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Extension of Holomorphic Functions with Growth Conditions

By

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§1. Introduction and Summary

In his paper [4], Y. Nishimura discussed the problem of extending holomorphic functions on a submanifold X of $\mathbb{C}^n$ to ones on the ambient space, with the order of growth in consideration. In this note we shall take up this problem in a slightly different setting. We consider a Stein manifold X with a strictly plurisubharmonic C^∞ exhaustion function ψ such that $\sup \psi = +\infty$. We also set $t_0 = \inf \psi$. For a real valued continuous function $\lambda(t)$ on $[t_0, \infty)$, we set

$$(1.1)\qquad \mathcal{H}(X, \lambda) = \{h \mid h \text{ is holomorphic on } X \text{ and } \sup |h(p)| \exp(-\lambda(\psi(p))) < \infty\}.$$

$\mathcal{H}(X, \lambda)$ is a Banach space with respect to the norm

$$(1.2)\qquad \|h\|_\lambda = \sup_{p \in X} |h(p)| \cdot \exp\{-\lambda(\psi(p))\}.$$

If Y is a closed analytic submanifold of X, Y is again a Stein manifold and $\psi|_Y$ is a strictly plurisubharmonic exhaustion function for Y. We shall use the similar notation as (1.1) and (1.2) for Y and $\psi|_Y$.

For two C^∞ functions $\mu(t)$ and $\lambda(t)$ on $[t_0, \infty)$, $\mu \ll \lambda$ shall mean $\mu(t) \leqq \lambda(t)$, $\mu'(t) \leqq \lambda'(t)$ and $\mu''(t) \leqq \lambda''(t)$ for all $t \in [t_0, \infty)$.

Now our main assertion sounds as

Theorem 1. *If X and ψ are as above and if Y is a closed analytic submanifold of codimension 1, then there exists a non-decreasing convex C^∞ function $\nu(t)$ of $t \in [t_0, \infty)$, such that for any C^∞ function $\lambda(t)$ with $\nu \ll \lambda^{(1)}$, there exists a constant C_λ such that any $h \in \mathcal{H}(Y, \lambda)$ has an extension $H \in \mathcal{H}(X, \lambda^{(2)} + \nu^{(1)})$,*

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with $||H||_{\lambda^{(2)}+\nu^{(1)}} \leqq C_\lambda ||h||_\lambda$. *Here* $\lambda^{(i)}$ *is defined by* $\lambda^{(i)}(t)=\lambda(t+i)$ *for* $i=1, 2$ *and similarly for* $\nu^{(i)}$.

There are many works in this direction. To mention some of them, we have [0] and [6]. They have their own general scope or concrete view-points. The author hopes that the present work is not completely covered by them.

Many thanks are due to the referee for valuable suggestions.

§2. Construction of ν

Let X, ψ and Y be as above. Let us take locally finite open covers $\mathcal{U}=\{U_i\}_{i\in I}$ and $\mathcal{U}'=\{U_i'\}_{i\in I}$ with the following properties:
(1) For each $i\in I$, there exists a system of local coordinates $(z_i^1, \cdots, z_i^n)$ which is valid on an neighborhood U_i'' containing $\bar{U}_i$, and in this coordinate system U_i and U_i' have forms

$$(2.1)\qquad \begin{cases} U_i = \{(z_i) \mid |z_i^\alpha|<1 \quad \text{for} \quad \alpha = 1, \cdots, n\}, \\ U_i' = \{(z_i) \mid |z_i^\alpha|<1-\varepsilon_i \quad \text{for} \quad \alpha = 1, \cdots, n\} \quad 0<\varepsilon_i<1 . \end{cases}$$

(2) I is the disjoint union of subsets I_1 and I_2. For $i\in I_1$, $U_i\cap Y\neq\phi$ and $z_i^n=0$ is a local equation for Y in U_i''. For $i\in I_2$, $\bar{U}_i\cap Y=\phi$.
(3) If p and q belong to U_i, then $|\psi(p)-\psi(q)|<1$.

We set $R_i=z_i^n$ for $i\in I_1$ and $R_i=1$ for $i\in I_2$. Then $\{R_i\}$ form a system of local equations for Y and

$$(2.2)\qquad b_{ij} = R_i/R_j$$

defines a system of transition functions for $[Y]$. It is clear that such coverings exist provided ε_i are suitably chosen. For $i\in I_1$, we define the projection π_i: $U_i''\rightarrow U_i''\cap Y$ by

$$(2.3)\qquad \pi_i(\tilde{z}_i, z_i^n) = \tilde{z}_i\,, \quad \text{where} \quad \tilde{z}_i = (z_i^1, \cdots, z_i^{n-1})\,.$$

Now there exists, for each $p\in U_j\cap U_l'$ $(j\in I_1)$, a positive number $s_{jl}(p)$ such that

$$|z_j^n(p)|\geqq s_{jl}(p) \quad \text{or} \quad (\tilde{z}_j(p), \zeta)\in U_j\cap U_l \quad \text{for} \quad |\zeta|<s_{jl}(p)\,.$$

If we set

$$\pi_j^{-1}(\pi_j(p))\cap U_j\cap U_l = \pi_j(p)\times W_p\,,$$
$$\pi_j^{-1}(\pi_j(p))\cap U_j\cap U_l' = \pi_j(p)\times W_p'\,,$$

W_p and W'_p being open sets on the z^n_j-plane, we have

$$(2.4)\qquad \begin{cases} r_{jl}(p) = \max\,[z^n_j(p), \sup\{s\,|\,\{|z^n_j|<s\}\subset W_p\}] & \text{if}\quad W_p\ni 0\,, \\ r_{jl}(p) = |z^n_j(p)| & \text{if}\quad W_p\not\ni 0\,. \end{cases}$$

(Note that $r_{jl}(p)\geqq |z^n_j(p)|$ in any cases.) Then we have the following:

Lemma 1. $\liminf r_{jl}(q)\geqq r_{jl}(p)$ *when* $q\to p$. r_{jl} *is bounded from below by a positive number.*

Proof. Suppose $W_p\ni 0$ and $r_{jl}(p)=\sup\{s\,|\,\{|z^n_j|<s\}\subset W_p$. For any s with $0<s<r_{jl}(p)$, $\pi_j(p)\times\{|z^n_j|\leqq s\}$ is a compact set in $U_j\times U_l$. Hence there exists an open neighborhood V of $\pi_j(p)$ such that $V\times\{|z^n_j|\leqq s\}\subset U_j\cap U_l$. Then for q with $q\in U_j\cap U'_l$ and $\pi_j(q)\in V$, $W_q\supset\{|z^n_j|\geqq s\}$. Hence (2.4) holds for q and we have $r_{jl}(q)\geqq s$. Hence $\liminf r_{jl}(q)\geqq s$

In other cases we have $\liminf r_{jl}(q)\geqq\liminf|z^n_j(q)|=|z^n_j(p)|=r_{jl}(p)$.

Next, set

$$\varepsilon_{jl}=\frac{1}{2}\inf\{|x-y|\,|\,x, y\in\mathbb{C},\ x\in\bar{W}'_p,\ y\notin W_p\}.$$

It is clear that $\varepsilon_{jl}>0$. We have $r_{jl}(p)\geqq\varepsilon_{jl}$ for $p\in U_j\cap U'_l$. In fact, if $W_p\not\ni 0$, then

$$r_{jl}(p)=|z^n_j(p)|\geqq|z^n_j(p)-0|\geqq 2\varepsilon_{jl}\,.$$

($z^n_j(p)$ is in W'_p always.) Suppose $W_p\not\ni 0$. If $\varepsilon_{jl}<\sup\{s\,|\,\{|z^n_j|<s\}\subset W_p\}$, then clearly $r_{jl}(p)>\varepsilon_{jl}$. Otherwise there is a $y\in\mathbb{C}-W_p$ with $|y|=\varepsilon_{jl}$. Hence $r_{jl}(p)\geqq|z^n_j(p)|\geqq 2\varepsilon_{jl}-|y|=\varepsilon_{jl}$.

For each p, there are only a finite number of pairs (j, l) such that $p\in U_j\cap U'_l$. Hence

$$(2.5)\qquad r(p)=\min_{(j,l)} r_{jl}(p)$$

is a well defined positive valued function on a neighborhood of Y. This can be extended to a function on X with the same properties, by setting $r(p)=1$ outside.

We choose a partition of unity $\{\rho_i\}$ subordinate to $\mathcal{U}'$. We also take a metric along the fibres of the bundle $[Y]^{-1}$. This is expressed by a system $\{a_i\}$ of positive valued C^∞ functions a_i: $U_i\to\mathbb{R}$, such that

$$(2.6)\qquad a_j/a_k=|b_{jk}|^2\quad\text{on}\quad U_j\cap U_k\,.$$

Because the coordinates (z_j) exist on U_j'' which contains $\overline{U}_j$ and because we can assume R_j is also defined on U_j'', we can assume that a_j is defined as a differentiable function on U_j''. This we shall assume to be the case. Set, for each $p\in U_j$, $b_j(p)=\max_l |\log|b_{jl}(p)|^2|$, l ranging over such indices that $p\in U_l$. Then we have

$$-b_l(p)\leqq \log a_j(p)-\log a_l(p)\leqq b_l(p)\,.$$

We put $b(p)=\sum_l \rho_l(p)b_l(p)$, then we have

$$-b(p)\leqq \log a_j(p)-\sum_l \rho_l(p)\log a_l(p)\leqq b(p)\,.$$

We replace $a_j(p)$ by $a_j(p)\exp(-\sum_l \rho_l(p)\log a_l(p))$, then we have the estimate

$$\exp(-b(p))\leqq a_j(p)\leqq \exp(b(p))\,. \tag{2.7}$$

Finally we choose a complete Kähler metric ds^2 on X. In terms of local coordinate system (z_j), ds^2 will be expressed as

$$ds^2=2\sum g_{j\alpha\bar{\beta}}dz_j^\alpha d\bar{z}_j^\beta\,.$$

We set

$$r_j=\det(g_{j\alpha\bar{\beta}})\,, \tag{2.8}$$

then $\{r_j\}$ defines a metric along the fibres of K_X^{-1}, where K_X denotes the canonical bundle of X.

All these being settled, we take a C^∞ non-decreasing convex function $\nu(t)$ on $[t_0,\infty)$, such that

$$\begin{cases} \text{(a)} \quad \int_X \exp\{-\nu(\psi(p))+3b(p)\}\dfrac{1}{r(p)^2}\max\{1,(\sum_l|\bar{\partial}\rho_l|)^2\}dv<\infty\,. \\ \text{(b)} \quad 3\left(\dfrac{\partial^2\nu(\psi)}{\partial z_j^\alpha\partial\bar{z}_j^\beta}\right)-\left(\dfrac{\partial^2\log a_j}{\partial z_j^\alpha\partial\bar{z}_j^\beta}\right)-\left(\dfrac{\partial^2\log r_j}{\partial z_j^\alpha\partial\bar{z}_j^\beta}\right)\geqq(g_{j\alpha\bar{\beta}})\,, \\ \text{(c)} \quad \exp\{-\nu(\psi(p))+b(p)\}\circ r_j(p)^{-1}\circ\varepsilon_j^{-2n}\leqq C\,, \end{cases} \tag{2.9}$$

where ε_j appeared in the definition of U_j', j is any index such that $p\in U_j'$, and C is a positive constant.

§3. The Proof of the Theorem

For a function $h\in\mathcal{H}(Y,\lambda)$, we extend it to the holomorphic function $\tilde{h}_j$ in U_j $(j\in I_1)$, by

$$(3.1) \qquad \tilde{h}_j(p) = h(\pi_j(p)) .$$

For $j \in I_2$, we simply set $\tilde{h}_j = 0$. In $U_j \cap U_k$ $\tilde{h}_j - \tilde{h}_k$ and R_j have the zero set $U_j \cap U_k \cap Y$ in common. Hence

$$(3.2) \qquad \tilde{h}_{jk} = (\tilde{h}_j - \tilde{h}_k)/R_j$$

is holomorphic in $U_j \cap U_k$ and $\{\tilde{h}_{jk}\}$ form a 1-cocycle in $Z^1(\mathcal{U}, \mathcal{O}([Y]^{-1}))$. As usual we consider the 1-cocycle $\{H_{jk}\}$ in $Z^1(\mathcal{U}, \Omega^n([Y]^{-1} \otimes K_X^{-1})$ defined by

$$H_{jk} = \tilde{h}_{jk} dz_j^1 \wedge \cdots \wedge dz_j^n$$

and set

$$(3.3) \qquad \eta_j = -\sum_m \rho_m e_{jm} H_{mj} ,$$

where $e_{jk} = (R_k/R_j) \det \dfrac{\partial(z_j)}{\partial(z_k)}$, then we have $\eta_j - e_{jk}\eta_k = H_{jk}$. We set

$$(3.4) \qquad f_j = \bar{\partial}\eta_j ,$$

then $f = \{f_j\}$ is a $[Y]^{-1} \otimes K_X^{-1}$-valued $(n, 1)$-form on X with $\bar{\partial} f = 0$.

With the aid of a function $\mu: \mathbb{R} \to \mathbb{R}$, we introduce the inner product (φ, χ) of $[Y]^{-1} \otimes K_X^{-1}$-valued (n, q)-forms $\varphi = \{\varphi_j\}$ and $\chi = \{\chi_j\}$ by

$$(3.5) \qquad (\varphi, \chi) = \int_X \exp(-\mu(\psi)) \circ a_j r_j \varphi_j \wedge \overline{*\chi_j} ,$$

and denote by $\mathcal{L}^q(X, \mu)$ the Hilbert space of $[Y]^{-1} \otimes K_X^{-1}$-valued measurable (n, q)-forms φ with $(\varphi, \varphi) < \infty$.

Lemma 2. *For any* $h \in \mathcal{H}(Y, \lambda)$, f *constructed as* (3.1)～(3.4) *belongs to* $\mathcal{L}^1(X, 2\lambda^{(1)} + \nu)$ *and the correspondence* $h \to f$ *is a continuous linear map. For* η_j *we also have*

$$(3.6) \qquad \|\eta_j\|_{U_j}^2 = \int_{U_j} \exp\{-(2\lambda^{(1)}(\psi) + \nu(\psi))\} a_j r_j \eta_j \wedge \overline{*\eta_j} \leqq C' \circ \|h\|_\lambda^2 .$$

Proof. For $h \in \mathcal{H}(Y, \lambda)$, $\tilde{h}_j$ satisfies $|\tilde{h}_j|(p) \leqq \|h\|_\lambda \circ \exp\{\lambda^{(1)}(\psi(p))\}$, because p and $\pi_j(p)$ belong to the same U_j. Consider $\tilde{h}_{jk}(p)$ for $p \in U_j \cap U_k'$, and in particular as a function of z_j^n, other z_j^α's being considered as parameters. If $|z_j^n(p)| \geqq r(p)$, then (3.7) below holds trivially. If $(\tilde{z}_j(p), \zeta) \in U_j \cap U_k$ for $|\zeta| \leqq r(p)$, then $\tilde{h}_{jk}(\tilde{z}_j, \zeta)$ has the maximum of its absolute values on the circle $|\zeta| = r(p)$. Hence

$$(3.7) \qquad |\tilde{h}_{jk}(p)| \leqq \frac{2}{r(p)} \|h\|_\lambda \circ \exp\{\lambda^{(1)}(\psi(p))\}$$

holds. Since supp ρ_m is contained in U'_m for any m, we can estimate η_j and f_j by virtue of (3.7), and see

$$
\begin{aligned}
(f, f) &= \int_X \exp\{-(2\lambda^{(1)}(\psi)+\nu(\psi))\} a_j r_j \big(\sum_l \bar\partial\rho_l e_{jl}\tilde h_{lj} dz_l^1\wedge\cdots\wedge dz_l^n\big) \\
&\qquad \wedge *\overline{\big(\sum \bar\partial\rho_m e_{jm}\tilde h_{mj} dz_m^1\wedge\cdots\wedge dz_m^n\big)} \\
&= \int_X \exp\{-(2\lambda^{(1)}(\psi)+\nu(\psi))\} a_j r_j \big(\sum_l \bar\partial\rho_l b_{lj}\tilde h_{lj} dz_j^1\wedge\cdots\wedge dz_j^n\big) \\
&\qquad \wedge *\overline{\big(\sum_m \bar\partial\rho_m b_{mj}\tilde h_{mj} dz_j^1\wedge\cdots\wedge dz_j^n\big)} \\
&\leqq C_0 \int_X \exp\{-(2\lambda^{(1)}(\psi)+\nu(\psi))\} a_j r_j \sum_{l,m} | b_{lj}\bar b_{mj}\tilde h_{lj}\bar{\tilde h}_{mj} u_{lm} | w_j dv,
\end{aligned}
$$

where u_{lm} and w_j denote the pointwise inner product of $\bar\partial\rho_l$ with $\bar\partial\rho_m$ and $dz_j^1\wedge\cdots\wedge dz_j^n$ with itself respectively, that is: $\bar\partial\rho_l\wedge*\overline{(\bar\partial\rho_m)}=u_{lm}dv$ and $(dz_j^1\wedge\cdots\wedge dz_j^n)\wedge*\overline{(dz_j^1\wedge\cdots\wedge dz_j^n)}=w_j dv$. C_0 is the constant which appears in [5], Lemma A (p. 18). (The same lemma is referred to as Lemma 11.1 in [3].) Hence we have

$$
\begin{aligned}
(f, f) &\leqq 4C_0\|h\|_\lambda^2 \int_X \exp\{-\nu(\psi)+3b\} r(p)^{-2} \sum |u_{lm}| dv \\
&\leqq 4C_0\|h\|_\lambda^2 \int_X \exp\{-\nu(\psi)+3b\} r(p)^{-2} \big(\sum_l |\bar\partial\rho_l|\big)^2 dv \\
&\leqq C_1\cdot\|h\|_\lambda^2,
\end{aligned}
$$

where C_1 is a constant independent of h. Thus we have

$$
(3.8) \qquad \|f\|_{\mathcal{L}} \leqq \sqrt{C_1}\cdot\|h\|_\lambda .
$$

The proof of (3.6) is similar.

Next we apply Hörmander's $\bar\partial$-theory. Since components of the curvature form Θ of the metric along the fibres are

$$
2\left[\frac{\partial^2\lambda^{(1)}(\psi)}{\partial z_j^\alpha\partial\bar z_j^\beta}\right]+\left[\frac{\partial^2\nu(\psi)}{\partial z_j^\alpha\partial\bar z_j^\beta}\right]-\left[\frac{\partial^2\log a_j}{\partial z_j^\alpha\,\partial\bar z_j^\beta}\right]-\left[\frac{\partial^2\log r_j}{\partial z_j^\alpha\partial\bar z_j^\beta}\right]\geqq(g_{j\alpha\bar\beta}),
$$

we have

$$
((e(\Theta)\Lambda-\Lambda e(\Theta))\varphi, \varphi)\geqq(\varphi, \varphi)
$$

for (n, q)-form φ with compact support ($q\geqq 1$). Hence

$$
(3.9) \qquad (\varphi, \varphi)\leqq(\bar\partial\varphi, \bar\partial\varphi)+(\bar\partial^*\varphi, \bar\partial^*\varphi)
$$

for such a φ, where $\bar\partial^*$ denotes the adjoint operator to $\bar\partial$. Because ds^2 is complete, (3.9) holds for any φ of $\mathcal{L}^q(X, 2\lambda^{(1)}+\nu)$, which belongs to the intersection of the domains of $\bar\partial$ and $\bar\partial^*$. ([5], Theorem 1.1, p. 18. See also [3], theorem 14.1.) Then Hörmander's theorem ([2], Theorem 1.1.4) tells that

(*) *For f defined by* (3.4), *there exists a $\xi\in\mathcal{L}^0(X, 2\lambda^{(1)}+\nu)$ such that $f=\bar\partial\xi$, $\|\xi\|_{\mathcal{L}}\leqq\|f\|_{\mathcal{L}}$ and ξ is C^∞.*

Set

$$\eta_j-\xi_j=g_j\,dz_j^1\wedge\cdots\wedge dz_j^n, \tag{3.10}$$

then g_j is a holomorphic function on U_j and we have

$$\tilde h_{jk}=g_j-(R_k/R_j)g_k \quad \text{on} \quad U_j\cap U_k. \tag{3.11}$$

Hence

$$H=\tilde h_j-R_jg_j=\tilde h_k-R_kg_k$$

is a global holomorphic function on X which extends h on Y.

We have only to estimate H. Take any point p of X. p is contained in U'_j for some j. Set $\zeta^\alpha=z_j^\alpha-z_j^\alpha(p)$, then the multi-disk $D=\{(\zeta)\mid|\zeta^\alpha|<\varepsilon_j\}$ is contained in U_j and g_j is holomorphic there. Hence we have, for $|\tau_\alpha|<\varepsilon_j$

$$\begin{aligned}g_j(p)&=\frac{1}{(2\pi\sqrt{-1})^n}\int_{|\zeta^1|=\tau_1}\cdots\int_{|\zeta^n|=\tau_n}\frac{g_j(z_j(p)+\zeta)}{\zeta^1\cdots\zeta^n}d\zeta^1\cdots d\zeta^n\\&=\frac{1}{(2\pi)^n}\int_{\theta_1=0}^{2\pi}\cdots\int_{\theta_n=0}^{2\pi}g_j(z_j^\alpha(p)+\tau_\alpha e^{\sqrt{-1}\theta_\alpha})d\theta_1\cdots d\theta_n.\end{aligned}$$

Multiplying both hand sides by $\tau_1\cdots\tau_n$ and integrating from $\tau_\alpha=0$ to $\tau_\alpha=\varepsilon_j$, we obtain

$$\frac{\varepsilon_j^{2n}}{2^n}g_j(p)=\frac{1}{(2\pi)^n}\int_D g_j(z)dz^1d\bar z^1\cdots dz^nd\bar z^n.$$

Hence

$$\frac{\varepsilon_j^{2n}}{2^n}|g_j(p)|\leqq\left|\frac{1}{(2\pi)^n}\int_D(\eta'_j(z)-\xi'_j(z))dz^1d\bar z^1\cdots dz^nd\bar z^n\right|, \tag{3.12}$$

where η'_j and ξ'_j denote the coefficients of $dz_j^1\wedge\cdots\wedge dz_j^n$ in η_j and ξ_j respectively. We have, by Schwarz inequality,

$$\begin{aligned}&\left|\int_D(\eta'_j(z)-\xi'_j(z))dz^1d\bar z^1\cdots dz^nd\bar z^n\right|\\&=\left|\frac{1}{2^n}\int_D a_j^{-1/2}r_j^{-1}\exp\left\{\lambda^{(1)}(\psi)+\frac{\nu(\psi)}{2}\right\}\cdot a_j^{1/2}\exp\left\{-\lambda^{(1)}(\psi)-\frac{\nu(\psi)}{2}\right\}(\eta'_j-\xi'_j)dv\right|\\&\leqq\frac{1}{2^n}\left\{\int_D a_j^{-1}r_j^{-2}\exp\{2\lambda^{(1)}(\psi)+\nu(\psi)\}dv\right\}^{1/2}\end{aligned}$$

$$\cdot\left\{\int_D a_j \exp\{-2\lambda^{(1)}(\psi)-\nu(\psi)\}\,|\eta'_j-\xi'_j|^2 dv\right\}^{1/2}$$

$$=2^{-n/2}\left|\int_D a_j^{-1}r_j^{-1}\exp(2\lambda^{(1)}(\psi)+\nu(\psi))dz^1d\bar{z}^1\cdots dz^n d\bar{z}^n\right|^{1/2}\|\eta'-\xi'\|_D,$$

where $\|\ \|_D$ denotes the latter factor in the preceding expression. Because of (2.7) and (2.9c), we have

$$\left|\int_D a_j^{-1}r_j^{-1}\exp\{2\lambda^{(1)}(\psi)+\nu(\psi)\}dz^1d\bar{z}^1\cdots dz^n d\bar{z}^n\right|$$

$$\leqq \exp\{2\lambda^{(2)}(\psi(p))+2\nu^{(1)}(\psi(p))\}\left|\int_D \exp\{-\nu(\psi)+b(\psi)\}r_j^{-1}dz^1\cdots d\bar{z}^n\right|$$

$$\leqq \frac{\varepsilon_j^{4n}}{2^n}C\exp\{2\lambda^{(2)}(\psi(p))+2\nu^{(1)}(\psi(p))\},$$

while

$$\|\eta'-\xi'\|_D \leqq \|\eta'-\xi'\|_{U_j}=\|\eta-\xi\|_{U_j}\leqq\{2\|\eta\|^2_{U_j}+2\|\xi\|^2_{\mathcal{L}}\}^{1/2}\leqq 2C_1\|h\|_\lambda.$$

Thus we have, from (3.12),

$$\frac{\varepsilon_j^{2n}}{2^n}|g_j(p)|\leqq\frac{1}{(2\pi)^n}\frac{\varepsilon_j^{2n}}{\sqrt{2^n}}\sqrt{\pi^n}\sqrt{C}\cdot(2C_1)\|h\|_\lambda\exp\{\lambda^{(2)}(\psi(p))+\nu^{(1)}(\psi(p))\}.$$

Putting $C_2=\dfrac{2\sqrt{C}\cdot C_1}{(2\pi)^{n/2}}$, we finally have

$$|g_j(p)|\cdot\exp\{-\lambda^{(2)}(\psi(p))-\nu^{(1)}(\psi(p))\}\leqq C_2\cdot\|h\|_\lambda. \tag{3.13}$$

Because $|R_j(p)|\leqq 1$ and $|\tilde{h}_j(p)|\leqq\|h\|_\lambda\exp\{\lambda^{(1)}(\psi(p))\}$, we have proved that

$$|H(p)|\cdot\exp\{-\lambda^{(2)}(\psi(p))-\nu^{(1)}(\psi(p))\}\leqq C_3\cdot\|h\|_\lambda. \tag{3.14}$$

§4. Nishimura's Case

In this section we shall sketch that we can deal with Nishimura's case by our method partly. In this case $X=\boldsymbol{C}^n$ with the standard cartesian coordinates $(\zeta^1,\cdots,\zeta^n)$ and $\psi(\zeta)=\{1+\sum_\alpha|\zeta^\alpha|^2\}^{1/2}$. Y is defined by an entire function F as $Y=\{(\zeta)\,|\,F(\zeta)=0\}$ and F satisfies the conditions

$$\begin{cases}|F(\zeta)|\exp\{-B\psi(\zeta)^\eta\}\leqq M & \zeta\in X,\\ |dF(\zeta)|\exp\{B'\psi(\zeta)^\eta\}\geqq M'>0 & \zeta\in Y,\end{cases} \tag{4.1}$$

where $|dF(\zeta)|^2=\sum_\alpha|\partial F/\partial\zeta^\alpha|^2$ and η, M, M', B and B' are positive constants.

Nishimura's main result is as follows:

Theorem N. *Suppose that above conditions are satisfied, then for any* $B_1>0$, *there exist positive constants* B_2 *and* C_2 *such that for any* $h\in\mathcal{H}(Y, B_1t^\eta)$, *there exists an* $H\in\mathcal{H}(\mathbf{C}^n, B_2t^\eta)$ *satisfying* $H|_Y=h$ *and* $\|H\|_{B_2t^\eta}\leqq C_2\|h\|_{B_1t^\eta}$.

We shall assume that $\eta\geqq 1$, and shall show in the following that

Theorem 2. *If* $X=\mathbf{C}^n$ *and* Y *is defined by an* F *satisfying* (4.1) *with* $\eta\geqq 1$, *then the function* $\nu(t)$ *mentioned in Theorem* 1 *can be taken as* $\nu(t)=B_{10}t^\eta$, *where* B_{10} *is a suitable positive constant.*

Since $(t+i)^\eta\ll At^\eta$ for a constant A and for $i=1, 2$, it is clear that Theorem N follows from Theorems 1 and 2.

We shall sketch direct construction of the covers $\mathcal{U}$ and $\mathcal{U}'$. First we recall a classical lemma in the theory of functions of a complex variable.

Lemma 3. *Suppose* $w=f(z)$ *is a function of a complex variable* z, *holomorphic in the disk* $|z|<R$ *and* $|f(z)|\leqq M$ *there. Moreover assume that* $f(0)=0$ *and* $f'(0)=c_1\neq 0$. *Then the inverse function* $z=g(w)$ *(with* $g(0)=0$*) is defined at least in the domain* $|w|<|c_1|^2R^2/6M$ *and we have* $|g(w)|\leqq |c_1|\circ R^2/4M$ *there.*

For the proof see, for example, [1] Lemma 17. 7.1 pp. 385–386. We shall also make use of the following*)

Lemma 4. ([4], Lemma 2) *If* H *is a function in* $\mathcal{H}(\mathbf{C}^n, Bt^\eta)$ *for some* $\eta>0$, $B>0$, *then* $|1/m!\,\partial^{|m|}H/\partial\zeta^m|\leqq\|H\|_{Bt^\eta}(\sqrt{n})^{|m|}\exp\{-KBt^\eta\}$, *where* m *denotes the multi-index* $(m_1, \cdots, m_n)$, $|m|=\sum m_\alpha$ *and* $K=2^\eta+1$.

These being settled, we first verify without difficulty, that if p and $q\in\mathbf{C}^n$ are such that $\psi(p)\geqq 2$ and $\sum_\alpha|\zeta^\alpha(p)-\zeta^\alpha(q)|^2<\frac{1}{9}\{\sum|\zeta^\alpha(p)|^2\}^{-1}$, then we have $|\psi(q)-\psi(p)|<1$. In the construction of $\mathcal{U}$ and $\mathcal{U}'$ we pay attentions to that part for which $\psi(p)\geqq 2$. Remaining part can be dealt with separately.

Take $p\in Y$, we change the coordinate into (ξ_p^α) by a unitary transformation;

$$\xi_p^\alpha=\sum_\beta c_\beta^\alpha(p)(\zeta^\beta-\zeta^\beta(p)), \tag{4.2}$$

*) Note that our $\{1+\sum|\zeta^\alpha|^2\}^{\eta/2}$ and Nishimura's $(\sum|\zeta^\alpha|^2)^{\eta/2}+1$ define the equivalent norms in the sense of (1.1).

so that $\partial F/\partial\xi_p^\alpha(p)=0$ for $\alpha<n$. Then $\partial F/\partial\xi_p^n(p)$, which we denote by c_1, has the absolute value

$$(4.3)\qquad |c_1| = \{\sum_\alpha |\frac{\partial F}{\partial \xi_p^\alpha}(p)|^2\}^{1/2} \geqq \exp\{-(A'+B'\psi(p)^\eta)\},$$

where B' appears in (4.1) and A' is a constant determined by M'.

For the moment we fix $\xi_p^1, \cdots, \xi_p^{n-1}$ to be $=0$ and consider F as a function $f(\xi)=w$ of a single variable $\xi=\xi_p^n$. If we set

$$(4.4)\qquad \begin{cases} R = \frac{1}{3}\psi(p)^{-1/2}, \\ M = \exp\{A+B\psi(p)^\eta\}, \end{cases}$$

then we can apply Lemma 3 to f with R, M in (4.4). (A is determined in connection with M in (4.1).) Thus we have the inverse function $\xi=g(w)$, defined in a domain containing $\{w \mid |w| < \exp\{-(A_3+B_3\psi(p)^\eta)\}\}$. (Here and in the following, A_i, B_i are suitably chosen constants independent of p.)

In order to take the change of $\xi_p^1, \cdots, \xi_p^{n-1}$ into account, we consider the relation

$$w' = H(\tilde{\xi}_p, \xi) = G_p(\tilde{\xi}_p, \xi) - G_p(\tilde{\xi}_p, 0),$$

where $\tilde{\xi}_p$ stands for $(\xi_p^1, \cdots, \xi_p^{n-1})$ and G_p is the function F expressed in terms of (ξ_p). Because of Lemma 4 we have

$$|H(\tilde{\xi}_p, \xi)| = |G_p(\tilde{\xi}_p, \xi) - G_p(\tilde{\xi}_p, 0)| = |\int_0^\xi \partial G_p/\partial\xi(\tilde{\xi}_p, \tau)d\tau|$$
$$\leqq \sqrt{n}\, M \exp\{KB(\psi(p)+1)^\eta\}\, |\xi|.$$
$$|G_p(\tilde{\xi}_p, 0)| = |\int_0^1 \frac{d}{d\tau}\{G_p(\tau\tilde{\xi}_p, 0)\}d\tau| = |\int_0^1 \sum_{\alpha=1}^{n-1} \xi_p^\alpha \frac{\partial G_p}{\partial \xi_p^\alpha}(\tau\tilde{\xi}_p, 0)d\tau|$$
$$\leqq \sum_{\alpha=1}^{n-1} |\xi_p^\alpha| \cdot \sqrt{n}\, M \exp\{KB(\psi(p)+1)^\eta\},$$
$$\left|\frac{\partial H}{\partial \xi}(\tilde{\xi}_p, 0)\right| = \left|\frac{\partial G_p}{\partial \xi}(\tilde{\xi}_p, 0)\right| = \left|\frac{\partial G_p}{\partial \xi}(0, 0) + \int_0^1 \frac{d}{d\tau}\left\{\frac{\partial G_p}{\partial \xi}(\tau\tilde{\xi}_p, 0)\right\}d\tau\right|$$
$$\geqq \left|\frac{\partial G_p}{\partial \xi}(0, 0)\right| - \left|\int_0^1 \sum_{\alpha=1}^{n-1} \xi_p^\alpha \frac{\partial^2 G_p}{\partial \xi \partial \xi_p^\alpha}(\tau\tilde{\xi}_p, 0)d\tau\right|$$
$$\geqq |c_1| - \sum_{\alpha=1}^{n-1} |\xi_p^\alpha| \cdot 2nM \exp\{KB(\psi(p)+1)^\eta\}.$$

Hence if we choose A_4 and B_4 suitably and set

$$V_p = \{(\xi_p) \mid |\xi_p^\alpha| < \exp\{-(A_4+B_4\psi(p)^\eta)\}\},$$

we have, in V_p,

$$(4.5)\qquad \begin{cases} |H| \leqq \exp\{-(A_5+B_5\psi(p)^\eta)\}, \\ \left|\dfrac{\partial H}{\partial \xi}(\tilde{\xi}_p, 0)\right| \geqq \dfrac{|c_1|}{2}, \quad |c_1| \geqq \exp\{-(A'+B'\psi(p)^\eta)\}. \end{cases}$$

We apply Lemma 3 to H. When $|\xi_p^\alpha| < \exp\{-(A_4+B_4\psi(p)^\eta)\}$ for $\alpha=1, \cdots, n-1$, the relation $w'=H(\tilde{\xi}_p, \xi)$ can be solved as $\xi=\psi(\tilde{\xi}_p, w')$ for $|w'| < \exp\{-(A_6+B_6\psi(p)^\eta)\}$. We change A_4 and B_4 if necessary and can achieve that $|G_p(\tilde{\xi}_p, 0)| < \frac{1}{2}\exp\{-(A_6+B_6\psi(p)^\eta)\}$, then

$$\xi = \Psi(\tilde{\xi}_p, w-G_p(\tilde{\xi}_p, 0))$$

is holomorphic for $|w| < \frac{1}{2}\exp\{-(A_6+B_6\psi(p)^\eta)\}$. In other words, we have constructed a neighborhood V_p' of p and local coordinates $(\tilde{\xi}_p, w)$ such that

$$V_p' = \{(\tilde{\xi}_p, w) \mid |\xi_p^\alpha| < \exp\{-(A_4+B_4\psi(p)^\eta)\},\ |w| < \frac{1}{2}\exp\{-(A_6+B_6\psi(p)^\eta\}\},$$

where $w(q)=F(q)$ for $q \in V_p'$.

Take $U_p=\{(\tilde{\xi}_p, w) \mid |\xi_p^\alpha| < \rho,\ |w| < \rho\}$ and $U_p'=\{(\tilde{\xi}_p, w) \mid |\xi_p^\alpha| < (1-\varepsilon)\rho,\ |w| < (1-\varepsilon)\rho\}$, where $\rho=\exp\{-(A_7+B_7\psi(p)^\eta)\}$, ε is small positive and independent of p, in such a way that $\bar{U}_p \subset V_p$. Choose a locally finite system $\{U_j'\}_{j\in I_1}$ from among $\{U_p'\}_{p\in Y}$ so that $\cup U_j' \supset Y$. As a matter of fact we choose $\{p_j\}$ and denote U_{p_j}' by U_j'. (Similarly for U_j.) We add $\{U_j'\}_{j\in I_2}$ and $\{U_j\}_{j\in I_2}$ to obtain covers of X, as described in §2. Those U_j' and U_j $(j\in I_2)$ which intersect $\bigcup_{j\in I_1} U_j$ shall be taken in similar forms as U_p', U_p above with the origin p_j of (ξ_j)-coordinates as the centre of U_j', and $w_j=F-F(p_j)$ instead of $w_p=F$. Coordinates for U_j which does not intersect any $U_k (k\in I_1)$ can be taken as linear coordinates unitary equivalent to (ζ). ε will be taken to be common all through.

These coordinate systems are not normalized as described in §2, but dependence of ρ_j on j is known and its effect on the choice of the function ν is clear.

We have only to estimate r_{jk} appeared in §2. Take a pair (j, k) of indices in I_1 such that $U_j \cap U_k' \neq \phi$. ξ_k^α $(\alpha=1, \cdots, n-1)$ are linear functions of $(\xi_j^1, \cdots, \xi_j^n)$:

$$\xi_k^\alpha = \sum_{\beta,\gamma} c_\beta^\alpha(p_k)\cdot \bar{c}_\beta^\gamma(p_j)\xi_j^\gamma + \sum c_\beta^\alpha(p_k)(\zeta^\beta(p_j)-\zeta^\beta(p_k)) . \tag{4.6}$$

$c_\beta^\alpha(p_k)$ and $c_\beta^\gamma(p_j)$ are entries of unitary matrices and hence coefficients of the linear terms in (4.6) are bounded when j and k vary. In U_j we take ξ_j^α ($\alpha=1, \cdots, n-1$) and $w_j=F$ as coordinates. ξ_k^α ($1\leqq\alpha\leqq n-1$) are holomorphic functions of these coordinates. We have

$$\frac{\partial \xi_k^\alpha}{\partial w_j} = \sum_\beta c_\beta^\alpha(p_k)\cdot \bar{c}_\beta^n(p_j)\left(\frac{\partial F}{\partial \xi_j^n}\right)^{-1} .$$

Hence

$$\left|\frac{\partial \xi_k^\alpha}{\partial w_j}(q)\right| < \exp\{A_8+B_8\psi(q)^\eta\} \qquad \text{for} \quad q\in U_j .$$

Consider a point $q\in U_j\cap U_k'$, for which

$$|w_j(q)| < \exp\{-(A_9+B_9\psi(q)^\eta)\} ,$$

where A_9 and B_9 are suitably chosen. We have $|\xi_k^\alpha(q)|<(1-\varepsilon)\rho_k$ for $1\leqq\alpha\leqq n-1$, hence for $q'\in\pi_j^{-1}(\pi_j(q))$ for which $|w_j(q')|\leqq \exp\{-(A_9+B_9\psi(q)^\eta)\}$, we have

$$|\xi_k^\alpha(q')| <\rho_k \qquad (\alpha = 1, \cdots, n-1) ,$$

and we see that $q'\in U_j\cap U_k$.

This shows that r_j is not less than $\exp\{-(A_9+B_9\psi^\eta)\}$, and combined with above considerations, shows that we can take $\nu(t)$ to be $B_{10}t^\eta$.

Note added in Proof. The condition $\eta\geqq 1$ in Theorem 2 can be replaced by $\eta>0$, if we take $\psi(\zeta)$ to be $\psi(\zeta)=\frac{1}{2}\log\{1+\sum|\zeta^\alpha|^2\}$ and $\nu(t)=B_{10}\exp(\eta t)$.

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