isometry of $\psi_V(U \cap V)$ onto $\psi_U(U \cap V)$. Since g_{UV} carries $\psi_{V'}(U' \cap V')$ onto $\psi_{U'}(U' \cap V')$ and since $\psi_{V'}(U' \cap V')$ (resp. $\psi_{U'}(U' \cap V')$) is a regular submanifold of $\psi_V(U \cap V)$ (resp. $\psi_U(U \cap V)$), the restriction $g_{U'V'}$ of g_{UV} to $\psi_{V'}(U' \cap V')$ is also an isometry. This shows that $\mathcal{F}$ induces a Riemannian foliation on M'.

By combining Lemma 10.1 with Lemma 11.1, Theorem 8.1 is immediately obtained.

12. For the sheaf Θ of germs of Killing vector fields on E^n, denote by $\mathcal{N}(\Theta)$ the normalizer of Θ in the sheaf of germs of all local vector fields on E^n.

Lemma 12.1. *X is a local cross-section of $\mathcal{N}(\Theta)$ if and only if*

$$X = \sum_i \left(\bar{c}u_i + \sum_{j \neq i} \alpha_{ij} u_j + \beta_i\right) \frac{\partial}{\partial u_i}$$

where α_{ij}, β_i and $\bar{c}$ are constants and the $n \times n$ matrix (α_{ij}) is skew-symmetric.

Proof. Y is a local cross-section of Θ if and only if

$$Y = \sum_i \left(\sum_{j \neq i} \alpha_{ij} u_j + \beta_i\right) \frac{\partial}{\partial u_i}$$

where α_{ij} and β_i are constants and $\alpha_{ij} = -\alpha_{ji}$. We set $X = \sum_h A_h(u) \frac{\partial}{\partial u_h}$ and calculate the blacket $[X, Y]$. Then

$$[X, Y] = \sum_i \left\{\sum_{j \neq i} \alpha_{ij} A_j(u) - \sum_h \left(\sum_{j \neq h} \alpha_{hj} u_j + \beta_h\right) \frac{\partial A_i}{\partial u_h}\right\} \frac{\partial}{\partial u_i}.$$

On the other hand since $[X, Y]$ is a local cross-section of Θ, $[X, Y]$ is written by

$$[X, Y] = \sum_i \left(\sum_{j \neq i} \gamma_{ij} u_j + \delta_i\right) \frac{\partial}{\partial u_i}.$$

Thus we get

$$\sum_{j \neq i} \alpha_{ij} A_j(u) - \sum_h \left(\sum_{j \neq h} \alpha_{hj} u_j + \beta_h\right) \frac{\partial A_i}{\partial u_h} = \sum_{j \neq i} \gamma_{ij} u_j + \delta_i. \tag{12.1}$$

For any skew-symmetric $n \times n$ matrix (α_{ij}) and any vector $(\beta_1, \cdots, \beta_n)$, the left-hand side of (12.1) is always reduced to the form of the right-hand side. This means that $A_j(u)$ must be of the form

$$A_j(u) = \sum_l \lambda_{jl} u_l + \eta_j$$

where λ_{jl} and η_j are constants. Then from (12.1), we get

$$\sum_{j \neq i} \alpha_{ij} \lambda_{jl} - \sum_{j \neq l} \alpha_{jl} \lambda_{ij} = \gamma_{il} . \tag{12.2}$$

In particular, since $\gamma_{ii}=0$, we get

$$\sum_{j \neq i} (\alpha_{ij} \lambda_{ji} - \alpha_{ji} \lambda_{ij}) = 0 \quad \text{i.e.} \tag{12.3}$$

$$\sum_{j<i} \alpha_{ij} (\lambda_{ij} + \lambda_{ji}) = 0 \tag{12.4}$$

from which we get $\lambda_{ij} = -\lambda_{ji}$ $(i \neq j)$ because α_{ij} $(i<j)$ is any constant.

Furthermore we shall show that $\lambda_{ii}=\lambda_{jj}$ for any i and j. The relation (12.2) is written by

$$\alpha_{il}(\lambda_{ll} - \lambda_{ii}) + \sum_{j \neq i,l} (\alpha_{ij} \lambda_{jl} - \alpha_{jl} \lambda_{ij}) = \gamma_{il} \tag{12.5}$$

and exchanging the role of i and l, we get

$$\alpha_{li}(\lambda_{ii} - \lambda_{ll}) + \sum_{j \neq i,l} (\alpha_{lj} \lambda_{ji} - \alpha_{ji} \lambda_{lj}) = \gamma_{li} . \tag{12.6}$$

Since $\alpha_{ij} = -\alpha_{ji}$, $\gamma_{ij} = -\gamma_{ji}$ and $\lambda_{ij} = -\lambda_{ji}$, (12.6) is written by

$$\alpha_{il}(\lambda_{ii} - \lambda_{ll}) + \sum_{j \neq i,l} (\alpha_{ij} \lambda_{jl} - \alpha_{jl} \lambda_{ij}) = \gamma_{il} . \tag{12.7}$$

From (12.5) and (12.7), we get $\alpha_{il}(\lambda_{ll} - \lambda_{ii})=0$ $(i \neq l)$. This proves that $\lambda_{ii}=\bar{c}$ for any i because α_{il} $(i<l)$ is any constant. This completes the proof of Lemma 12.1.

Denote by $C^k(\mathcal{P}, r)$ the set of differential equations $\mathcal{E} \subset J^k(R^m, E^n)$ such that $\mathcal{A}(\mathcal{E})=\mathcal{P}$ and $S_r^k(\mathcal{E})$ is a regular submanifold of J_r^k.

Theorem 12.2. *We can naturally associate to any differential equation $\mathcal{E} \in C^k(\mathcal{P}, r)$ a $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}) \rangle$-subbundle $P_r(\mathcal{E})$ on $S_r^k(\mathcal{E})$. This correspondence is compatible with respective isomorphisms in the following sense: If ϕ is a local isomorphism of $\mathcal{E}_1$ to $\mathcal{E}_2$ which is near the identity and satisfies $\phi(z_0)=z_0$, then ϕ induces an isomorphism ϕ^k of the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}_1) \rangle$-subbundle $P_r(\mathcal{E}_1)$ on $S_r^k(\mathcal{E}_1)$ with the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}_2) \rangle$-subbundle $P_r(\mathcal{E}_2)$ on $S_r^k(\mathcal{E}_2)$. If* codim $\mathcal{F}_r^k(\mathcal{E}) \geq 3$, *the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}) \rangle$-subbundle $P_r(\mathcal{E})$ is prolonged to a generalized Lie foliation $(\mathcal{F}_r(\mathcal{E}), \omega_r(\mathcal{E}))$ on $M_r(\mathcal{E})=(P_r(\mathcal{E}))^{(2)}$. The correspondence $\mathcal{E} \to (\mathcal{F}_r(\mathcal{E}), \omega_r(\mathcal{E}))$ is also compatible with respective isomorphism where $\omega_r(\mathcal{E})$ is the basic* *form of* $P_r(\mathcal{E})$.

Proof. By Theorem 8.1 $D^{(k)}$ induces a Riemannian foliation $\mathcal{F}_r^k(\mathcal{E})$ on $S_r^k(\mathcal{E})$. Then there is associated to $(S_r^k(\mathcal{E}), \mathcal{F}_r^k(\mathcal{E}))$ an $\langle O(q), \mathcal{F}_r^k(\mathcal{E})\rangle$-subbundle $P_r'(\mathcal{E})$ on $S_r^k(\mathcal{E})$ where $q = \operatorname{codim} \mathcal{F}_r^k(\mathcal{E})$. Thus we can associate to $\mathcal{E}$ the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E})\rangle$-subbundle $P_r(\mathcal{E}) \supset P_r'(\mathcal{E})$ on $S_r^k(\mathcal{E})$. We shall show that this structure is an invariant of local isomorphism classes of differential equations in $C^k(\mathcal{P}, r)$ in the sense stated above.

If ϕ is a local isomorphism of $\mathcal{E}_1$ to $\mathcal{E}_2$ with $\phi(z_0) = z_0$, then $\phi \mathcal{A}(\mathcal{E}_1)\phi^{-1} = \mathcal{A}(\mathcal{E}_2)$ on a neighbourhood of z_0. Therefore ϕ induces a transformation ϕ^k on $J^k(E^n)$ such that $\phi_*^k D^{(k)} = D^{(k)}$ and $\phi^k(S_r^k(\mathcal{E}_1)) = S_r^k(\mathcal{E}_2)$. This shows that ϕ^k is an isomorphism of $\mathcal{F}_r^k(\mathcal{E}_1)$ to $\mathcal{F}_r^k(\mathcal{E}_2)$.

Since $\mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2) = \mathcal{P}$, ϕ belongs to the normalizer $\mathcal{N}(\mathcal{P})$ of $\mathcal{P}$ in the pseudogroup of all local transformations on E^n. We shall prove that ϕ^k is a conformal foliation isomorphism.

Since $\phi_* \Theta = \Theta$ and ϕ is near the identity, there exists a local cross-section Z of $\mathcal{N}(\Theta)$ such that $\phi_{t_0} = \phi$ where ϕ_t is the local 1-parameter group of local transformations generated by Z. By Lemma 12.1, Z is written by

$$Z = \sum_i (\bar{c}u_i + \sum_{j \neq i} \alpha_{ij} u_j + \beta_i) \frac{\partial}{\partial u_i} .$$

This shows that ϕ is a local transformation of the form

$$\phi_i(u) = cu_i + \sum_j a_{ij} u_j + b_i$$

where a_{ij}, b_i and c are constants and the $n \times n$ matrix (a_{ij}) belongs to $SO(n)$. Then

$$\frac{\partial^l \phi_i(u)}{\partial x_{j_1}^{l_1} \cdots \partial x_{j_p}^{l_p}} = \sum_{h=1}^n \frac{\partial \phi_i(u)}{\partial u_h} \frac{\partial^l u_h}{\partial x_{j_1}^{l_1} \cdots \partial x_{j_p}^{l_p}} \quad (l = l_1 + \cdots + l_p) .$$

This shows that ϕ^k is a transformation on $J^k(E^n)$ such that, if we set

$$\overline{p_{j_1,\cdots,j_p}^i} = p_{j_1,\cdots,j_p}^i(\phi^k) ,$$

we have

$$\overline{p_{j_1,\cdots,j_p}^i} = \sum_{h=1}^n r_{ih} p_{j_1,\cdots,j_p}^h$$

where $r_{ij} = a_{ij} + c\delta_{ij}$. Therefore ϕ^k is a conformal transformation of the Euclidean space $J^k(E^n)$. Since $\mathcal{F}_r^k(\mathcal{E}_i)$ is a Riemannian and so conformal foliation with respect to the Riemannian metric on $S_r^k(\mathcal{E}_i)$ induced from the Euclidean metric on $J^k(E^n)$, ϕ^k is a foliation isomorphism as a conformally foliated structure.

Now for $\mathcal{E}\in\mathcal{C}^k(\mathcal{P}, r)$ with codim $\mathcal{F}_r^k(\mathcal{E})\geq 3$, by the 2-nd prolongation of $P_r(\mathcal{E})$ we get a $\langle CO(q)^{(2)}, (\mathcal{F}_r^k(\mathcal{E}))^{(1)}\rangle$-subbundle $P_r(\mathcal{E})^{(2)}$ on $P_r(\mathcal{E})^{(1)}$ and we know that $CO(q)^{(2)}=\{e\}$. Therefore by Proposition 5.4 the foliation $(\mathcal{F}_r^k(\mathcal{E}))^{(2)}$ on $P_r(\mathcal{E})^{(2)}$ is a generalized Lie foliation $(\mathcal{F}_r(\mathcal{E}), \omega_r(\mathcal{E}))$ where $\mathcal{F}_r(\mathcal{E})=(\mathcal{F}_r^k(\mathcal{E}))^{(2)}$ and the 1-form $\omega_r(\mathcal{E})$ is given by the basic form $\omega_r^{(2)}$ on $P_r(\mathcal{E})^{(2)}$. Since ϕ^k is an isomorphism of the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}_1)\rangle$-subbundle $P_r(\mathcal{E}_1)$ with the $\langle CO(q), \mathcal{F}_r^k(\mathcal{E}_2)\rangle$-subbundle $P_r(\mathcal{E}_2)$, by Proposition 5.3 ϕ^k induces the isomorphism $(\phi^k)^{(1)}$ of $\langle CO(q)^{(2)}, (\mathcal{F}_r^k(\mathcal{E}_1))^{(1)}\rangle$-subbundle $P_r(\mathcal{E}_1)^{(2)}$ to $\langle CO(q)^{(2)}, (\mathcal{F}_r^k(\mathcal{E}_2))^{(1)}\rangle$-subbundle $P_r(\mathcal{E}_2)^{(2)}$ i.e. if codim $\mathcal{F}_r^k(\mathcal{E}_i)\geq 3$, $(\phi^k)^{(2)}$ is an isomorphism of $(\mathcal{F}_r(\mathcal{E}_1), \omega_r(\mathcal{E}_1))$ to $(\mathcal{F}_r(\mathcal{E}_2), \omega_r(\mathcal{E}_2))$. This completes the proof of Theorem 12.2.

13. Let us give an example of a differential equation $\mathcal{E}$ such that codim $\mathcal{F}_r^k(\mathcal{E})>2$.

Let $\mathcal{E}\subset J^2(R^m, E^n)$ be a differential equation defined by

$$\rho_1=\lambda_1 \text{ and } \rho_2=\lambda_2 \tag{13.1}$$

and with $\mathcal{A}(\mathcal{E})=\mathcal{P}$ where $\rho_1=\sum_{i=1}^{n}\left(\frac{\partial u_i}{\partial x_1}\right)^2$, $\rho_2=\sum_{i=1}^{n}\left(\frac{\partial^2 u_i}{\partial x_1^2}\right)^2$, $\lambda_1(x)=\rho_1(j_x^1(f))$ and $\lambda_2(x)=\rho_2(j_x^2(f))$.

By regarding $x_2, \cdots, x_m$ as parameters, (13.1) is a system of ordinary differential equations. Assume that $n\geq 3$ and $p_1^1(p)\neq 0$ for any $p\in\mathcal{E}$. Then it is easy to check that the differential equation $\mathcal{E}$ is defined by

$$\tag{13.2}\left\{\begin{aligned}&\left(\frac{\lambda_1'(t)}{2}-\sum_{i=2}^{n}\frac{d^2u_i}{dt^2}\frac{du_i}{dt}\right)^2\\ &\qquad=\left(\lambda_1(t)-\sum_{i=2}^{n}\left(\frac{du_i}{dt}\right)^2\right)\left(\lambda_2(t)-\sum_{i=2}^{n}\left(\frac{d^2u_i}{dt^2}\right)^2\right),\\ &\left(\frac{du_1}{dt}\right)^2=\lambda_1(t)-\sum_{i=3}^{n}\left(\frac{du_i}{dt}\right)^2\end{aligned}\right.$$

where $t\equiv x_1$. For any $n-2$ functions $u_3(t, x_2, \cdots, x_m), \cdots, u_n(t, x_2, \cdots, x_m)$, the differential equation (13.2) admits a solution $(u_1(t, x_2, \cdots, x_m), u_2(t, x_2, \cdots, x_m))$. We shall show that there always exists a solution (u_1, u_2) such that u_1 is not constant which means that $(u_1, u_2, \cdots, u_n)\in\mathcal{S}(\mathcal{E})$.

Given any functions $u_3, \cdots, u_n$, any solution u_2 of the differential equation

$$\left(\frac{du_2}{dt}\right)^2+\sum_{i=3}^{n}\left(\frac{du_i}{dt}\right)^2-\lambda_1(t)=0 \tag{13.3}$$

is also a solution of the differential equation

$$\frac{d^2u_2}{dt^2}\frac{du_2}{dt}+\sum_{i=3}^{n}\frac{d^2u_i}{dt^2}\frac{du_i}{dt}-\frac{\lambda_1'(t)}{2}=0. \tag{13.4}$$

Therefore the solution space $\mathcal{S}'$ of (13.3) is contained in the solution space $\mathcal{S}''$ of the differential equation

$$\begin{aligned}\left(\frac{d^2u_2}{dt^2}\frac{du_2}{dt}+\sum_{i=3}^{n}\frac{d^2u_i}{dt^2}\frac{du_i}{dt}-\frac{\lambda_1'(t)}{2}\right)^2 &= \left(\left(\frac{du_2}{dt}\right)^2+\sum_{i=3}^{n}\left(\frac{du_i}{dt}\right)^2-\lambda_1(t)\right)\\ &\quad\times\left(\left(\frac{d^2u_2}{dt^2}\right)^2+\sum_{i=3}^{n}\left(\frac{d^2u_i}{dt^2}\right)^2-\lambda_2(t)\right).\end{aligned}$$

Choose any solution $u_2\in\mathcal{S}''\backslash\mathcal{S}'$ and consider the differential equation

$$\left(\frac{du_1}{dt}\right)^2=\lambda_1(t)-\sum_{i=2}^{n}\left(\frac{du_i}{dt}\right)^2. \tag{13.5}$$

Then because $u_2\notin\mathcal{S}'$, $\lambda_1(t)-\sum_{i=2}^{n}\left(\frac{du_i}{dt}\right)^2\neq 0$. Therefore any solution u_1 of (13.5) is not constant.

Now the involutive distribution $D^{(k)}$ on $J^k(E^n)$ is generated by the vector fields

$$\begin{aligned}Z_{ij}^{(k)} &= \sum_{j_1}\left(p^j_{j_1}\frac{\partial}{\partial p^i_{j_1}}-p^i_{j_1}\frac{\partial}{\partial p^j_{j_1}}\right)+\cdots\\ &\quad+\sum_{j_1,\cdots,j_k}\left(p^j_{j_1,\cdots,j_k}\frac{\partial}{\partial p^i_{j_1,\cdots,j_k}}-p^i_{j_1,\cdots,j_k}\frac{\partial}{\partial p^j_{j_1,\cdots,j_k}}\right).\end{aligned}$$

Therefore the dimension r of the orbit of $D^{(k)}$ through a point $p\in J^k(E^n)$ is at most $\frac{n(n-1)}{2}$.

On the other hand, for any functions $u_3, \cdots, u_n$, we have a solution $(u_1, u_2, u_3, \cdots, u_n)$ of $\mathcal{E}$. Then by considering the $(k-2)$-th prolongation $p^{k-2}(\mathcal{E})$ of $\mathcal{E}$ and by choosing an open subvariety $\mathcal{E}^k$ of $p^{k-2}(\mathcal{E})$ such that $S_r^k(\mathcal{E}^k)$ is a regular submanifold of J_r^k, we have the inequality $\dim S_r^k(\mathcal{E}^k)+m+n> \dim J^k(R^m, E^{n-2})$ and by taking an integer k such that $\dim J^k(R^m, E^{n-2})>\frac{n(n-1)}{2}+m+n+2$ we get $\dim S_r^k(\mathcal{E}^k)>\frac{n(n-1)}{2}+2>r+2$ i.e. $\operatorname{codim}\mathcal{F}_r^k(\mathcal{E}^k)>2$. Therefore there corresponds to $\mathcal{E}^k$ a generalized Lie foliation $(\mathcal{F}_r(\mathcal{E}^k), \omega_r(\mathcal{E}^k))$ on a manifold $M(\mathcal{E}^k)$ in the sense of Theorem 12.2.

14. In this section we refer to foliations on a subvariety of a manifold. These foliations naturally correspond to involutive systems of differential equations of some regular type.

Let A be a subvariety of a manifold M i.e. a subset of M such that, for each point $p_0 \in A$, there exist a neighbourhood U of p_0 in M and smooth functions $f_1, \cdots, f_\alpha$ on U satisfying $A \cap U = \{p \in U; f_1(p) = \cdots = f_\alpha(p) = 0\}$. A subvariety A is said to be almost regular if the set A' of points $p \in A$ such that $A \cap U$ is a submanifold of M for a neighbourhood U of p is open and dense in A. Let $A(d)$ be the union of d-dimensional connected components of A'. Then A' is the disjoint union $\bigcup_d A(d)$.

Definition 14.1. A family of connected subsets $\mathcal{F} = \{L_\beta\}_{\beta \in K}$ of an almost regular subvariety A of a manifold is called a foliation on A if the restriction $\mathcal{F}(d)$ of $\mathcal{F}$ to $A(d)$ is a foliation on the manifold $A(d)$ for each d.

Definition 14.2. Let A and B be almost regular subvarieties of a manifold and let $\mathcal{F}^A$ and $\mathcal{F}^B$ be foliations on A and B, respectively. A homeomorphism $\phi: A \to B$ is called an isomorphism of $\mathcal{F}^A$ to $\mathcal{F}^B$ if it satisfies the following conditions:

(1) $\phi(A(d)) = B(d)$ and the restriction ϕ_d of ϕ to $A(d)$ is smooth for each d.

(2) ϕ_d is an isomorphism of $\mathcal{F}^A(d)$ to $\mathcal{F}^B(d)$ for each d.

Now let us consider differential equations in $J^k(R^m, E^n)$.

Definition 14.3. A differential equation $\mathcal{E} \subset J^k(R^m, E^n)$ is said to be pseudo-involutive if, for any $p \in \mathcal{E}$, there exists a solution $s \in \mathcal{S}(\mathcal{E})$ such that $j^k_x(s) = p$.

If $\mathcal{E} \subset J^k(R^m, E^n)$ is pseudo-involutive, then for each $p_0 \in \mathcal{E}$, $\mathcal{E}$ is defined on a neighbourhood $\tilde{U}$ of p_0 in $J^k(R^m, E^n)$ by smooth functions $F_1, \cdots, F_\alpha$ and, if $U = \tilde{U} \cap J^k(E^n) \neq \phi$, $S^k(\mathcal{E})$ is the common zeros on U of the smooth functions $f_1, \cdots, f_\alpha$ where $f_i = F_i | U$. Therefore $S^k(\mathcal{E})$ is a subvariety of $J^k(E^n)$.

Corollary 14.1. *Let $\mathcal{E}$ be a pseudo-involutive differential equation in $J^k(R^m, E^n)$ with $\mathcal{A}(\mathcal{E}) = \mathcal{P}$ and set $A_r = S^k_r(\mathcal{E})$. Then A_r is a subvariety of J^k_r. If A_r is almost regular, then there is induced a foliation $\mathcal{F}_r$ on A_r such that $\mathcal{F}_r(d)$ is a Riemannian foliation on $A_r(d)$ for each r and d.*

Proof. Since $S^k(\mathcal{E})$ is a subvariety of $J^k(E^n)$ and J^k_r is open in $J^k(E^n)$, $S^k_r(\mathcal{E}) = S^k(\mathcal{E}) \cap J^k_r$ is also a subvariety of J^k_r.

Let γ^k denote the projection of $J^k(R^m, E^n) = R^m \times E^n \times R^{n(1)} \times \cdots \times R^{n(k)}$

onto $J^k(E^n)=R^{n(1)}\times \cdots \times R^{n(k)}$ defined by $r^k(x, z, p)=p$ where $(x, z)\in R^m\times E^n$ and $p\in R^{n(1)}\times \cdots \times R^{n(k)}$. For any open subset K of $A_r=S_r^k(\mathcal{E})$, $\mathcal{E}_K=\mathcal{E}\cap \tilde{K}$ where $\tilde{K}=(r^k)^{-1}(K)$ is an open subset of $\mathcal{E}$ and $S_r^k(\mathcal{E}_K)=K$. In particular for $K=A_r(d)$ we have a pseudo-involutive differential equation $\mathcal{E}_0$ being open in $\mathcal{E}$ and satisfying $S_r^k(\mathcal{E}_0)=A_r(d)$. Therefore $D^{(k)}$ induces a foliation $\mathcal{F}_r(d)$ on $A_r(d)$ of codimension $d-r$ such that any leaf of $\mathcal{F}_r(d)$ is an orbit of $D^{(k)}$. Thus if we denote by $\mathcal{F}_r$ the family of orbits of $D^{(k)}$ in A_r, $\mathcal{F}_r$ is a foliation on A_r. By Theorem 8.1, $\mathcal{F}_r(d)$ is a Riemannian foliation on $A_r(d)$. This completes the proof of Corollary 14.1.

Corollary 14.2. *Let $\mathcal{E}^i$ ($i=1, 2$) be pseudo-involutive differential equations in $J^k(R^m, E^n)$ with $\mathcal{A}(\mathcal{E}^i)=\mathcal{P}$ and set $A_r^i=S_r^k(\mathcal{E}^i)$. Assume that A_r^i, $i=1, 2$, are almost regular. If there exists a local isomorphism ϕ of $\mathcal{E}^1$ to $\mathcal{E}^2$ satisfying $\phi(z_0)=z_0$ and being near the identity, then ϕ induces a foliation isomorphism ϕ_r of $(A_r^1, \mathcal{F}_r^1)$ to $(A_r^2, \mathcal{F}_r^2)$ such that the restriction of ϕ_r to $A_r^1(d)$ is a conformal foliation isomorphism of $(A_r^1(d), \mathcal{F}_r^1(d))$ to $(A_r^2(d), \mathcal{F}_r^2(d))$ for each r and d.*

Proof. By Corollary 14.1, on the subvariety $A_r^i=S_r^k(\mathcal{E}^i)$ there is induced a foliation $\mathcal{F}_r^i$ such that the restriction $\mathcal{F}_r^i(d)$ of $\mathcal{F}_r^i$ to $A_r^i(d)$ is Riemannian and $A_r^i(d)=S_r^k(\mathcal{E}_0^i)$ for some differential equation $\mathcal{E}_0^i$ which is open in $\mathcal{E}^i$. ϕ induces a homeomorphism ϕ_r of A_r^1 to A_r^2 and since ϕ is also a local isomorphism of $\mathcal{E}_0^1$ to $\mathcal{E}_0^2$ at $(z_0, z_0)\in E^n\times E^n$, it induces a foliation isomorphism of $(S_r^k(\mathcal{E}_0^1), \mathcal{F}_r^k(\mathcal{E}_0^1))$ to $(S_r^k(\mathcal{E}_0^2), \mathcal{F}_r^k(\mathcal{E}_0^2))$ which is conformal by Theorem 12.2. Since $(S_r^k(\mathcal{E}_0^i), \mathcal{F}_r^k(\mathcal{E}_0^i))=(A_r^i(d), \mathcal{F}_r^i(d))$, this completes the proof.

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