

# The Completion of the Maximal $\text{Op}^*$ -Algebra on a Frechet Domain

By

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## Abstract

This paper investigates the completion of the maximal  $\text{Op}^*$ -algebra  $L^+(D)$  of (possibly) unbounded operators on a dense domain  $D$  in a Hilbert space. It is assumed that  $D$  is a Frechet space with respect to the graph topology. Let  $D^+$  denote the strong dual of  $D$ , equipped with the complex conjugate linear structure. It is shown that the completion of  $L^+(D)$  (endowed with the uniform topology) is the space of continuous linear operators  $\mathcal{L}(D, D^+)$ . This space is studied as an ordered locally convex space with an involution and a partially defined multiplication. A characterization of bounded subsets of  $D$  in terms of self-adjoint operators is given. The existence of special factorizations for several kinds of operators is proved. It is shown that the bounded operators are uniformly dense in  $L^+(D)$ .

## § 1. Introduction

Non-normable topological  $*$ -algebras satisfying various completeness conditions have been studied in several papers (see, e. g., [7, 8, 10, 12, 22, 23, 31, 34]). However, these conditions are not fulfilled for the maximal  $*$ -algebra  $L^+(D)$  of (possibly unbounded) operators on a dense linear subspace  $D$  of some Hilbert space  $H$  (for precise definitions, see Section 2). On the other hand,  $L^+(D)$  is one of the most important unbounded operator algebras because it contains all  $*$ -algebras of operators on a fixed domain  $D$ .

It is the aim of this paper to study the completion of  $L^+(D)$  with respect to the uniform topology. We assume throughout that  $D$  is a Frechet space in the topology defined by the graph norms of operators belonging to  $L^+(D)$ . However, some of the results can be obtained for more general domains  $D$  by the same proofs (see Remark 3 after Proposition 3.8 and the remarks after Proposition 5.1 and Corollary 5.6).

Among others we show that the completion of  $L^+(D)$  is the space of continuous linear operators  $\mathcal{L}(D, D^+)$ . Here  $D^+$  denotes the strong dual of  $D$ , equipped with the complex conjugate linear structure. This completion is not

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an algebra if  $D \neq H$ . However, it has the structure of an ordered locally convex space with continuous involution and with a partially defined multiplication.

Note that the question whether or not  $\mathcal{L}(D, D^+)$  is the completion of  $L^+(D)$  arose in [24] in connection with the study of the time development of thermodynamical systems in quantum statistics. It was explained in [25] that the problem of defining products on  $\mathcal{L}(D, D^+)$  is connected with quantization procedures, if  $D$  is the Schwartz space  $\mathcal{S}$  of test functions. Our definition of the partial product is more general than that of [24, 25]. However, it is closely related to the product of operators on partial inner product spaces which was defined in [3]. Linear spaces with a partially defined multiplication were previously considered also in [4, 5, 6, 11].

The pattern of the paper is as follows. In Section 2, we recall some definitions, notations, and some known or easy results. In particular, we endow the space  $\mathcal{L}(D, D^+)$  with the topology of uniform convergence on bounded sets.  $\mathcal{L}(D, D^+)$  contains both the algebra  $L^+(D)$  and the algebra  $\mathcal{C}(H, H)$  which is isomorphic to the algebra of all norm continuous linear operators on  $H$ . In Section 3, we define a partial multiplication on  $\mathcal{L}(D, D^+)$  which generalizes the familiar multiplication of  $L^+(D)$  and  $\mathcal{C}(H, H)$ . Moreover, we give examples which indicate some of the difficulties connected with the definition of such a partial multiplication. In Section 4, we characterize bounded subsets of  $D$  in terms of self-adjoint operators. Some applications of this characterization are given. In Section 5, we prove the existence of special factorizations for several kinds of operators. In Section 6, we show that  $\mathcal{L}(D, D^+)$  is the completion of  $L^+(D)$ .

The study of the space  $\mathcal{L}(D, D^+)$  will be continued in [21]. In particular, we show there that  $\mathcal{L}(D, D^+)$  is the second strong dual of its subspace of completely continuous operators. In [20, 33], the methods of the present paper are applied to the investigation of closed ideals in  $L^+(D)$ .

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### §2. Notations and Preliminary Results

In this section, we fix some definitions and notations. Moreover, we collect some well-known or simple facts for later use.

Suppose that  $D$  is a dense linear subspace of a complex Hilbert space  $H$ . We denote the norm, the unit ball, and the scalar product of  $H$  by  $\|\cdot\|$ ,  $U_H$ , and  $\langle \cdot, \cdot \rangle$ , respectively. We assume the scalar product to be linear in the

second argument. For an operator  $A$  on  $H$ , let  $\bar{A}$ ,  $D(A)$ , and  $\|A\|$  denote the closure, domain, and the norm of  $A$  (provided the later exists), respectively.

The following definition was introduced in [22]. Let

$$L^+(D) = \{A \in \text{End}(D) : D \subset D(A^*) \text{ and } A^*(D) \subset D\}.$$

Then  $L^+(D)$  is a \*-algebra of closable operators with involution  $A \rightarrow A^+ := A^* \upharpoonright D$ .

We endow  $D$  with the locally convex topology defined by the system of seminorms  $\varphi \rightarrow \|A\varphi\|$  where  $A \in L^+(D)$ . Throughout this paper, we assume that  $D$  is a Frechet space. In this case we simply say that  $D$  is a Frechet domain. Then  $D$  is reflexive [9, 29]. Furthermore, there exists a sequence  $(A_n)$  in  $L^+(D)$  such that the following conditions are satisfied (see, e.g., [6, 22]):

- a) The topology of  $D$  is generated by the sequence of seminorms  $(\|A_n(\cdot)\|)$ . Moreover,  $D = \bigcap D(\bar{A}_n)$ .
- b) For each  $A \in L^+(D)$  there exists  $n \in \mathbb{N}$  such that  $|\langle A\varphi, \varphi \rangle| \leq \langle A_n\varphi, \varphi \rangle$  for all  $\varphi \in D$ .
- c)  $A_1\varphi = \varphi$ ,  $\langle A_n{}^2\varphi, \varphi \rangle \leq \langle A_{n+1}\varphi, \varphi \rangle$ , and  $\|A_n\varphi\| \leq \|A_{n+1}\varphi\|$  for all  $\varphi \in D$ .

We fix a sequence  $(A_n)$  satisfying the conditions a), b), and c).

Let  $D'$  denote the strong dual of  $D$ . Replacing the multiplication with scalars in a locally convex space by the mapping  $(\lambda, x) \rightarrow \bar{\lambda} \cdot x$ , we obtain a new locally convex space which is called the complex conjugate space. Let  $\bar{D}$  and  $D^+$  denote the complex conjugate spaces of  $D$  and  $D'$ , respectively.

We always identify  $f \in H$  with the linear functional  $\langle f, \cdot \rangle$  on  $D$ . Then we have the continuous inclusions  $D \subset H \subset D^+$ . Elements of  $D$  are denoted by greek letters  $\varphi, \psi, \dots$ . Elements of  $D^+$  are denoted by  $f, g, h, \dots$ , or  $\langle f, \cdot \rangle, \langle g, \cdot \rangle, \langle h, \cdot \rangle, \dots$ . The complex conjugate number of  $\langle f, \varphi \rangle$  is denoted by  $\langle \varphi, f \rangle$ . The pair of locally convex spaces  $(\bar{D}, D^+)$  is a reflexive pairing with respect to the bilinear functional

$$\bar{D} \times D^+ \ni (\varphi, f) \rightarrow \langle \varphi, f \rangle.$$

Note that  $D$  is a dense linear subspace of  $D^+$  because it is weakly dense.

If  $E$  and  $F$  are locally convex spaces, we denote by  $\mathcal{L}(E, F)$  the linear space of all continuous linear operators mapping  $E$  into  $F$ . We use the abbreviation  $\mathcal{L}$  for the space  $\mathcal{L}(D, D^+)$  equipped with the topology of uniform convergence on bounded sets. This topology is generated by the system of seminorms

$$q_M(T) = \sup\{|\langle T\varphi, \psi \rangle| : \varphi, \psi \in M\},$$

where  $M$  runs through the system of bounded subsets of  $D$ .

If  $T \in \mathcal{L}$ , then the adjoint operator  $T'$  belongs to  $\mathcal{L}(\bar{D}, D')$  and satisfies  $(T'\varphi)(\psi) = \langle \varphi, T\psi \rangle$  for all  $\varphi \in \bar{D}$  and  $\psi \in D$ . Note that  $D$  and  $\bar{D}$  coincide as real linear topological spaces. The same is true for  $D'$  and  $D^+$ . Hence, there exists a unique operator  $T^+ \in \mathcal{L}$  satisfying

$$\langle T^+\varphi, \psi \rangle = (T'\varphi)(\psi) = \langle \varphi, T\psi \rangle$$

for all  $\varphi, \psi \in D$ .

The mapping  $T \rightarrow T^+$  is an involution of  $\mathcal{L}$ . An element  $T \in \mathcal{L}$  is said to be hermitian if  $T = T^+$ . Let  $\mathcal{L}_h$  denote the set of all hermitian elements of  $\mathcal{L}$ . The formula

$$T = \frac{1}{2}(T + T^+) - \frac{1}{2}i(iT - iT^+)$$

shows that  $\mathcal{L} = \mathcal{L}_h + i\mathcal{L}_h$ .

We define a partial order relation on  $\mathcal{L}_h$  as follows:

$$T_1 \leq T_2 \text{ if and only if } \langle T_1\varphi, \varphi \rangle \leq \langle T_2\varphi, \varphi \rangle \text{ for all } \varphi \in D.$$

Recall that  $D$  is a Frechet space and hence bornological. Therefore the following proposition follows from the theory of locally convex spaces (see e. g., [17], § 40, 2).

**Proposition 2.1.** *For a bilinear form  $t(\varphi, \psi)$  defined on  $\bar{D} \times D$ , the following properties are equivalent:*

- a) *There exists  $T$  in  $\mathcal{L}$  such that  $t(\varphi, \psi) = \langle T\varphi, \psi \rangle$  for all  $\varphi, \psi \in D$ .*
- b) *There exists  $n \in \mathbb{N}$  such that  $|t(\varphi, \psi)| \leq \|A_n\varphi\| \|A_n\psi\|$  for all  $\varphi, \psi \in D$ .*
- c) *For any bounded subset  $M \subset D$ , the form  $t$  is bounded on  $M \times M$ .*

Moreover, the sets

$$\mathfrak{B}_n = \{T \in \mathcal{L} : |\langle T\varphi, \psi \rangle| \leq \|A_n\varphi\| \|A_n\psi\| \text{ for all } \varphi, \psi \in D\}$$

form a fundamental system of bounded subsets of  $\mathcal{L}$ .

In particular, the space  $\mathcal{L}$  and the space of continuous sesquilinear forms, considered in [6], are isomorphic as linear spaces.

In the sequel, we are concerned with locally convex spaces  $E$  fulfilling the condition:

- d) *There are continuous inclusions  $D \subset E$  and  $E \subset D^+$ .*

If  $E$  and  $F$  are such spaces, we define

$$\mathcal{C}(E, F) = \{T \in \mathcal{L} : \text{There exists } S \in \mathcal{L}(E, F) \\ \text{such that } T\varphi = S\varphi \text{ for all } \varphi \in D\}.$$

We abbreviate  $\mathcal{C}(D^+, D)$  by  $\mathcal{C}$ . If  $T \in \mathcal{C}(D^+, D^+)$ ,  $T \in \mathcal{C}(D^+, H)$ , or  $T \in \mathcal{C}(D^+, D)$ , then the continuous extension of  $T$  is denoted by  $\tilde{T}$ .

From now on, we regard  $L^+(D)$  as a subspace of  $\mathcal{L}$ . This is possible since  $L^+(D) \subset \mathcal{L}(D, D)$  by [22].

We wanted to remark that our definition of the involution on  $\mathcal{L}$  coincides for operators in  $L^+(D)$  with the familiar definition of the involution in  $L^+(D)$ .

We always equip the spaces  $\mathcal{C}(E, F)$  and  $L^+(D)$  with the topology induced by  $\mathcal{L}$ . On  $L^+(D)$ , this topology coincides with the uniform topology  $\tau_D$  defined in [22].

The following proposition was formulated in [23, 24].

**Proposition 2.2.**  $L^+(D) = \mathcal{C}(D, D) \cap \mathcal{C}(D^+, D^+)$ .

*Proof.* If  $T \in \mathcal{C}(D, D) \cap \mathcal{C}(D^+, D^+)$ , then  $T$  has a continuous extension  $\tilde{T} \in \mathcal{L}(D^+, D^+)$ . The adjoint operator  $\tilde{T}'$  belongs to  $\mathcal{L}(\bar{D}, \bar{D})$  ( $\cong \mathcal{L}(D, D)$ ) and satisfies

$$\langle \psi, \tilde{T}'\varphi \rangle = \langle \tilde{T}\psi, \varphi \rangle = \langle T\psi, \varphi \rangle$$

for all  $\varphi, \psi \in D$ . This means  $T^* \supset \tilde{T}'$ , which implies  $T \in L^+(D)$ .

Conversely, assume  $T \in L^+(D)$ . Then  $T$  and  $T^+ := T^* \upharpoonright D$  are in  $\mathcal{C}(D, D)$ . The adjoint  $(T^+)'$  belongs to  $\mathcal{L}(D', D')$  ( $\cong \mathcal{L}(D^+, D^+)$ ) and satisfies

$$\langle (T^+)' \varphi, \psi \rangle = \langle \varphi, T^+ \psi \rangle = \langle T\varphi, \psi \rangle$$

for all  $\varphi, \psi \in D$ . This implies  $T \in \mathcal{C}(D^+, D^+)$ , which completes the proof.

Concrete Frechet domains have been investigated, e. g., in [22, 25, 30, 32].

We refer to [1] for the theory of operators in a Hilbert space and to [16, 17] for the theory of locally convex spaces.

### § 3. The Partial Multiplication

In this section, we define a partial multiplication on  $\mathcal{L}$  which generalizes the familiar multiplications defined on  $L^+(D)$  and on  $\mathcal{C}(H, H)$  ( $\cong B(H)$ ). Furthermore, we give examples which indicate some of the difficulties connected with the definition of such a partial multiplication.

Consider a class  $\mathfrak{R}$  of locally convex spaces  $E, F, \dots$ , each of which satisfies condition d) of Section 2. We assume that the following property is satisfied:

For  $E, F \in \mathfrak{R}$ , the intersection  $E \cap F$  equipped with the topology of the locally convex kernel contains  $D$  as a dense linear subspace.

Next we define products with respect to the class  $\mathfrak{R}$ .

**Definition 3.1.** The product  $T_n \circ T_{n-1} \circ \dots \circ T_1$  of elements of  $\mathcal{L}$  is said to be defined with respect to the class  $\mathfrak{R}$ , if there are spaces  $E_0, E_1, \dots, E_n$  in  $\mathfrak{R}$  such that  $T_j \in \mathcal{C}(E_{j-1}, E_j)$ . If  $S_j \in \mathcal{L}(E_{j-1}, E_j)$  is the unique extension of  $T_j$ , the product  $T_n \circ T_{n-1} \circ \dots \circ T_1$  is defined by

$$T_n \circ \dots \circ T_2 \circ T_1 \varphi = S_n(\dots (S_2(S_1\varphi)) \dots) \quad (\varphi \in D).$$

**Proposition 3.2.** *The product  $T_n \circ \dots \circ T_1$  does not depend on the special choice of the spaces  $E_0, \dots, E_n$  in  $\mathfrak{R}$ .*

*Proof.* We proceed by induction on  $n$ . For  $n=1$ , the assertion is obvious.

Suppose that the assertion is true for  $n$  factors. Consider elements  $T_1, \dots, T_{n+1}$  of  $\mathcal{L}$  and spaces  $E_0, \dots, E_{n+1}, F_0, \dots, F_{n+1}$  belonging to  $\mathfrak{R}$  such that

$$T_j \in \mathcal{C}(E_{j-1}, E_j) \cap \mathcal{C}(F_{j-1}, F_j) \quad (j \in \{1, \dots, n+1\}).$$

Let  $S_j \in \mathcal{L}(E_{j-1}, E_j)$  and  $R_j \in \mathcal{L}(F_{j-1}, F_j)$  denote the continuous extensions of  $T_j$ . We have to show that

$$S_{n+1}(S_n \cdots S_2(S_1\varphi) \cdots) = R_{n+1}(R_n \cdots R_2(R_1\varphi) \cdots) \tag{1}$$

for all  $\varphi \in D$ .

To do this, we note that the embeddings of  $E_n \cap F_n$  into  $E_n$  and into  $F_n$  are continuous if  $E_n \cap F_n$  is endowed with the topology of the locally convex kernel. Therefore the restrictions to  $E_n \cap F_n$  of  $R_{n+1}$  and  $S_{n+1}$  belong to  $\mathcal{L}(E_n \cap F_n, D^+)$ . Since these restrictions coincide on the dense subset  $D$ , they coincide everywhere on  $E_n \cap F_n$ . Now equation (1) follows from the fact that

$$T_n \circ \cdots \circ T_1 \varphi = S_n \cdots S_2 S_1 \varphi = R_n \cdots R_2 R_1 \varphi \in E_n \cap F_n$$

by assumption. This completes the proof.

*Remark.* Definition 3.1 is closely related to the definition of products of operators acting on partial inner product spaces, which was given in [3]. If the space  $D^+$  has the structure of a partial inner product space in the sense of [3] such that  $(D^+)^* = D$ , then there is defined a set  $\{V_\tau\}$  of assaying subspaces of  $D^+$  (see [3]). In Definition 3.1 one can use the set

$$\mathfrak{R} = \{V_\tau\},$$

where each space  $V_\tau$  is equipped with its Mackey topology  $\tau(V_\tau, V_{\hat{\tau}})$ .

Now, we give an example which shows that it is impossible to omit the condition on  $\mathfrak{R}$  concerning the density of  $D$  in  $E \cap F$ . The following proposition collects some consequences of this example.

**Proposition 3.3.** *Let  $D \subset l_2$  be the Schwartz space of rapidly decreasing sequences. There exist locally convex spaces  $E$  and  $F$  satisfying condition d) of Section 2 and operators  $T_1 \in \mathcal{C}(D, E) \cap \mathcal{C}(D, F)$ ,  $T_2 \in \mathcal{C}(E, D^+) \cap \mathcal{C}(F, D^+)$  such that the following assertion is true:*

*The continuous extensions  $R \in \mathcal{L}(E, D^+)$  and  $S \in \mathcal{L}(F, D^+)$  of  $T_2$  satisfy*

$$R(T_1\varphi) \neq S(T_1\varphi)$$

*for some  $\varphi \in D$ .*

*Moreover,  $E$  and  $F$  can be specified to be Hilbert spaces or Frechet domains.*

*Proof.* We denote the canonical orthonormal basis of  $D$  by  $(\varphi_n)$ . Then

$$D = \left\{ \sum_{n=1}^{\infty} x_n \varphi_n : \sum_{n=1}^{\infty} |x_n|^2 n^{2k} < \infty \text{ for all } k \in \mathbb{N} \right\}.$$

Defining

$$A\varphi = \sum_{n=1}^{\infty} \langle -2n\varphi_{2n} + (2n+1)\varphi_{2n+1}, \varphi \rangle (-2n\varphi_{2n} + (2n+1)\varphi_{2n+1}),$$

$$B\varphi = \sum_{n=1}^{\infty} \langle (2n-1)\varphi_{2n-1} - 2n\varphi_{2n}, \varphi \rangle ((2n-1)\varphi_{2n-1} - 2n\varphi_{2n})$$

for  $\varphi \in D$ , we obtain essential self-adjoint operators  $A$  and  $B$  belonging to  $L^+(D)$ . Let  $E$  and  $F$  denote the normed spaces  $D(\bar{A})$  and  $D(\bar{B})$ , respectively, equipped with the norms

$$\|\varphi\|_E = (\|\bar{A}\varphi\|^2 + \|\varphi\|^2)^{1/2}, \quad \|\varphi\|_F = (\|\bar{B}\varphi\|^2 + \|\varphi\|^2)^{1/2}.$$

Obviously,  $E$  and  $F$  fulfil condition d) of Section 2.

We set

$$\phi_k = \sum_{n=1}^k n^{-1} \varphi_n, \quad f = \sum_{n=1}^{\infty} n^{-1} \varphi_n.$$

Since  $A\phi_{2k+1} = B\phi_{2k} = 0$ , the sequences  $(\phi_{2k+1})$  and  $(\phi_{2k})$  are Cauchy sequences in  $E$  and  $F$ , respectively. Hence, their common limit  $f$  belongs to  $E \cap F$ . Thus, the operator  $T_1$  defined by

$$T_1\varphi = \langle \varphi_1, \varphi \rangle f$$

is a rank one operator belonging to  $\mathcal{C}(D, E) \cap \mathcal{C}(D, F)$ .

We define an operator  $R$  on  $E$  by

$$R\left(\sum_{n=1}^{\infty} x_n \varphi_n\right) = \left(x_1 + \sum_{n=1}^{\infty} (-2n x_{2n} + (2n+1)x_{2n+1})\right) \varphi_1.$$

Since

$$\begin{aligned} \left| x_1 + \sum_{n=1}^{\infty} (-2n x_{2n} + (2n+1)x_{2n+1}) \right| &\leq |x_1| \\ &+ \left( \sum_{n=1}^{\infty} |-2n x_{2n} + (2n+1)x_{2n+1}|^2 ((2n)^2 + (2n+1)^2) \right)^{1/2} \\ &\cdot \left( \sum_{n=1}^{\infty} ((2n)^2 + (2n+1)^2)^{-1} \right)^{1/2} \\ &\leq \|\sum x_n \varphi_n\| + (\sum n^{-2})^{1/2} \|A \sum x_n \varphi_n\| \leq 3 \|\sum x_n \varphi_n\|_E, \end{aligned}$$

$R$  is well defined and belongs to  $\mathcal{L}(E, D^+)$ .

Similarly, we obtain an operators  $S \in \mathcal{L}(F, D^+)$  by the definition

$$S\left(\sum_{n=1}^{\infty} x_n \varphi_n\right) = \sum_{n=1}^{\infty} ((2n-1)x_{2n-1} - 2n x_{2n}) \varphi_1.$$

It is easy to see that  $R\varphi = S\varphi$  for all  $\varphi \in D$ . Therefore  $R$  and  $S$  are extensions of the same operator, say  $T_2$ , which belongs to  $\mathcal{C}(E, D^+) \cap \mathcal{C}(F, D^+)$ . Moreover,  $R(T_1\varphi) = \langle \varphi_1, \varphi \rangle \varphi_1$  and  $S(T_1\varphi) = 0$  for all  $\varphi \in D$ . In particular,

$$S(T_1\varphi_1) \neq R(T_1\varphi_1).$$

Since  $f = \sum n^{-1}\varphi_n \in D(\bar{A}^k) \cap D(\bar{B}^k)$  for each  $k \in \mathbb{N}$ , it is possible to modify the above example by setting

$$E = \bigcap_{n=1}^{\infty} D(\bar{A}^n), \quad F = \bigcap_{n=1}^{\infty} D(\bar{B}^n).$$

This completes the proof.

*Remarks 1.* The operators  $T_1$  and  $T_2$  satisfy even the conditions

$$T_1 \in \mathcal{C}(D^+, E) \cap \mathcal{C}(D^+, F), \quad T_2 \in \mathcal{C}(E, D) \cap \mathcal{C}(F, D).$$

2. Using the notations of the preceding proof, we obtain the following counterexample to [4] Proposition 2.4 and [5] Proposition 3.1. Let  $T \in \mathcal{L}(H, H)$  be the operator defined by  $Tg = \langle f, g \rangle f$ . The operators  $\bar{A}^2$  and  $\bar{B}^2$  are left multipliers of  $T$  in the sense of [4]. We show that their strong sum, i.e., the closure of  $A^2 + B^2$ , is not a left multiplier of  $T$ . Suppose, on the contrary, that it is a left multiplier of  $T$ . Then  $f$  belongs to the domain of the closure of  $A^2 + B^2$ . This means that there is a sequence  $(\eta_n)$  in  $D$  such that

$$\lim_{n \rightarrow \infty} \|\eta_n - f\| = 0, \quad \lim_{n, m \rightarrow \infty} \|(A^2 + B^2)(\eta_n - \eta_m)\| = 0.$$

Since

$$\|\varphi\|_E^2 + \|\varphi\|_F^2 = \langle (A^2 + B^2)\varphi, \varphi \rangle + 2\|\varphi\|^2 \leq \|(A^2 + B^2)\varphi\|^2 + 3\|\varphi\|^2,$$

$(\eta_n)$  is a Cauchy sequence in both spaces  $E$  and  $F$ . Consequently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\eta_n - f\|_E &= \lim_{n \rightarrow \infty} \|\eta_n - f\|_F = 0, \\ \lim_{n \rightarrow \infty} \|R(\eta_n - f)\| &= \lim_{n \rightarrow \infty} \|S(\eta_n - f)\| = 0. \end{aligned}$$

This gives a contradiction because  $R\eta_n = S\eta_n = T_2\eta_n$ ,  $Rf = \varphi_1$ , and  $Sf = 0$ .

3. In [25], there is defined a multiplication of certain classes of pairs of operators in  $\mathcal{L}(\mathfrak{S}, \mathfrak{S}')$ , where  $\mathfrak{S}'$  is the space of tempered distribution and  $\mathfrak{S}$  is the subspace of test functions. This definition corresponds (but is not equivalent) to our Definition 3.1 in the case that  $\mathfrak{R} = \{\mathfrak{S}, V, \mathfrak{S}'\}$  and  $V$  satisfies some additional conditions. Note that  $\mathfrak{S}$  is isomorphic to the space  $\mathcal{S}$  of rapidly decreasing sequences. The spaces  $E$  and  $F$  constructed in the proof of Proposition 3.3 are  $F$ -domains in the sense of [25]. Therefore the product in the sense of [25] Definition 4.3 is not independent of the choice of  $V$ .

4. Using the notations of the proof of Proposition 3.3, we consider the functionals  $g_1 \in E'$ ,  $g_2 \in F'$ , and  $g \in D^+$  defined by  $g_1(h) = \langle Rh, \varphi_1 \rangle$ ,  $g_2(h) = \langle Sh, \varphi_1 \rangle$ , and  $\langle \varphi, g \rangle = \langle T_2\varphi, \varphi_1 \rangle$ . It is clear that  $g_1$  and  $g_2$  are extensions of  $g$  and that  $g_1(f)$  is different from  $g_2(f)$ .

Therefore it is not possible to define the structure of a partial inner product



space on  $D^+$  in the sense of [3] such that the following condition is satisfied:

If for some  $A \in L^+(D)$  with  $A \geq Id$  and for some  $g \in D^+$  the inequality  $|\langle \varphi, g \rangle| \leq \|A\varphi\|$  is satisfied for all  $\varphi \in D$ , then the partial inner product  $\langle f, g \rangle$  is defined for all  $f \in D(\bar{A})$  and satisfies the inequality  $|\langle f, g \rangle| \leq \|\bar{A}f\|$ .

In particular the up- and downward directed set  $(D_\alpha)$  of Hilbert spaces, which was defined in [9] (see also [23]) does not define a structure of a partial inner product space such that the partial inner product  $\langle f, g \rangle$  exists for all  $f \in D_\alpha$  and  $g \in D_{\alpha^-}$ .

From now on, we restrict ourselves to the class  $\mathfrak{R} = \{D, H, D^+\}$ . We repeat the definition of the partial multiplication in this case.

**Definition 3.4.** We say that the product  $T_n \circ \dots \circ T_1$  of elements of  $\mathcal{L}$  is defined if there are spaces  $E_0, \dots, E_n$  belonging to  $\{D, H, D^+\}$  such that  $T_j \in \mathcal{C}(E_{j-1}, E_j)$ . Let  $S_j \in \mathcal{L}(E_{j-1}, E_j)$  denote the continuous extension of  $T_j$ . Then the product  $T_n \circ \dots \circ T_1$  is defined by

$$T_n \circ \dots \circ T_2 \circ T_1 \varphi = S_n(\dots (S_2(S_1 \varphi)) \dots) \quad (\varphi \in D).$$

By Proposition 3.2, this definition is correct.

The following two propositions are simple consequences of the definitions and notations introduced above. We therefore omit the details of the proofs. We use the notations  $(D^+)^+ = D$  and  $H^+ = H$ .

**Proposition 3.5.** Let  $T_1, \dots, T_n$  be operators such that the product  $T_n \circ \dots \circ T_1$  is defined. Let  $E_0, \dots, E_n$  be elements of  $\{D, H, D^+\}$  such that  $T_j \in \mathcal{C}(E_{j-1}, E_j)$ . Then

- a)  $T_j^+ \in \mathcal{C}(E_j^+, E_{j-1}^+)$ ,
- b)  $(T_n \circ \dots \circ T_1)^+ = T_1^+ \circ \dots \circ T_n^+$ ,
- c)  $T_n \circ \dots \circ T_1 \in \mathcal{C}(E_0, E_n)$ .

**Proposition 3.6.** The product  $T_2 \circ T_1$  is defined if and only if at least one of the following conditions is satisfied:

- a)  $T_1 \in \mathcal{C}(D, D)$  and  $T_2 \in \mathcal{L}$ .
- b)  $T_1 \in \mathcal{C}(D, H)$  and  $T_2 \in \mathcal{C}(H, D^+)$ .
- c)  $T_1 \in \mathcal{L}$  and  $T_2 \in \mathcal{C}(D^+, D^+)$ ,

If a) is satisfied, then

$$T_2 \circ T_1 \varphi = T_2(T_1 \varphi)$$

for all  $\varphi \in D$ . If a), b), or c) is satisfied, then

$$\langle T_2 \circ T_1 \varphi, \psi \rangle = \langle T_1 \varphi, T_2^+ \psi \rangle$$

for all  $\varphi, \psi \in D$ .

Note that the partial product is associative in the sense that  $(R \circ S) \circ T = R \circ (S \circ T) = R \circ S \circ T$  if the last product exists in the sense of Definition 3.4. We shall see in a moment that the partial product is not associative in the stronger sense that  $(R \circ S) \circ T = R \circ (S \circ T)$  if both  $(R \circ S) \circ T$  and  $R \circ (S \circ T)$  exist. To prove this, we need the following lemma.

**Lemma 3.7.** *Let  $A$  be a densely defined closable unbounded linear operator on  $H$  such that  $\langle A\varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi \in D(A)$ . Then there exists a bounded linear operator  $R$  defined on  $H$  such that  $R(H)$  is not dense in  $H$  and  $\langle RA\varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi \in D(A)$ .*

*Proof.* First note that upon replacing  $A$  by its Friedrichs extension, we can assume that  $A$  is self-adjoint. We fix positive numbers  $\varepsilon_n$  and  $\varepsilon$  such that

$$\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon < 1/4.$$

Let  $\chi: (\varepsilon, \infty) \rightarrow [1, 2]$  be the function which is defined on  $(n\varepsilon, (n+1)\varepsilon]$  by  $\chi(\lambda) = \lambda^{-1}(n+1)\varepsilon$ . Then there is an orthonormal basis of  $H$  consisting of eigenvalues of the self-adjoint operator  $B := \chi(A) \cdot A$ .

The operator  $R$  will be constructed such that

$$RA\varphi_n = \lambda_{n-1}\varphi_{n-1} + a_n\varphi_n + \lambda_n\varphi_{n+1},$$

where  $(\varphi_n)$  is a certain sequence of eigenvectors of  $B$ .

We put  $\lambda_0 = 0$  and  $\varphi_0 = 0$ . By induction, we define a real sequence  $(a_n)$  and an orthonormal sequence of eigenvectors  $(\varphi_n)$  with  $B\varphi_n = \lambda_n\varphi_n$  such that the following inequalities are satisfied:

$$a_1 > 1. \tag{2}$$

$$\lambda_n > (\varepsilon_n)^{-1}(|a_n| + \lambda_{n-1}). \tag{3}$$

$$\det(a_{k,l})_{k,l=1}^{n+1} > 0, \tag{4}$$

where

$$a_{k,k} = a_k - 1, \quad a_{k,k+1} = a_{k+1,k} = \lambda_k, \quad \text{and} \\ a_{k,l} = 0 \quad \text{if } |k-l| > 1.$$

Note that

$$\det(a_{k,l})_{k,l=1}^{n+1} - (a_{n+1} - 1)\det(a_{k,l})_{k,l=1}^n$$

depends only on the numbers  $a_1, \dots, a_n, \lambda_1, \dots, \lambda_n$ . If these numbers are fixed and if

$$\det(a_{k,l})_{k,l=1}^n > 0,$$

then (4) is satisfied for sufficiently large numbers  $a_{n+1}$ .

Now, let  $H_1$  be the closed linear hull of  $\{\varphi_n\}_{n=1}^{\infty}$ . Set

$$\phi_n = \lambda_n^{-1}(\lambda_n \varphi_{n+1} + a_n \varphi_n + \lambda_{n-1} \varphi_{n-1}).$$

Then it follows that

$$\|\phi_n - \varphi_{n+1}\| = \lambda_n^{-1} \|a_n \varphi_n + \lambda_{n-1} \varphi_{n-1}\| \leq \varepsilon_n.$$

This implies that  $\varphi_1, \phi_1, \phi_2, \dots$  is a basis of  $H_1$  which is equivalent to an orthonormal basis (see, e.g., [27] Proposition 1.a.9). Define an operator  $S_1$  on  $H_1$  by

$$S_1\left(\sum_{n=1}^{\infty} x_n \varphi_n\right) = \sum_{n=1}^{\infty} x_n \phi_n.$$

Then  $S_1$  is bounded and its range is not dense in  $H_1$ .

It follows from the Sylvester criterion, from (2) and (4) and from the equation

$$\begin{aligned} & \left\langle S_1 B \sum_{k=1}^n x_k \varphi_k, \sum_{l=1}^n x_l \varphi_l \right\rangle - \left\langle \sum_{k=1}^n x_k \varphi_k, \sum_{l=1}^n x_l \varphi_l \right\rangle \\ &= \left\langle \sum_{k=1}^n x_k (\lambda_k \varphi_{k+1} + a_k \varphi_k + \lambda_{k-1} \varphi_{k-1} - \varphi_k), \sum_{l=1}^n x_l \varphi_l \right\rangle \\ &= \sum_{k,l=1}^n a_{k,l} \bar{x}_k x_l, \end{aligned}$$

that  $\langle S_1 B \varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi$  in the linear hull of  $\{\varphi_n\}$ . This linear hull is dense in  $D(B) \cap H_1$  with respect to the graph norm  $\varphi \rightarrow \|B\varphi\|$ . Hence,  $\langle S_1 B \varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi \in D(B) \cap H_1$ .

Finally, we define  $R := (S_1 \oplus S_2)\mathcal{X}(A)$ , where  $S_2$  is the identity map of  $H \ominus H_1$ . Then  $R(H)$  is not dense in  $H$  and

$$\langle R A \varphi, \varphi \rangle = \langle (S_1 \oplus S_2) B \varphi, \varphi \rangle \geq \|\varphi\|^2$$

for all  $\varphi \in D(B) = D(A)$ . This completes the proof.

*Remarks 1.* It is easy to construct the operator  $R$  such that the orthogonal complement of  $R(H)$  has infinite dimension. It suffices to write the operator  $B$  as an orthogonal direct sum

$$\sum_{n=1}^{\infty} \oplus B_n,$$

where every  $B_n$  is a self-adjoint unbounded operator, and to apply the preceding proof to each operator  $B_n$ .

2. The author does not know whether or not  $R$  can be chosen to be a partial isometry. But the preceding proof shows that the following is true: Let  $\varepsilon > 0$  be fixed. Then the operator  $R$  in Lemma 3.7 can be chosen such that there exists a partial isometry  $U$  with  $\|U - R\| < \varepsilon$ .

**Proposition 3.8.** *If  $L^+(D)$  contains unbounded operators, then there exist elements  $A, B, R, S, T$  in  $\mathcal{L}$  such that the following assertions are satisfied:*

- a) The expressions  $R \circ (S \circ T)$ ,  $(R \circ S) \circ T$ ,  $S \circ A$ ,  $(R \circ S) \circ A$ ,  $B \circ (R \circ S)$  are defined in the sense of Definition 3.4.  
 b)  $(R \circ S) \circ T \neq R \circ (S \circ T)$ .  
 c) The products  $R \circ (S \circ A)$  and  $B \circ R$  are not defined.

*Proof.* Let  $W \in L^+(D)$  be an unbounded operator. Then Lemma 3.7 applies to the operator  $W^+W + Id$ , where  $Id$  is the identity map of  $D$ . Therefore there exists an operator  $S \in \mathcal{C}(D, H)$  such that  $S(D)$  is not dense in  $H$  and  $\langle S\varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi \in D$ . Let  $R \in \mathcal{C}(H, H)$  be the restriction to  $D$  of the inverse of Friedrichs extension of  $S$ . It follows from Proposition 3.5 that  $S = S^+ \in \mathcal{C}(H, D^+)$ .

Note that the range of the continuous extension  $\tilde{S} \in \mathcal{L}(H, D^+)$  is not contained in  $H$ . Indeed,  $S(H) \subset H$  would imply  $\tilde{S} \in \mathcal{L}(H, H)$  by the closed graph theorem.

Take  $g \in H$  such that  $\tilde{S}g \in H$ . Fix an element  $f$  in  $H \setminus \{0\}$  which is orthogonal to  $S(D)$  and an arbitrary  $h$  in  $D^+ \setminus H$ . Now, the operators  $A \in \mathcal{C}(H, H)$ ,  $B \in \mathcal{L}$ , and  $T \in \mathcal{C}(H, H)$  are defined by

$$A\varphi = \langle g, \varphi \rangle g,$$

$$B\varphi = \langle h, \varphi \rangle h,$$

$$T\varphi = \langle f, \varphi \rangle f.$$

The products  $R \circ S$  and  $S \circ T$  exist. We have

$$\langle R \circ S\varphi, \psi \rangle = \langle S\varphi, R\psi \rangle = \langle \bar{R}S\varphi, \psi \rangle = \langle \varphi, \psi \rangle,$$

$$\langle S \circ T\varphi, \psi \rangle = \langle T\varphi, S\psi \rangle = 0$$

for all  $\varphi, \psi \in D$ . Hence,  $R \circ S = Id$  (the unit of  $L^+(D)$ ) and  $S \circ T = 0$ . Consequently, all products mentioned in a) are defined. Moreover,  $(R \circ S) \circ T = T \neq 0 = R \circ (S \circ T)$ .

We show that  $R \in \mathcal{C}(D, D) \cup \mathcal{C}(D^+, D^+)$ . Suppose, on the contrary, that  $R \in \mathcal{C}(D, D)$  or  $R \in \mathcal{C}(D^+, D^+)$ . Then  $R = R^+$  implies  $R \in \mathcal{C}(D, D) \cap \mathcal{C}(D^+, D^+) = L^+(D)$ . Hence, the product  $R \circ S \circ T$  exists. This is a contradiction to b), which shows that  $R \in \mathcal{C}(D, D) \cup \mathcal{C}(D^+, D^+)$ .

On the other hand,  $S \circ A \in \mathcal{C}(D, H)$  since the image of  $S \circ A$  contains  $\tilde{S}g$ . Clearly,  $B \in \mathcal{C}(H, D^+)$ . Now, it follows from Proposition 3.6 that the products  $R \circ (S \circ A)$  and  $B \circ R$  are not defined. This completes the proof.

*Remarks 1.* The multiplication  $R * S$ , defined in [5], is not associative. Indeed, it follows from Proposition 3.6 that the operators  $R, S, T$  constructed in the preceding proof satisfy the following conditions:

- a)  $S^*$  is a left  $*$ -multiplier of  $T^*$  (in the sense of [5]) and  $(S^*) * (T^*) = 0$ .  
 b)  $R^*$  is a left  $*$ -multiplier of  $S^*$  and  $(R^*) * (S^*) = Id^*$ .  
 c)  $R^*$  is a left  $*$ -multiplier of  $(S^*) * (T^*)$ ,  $(R^*) * (S^*)$  is a left  $*$ -multiplier of  $T^*$  and  $(R^*) * ((S^*) * (T^*)) \neq ((R^*) * (S^*)) * (T^*)$ .

2. It follows from Corollary 6.4 below that  $(\mathcal{L}, L^+(D))$  is a quasi-algebra in the sense of [24]. By definition, the product  $S \cdot T$  exists in the quasi-algebra if and only if  $S \in L^+(D)$  or  $T \in L^+(D)$ . In this case the product  $S \cdot T$  coincides with the product  $S \circ T$  in the sense of Definition 3.4. The proof of Proposition 3.8 shows that the quasi-algebra  $(\mathcal{L}, L^+(D))$  is not associative if there exists an operator  $S \geq \text{Id}$  in  $L^+(D)$  which is not essential self-adjoint. Such an operator exists, e. g., if  $D$  is the Schwartz spaces  $\mathcal{S} \subset l_2$ .

3. In this remark, we wanted to mention a generalization of the results obtained up to now to more general domains than Frechet spaces.

Assume that  $D$  is a semi-reflexive space with respect to the topology introduced in section 2. Denote by  $D_\tau$  the space  $D$  endowed with the Mackey topology  $\tau(D, D')$ .

Since  $\mathcal{L}(D_\tau, D^+)$  coincides with the space of all weakly continuous operators from  $D$  into  $D^+$ , it follows from [17] §40.1 that  $\mathcal{L}(D_\tau, D^+)$  is isomorphic to the space of all separately continuous bilinear forms defined on  $\bar{D} \times D$ . By [17] §39.6 (2a) and (3), the space  $\mathcal{L}(D_\tau, D^+)$  is complete with respect to the topology of uniform convergence on bounded sets if and only if  $D'$  is complete.

Moreover, the Propositions 2.2, 3.2, 3.5, 3.6, and 3.8 remain true if the following modifications are undertaken:

- The space  $\mathcal{L}$  is always replaced by  $\mathcal{L}(D_\tau, D^+)$ .
- The space  $D$  is replaced by  $D_\tau$  in condition d) of Section 2, in Definition 3.4, in Proposition 3.5, and in Proposition 3.6.a).
- In Proposition 3.5, the notations  $(D^+)^+ = D_\tau$  and  $(D_\tau)^+ = D^+$  are used.

This can be shown by using the same proofs as for Frechet domains. In the proof of Proposition 2.2, the operator  $\tilde{T}'$  belongs to  $\mathcal{L}(D_\tau, D_\tau)$  only. But this suffices to show that  $T$  belongs to  $L^+(D)$ .

#### § 4. Bounded Sets in $D$

In this section, we characterize the bounded subsets of  $D$  in terms of bounded self-adjoint operators. This result is an important tool for the study of the spaces  $\mathcal{L}$  and  $L^+(D)$ .

**Theorem 4.1.** *If  $M \subset D$  is a bounded set, then there exists  $A$  in  $\mathcal{C}$  such that  $A \geq 0$  and  $M \subset \bar{A}(U_H)$ .*

*Proof.* Since  $M$  is bounded, there exists a sequence of positive numbers  $(\epsilon_n)$  such that

$$\sum_{n=1}^{\infty} \epsilon_n \sup\{\|A_n \varphi\| : \varphi \in M\} < 1.$$

On the domain

$$D_t := \left\{ \varphi \in D : \sum_{n=1}^{\infty} \varepsilon_n \|A_n \varphi\|^2 < \infty \right\}$$

we define a hermitian sesquilinear form  $t$  by

$$t(\varphi, \psi) = \sum_{n=1}^{\infty} \varepsilon_n \langle A_n \varphi, A_n \psi \rangle.$$

Because of the inequality

$$\sum_{n=1}^{\infty} \varepsilon_n \|A_n(\varphi + \psi)\|^2 \leq \sum_{n=1}^{\infty} 2\varepsilon_n (\|A_n \varphi\|^2 + \|A_n \psi\|^2),$$

$D_t$  is a linear space. Furthermore,

$$t(\varphi, \varphi) \geq \varepsilon_1 \|\varphi\|^2 \quad (5)$$

for all  $\varphi \in D_t$ .

We prove that  $t$  is closed. For, let  $(\varphi_k)$  be a sequence in  $D_t$  such that

$$\lim_{k, l \rightarrow \infty} t(\varphi_k - \varphi_l, \varphi_k - \varphi_l) = 0.$$

Since

$$\|A_n(\varphi_k - \varphi_l)\|^2 \leq (\varepsilon_n)^{-1} t(\varphi_k - \varphi_l, \varphi_k - \varphi_l)$$

for all  $k, l, n \in \mathbb{N}$ , the sequence  $(\varphi_k)$  is a Cauchy sequence in  $D$ . Hence, there exists  $\varphi_0 \in D$  such that the sequence  $(\varphi_k)$  converges to  $\varphi_0$  in  $D$ .

We show that

$$\lim_{k \rightarrow \infty} t(\varphi_k - \varphi_0, \varphi_k - \varphi_0) = 0. \quad (6)$$

Given  $\varepsilon > 0$ , there exists  $k_0$  in  $\mathbb{N}$  such that

$$t(\varphi_k - \varphi_l, \varphi_k - \varphi_l) = \sum_{n=1}^{\infty} \varepsilon_n \|A_n(\varphi_k - \varphi_l)\|^2 < \varepsilon$$

if  $k > k_0$  and  $l > k_0$ . Keeping  $k > k_0$  fixed and letting  $l \rightarrow \infty$ , we get

$$t(\varphi_k - \varphi_0, \varphi_k - \varphi_0) = \sum_{n=1}^{\infty} \varepsilon_n \|A_n(\varphi_k - \varphi_0)\|^2 \leq \varepsilon.$$

In particular, this implies  $\varphi_k - \varphi_0 \in D_t$ . Since  $\varphi_k \in D_t$ ,  $\varphi_0 \in D_t$ . Moreover, (6) is satisfied.

Hence,  $D_t$  is complete with respect to the norm  $\varphi \rightarrow (t(\varphi, \varphi))^{1/2}$ . But this means that  $t$  is closed.

Let  $H_1$  denote the closure of  $D_t$  in  $H$ . By the representation theorem for closed positive sesquilinear forms (see, e.g., [15] Chap. VI Theorem 2. 23), there exists a positive self-adjoint operator  $S$  acting on  $H_1$ , with domain  $D_t$ , such that

$$\langle S\varphi, S\psi \rangle = t(\varphi, \psi)$$

for all  $\varphi, \psi \in D_t$ .

Define  $T = S^{-1} \oplus 0$  with respect to the orthogonal direct sum  $H_1 \oplus (H \ominus H_1)$ .

It follows from (5) that  $T$  is a bounded self-adjoint operator.

We check that  $T \in \mathcal{L}(H, D)$ . Fix  $k \in \mathbb{N}$ . Since the range of  $T$  coincides with  $D_t \subset D$ ,  $A_k T$  is defined on all of  $H$ . Because  $A_k T$  is the adjoint of  $T A_k$ ,  $A_k T$  is closed and hence bounded. This implies  $T \in \mathcal{L}(H, D)$ .

Therefore  $R := T \upharpoonright D \in \mathcal{C}(H, D)$ . By Proposition 3.5,  $A := R \circ R^+ \in \mathcal{C}$ . From Proposition 3.6 we conclude that  $\bar{A} = T^2$  and  $A \geq 0$ .

We prove  $M \subset \bar{A}(U_H)$ . It suffices to show that  $M \subset D(S^2)$  and  $S^2(M) \subset U_H$ . For, take  $\varphi \in M$  and  $\psi \in D_t$ . The inequalities

$$\begin{aligned} \|A_n \varphi\|^2 &= \langle A_n^2 \varphi, \varphi \rangle \leq \|A_n^2 \varphi\| \|\varphi\|, \\ \sum_{n=1}^{\infty} \varepsilon_n \|A_n^2 \varphi\| &\leq 1 \end{aligned}$$

imply  $\varphi \in D_t$ . Now, it follows from

$$|\langle S\varphi, S\psi \rangle| = \left| \sum_{n=1}^{\infty} \varepsilon_n \langle A_n^2 \varphi, \psi \rangle \right| \leq \|\psi\|$$

that  $S\varphi \in D(S^*) = D(S)$  and  $\|S^2\varphi\| \leq 1$ , which yields our assertion.

*Remarks 1.* If the Hilbert space  $H$  is separable, then the operator  $A$  in Theorem 4.1 can be chosen such that  $\bar{A}$  is an invertible self-adjoint operator, or equivalently, that  $\bar{A}$  is injective. But, this is not possible for general Frechet domains in non-separable Hilbert spaces. A counterexample may be obtained by modifying I. Amemiyas construction of a Frechet space without any total bounded subset (see [2] or [16] §29, 6). Indeed, replacing the index set  $R$  by

$$\{(\eta_n) : \eta_n \geq 1 \text{ and } \eta_{n+1} \geq \eta_n^2\}$$

in this construction, we get a Frechet domain without any total bounded subset.

2. Theorem 4.1 and the following corollary generalize [14] Proposition 1.4 and [23] Lemma 5.

**Corollary 4.2.** *The system*

$$\{\bar{A}U_H : A \in \mathcal{C} \text{ and } A \geq 0\}$$

*is a fundamental system of bounded subsets of  $D$ .*

*Proof.* Since  $\bar{A} \in \mathcal{L}(H, D)$ ,  $\bar{A}U_H$  is bounded in  $D$ . Thus, the assertion follows immediately from Theorem 4.1.

**Corollary 4.3.** *The strong topology of  $D^+$  can be defined by the system of seminorms*

$$\{\|\tilde{A} \cdot\| : A \in \mathcal{C} \text{ and } A \geq 0\}.$$

*Proof.* By Corollary 4.2, the strong topology on  $D^+$  can be defined by the

system of seminorms

$$\|f\|_A = \sup\{|\langle f, \bar{A}g \rangle| : g \in U_H\},$$

where  $A \in \mathcal{C}$  and  $A \geq 0$ . Clearly,

$$\begin{aligned} \|f\|_A &= \sup\{|\langle f, A\varphi \rangle| : \varphi \in U_H \cap D\} \\ &= \sup\{|\langle \tilde{A}f, \varphi \rangle| : \varphi \in U_H \cap D\} = \|\tilde{A}f\|. \end{aligned}$$

**Corollary 4.4.** *For  $A \in \mathcal{C}$ , let  $\|\cdot\|_A$  denote the seminorm defined on  $\mathcal{L}$  by  $\|T\|_A = \|A \circ T \circ A\|$ . Then the topology of  $\mathcal{L}$  can be defined by the system of seminorms*

$$\{\|\cdot\|_A : A \in \mathcal{C} \text{ and } A \geq 0\}.$$

*Proof.* For  $A \in \mathcal{C}$ , we take  $M = \bar{A}U_H$ . Using Proposition 3.6, we get

$$\begin{aligned} q_M(T) &= \sup\{|\langle T\bar{A}f, \bar{A}g \rangle| : f, g \in U_H\} \\ &= \sup\{|\langle TA\varphi, A\psi \rangle| : \varphi, \psi \in U_H \cap D\} \\ &= \sup\{|\langle A \circ T \circ A\varphi, \psi \rangle| : \varphi, \psi \in U_H \cap D\} = \|A \circ T \circ A\|. \end{aligned}$$

Thus, Corollary 4.4 is a consequence of Corollary 4.2.

The knowledge of the bounded sets of  $D$  enables us to give a more explicit description of the so-called quasi-uniform topologies  $\tau^D$ ,  $\tau^{D'}$ , and  $\tau_{\#}^D$  defined on  $\mathcal{C}(D, D)$ ,  $\mathcal{C}(D^+, D^+)$ , and on  $L^+(D)$ , respectively. We refer to [23] for the definition and properties of these topologies. Recall that  $D$  is assumed to be a Frechet domain.

**Corollary 4.5.** *We denote*

$$\begin{aligned} \|T\|_{n,A} &= \|A_n \circ T \circ A\|, \quad \|T\|_{A,n} = \|A \circ T \circ A_n\|, \\ \|T\|_{A,n}^* &= \|A_n \circ T \circ A\| + \|A \circ T \circ A_n\|. \end{aligned}$$

*Then the quasi-uniform topologies  $\tau^D$ ,  $\tau^{D'}$ , and  $\tau_{\#}^D$  can be defined by the systems of seminorms*

$$\begin{aligned} \{\|\cdot\|_{n,A} : n \in \mathbf{N}, A \in \mathcal{C} \text{ and } A \geq 0\}, \\ \{\|\cdot\|_{A,n} : n \in \mathbf{N}, A \in \mathcal{C} \text{ and } A \geq 0\}, \\ \{\|\cdot\|_{A,n}^* : n \in \mathbf{N}, A \in \mathcal{C} \text{ and } A \geq 0\}. \end{aligned}$$

*respectively.*

At the end of this section, we prove the existence of certain orthonormal projections in  $\mathcal{C}$ .

**Proposition 4.6.** *Let  $\mathcal{X} \subset \mathbf{R}$  be a Borel measurable set the closure of which does not contain zero. Suppose  $A \in \mathcal{C}$  and  $A \geq 0$ . Let  $\bar{A} = \int \lambda dP_\lambda$  be the spectral*



representation of  $\bar{A}$ . Define

$$P = \int_{\mathcal{X}} dP_{\lambda}.$$

Then  $P \upharpoonright D \in \mathcal{C}$ .

*Proof.* We denote

$$B = \int_{\mathcal{X}} \lambda^{-2} dP_{\lambda}.$$

Then the continuous extension  $\tilde{P} \in \mathcal{L}(D^+, D)$  can be defined by

$$\tilde{P}f = \tilde{A}B\tilde{A}f.$$

### § 5. Factorization of Operators

In this section, we obtain special factorizations for several kinds of operators. In particular, these factorizations are useful for the study of the order structure of  $\mathcal{L}_h$ .

**Proposition 5.1.** *If  $T \in \mathfrak{B}_n$ , then there exists  $R$  in  $\mathcal{C}(H, H)$  such that  $\|R\| \leq 1$  and  $T = A_n \circ R \circ A_n$ . Under the additional assumption  $T \geq 0$ , it is possible to choose  $R \geq 0$ , as well.*

*Proof.* By the definition of  $\mathfrak{B}_n$ , we have

$$|\langle T\varphi, \psi \rangle| \leq \|A_n\varphi\| \|A_n\psi\|$$

for all  $\varphi, \psi \in D$ . Thus, the sesquilinear form  $t(\varphi, \psi) := \langle T\varphi, \psi \rangle$  is densely defined and continuous on the Hilbert space  $D(\bar{A}_n)$  equipped with the new scalar product  $(\varphi, \psi) := \langle \bar{A}_n\varphi, \bar{A}_n\psi \rangle$ . By the representation theorem for continuous sesquilinear forms, there exists an operator  $S$  on  $D(\bar{A}_n)$  satisfying

$$\langle T\varphi, \psi \rangle = (S\varphi, \psi)$$

for all  $\varphi, \psi \in D$  and

$$|(Sf, g)|^2 \leq (f, f)(g, g)$$

for all  $f, g \in D(\bar{A}_n)$ . This implies

$$\|\bar{A}_n S f\|^2 = (Sf, Sf)^2 \leq (f, f)(Sf, Sf) \leq (f, f)^2 = \|\bar{A}_n f\|^2$$

for all  $f \in D(\bar{A}_n)$ . Consequently, we can define an operator  $R_1$  with domain  $A_n(D)$  such that  $\|R_1\| \leq 1$  and  $R_1(A_n\varphi) = \bar{A}_n(S\varphi)$  for all  $\varphi \in D$ . This operator can be extended to an operator  $R \in \mathcal{C}(H, H)$  with  $\|R\| \leq 1$ . According to Proposition 3.6,

$$\langle T\varphi, \psi \rangle = (S\varphi, \psi) = \langle \bar{A}_n S\varphi, A_n\psi \rangle = \langle R A_n\varphi, A_n\psi \rangle = \langle A_n \circ R \circ A_n\varphi, \psi \rangle$$

for  $\varphi, \psi \in D$ . This proves the first assertion.

Now we assume that  $T \geq 0$ . In this case we define  $R$  in a different way.

The operator  $S$  defined above satisfies

$$\langle S\varphi, \varphi \rangle = \langle T\varphi, \varphi \rangle \geq 0.$$

Let  $S_1$  denote the non-negative square root of  $S$  in the Hilbert space  $D(\bar{A}_n)$ . Then

$$\|\bar{A}_n S_1 f\|^2 = (S_1 f, S_1 f) = (Sf, f) \leq \|\bar{A}_n f\|^2$$

for all  $f \in D(\bar{A}_n)$ . Consequently, there exists an operator  $R_2$  with domain  $A_n(D)$  such that  $\|R_2\| \leq 1$  and  $R_2(A_n \varphi) = \bar{A}_n(S_1 \varphi)$  for all  $\varphi \in D$ . Let  $R_3 \in \mathcal{C}(H, H)$  be an extension of  $R_2$  with the same norm. Taking  $R := R_3^+ \circ R_2$ , we see that  $R \in \mathcal{C}(H, H)$ ,  $\|R\| \leq 1$ , and  $R \geq 0$ . For  $\varphi, \psi \in D$ , we have

$$\begin{aligned} \langle T\varphi, \psi \rangle &= \langle S\varphi, \psi \rangle = \langle S_1 \varphi, S_1 \psi \rangle = \langle \bar{A}_n S_1 \varphi, \bar{A}_n S_1 \psi \rangle \\ &= \langle R_3 A_n \varphi, R_3 A_n \psi \rangle = \langle R A_n \varphi, A_n \psi \rangle = \langle A_n \circ R \circ A_n \varphi, \psi \rangle \end{aligned}$$

by Proposition 3.6. This completes the proof.

*Remark.* Actually the proof of Proposition 5.1 shows that the following proposition is true:

Let  $A$  be an arbitrary closable operator on  $H$  with domain of definition  $D(A)$  such that  $\|A\varphi\| \geq \|\varphi\|$  for all  $\varphi \in D(A)$  and let  $t$  be a bilinear form on  $\overline{D(A)} \times D(A)$  such that  $|t(\varphi, \psi)| \leq \|A\varphi\| \|A\psi\|$  for all  $\varphi, \psi \in D(A)$ . (Here  $\overline{D(A)}$  denotes the complex conjugate space of  $D(A)$ ). Then there exists an operator  $R \in \mathcal{L}(H, H)$  with  $\|R\| \leq 1$  such that  $t(\varphi, \psi) = \langle R A \varphi, A \psi \rangle$  for all  $\varphi, \psi \in D(A)$ . If additionally  $t(\varphi, \varphi) \geq 0$  for all  $\varphi \in D(A)$ , then it is possible to choose  $R \geq 0$ .

**Corollary 5.2.** *If  $T \in \mathcal{L}$  and  $T \geq 0$ , then there exists  $S$  in  $\mathcal{C}(D, H)$  satisfying  $T = S^+ \circ S$ .*

*Proof.* We represent  $T$  as  $A_n \circ R \circ A_n$  with  $R \geq 0$  and  $R \in \mathcal{C}(H, H)$ . Then we define  $S$  in  $\mathcal{C}(D, H)$  by

$$S\varphi = (\bar{R})^{1/2} A_n \varphi.$$

**Corollary 5.3.** *The linear hull of the positive cone of  $\mathcal{L}_n$  coincides with  $\mathcal{L}$ .*

For  $S, T \in \mathcal{L}_n$ , let  $[S, T]$  denote the order interval

$$[S, T] = \{R \in \mathcal{L}_n : S \leq R \leq T\}.$$

**Corollary 5.4.** *For  $n \in \mathbb{N}$ , the following inclusions are satisfied.*

$$\mathfrak{B}_n \cap \mathcal{L}_n \subset [-A_n^2, A_n^2] \subset 2\mathfrak{B}_n, \tag{7}$$

$$\mathfrak{B}_n \subset [-A_n^2, A_n^2] + i[-A_n^2, A_n^2]. \tag{8}$$

Consequently, the absolutely convex hulls of the order intervals of  $\mathcal{L}_n$  form a

fundamental system of bounded subsets of  $\mathcal{L}$ .

*Proof.* According to Proposition 5.1, each operator  $T \in \mathfrak{B}_n$  can be represented as  $T = A_n \circ R \circ A_n$  whereas  $\|R\| \leq 1$ . Now (8) is a consequence of the formula

$$T = \frac{1}{2} A_n \circ (R + R^+) \circ A_n - \frac{1}{2} i A_n \circ (iR - iR^+) \circ A_n.$$

If we assume in addition  $T \in \mathcal{L}_h$ , we get

$$A_n \circ (R - R^+) \circ A_n = T - T^+ = 0.$$

In this case,  $T$  belongs to  $[-A_n^2, A_n^2]$ , which proves the first inclusion of (7).

We now verify the second one. Put  $T \in [-A_n^2, A_n^2]$ . This means that

$$-\|A_n \varphi\|^2 \leq \langle T \varphi, \varphi \rangle \leq \|A_n \varphi\|^2$$

for all  $\varphi \in D$ . Taking  $\varphi, \psi \in D \setminus \{0\}$ , setting  $c := \|A_n \psi\|^{1/2} \|A_n \varphi\|^{-1/2}$ , and using the polarization formula, we get

$$\begin{aligned} |\langle T \varphi, \psi \rangle| &= \left| \frac{1}{4} \sum_{k=0}^3 i^{-k} \langle T(c\varphi + i^k c^{-1} \psi), c\varphi + i^k c^{-1} \psi \rangle \right| \\ &\leq \frac{1}{4} \sum_{k=0}^3 \|A_n(c\varphi + i^k c^{-1} \psi)\|^2 = c^2 \|A_n \varphi\|^2 + c^{-2} \|A_n \psi\|^2 \\ &= 2 \|A_n \varphi\| \|A_n \psi\|. \end{aligned}$$

This proves  $T \in 2\mathfrak{B}_n$ .

By Proposition 2.1, (7) and (8) imply that the absolutely convex hulls of the order intervals  $[-A_n^2, A_n^2]$  form a fundamental system of bounded subsets of  $\mathcal{L}$ . The assertion now follows from the fact that each order interval  $[S, T]$  is contained in  $[-A_n^2, A_n^2]$  for some  $n \in \mathbb{N}$ .

*Remark.* On  $\mathcal{L}$ , the associated bornological topology coincides with the order topology, or equivalently, the  $\rho$ -topology considered in [6]. It was shown in [32] that this topology is different from the uniform topology, in general.

Next, we give factorizations for operators in  $\mathcal{C}(D, H)$  and  $\mathcal{C}(H, D^+)$ .

**Proposition 5.5.** *For each  $T$  in  $\mathcal{C}(D, H)$  satisfying  $T^+ \circ T \leq A_n^2$ , there exists  $R$  in  $\mathcal{C}(H, H)$  such that  $T = R \circ A_n$  and  $\|R\| \leq 1$ .*

*Proof.* By Proposition 3.5, the product  $T^+ \circ T$  exists if  $T \in \mathcal{C}(D, H)$ . We have

$$\|T \varphi\|^2 = \langle T^+ \circ T \varphi, \varphi \rangle \leq \langle A_n^2 \varphi, \varphi \rangle = \|A_n \varphi\|^2$$

for all  $\varphi \in D$ . We define an operator  $R_1$  with domain  $A_n(D)$  by setting  $R_1(A_n \varphi) = T \varphi$ . Obviously,  $\|R_1\| \leq 1$ . Let  $R \in \mathcal{C}(H, H)$  be an extension of  $R_1$  with the

same norm. Then the proposition follows from

$$R(A_n\varphi) = R_1(A_n\varphi) = T\varphi \quad (\varphi \in D).$$

**Corollary 5.6.** *For each  $T \in \mathcal{C}(H, D^+)$  satisfying  $T \circ T^+ \leq A_n^2$ , there exists  $R$  in  $\mathcal{C}(H, H)$  such that  $T = A_n \circ R$  and  $\|R\| \leq 1$ .*

*Proof.* Since  $T^+$  satisfies the assumptions of Proposition 5.5, we find  $S \in \mathcal{C}(H, H)$  such that  $T^+ = S \circ A_n$  and  $\|S\| \leq 1$ . Let  $R = S^+$ . By Proposition 3.5,  $T = A_n \circ S^+ = A_n \circ R$ , which proves the corollary.

*Remark.* It is obvious from the proofs that Proposition 5.5 and Corollary 5.6 are valid for every semi-reflexive domain  $D$ , if the operator  $A_n$  is replaced by an arbitrary operator  $A \in L^+(D)$  such that  $\langle A\varphi, \varphi \rangle \geq \|\varphi\|^2$  for all  $\varphi \in D$ .

We conclude this section by factorizing operators belonging to  $\mathcal{C}$ ,  $\mathcal{C}(H, D)$ , or  $\mathcal{C}(D^+, H)$ .

**Proposition 5.7.** *For each  $S \in \mathcal{C}$ , there exist operators  $A \in \mathcal{C}$  and  $R \in \mathcal{C}(H, H)$  such that  $S = A \circ R \circ A$  and  $A \geq 0$ . Under the additional assumption  $S \geq 0$ , it is possible to choose  $R \geq 0$ , as well.*

*Proof.* According to [13] Proposition 6.2.1, there exist a neighbourhood of zero  $U$  in  $D^+$  and a bounded subset  $M$  of  $D$  such that  $\tilde{S}(U) \subset M$  and  $\tilde{S}^+(U) \subset M$ . By the Corollaries 4.2 and 4.3, we find  $A$  in  $\mathcal{C}$  with  $A \geq 0$  such that

$$U \supset \{f \in D^+ : \|\tilde{A}f\| \leq 1\} \quad \text{and} \quad M \subset \bar{A}(U_H).$$

Let  $\bar{A} = \int \lambda dP_\lambda$  be the spectral representation. We define a self-adjoint operator by

$$T := \int_{(0, \infty)} \lambda^{-1} dP_\lambda.$$

We show that the mapping  $\varphi \rightarrow T(\tilde{S}(T\varphi))$  with domain  $\bar{A}(H) \oplus \text{Ker } \bar{A}$  is a densely defined bounded operator on  $H$ . For, let  $\varphi \in (\bar{A}(H) \oplus \text{Ker } \bar{A}) \cap U_H$ . Then we get

$$\|\tilde{A}T\varphi\| = \|\bar{A}T\varphi\| \leq \|\varphi\| \leq 1,$$

$$T\varphi \in U,$$

$$\tilde{S}T\varphi \in M \subset \bar{A}(U_H),$$

$$\|T\tilde{S}T\varphi\| \leq 1.$$

Let  $R \in \mathcal{C}(H, H)$  be the restriction to  $D$  of the closure of the mapping

$$\varphi \rightarrow T\tilde{S}T\varphi \quad (\varphi \in \bar{A}(H) \oplus \text{Ker } \bar{A}).$$

Since both  $\tilde{S}(D^+)$  and  $\tilde{S}^+(D^+)$  are contained in  $\bar{A}(H)$ , we get

$$\begin{aligned} \langle \bar{A}RA\varphi, \psi \rangle &= \langle \bar{A}T\tilde{S}TA\varphi, \psi \rangle = \langle \bar{S}TA\varphi, \psi \rangle \\ &= \langle TA\varphi, S^+\psi \rangle = \langle \varphi, \bar{A}TS^+\psi \rangle = \langle \varphi, S^+\psi \rangle = \langle S\varphi, \psi \rangle. \end{aligned}$$

for all  $\varphi, \psi \in D$ . This implies  $S = A \circ R \circ A$  by Proposition 3.6.

It is an obvious consequence of the definition of  $R$  that  $\|R\| \leq 1$  and that  $R \geq 0$  if  $S \geq 0$ . This completes the proof.

**Proposition 5.8.** *For each  $S \in \mathcal{C}(D^+, H)$  there exist operators  $R \in \mathcal{C}(H, H)$  and  $A \in \mathcal{C}$  such that  $S = R \circ A$  and  $A \geq 0$ .*

*Proof.* According to Corollary 4.3, there exists  $A$  in  $\mathcal{C}$  such that  $A \geq 0$  and  $\|\tilde{S}f\| \leq \|\tilde{A}f\|$  for all  $f \in D^+$ . Therefore, the formula

$$R_1(\tilde{A}f) = \tilde{S}f$$

defines an operator  $R_1$  with domain  $\tilde{A}(D^+)$ . It is clear that  $\|R_1\| \leq 1$ . Let  $R \in \mathcal{C}(H, H)$  be an extension of  $R_1$  with  $\|R\| \leq 1$ . Then  $R(A\varphi) = R_1(A\varphi) = S\varphi$  for all  $\varphi \in D$ , which completes the proof.

**Corollary 5.9.** *For each  $S \in \mathcal{C}(H, D)$ , there exist operators  $A \in \mathcal{C}$  and  $R \in \mathcal{C}(H, H)$  such that  $S = A \circ R$  and  $A \geq 0$ .*

The proof is similar to the proof of Corollary 5.6. We therefore omit the details.

### § 6. The Density Theorem

The main result of this section is the following density theorem which implies that  $\mathcal{C}$  is dense in  $\mathcal{L}$  and that  $\mathcal{L}$  is the completion of  $L^+(D)$ .

**Theorem 6.1.** *Consider an arbitrary continuous seminorm  $q$  on  $\mathcal{L}$  and a bounded subset  $\mathfrak{B} \subset \mathcal{L}$ . There exists  $P$  in  $\mathcal{C}$  such that  $\bar{P}$  is an orthogonal projection on  $H$  and*

$$q(T - P \circ T \circ P) \leq 1$$

for all  $T \in \mathfrak{B}$ .

*Proof.* Because of Proposition 2.1 and Corollary 4.4, we can assume that  $\mathfrak{B} = \mathfrak{B}_n$  and  $q(T) = \|A \circ T \circ A\|$ , where  $A \in \mathcal{C}$  and  $A \geq 0$ . Let  $\bar{A} = \int \lambda dP_\lambda$  be the spectral representation of the self-adjoint operator  $\bar{A}$ . Set  $c = \|A_n \bar{A}\| + \|A_n^2 \bar{A}\|$  and  $\varepsilon = (3c)^{-3}$ . The operator

$$P := \left( \int_{(\varepsilon, \infty)} dP_\lambda \right) \upharpoonright D$$

belongs to  $\mathcal{C}$  by Proposition 4.6. Define  $Q := \text{Id} - P$ .

For  $f, g \in U_H$ , we have

$$\|A_n Q \bar{A} f\|^2 = \langle A_n^2 \bar{A} Q f, Q \bar{A} f \rangle \leq \|A_n^2 \bar{A}\| \|Q \bar{A}\| \leq c \varepsilon.$$

For  $T \in \mathfrak{B}_n$ , this implies

$$\begin{aligned} |\langle A \circ (T - P \circ T \circ P) \circ A f, g \rangle| &= |\langle T Q \bar{A} f, \bar{A} g \rangle + \langle A \circ Q \circ T P \bar{A} f, g \rangle| \\ &= |\langle T Q \bar{A} f, \bar{A} g \rangle + \langle T A \bar{P} f, Q \bar{A} g \rangle| \leq \|A_n Q \bar{A}\| \|A_n \bar{A}\| \\ &\quad + \|A_n \bar{A}\| \|A_n Q \bar{A}\| \leq 2c(c\varepsilon)^{1/2} \leq 1. \end{aligned}$$

Therefore  $\|A \circ (T - P \circ T \circ P) \circ A\| \leq 1$ , which completes the proof.

Let  $\mathcal{P}$  denote the set

$$\mathcal{P} = \{P \in \mathcal{C} : \bar{P} \text{ is an orthogonal projection on } H\}.$$

**Corollary 6.2.** *The convex cone generated by  $\mathcal{P}$  is dense in the positive cone of  $\mathcal{L}_n$ . The linear hull of  $\mathcal{P}$  is dense in  $\mathcal{L}$ .*

*Proof.* According to Theorem 6.1, the positive cone of  $\mathcal{L}_n \cap \mathcal{C}$  is dense in the positive cone of  $\mathcal{L}_n$ . Fix  $A \in \mathcal{C}$  with  $A \geq 0$ . Let

$$\bar{A} = \int_{(0, c)} \lambda dP_\lambda$$

be the spectral representation of  $\bar{A}$ . Define

$$P_{n, k} = \int_{(n^{-1}ck, n^{-1}c(k+1))} dP_\lambda$$

for  $n, k \in \mathbb{N}$ ,  $k < n$ . By Proposition 4.6,  $P_{n, k} \upharpoonright D$  is in  $\mathcal{P}$ . Since  $\bar{A}$  is the norm limit of the operators

$$B_n := \sum_{k=1}^{n-1} n^{-1}ck P_{n, k}$$

and the operator norm topology is stronger than the topology of  $\mathcal{L}$ ,  $A$  is in the closure of the cone generated by  $\mathcal{P}$ . This gives the first assertion. The second assertion is now a consequence of Corollary 5.3.

**Corollary 6.3.** *The set  $\mathfrak{B}_n \cap \mathcal{C}$  is dense in  $\mathfrak{B}_n$ .*

*Proof.* By Proposition 5.1, each  $T \in \mathfrak{B}_n$  has a representation  $T = A_n \circ R \circ A_n$ , where  $R \in \mathcal{C}(H, H)$  and  $\|R\| \leq 1$ . By Theorem 6.1,  $R$  belongs to the closure of

$$\{P \circ R \circ P : P \in \mathcal{P}\}.$$

Since the map  $\mathcal{L} \ni S \rightarrow A_n \circ S \circ A_n \in \mathcal{L}$  is continuous,  $T$  belongs to the closure of

$$\{A_n \circ P \circ R \circ P \circ A_n : P \in \mathcal{P}\} \subset \mathfrak{B}_n \cap \mathcal{C},$$

which proves the corollary.

Let us note that a special case of the following corollary was already formulated in [24].

**Corollary 6.4.** *The completion of  $L^+(D)$  is  $\mathcal{L}$ .*

*Proof.*  $L^+(D)$  is dense in  $\mathcal{L}$  because its subset  $\mathcal{C}$  is already dense in  $\mathcal{L}$  by Corollary 6.2. On the other hand, it is well-known that  $\mathcal{L}$  is complete (see, e.g., [17] §39, 6).

*Remarks.* 1. The author does not know whether or not  $L^+(D)$  is always dense in  $\mathcal{L}$  with respect to the order topology of  $\mathcal{L}$ . However,  $\mathcal{C}$  is not always dense in  $\mathcal{L}$  with respect to the order topology of  $\mathcal{L}$ . To give an example, we use a construction taken from [32].

Let  $\{\varphi_{kl} : k, l \in \mathbb{N}\}$  be an orthonormal basis of  $H$ . Let  $\bar{A}_n$  be the self-adjoint operator on  $H$  defined by

$$\bar{A}_n \left( \sum_{k,l=1}^{\infty} x_{kl} \varphi_{kl} \right) = 2^{(2^n - 2)} \sum_{l=1}^{\infty} \left( \sum_{k=1}^{n-1} 2^{(2^l)} x_{kl} \varphi_{kl} + \sum_{k=n}^{\infty} x_{kl} \varphi_{kl} \right).$$

Let

$$D := \bigcap_{n=1}^{\infty} D(\bar{A}_n) \quad \text{and} \quad A_n := \bar{A}_n \upharpoonright D.$$

Then  $D$  is a Frechet domain and the sequence  $(A_n)$  fulfils the conditions a), b), and c) of Section 2.

For each real sequence  $(a_n)$ , we define

$$M((a_n)) := \{(k, l) \in \mathbb{N} \times \mathbb{N} : k > a_1, l > a_{k+1}\}.$$

Let  $\mathfrak{U}$  be an ultrafilter on  $\mathbb{N} \times \mathbb{N}$  which contains all sets  $M((a_n))$ .

Let  $T \in \mathfrak{B}_n$ . By Proposition 5.1,  $T$  has a representation  $A_n \circ R \circ A_n$  with  $\|R\| \leq 1$ . Consequently,

$$|\langle T\varphi_{kl}, \varphi_{kl} \rangle| = |\langle RA_n\varphi_{kl}, A_n\varphi_{kl} \rangle| \leq \|A_n\varphi_{kl}\|^2 \leq 2^{(2^{n+1})}$$

if  $k > n$ . Therefore the limit with respect to the ultrafilter

$$\lim_{\mathfrak{U}} \langle T\varphi_{kl}, \varphi_{kl} \rangle =: \omega(T)$$

exists and defines a positive linear functional on  $\mathcal{L}$ . Moreover,  $\omega$  is bounded on bounded subsets of  $\mathcal{L}$ .

We show that  $\mathcal{C} \subset \text{Ker } \omega$ . For  $T \in \mathcal{C}$ , let  $(a_n)$  be a sequence such that  $a_n > \|A_n \circ T\|$  for all  $n \in \mathbb{N}$ . Since

$$\begin{aligned} |\langle T\varphi_{kl}, \varphi_{kl} \rangle| &= 2^{(-2^k + 1 + 2^{-2^l})} |\langle T\varphi_{kl}, A_{k+1}\varphi_{kl} \rangle| \\ &\leq 2^{(-2^k - 2^l)} \|A_{k+1} \circ T\| \leq 2^{-(2^k)} \leq (a_1)^{-1} \end{aligned}$$

for all  $(k, l) \in M((a_n))$ , it follows that  $\omega(T) = 0$ .

But  $\text{Ker } \omega$  is certainly not dense in  $\mathcal{L}$  with respect to the order topology because  $\omega$  is continuous in this topology.

2. The author does not know whether or not  $L^+(D)$  is uniformly dense in  $\mathcal{L}$  for non-metrizable complete domains of  $\text{Op}^*$ -algebras. The example in [18, 19] gives a complete non-metrizable domain  $D$  for which  $\mathcal{L}(D, D^+)$  is not complete. This was already conjectured in [24].

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