Publ. RIMS, Kyoto Univ.
22 (1986), 259–327

Fock Space Representations of the Virasoro Algebra

— Intertwining Operators —

By

Akihiro TSUCHIYA*) and Yukihiro KANIE**)

Contents

Introduction

The *Virasoro algebra* $\mathcal{L}$ is the Lie algebra over the complex number field $\boldsymbol{C}$ of the following form:

$$\mathcal{L} = \sum_{n\in\boldsymbol{Z}} \boldsymbol{C}e_n \oplus \boldsymbol{C}e_0',$$

with the relations: for any $n, m \in \boldsymbol{Z}$

$$\begin{cases} [e_n, e_m] = (m-n)e_{n+m} + \dfrac{m^3-m}{12}\delta_{n+m,0}e_0', \\ [e_0', e_n] = 0. \end{cases}$$

This type of the Lie algebras was first introduced by a physist M.A. Virasoro (cf. S. Mandelstam [1974]), as the gauge group of the string model of elementary particle physics.

Consider the Lie algebra $\mathcal{L}'$ of trigonometric polynomial vector fields

Communicated by M. Kashiwara, September 11, 1985.
*) Department of Mathematics, Nagoya University, Nagoya 464, Japan
**) Department of Mathematics, Mie University, Tsu 514, Japan

on the circle:

$$\mathcal{L}' = \sum_{n\in \boldsymbol{Z}} \boldsymbol{C} l_n;\ [l_n, l_m] = (m-n) l_{n+m} \qquad (n, m \in \boldsymbol{Z});$$

$$l_n = z^{n+1}\frac{d}{dz} \qquad (z = e^{i\theta}).$$

Then the 2-dimensional Gel'fand-Fuks cohomology of $\mathcal{L}'$ is known as

$$H^2(\mathcal{L}'; \boldsymbol{C}) \cong \boldsymbol{C},$$

and its generator ϕ can be taken as

$$\phi(e_n, e_m) = \frac{m^3 - m}{12}\delta_{n+m,0}.$$

The Virasoro algebra $\mathcal{L}$ is the central extension of the Lie algebra $\mathcal{L}'$ defined by the cocycle ϕ.

Quite recently the Virasoro algebra was used to analyze the critical phenomena in the two dimensional statistical physics (cf. A.A. Belavin-A.M. Polyakov-A.B. Zamolodchikov [1984]). In that situation, the Virasoro algebra plays the symmetry group of the theory.

In the mathematical side, V.G. Kac ([1979]) studied the left $\mathcal{L}$-module $M(h, c)$ parametrized by $\boldsymbol{C}^2 \ni (h, c)$, called the Verma module. The Verma module $M(h, c)$ is the left $\mathcal{L}$-module with a cyclic vector $|h, c\rangle$ with the fundamental relations:

$$e_{-n}|h, c\rangle = 0 \quad (n \geqq 1);\ e_0|h, c\rangle = h|h, c\rangle, \quad e_0'|h, c\rangle = c|h, c\rangle.$$

V.G. Kac obtained the formula concerning the determinant of the matrix of the vacuum expectation values of $M(h, c)$ (cf. §2). Using this Kac's determinant formula, F.L. Feigin and D.B. Fuks ([1983]) determined completely the composition series of $M(h, c)$ for each $(h, c) \in \boldsymbol{C}^2$.

In this paper, we construct another kind of representations of $\mathcal{L}$ parametrized by $\boldsymbol{C}^2 \ni (w, \lambda)$, which we call the Fock space representation $(\mathcal{F}(w, \lambda), \mathcal{L})$, and intertwining operators between them, and investigate these $\mathcal{L}$-modules $\mathcal{F}(w, \lambda)$ and $M(h, c)$ and their relationship.

At first consider the associative algebra $\mathcal{A}$ over $\boldsymbol{C}$ generated by p_n $(n \in \boldsymbol{Z})$ and Λ with the following Bose commutation relations:

$$[p_n, p_m] = m\delta_{n+m,0};\ [\Lambda, p_n] = 0 \qquad (n, m \in \boldsymbol{Z}).$$

For each $(w, \lambda) \in \boldsymbol{C}^2$, we consider the left $\mathcal{A}$-module $\mathcal{F}(w, \lambda)$ with a cyclic vector $|w, \lambda\rangle$ with the fundamental relations:

$$p_{-n}|w, \lambda\rangle = 0 \quad (n \geqq 1); \; p_0|w, \lambda\rangle = w|w, \lambda\rangle; \; \Lambda|w, \lambda\rangle = \lambda|w, \lambda\rangle .$$

Now consider the following operators which can act on the Fock spaces $\mathscr{F}(w, \lambda)$:

$$L_n = (p_0 - n\Lambda)p_n + \frac{1}{2}\sum_{j=1}^{n-1} p_j p_{n-j} + \sum_{j\geqq 1} p_{n+j}p_{-j} \qquad (n \geqq 1);$$

$$L_{-n} = (p_0 + n\Lambda)p_{-n} + \frac{1}{2}\sum_{j=1}^{n-1} p_{-j}p_{j-n} + \sum_{j\geqq 1} p_j p_{-n-j} \qquad (n \geqq 1);$$

$$L_0 = \frac{1}{2}(p_0^2 - \Lambda^2) + \sum_{j\geqq 1} p_j p_{-j};$$

$$L_0' = (-12\Lambda^2 + 1)id .$$

Then the first result is the following (Proposition 1.3):

Theorem 0.1. *The operators L_n $(n \in \mathbb{Z})$ and L_0' satisfy the commutation relations of the Virasoro algebra: for $n, m \in \mathbb{Z}$*

$$\begin{cases} [L_n, L_m] = (m-n)L_{n+m} + \dfrac{m^3 - m}{12}\delta_{n+m,0}L_0'; \\ [L_0', L_n] = 0 . \end{cases}$$

In the special case where $\Lambda = 0$, these operators are the ones which were introduced by M.A. Virasoro (cf. S. Mandelstam [1974]).

By using the canonical homomorphism π (i.e. $\pi(e_n) = L_n$ $(n \in \mathbb{Z})$; $\pi(e_0') = L_0'$), we get the left $\mathcal{L}$-module $(\mathscr{F}(w, \lambda), \pi_{(w,\lambda)}, \mathcal{L})$ which is called the Fock space representation, and by the explicit formulae of L_n and L_0',

$$\begin{cases} L_0|w, \lambda\rangle = \dfrac{1}{2}(w^2 - \lambda^2)|w, \lambda\rangle; \quad L_0'|w, \lambda\rangle = (1 - 12\lambda^2)|w, \lambda\rangle; \\ L_{-n}|w, \lambda\rangle = 0 \qquad \text{for} \quad n \geqq 1 . \end{cases}$$

By the universal property of the Verma module $M(h, c)$ as an $\mathcal{L}$-module, for each $(w, \lambda) \in \mathbb{C}^2$ we get the unique $\mathcal{L}$-module mapping

$$\pi_{w,\lambda}: \; M(h(w, \lambda), c(\lambda)) \to \mathscr{F}(w, \lambda)$$

which sends the vacuum vector $|h(w, \lambda), c(\lambda)\rangle \in M(h(w, \lambda), c(\lambda))$ to the vacuum vector $|w, \lambda\rangle \in \mathscr{F}(w, \lambda)$, where

$$h(w, \lambda) = \frac{1}{2}(w^2 - \lambda^2) \quad \text{and} \quad c(\lambda) = 1 - 12\lambda^2 .$$

Then by using Theorems 0.3 and 0.6, we get the following (cf. Propositions 2.7 and 2.8).

Proposition 0.2. *Fix a pair* $(w, \lambda)\in \boldsymbol{C}^2$.

1) *The canonical* $\mathcal{L}$*-module mapping*

$$\pi_{w,\lambda}\colon M(h(w, \lambda), c(\lambda)) \to \mathcal{F}(w, \lambda)$$

is isomorphic, if and only if the equation

$$w+\frac{a}{2}s-\frac{b}{s}=0$$

has no integral solutions $(a, b)\in \boldsymbol{Z}^2$ *with* $a\geqq 1$ *and* $b\geqq 1$, *where* $s\in \boldsymbol{C}^*$ *is a root of the equation* $\lambda=\frac{1}{s}-\frac{s}{2}$.

2) *The* $\mathcal{L}$*-module* $\mathcal{F}(w, \lambda)$ *is irreducible, if and only if the equation*

$$w+\frac{a}{2}s-\frac{b}{s}=0$$

has no integral solutions $(a, b)\in \boldsymbol{Z}^2$ *with* $ab\geqq 1$, *where* $s\in \boldsymbol{C}^*$ *is a root of the equation* $\lambda=\frac{1}{s}-\frac{s}{2}$.

And this condition is equivalent to the fact that the corresponding Verma module $M(h(w, \lambda), c(\lambda))$ *is irreducible.*

To construct intertwining operators between Fock spaces, we introduce the operators of following type acting on $\mathcal{F}(w, \lambda)$. Fix $s\in \boldsymbol{C}^*$, and consider

$$X(s, \zeta) = \exp\left(s\sum_{n=1}^{\infty}\zeta^n\frac{p_n}{n}\right)\exp\left(-s\sum_{n=1}^{\infty}\zeta^{-n}\frac{p_{-n}}{n}\right)\zeta^{sp_0-(s^2/2)}T_s\,,$$

and for any $a\geqq 1$

$$Z(s;\zeta_1,\cdots,\zeta_a) = F\left(\frac{s^2}{2};\zeta_1,\cdots,\zeta_a\right)\exp\left(s\sum_{n=1}^{\infty}(\zeta_1^n+\cdots+\zeta_a^n)\frac{p_n}{n}\right)$$
$$\times\exp\left(-s\sum_{n=1}^{\infty}(\zeta_1^{-n}+\cdots+\zeta_a^{-n}\frac{p_{-n}}{n}\right)T_{as}\,,$$

where

$$T_s\colon \mathcal{F}(w,\lambda) \to \mathcal{F}(w+s, \lambda)$$

is the operator such that

$$T_s|w, \lambda\rangle = |w+s, \lambda\rangle;\ [T_s, p_n] = 0\quad (n\neq 0);\ [T_s, \Lambda] = 0\,,$$

and

$$F(\alpha;\zeta_1,\cdots,\zeta_a)=\prod_{j=1}^{a}\zeta_j^{-(a-1)\alpha}\prod_{1\leqq i<j\leqq a}(\zeta_i-\zeta_j)^{2\alpha}.$$

Operators of this type are called Vertex Operators (cf. S. Mandelstam [1974] and E. Date et al. [1983]).

Then $X(s;\zeta)$ and $Z(s;\zeta_1,\cdots,\zeta_a)$ are multi-valued holomorphic functions of $\zeta\in\boldsymbol{C}^*$ and $(\zeta_1,\cdots,\zeta_a)\in M_a$ respectively with valued in the operators acting on $\mathcal{F}(w,\lambda)$'s, where M_a is the manifold defined by

$$M_a=\{(\zeta_1,\cdots,\zeta_a)\in(\boldsymbol{C}^*)^a;\ \zeta_i\neq\zeta_j\ (1\leqq i<j\leqq a)\}.$$

For each $\alpha\in\boldsymbol{C}^*$, denote by $\mathcal{S}_\alpha^*$ the local coefficient system with values in $\boldsymbol{C}$ which is determined by the monodromy of the multi-valued holomorphic function $F(\alpha;\zeta_1,\cdots,\zeta_a)$ on M_a, and denote by $\mathcal{S}_\alpha$ the dual local system of $\mathcal{S}_\alpha^*$.

Fix $s\in\boldsymbol{C}^*$ and an integer $a\geqq 1$, and take an element $\Gamma\in H_a(M_a;\mathcal{S}_\alpha)$. For each integer $b\in\boldsymbol{Z}$, we consider the operator

$$O(s,\Gamma;a,b)=\int_\Gamma Z(s;\zeta_1,\cdots,\zeta_a)\zeta_1^{-b-1}\cdots\zeta_a^{-b-1}d\zeta_1\cdots d\zeta_a.$$

Main Theorem 0.3 (Theorem 3.8).

1) *For each* $(w,\lambda)\in\boldsymbol{C}^2$, *the operator* $O(s,\Gamma;a,b)$ *acts as*

$$O(s,\Gamma;a,b):\ \mathcal{F}(w,\lambda)\to\mathcal{F}(w+as,\lambda).$$

2) *Take* $s\in\boldsymbol{C}^*$ *and* $a,b\in\boldsymbol{Z}$ *with* $a\geqq 1$. *Put* $\lambda=\lambda(s)=\frac{1}{s}-\frac{s}{2}$, *then the operator*

$$O(s,\Gamma;a,b):\ \mathcal{F}\Big(-\frac{a}{2}s-\frac{b}{s},\lambda\Big)\to\mathcal{F}\Big(\frac{a}{2}s-\frac{b}{s},\lambda\Big)$$

commutes with the action of $\mathcal{L}$.

For suitable $s\in\boldsymbol{C}^*$ and $w\in\boldsymbol{C}$, the equation

$$w=\frac{a}{2}s-\frac{b}{s}$$

has a countable number of integral solutions $(a,b)\in\boldsymbol{Z}^2$, if and only if $\alpha=s^2/2$ is a rational number. This index α characterizes the property of the monodromy of the function

$$F(\alpha;\zeta_1,\cdots,\zeta_a)=\prod_{j=1}^{a}\zeta_j^{-(a-1)\alpha}\prod_{1\leqq i<j\leqq a}(\zeta_i-\zeta_j)^{2\alpha}.$$

If α is irrational, then the monodromy of the function F is of logarithmic type, and if α is rational, then the monodromy of F is of algebraic type.

The essential points to prove this theorem are the following two formulae (Propositions 3.3 and 3.4).

Proposition 0.4. *For any* $\zeta\in M_a$ *and* $s\in \boldsymbol{C}^*$,

$$X(s, \zeta_1)\cdots X(s, \zeta_a) = Z(s; \zeta_1, \cdots, \zeta_a)\,(\zeta_1\cdots\zeta_a)^{sp_0+(a/2)s^2}.$$

Proposition 0.5 (Conformal Covariance).

$$[L_m, X(s, \zeta)] = \zeta^{-m}\Big(\zeta\frac{d}{d\zeta}-m\Big(sA+\frac{s^2}{2}\Big)\Big)X(s, \zeta)$$

for each $m\in \boldsymbol{Z}$ *and* $s, \zeta\in \boldsymbol{C}^*$.

In the final step, we must construct a cycle $\Gamma(\alpha)\in H_a(M_a; \mathcal{S}_\alpha)$ which gives a nontrivial intertwining operator $O(s, \Gamma; a, b)$.

If we expand

$$\exp\Big(s\sum_{n=1}^{\infty}(\zeta_1^n+\cdots+\zeta_a^n)\frac{P_n}{n}\Big)\exp\Big(-s\sum_{n=1}^{\infty}(\zeta_1^{-n}+\cdots+\zeta_a^{-n})\frac{P_{-n}}{n}\Big)$$

as a Laurent series of $(\zeta_1, \cdots, \zeta_a)$, then the coefficient of the each term of the operator

$$\int_\Gamma Z(s; \zeta_1, \cdots, \zeta_a)\zeta_1^{-b-1}\cdots\zeta_a^{-b-1}d\zeta_1\cdots d\zeta_a$$

is written as

$$\int_\Gamma F\Big(\frac{s^2}{2}; \zeta_1, \cdots, \zeta_a\Big)\zeta_1^{-l_1-1}\cdots\zeta_a^{-l_a-1}d\zeta_1\cdots\zeta_a,$$

and this integral is reduced to

$$\int_{\Gamma_2}\prod_{j=1}^{a-1}k_j^{-(a-1)\alpha}(1-k_j)^{2\alpha}\prod_{1\leq i<j\leq a-1}(k_i-k_j)^{2\alpha}k_1^{-l_1-1}\cdots k_{a-1}^{-l_{a-1}-1}dk_1\cdots dk_{a-1},$$

where $\Gamma=\Gamma_1\times\Gamma_2\in H_a(M_a; \mathcal{S}_\alpha)\cong H_1(\boldsymbol{C}^*; \boldsymbol{C})\otimes H_{a-1}(Y_{a-1}; \mathcal{S}_\alpha)$, Γ_1 is a generator of $H_1(\boldsymbol{C}^*; \boldsymbol{C})$, and Y_{a-1} is the manifold defined by

$$\begin{aligned}Y_{a-1} = \{(k_1, \cdots, k_{a-1})\in \boldsymbol{C}^{a-1};\ & k_i\neq 0, 1 \quad (1\leq i\leq a-1),\\ & k_i\neq k_j \quad (1\leq i<j\leq a-1)\}.\end{aligned}$$

In §4 and §5, we construct the cycle $\Gamma_2(\alpha)\in H_{a-1}(Y_{a-1}; \mathcal{S}_\alpha)$ which regularizes the divergent integral

$$\int_{\Delta(a-1)}\prod_{j=1}^{a-1}k_j^{-(a-1)\alpha}(1-k_j)^{2\alpha}\prod_{1\leq i<j\leq a-1}(k_i-k_j)^{2\alpha}k_1^{-l_1-1}\cdots k_{a-1}^{-l_{a-1}-1}dk_1\cdots dk_{a-1},$$

where $\Delta(a-1)$ is the $(a-1)$-simplex defined by

$$\Delta(a-1) = \{(k_1, \cdots, k_{a-1}) \in \mathbb{R}^{a-1};\ 1 > k_1 > \cdots > k_{a-1} > 0\}.$$

In the construction of the cycle, we use the technique of resolutions of singularities. We think that the results in §5 are interesting for further study of the above integral.

Consider the set

$$\Omega = \{\alpha \in \mathbb{C};\ d(d+1)\alpha \notin \mathbb{Z},\ d(a-d)\alpha \notin \mathbb{Z}\ (1 \leqq d \leqq a-1)\},$$

then we get

Theorem 0.6. *There exists $\Gamma(\alpha) \in H_a(M_a; \mathcal{S}_\alpha)$ which depends holomorphically on $\alpha \in \Omega$ such that the operator*

$$O(s; a, b) = O\left(s, \Gamma\left(\frac{s^2}{2}\right); a, b\right) \colon \mathcal{F}\left(w-as, \frac{1}{s}-\frac{s}{2}\right) \to \mathcal{F}\left(w, \frac{1}{s}-\frac{s}{2}\right)$$

is nontrivial in the sense that for any $w \in \mathbb{C}$

1) *for $b \geqq 0$, the image $O(s; a, b)\,|w-as, \frac{1}{s}-\frac{s}{2}\rangle$ is a nonzero vector.*

2) *for $b<0$, there is a vector $|v\rangle \in \mathcal{F}\left(w-as, \frac{1}{s}-\frac{s}{2}\right)$ such that $O(s; a, b)|v\rangle = |w, \frac{1}{s}-\frac{s}{2}\rangle$.*

In the appendix, we construct the Fock space representations by using the charged Fermion operators and explain their relationship to the Bose formalism. In mathematical languages, the Fermi formalism corresponds to the spinor representations and the Bose formalism corresponds to the Weil-Segal representations.

Recently appeared the paper of Vl.S. Dotsenko and V.A. Fateev [1984] which seems to be very closely related to our results.

Finally we express our thanks to Professor K. Aomoto for valuable discussions.

Notations

$\#S$: the cardinal number of the set S
$\mathbb{Z}$: the ring of rational integers
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers

$\boldsymbol{C}^*$: the group of non-zero complex numbers
$\boldsymbol{C}[x_\gamma; \gamma\in\Gamma]$: the ring of polynomials in variables $\{x_\gamma; \gamma\in\Gamma\}$ over $\boldsymbol{C}$
$\boldsymbol{C}[[x_\gamma; \gamma\in\Gamma]]$: the ring of formal power series in variables $\{x_\gamma; \gamma\in\Gamma\}$ over $\boldsymbol{C}$
$\boldsymbol{C}[x_\gamma, x_\gamma^{-1}; \gamma\in\Gamma]$: the ring of Laurent polynomials in variables $\{x_\gamma; \gamma\in\Gamma\}$ over $\boldsymbol{C}$
$\boldsymbol{C}[[x_\gamma, x_\gamma^{-1}; \gamma\in\Gamma]]$: the ring of formal Laurent series in variables $\{x_\gamma; \gamma\in\Gamma\}$ over $\boldsymbol{C}$
$\Lambda(x_\gamma; \gamma\in\Gamma)$: the exterior algebra in variables $\{x_\gamma; \gamma\in\Gamma\}$ over $\boldsymbol{C}$
$\delta_{i,j}$: Kronecker's delta, that is, $\delta_{i,j}=0$ $(i\neq j)$, or 1 $(i=j)$
$\varprojlim_n A_n$: the projective limit of a projective system $\{A_n\}$
$a|b$: means that an integer b is divisible by an integer a

Let S be a subset of an ambient topological space X. Denote by $\bar{S}$ the closure of S in X, and by $\operatorname{int} S$ the interior of S in X.

§1. Canonical Commutation Relations and Fock Space Representations

1.1) Canonical Commutation Relations

The purpose of this paragraph is to present some facts about canonical commutation relations and Fock space representations. Consider operators p_n $(n\in\boldsymbol{Z})$ and Λ with the following commutation relations:

$$\begin{cases} [p_n, p_m] = m\delta_{n+m,0}id & (n, m\in\boldsymbol{Z}), \\ [\Lambda, p_n] = 0 . \end{cases} \tag{1.1.1}$$

We denote by $\mathcal{A}$ the associative algebra over $\boldsymbol{C}$ generated by operators p_n $(n\in\boldsymbol{Z})$ and Λ with the defining relations (1.1.1). The algebra $\mathcal{A}$ is made $\boldsymbol{Z}$-graded algebra by defining degrees as $\deg p_n=n$ and $\deg \Lambda=0$. Then $\mathcal{A}$ is decomposed to the sum of homogeneous components as

$$\mathcal{A} = \sum_{d\in\boldsymbol{Z}} \mathcal{A}(d). \tag{1.1.2}$$

Consider an index set $M=(m_1, m_2, \cdots)$ with non-negative integers m_j satisfying $||M||:=\sum_j jm_j<\infty$. We denote by $P_+(M)$ and $P_-(M)$ the elements $\cdots p_2^{m_2}p_1^{m_1}$ and $p_{-1}^{m_1}p_{-2}^{m_2}\cdots$ of the algebra $\mathcal{A}$ respectively. Then any homogeneous element of $\mathcal{A}$ of degree d is uniquely represented in the following form:

$$\sum_{||M||-||N||=d} c_{M,N}(p_0, \Lambda)P_+(M)\, P_-(N), \tag{1.1.3}$$

where $c_{M,N}(p_0, \Lambda)\in\boldsymbol{C}[p_0, \Lambda]$ and the above summation is finite.

Define a decreasing filtration

$$\mathcal{A}(d) = \mathcal{A}_0(d)\supset\mathcal{A}_1(d)\supset\mathcal{A}_2(d)\supset\cdots \tag{1.1.4}$$

by the following rules; the element

$$\sum_{||M||-||N||=d} c_{M,N}(p_0, \Lambda)P_+(M)P_-(N) \in \mathcal{A}(d)$$

belongs to $\mathcal{A}_n(d)$, if and only if $c_{M,N}(p_0, \Lambda)=0$ for $||N||<n$. Then we get the composition

$$\mathcal{A}_{n_1}(d_1) \times \mathcal{A}_{n_2}(d_2) \ni (a_1, a_2) \mapsto a_1 \circ a_2 \in \mathcal{A}_n(d_1+d_2) \tag{1.1.5}$$

with $n=\max(n_1-d_1, n_2)$. Define the completed vector space $\hat{\mathcal{A}}(d)$ by this filtration as

$$\hat{\mathcal{A}}(d) = \varprojlim_n \mathcal{A}(d)/\mathcal{A}_n(d) . \tag{1.1.6}$$

Then any element in $\hat{\mathcal{A}}(d)$ is represented in the following form:

$$\sum_{||M||-||N||=d} c_{M,N}(p_0, \Lambda)P_+(M)P_-(N) \in \hat{\mathcal{A}}(d) , \tag{1.1.7}$$

where $c_{M,N}(p_0, \Lambda) \in \mathbf{C}[p_0, \Lambda]$ and the summation may be infinite in this time. Then

$$\hat{\mathcal{A}}(d) = \hat{\mathcal{A}}_0(d) \supset \hat{\mathcal{A}}_1(d) \supset \hat{\mathcal{A}}_2(d) \supset \cdots \tag{1.1.8}$$

defines a completed linear Hausdorff topology and the embedding

$$\mathcal{A}(d) \supset \hat{\mathcal{A}}(d) \tag{1.1.9}$$

has a dense image and the composition (1.1.5) can be extended to a continuous map

$$\hat{\mathcal{A}}_{n_1}(d_1) \times \hat{\mathcal{A}}_{n_2}(d_2) \to \hat{\mathcal{A}}_n(d_1+d_2) \tag{1.1.10}$$

in a unique manner. And the sum of these spaces

$$\hat{\mathcal{A}} = \sum_{d \in \mathbf{Z}} \hat{\mathcal{A}}(d) \supset \mathcal{A} \tag{1.1.11}$$

becomes a topological graded algebra.

For each $(w, \lambda) \in \mathbf{C}^2$, consider the left $\mathcal{A}$-module $\mathcal{F}(w, \lambda) = \mathcal{A}|w, \lambda\rangle$ with the cyclic vector $|w, \lambda\rangle$ which has the following defining relations:

$$\begin{cases} p_{-n}|w, \lambda\rangle = 0 \qquad (n \geqq 1) \\ p_0|w, \lambda\rangle = w|w, \lambda\rangle \\ \Lambda|w, \lambda\rangle = \lambda|w, \lambda\rangle . \end{cases} \tag{1.1.12}$$

We call that this vector $|w, \lambda\rangle$ is the vacuum vector of $\mathcal{F}(w, \lambda)$. Similarly we define the right $\mathcal{A}$-module $\mathcal{F}^\dagger(w, \lambda) = \langle\lambda, w|\mathcal{A}$ with the vacuum vector

$\langle\lambda, w|$,

$$(1.1.13)\qquad \begin{cases} \langle\lambda, w|p_n = 0 & (n\geqq 1) \\ \langle\lambda, w|p_0 = \langle\lambda, w|w \\ \langle\lambda, w|\Lambda = \langle\lambda, w|\lambda . \end{cases}$$

The *vacuum expectation value*

$$(1.1.14)\qquad \langle\ |\ \rangle\colon\ \mathcal{F}^{\dagger}(w, \lambda)\times\mathcal{F}(w, \lambda)\to \boldsymbol{C}$$

is uniquely defined by the following conditions:

i) $\langle\ |\ \rangle$ is $\boldsymbol{C}$-bilinear; ii) $\langle\lambda, w|w, \lambda\rangle=1$;

iii) $\langle v\, a|v'\rangle=\langle v|a\, v'\rangle$ for any $\langle v|\in\mathcal{F}^{\dagger}(w, \lambda)$, $|v'\rangle\in\mathcal{F}(w, \lambda)$ and $a\in\mathcal{A}$.

A basis of $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^{\dagger}(w, \lambda)$ over $\boldsymbol{C}$ is given by

$$(1.1.15)\qquad \begin{cases} |M, w, \lambda\rangle = P_+(M)|w, \lambda\rangle & \text{with}\quad M=(m_1, m_2, \cdots) \quad\text{and} \\ \langle\lambda, w, N| = \langle\lambda, w|P_-(N) & \text{with}\quad N=(n_1, n_2, \cdots) \end{cases}$$

respectively. Then the vacuum expectation values are explicitly given by the following formula

$$(1.1.16)\qquad \langle\lambda, w, N|M, w, \lambda\rangle = \delta_{N,M}n^M M!\,,$$

where

$$(1.1.17)\qquad \begin{cases} \delta_{N,M} = \delta_{n_1,m_1}\delta_{n_2,m_2}\cdots\delta_{n_j,m_j}\cdots\,, \\ n^M = 1^{m_1}2^{m_2}\cdots j^{m_j}\cdots & \text{and} \\ M! = m_1!m_2!\cdots m_j!\cdots\,. \end{cases}$$

Define the grading in $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^{\dagger}(w, \lambda)$ by setting

$$(1.1.18)\qquad \deg|M, w, \lambda\rangle = \|M\| \quad\text{and}\quad \deg\langle\lambda, w, N| = \|N\|\,,$$

and decompose them into the sum of homogeneous components:

$$(1.1.19)\qquad \mathcal{F}(w, \lambda) = \sum_{d\geqq 0}\mathcal{F}_d(w, \lambda) \quad\text{and}\quad \mathcal{F}^{\dagger}(w, \lambda) = \sum_{d\geqq 0}\mathcal{F}^{\dagger}_d(w, \lambda)\,,$$

where $\mathcal{F}_d(w, \lambda)$ and $\mathcal{F}^{\dagger}_d(w, \lambda)$ is the space of homogeneous elements of degree d. This gives $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^{\dagger}(w, \lambda)$ the graded $\hat{\mathcal{A}}$-module structure. By enumerating the number of the basis (1.1.15), we get

$$(1.1.20)\qquad \dim\mathcal{F}_d(w, \lambda) = \dim\mathcal{F}^{\dagger}_d(w, \lambda) = p(d)\,,$$

where $p(d)$ is the number of partitions of the integer d.

By the formula (1.1.16), we get the following proposition.

Proposition 1.1. *Denote by* $\langle\ |\ \rangle_{d_1,d_2}$ *the restriction of the vacuum expecta-*

tion value to the homogeneous components $\mathcal{F}^{\dagger}_{d_1}(w, \lambda) \times \mathcal{F}_{d_2}(w, \lambda)$. *Then* $\langle \mid \rangle_{d_1,d_2}$ *is trivial unless* $d_1 = d_2$ *and* $\langle \mid \rangle_d = \langle \mid \rangle_{d,d}$ *is always nonsingular.*

Note that the restriction $\langle \mid \rangle_d$ is independent of the parameters w and λ, so denote by D_d the matrix determined by the restriction $\langle \mid \rangle_d$, that is, D_d is the diagonal matrix whose diagonal components are given in (1.1.16).

For each integer d, the submodule $\mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))(d)$ of $\mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))$ is defined by

$$\text{(1.1.21)} \quad \mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))(d) = \prod_{n=0}^{\infty} \mathrm{Hom}_C(\mathcal{F}_n(w, \lambda), \mathcal{F}_{n+d}(w, \lambda)) .$$

And the filtration of $\mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))(d)$ is defined by

$$\text{(1.1.22)} \quad G_0 \supseteqq G_1 \supseteqq G_2 \supseteqq \cdots ,$$

$$G_n = \{\phi \in \mathrm{Hom}_C(\mathcal{F}(w,\lambda), \mathcal{F}(w, \lambda))(d);\ \phi = 0 \quad \text{on} \quad \sum_{k=0}^{n-1} \mathcal{F}_k(w, \lambda)\} .$$

Then this filtration defines a complete Hausdorff linear topology on the space $\mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))(d)$ and the composition of mappings is continuous in this topology.

Consider the canonical mapping

$$\text{(1.1.23)} \quad \Phi(w, \lambda):\ \hat{\mathcal{A}}(d) \ni a \mapsto \Phi(w, \lambda)(a) \in \mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w,\lambda))(d)$$

defined by

$$\Phi(w, \lambda)(a) \mid v\rangle = a \mid v\rangle \quad \text{for} \quad \mid v\rangle \in \mathcal{F}(w, \lambda) .$$

Proposition 1.2. *The mapping* (1.1.23)

$$\Phi(w, \lambda):\ \hat{\mathcal{A}}(d) \to \mathrm{Hom}_C(\mathcal{F}(w, \lambda), \mathcal{F}(w, \lambda))(d)$$

is surjective and preserves the filtrations for any $d \in \mathbb{Z}$.

For two elements

$$a = \sum_{\|M\| - \|N\| = d} c_{M,N}(p_0, \Lambda) P_+(M) P_-(N) \quad \text{and}$$

$$a' = \sum_{\|M\| - \|N\| = d} c'_{M,N}(p_0, \Lambda) P_+(M) P_-(N)$$

the equality $\Phi(w, \lambda)(a) = \Phi(w, \lambda)(a')$ holds, if and only if $c_{M,N}(w, \lambda) = c'_{M,N}(w, \lambda)$ for any M and N with $\|M\| - \|N\| = d$.

By the commutation relations (1.1.1), we get the isomorphism

$$\mathcal{A} = \mathbb{C}[p_n(n \geqq 1)] \otimes \mathbb{C}[p_0, \Lambda] \otimes \mathbb{C}[p_n(n \leqq -1)]$$

as left $\boldsymbol{C}[p_n(n\geqq 1)]$- and right $\boldsymbol{C}[p_n(n\leqq -1)]$-modules. Hence we can define the *normal product*

$$:\ :\ :\boldsymbol{C}[p_n(n\in\boldsymbol{Z}),\ \Lambda]\rightarrow\mathcal{A} \tag{1.1.24}$$

as the uniquely defined $\boldsymbol{C}$-linear isomorphism under the condition that the mapping (1.1.24) is a left $\boldsymbol{C}[p_n(n\geqq 1)]$- and right $\boldsymbol{C}[p_n(n\leqq -1)]$-module map and satisfies the equalities $:\Lambda^n p_0^m:=\Lambda^n p_0^m$ for any $n, m\geqq 0$. Moreover this normal product on $\mathcal{A}$ can be naturally extended to the completion $\hat{\mathcal{A}}$.

Remark. The Fock space representations $\mathcal{F}(w, \lambda)$ of the algebra $\hat{\mathcal{A}}$ can be constructed on a function space of infinitely many variables $x_0', x_0, x_1, x_2, \cdots$. In fact, let

$$\Psi:\ \mathcal{F}(w, \lambda)\rightarrow V=\boldsymbol{C}[x_1, x_2, \cdots]e^{wx_0+\lambda x_0'}) \tag{1.1.25}$$

be the $\boldsymbol{C}$-linear mapping defined by

$$\Psi(|M, w, \lambda\rangle)=n^M x^M\ e^{wx_0+\lambda x_0'}\,, \tag{1.1.26}$$

where

$$x^M=x_1^{m_1}x_2^{m_2}\cdots\ . \tag{1.1.27}$$

Define the action of the operators in the algebra $\hat{\mathcal{A}}$ on the space V as

$$p_n=nx_n\ \ (n\geqq 1)\,,\quad p_{-n}=\frac{\partial}{\partial x_n}\ \ (n\geqq 0)\quad\text{and}\quad \Lambda=\frac{\partial}{\partial x_0'}\,. \tag{1.1.28}$$

Then it is easily seen that the mapping Ψ is an $\hat{\mathcal{A}}$-module isomorphism.

1.2) Fock Space Representations of the Virasoro Algebra

For a formal vaiiable $z\in\boldsymbol{C}^*$, define the operator

$$p(z)=\sum_{n\in\boldsymbol{Z}}p_n z^n\,, \tag{1.2.1}$$

and

$$\begin{aligned} L(z) &= \frac{1}{2}:p(z)^2:\ -\Lambda z\frac{d}{dz}p(z)-\frac{1}{2}\Lambda^2 \\ &= \sum_{n\in\boldsymbol{Z}}L_n z^n\,, \end{aligned} \tag{1.2.2}$$

then $L_n\in\hat{\mathcal{A}}(n)$ and the explicit forms of the operators L_n are given as follows: for $n\geqq 1$

(1.2.3)
$$\begin{cases} L_n = (p_0 - nA)p_n + \dfrac{1}{2}\sum_{j=1}^{n-1} p_j p_{n-j} + \sum_{j\geq 1} p_{n+j} p_{-j}, \\ L_{-n} = (p_0 + nA)p_{-n} + \dfrac{1}{2}\sum_{j=1}^{n-1} p_{-j} p_{j-n} + \sum_{j\geq 1} p_j p_{-n-j}, \\ L_0 = \dfrac{1}{2}(p_0^2 - A^2) + D_0, \end{cases}$$

where

(1.2.4)
$$D_0 = \sum_{j\geq 1} p_j p_{-j}$$

is the Euler operator, so the homogeneous decompositions of the Fock spaces are eigen spaces for D_0, that is,

(1.2.5)
$$\begin{cases} \mathcal{F}_d(w, \lambda) = \{|v\rangle \in \mathcal{F}(w, \lambda);\ D_0|v\rangle = d|v\rangle\}, \\ \mathcal{F}_d^\dagger(w, \lambda) = \{\langle v| \in \mathcal{F}^\dagger(w, \lambda);\ \langle v|D_0 = \langle v|d\}. \end{cases}$$

By elementary but long calculations we get the following proposition.

Proposition 1.3 (Commutation Relations for L_n).

(1.2.6)
$$[L_n, L_m] = (m-n)L_{n+m} + \frac{m^3-m}{12}\delta_{n+m,0}(-12A^2+1)id.$$

(1.2.7)
$$\begin{cases} [L_n, p_m] = mp_{n+m} + m^2 A\delta_{n+m,0}, \\ [L_n, p(z)] = z^{-n}\left(z\dfrac{d}{dz} - n\right)p(z) + n^2 A z^{-n}. \end{cases}$$

The *Virasoro algebra* $\mathcal{L}$ is the Lie algebra over $\mathbb{C}$ of the following form:

(1.2.8)
$$\mathcal{L} = \sum_{n\in\mathbb{Z}} \mathbb{C}e_n \oplus \mathbb{C}e_0',$$

with the relations

(1.2.9)
$$\begin{cases} [e_n, e_m] = (m-n)e_{n+m} + \dfrac{m^3-m}{12}\delta_{n+m,0}e_0', \\ e_0' \in \text{the center of the Lie algebra } \mathcal{L}. \end{cases}$$

Let $U(\mathcal{L})$ be denoted the universal enveloping algebra of $\mathcal{L}$. Put

(1.2.10)
$$\pi(e_n) = L_n \quad (n\in\mathbb{Z}) \quad \text{and} \quad \pi(e_0') = (1-12A^2)id,$$

then by Proposition 1.3, we can define the homomorphism

(1.2.11)
$$\pi:\ U(\mathcal{L}) \to \hat{\mathcal{A}},$$

and the action of $\hat{\mathcal{A}}$ on $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^\dagger(w, \lambda)$ gives the representation $\pi_{(w,\lambda)}$ of the Lie algebra $\mathcal{L}$ on $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^{\dagger}(w, \lambda)$ respectively. This representation is called *the Fock space representation* of the Virasoro algebra $\mathcal{L}$.

Introduce the *polarization* of the Virasoro algebra $\mathcal{L}$ as

(1.2.12) $$\mathcal{L} = \mathfrak{n}_{+} \oplus \mathfrak{h} \oplus \mathfrak{n}_{-} ,$$

where

$$\begin{cases} \mathfrak{n}_{+} = \sum_{n \geqq 1} \boldsymbol{C} e_n , \quad \mathfrak{n}_{-} = \sum_{n \leqq -1} \boldsymbol{C} e_n \\ \mathfrak{h} = \boldsymbol{C} e_0 + \boldsymbol{C} e_0' . \end{cases}$$

The dual $\mathfrak{h}^* = \mathrm{Hom}_{\boldsymbol{C}}(\mathfrak{h}, \boldsymbol{C})$ of the abelian subalgebra $\mathfrak{h}$ is identified with $\boldsymbol{C}^2$ by setting for $(h, c) \in \boldsymbol{C}^2$,

(1.2.13) $$(h, c)(e_0) = h \quad \text{and} \quad (h, c)(e_0') = c .$$

The homogeneous decompositions (1.1.19) of the left and right $\mathcal{L}$-modules $\mathcal{F}(w, \lambda)$ and $\mathcal{F}^{\dagger}(w, \lambda)$ respectively coincide with the weight space decomposition w.r.t. the subalgebra $\mathfrak{h}$, and the homogeneous components $\mathcal{F}_d(w, \lambda)$ and $\mathcal{F}_d^{\dagger}(w, \lambda)$ are the weight spaces belonging to the same weight $(h(w, \lambda)+d,\ c(\lambda))$, where

(1.2.14) $$h(w, \lambda) = \frac{1}{2}(w^2 - \lambda^2) \quad \text{and} \quad c(\lambda) = -12\lambda^2 + 1 ,$$

that is,

(1.2.15)
$$\begin{cases} \mathcal{F}_d(w, \lambda) = \{ |v\rangle \in \mathcal{F}(w, \lambda);\ L_0 |v\rangle = (h(w, \lambda)+d)|v\rangle,\ L_0' |v\rangle = c(\lambda)|v\rangle \} , \\ \mathcal{F}_d^{\dagger}(w, \lambda) = \{ \langle v| \in \mathcal{F}^{\dagger}(w, \lambda);\ \langle v| L_0 = \langle v|(h(w, \lambda)+d),\ \langle v| L_0' = \langle v| c(\lambda) \} . \end{cases}$$

1.3) Here we set up the fundamental problems for the Fock space representations $\mathcal{F}(w, \lambda)$ of the Virasoro algebra $\mathcal{L}$. In this and the succeeding articles, we will discuss these problems.

Firstly we give the definition of *singular vectors.* A vector $v \in \mathcal{F}(w, \lambda)$ is called singular if it satisfies the equalities

(1.3.1) $$\pi_{(w,\lambda)}(e_{-n})v = 0 \qquad (n = 1, 2, \cdots) .$$

Denote by $\mathcal{S}(w, \lambda)$ the set of singular vectors in $\mathcal{F}(w, \lambda)$ and by $\mathcal{S}_d(w, \lambda)$ the set of singular vectors of degree d.

Fundamental Problems *Take a pair* $(w, \lambda) \in \boldsymbol{C}^2$.

(1) *When the* $\mathcal{L}$*-module* $\mathcal{F}(w, \lambda)$ *is irreducible?*

(2) *Then the map* $U(\mathcal{L}) \ni a \mapsto a|w, \lambda\rangle \in \mathcal{F}(w, \lambda)$ *is surjective?*

(3) *Determine the subspace* $\mathcal{S}(w, \lambda)$ *of* $\mathcal{F}(w, \lambda)$. *In particular, is* $\mathcal{S}(w, \lambda)$ *bigger than* $\mathbf{C}|w,\lambda\rangle$?

(3') *Are there singular vectors of* $\mathcal{F}(w, \lambda)$ *of positive deggree d? that is, is* $\mathcal{S}_d(w, \lambda)$ *trivial or not?*

(4) *in the case when* $\mathcal{F}(w, \lambda)$ *is reducible, determine its composition tequence.*

Note. We can set up the similar problems for the dual module ${}^{\dagger}\mathcal{F}(w, \lambda)$. However the problems for $\mathcal{F}^{\dagger}(w, \lambda)$ will be solved in the same moment as for $\mathcal{F}(w, \lambda)$.

§2. The Verma Module of the Virasoro Algebra

2.1) Verma Modules and Kac's Determinant

For convenience, we summarize here some results about the Verma modules for the Virasoro algebra $\mathcal{L}$.

Fix the polarization of $\mathcal{L}$

(2.1.1) $$\mathcal{L} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- ,$$

as (1.2.12). Take an element (h, c) of the dual $\mathfrak{h}^*$ of the abelian subalgebra $\mathfrak{h}$. Similarly as the Fock space representations (§1.1), we can define the left $\mathcal{L}$-module $M(h, c)$ and the right $\mathcal{L}$-module $M^{\dagger}(h, c)$, with the vacuum vectors $|h, c\rangle \in M(h, c)$ and $\langle c, h| \in M^{\dagger}(h, c)$ respectively. The vacuum vectors are defined by the following relations:

(2.1.2) $$\begin{cases} e_{-n}|h, c\rangle = 0 & (n \geqq 1) \\ e_0|h, c\rangle = h|h, c\rangle; \quad e_0'|h, c\rangle = c|h, c\rangle, \end{cases}$$

(2.1.3) $$\begin{cases} \langle c, h|e_n = 0 & (n \geqq 1) \\ \langle c, h|e_0 = \langle c, h|h; \quad \langle c, h|e_0' = \langle c, h|c . \end{cases}$$

The Verma modules $M(h, c)$ and $M^{\dagger}(h, c)$ are generated by the vacuum vectors $|h, c\rangle$ and $\langle c, h|$ respectively.

Then by Birkhoff-Witt's theorem, the universal enveloping algebra $U(\mathcal{L})$ has the basis

(2.1.4) $$e_+(M)e_0^{l_0}e_0'^{l_1}e_-(N)$$

for multi-indices $M=(m_1, m_2, \cdots)$ and $N=(n_1, n_2, \cdots)$ of non-negative integers with $\|M\|=\sum_j jm_j$, $\|N\|=\sum_j jn_j<\infty$, and non-negative integers l_0 and l_1, where

(2.1.5) $$e_+(M) = \cdots e_2^{m_2}e_1^{m_1} \quad \text{and} \quad e_-(N) = e_{-1}^{n_1}e_{-2}^{n_2}\cdots .$$

And $U(\mathcal{L})$ is also a $\boldsymbol{Z}$-graded algebra as

$$U(\mathcal{L})=\sum_{d\in\boldsymbol{Z}} U(\mathcal{L})(d)\,, \tag{2.1.6}$$

by setting

$$\deg e_n = n \quad (n\in\boldsymbol{Z}) \quad \text{and} \quad \deg e_0' = 0\,, \tag{2.1.7}$$

and any element of $U(\mathcal{L})(d)$ is uniquely represented as

$$\sum_{\|M\|-\|N\|=d} c_{M,N}(e_0, e_0')e_+(M)e_-(N)\,, \tag{2.1.8}$$

where $c_{M,N}(e_0, e_0')$ is a polynomial of e_0 and e_0'. The Verma modules $M(h, c)$ and $M^\dagger(h, c)$ have the basis over $\boldsymbol{C}$:

$$\begin{cases} |M, h, c\rangle = e_+(M)|h, c\rangle\,, & \deg|M, h, c\rangle = \|M\| \quad \text{and} \\ \langle c, h, N| = \langle c, h|e_-(N)\,, & \deg\langle c, h, N| = \|N\| \end{cases} \tag{2.1.9}$$

for multi-indices M and N with $\|M\|, \|N\|<\infty$, and they have also a structure of graded $U(\mathcal{L})$-modules:

$$M(h, c)=\sum_{d\geqq 0} M_d(h, c) \quad \text{and} \quad M^\dagger(h, c)=\sum_{d\geqq 0} M_d^\dagger(h, c) \tag{2.1.10}$$

where

$$\begin{cases} M_d(h, c) = \sum_{\|M\|=d} \boldsymbol{C}|M, h, c\rangle \quad \text{and} \\ M_d^\dagger(h, c) = \sum_{\|M\|=d} \boldsymbol{C}\langle c, h, M|\,. \end{cases} \tag{2.1.11}$$

Then this homogeneous decomposition (2.1.10) is also the weight space decompositions of the Verma modules $M(h, c)$ and $M^\dagger(h, c)$ with respect to the abelian subalgebra $\mathfrak{h}$ respectively. And the subspaces $M_d(h, c)$ and $M_d^\dagger(h, c)$ belong to the same weight $(h+d, c)$, and their dimensions are also the same number $p(d)$.

The *vacuum expectation value*

$$\langle\ |\ \rangle_{h,c}\colon M^\dagger(h, c)\times M(h, c)\to \boldsymbol{C} \tag{2.1.12}$$

is uniquely defined by the following conditions:

$$\begin{cases} \text{i) } \boldsymbol{C}\text{-bilinear;} \quad \text{ii) } \langle c, h|h, c\rangle_{h,c}=1; \\ \text{iii) } \langle va|w\rangle_{h,c}=\langle v|aw\rangle_{h,c} \\ \qquad \text{for any } \langle v|\in M^\dagger(h, c),\ |w\rangle\in M(h, c) \text{ and } a\in U(\mathcal{L}). \end{cases}$$

Then the restriction of the vacuum expectation values to the subspace $M_d^\dagger(h, c)\times M_{d'}(h, c)$ is trivial unless $d=d'$.

Let τ be an automorphism (or anti-automorphism) of an algebra $\mathcal{G}$ and let ϕ be a $\boldsymbol{C}$-linear mapping between a left $\mathcal{G}$-modules $\mathcal{F}_1$ and a left (or right, respectively) $\mathcal{G}$-module $\mathcal{F}_2$. Then we say the map $\phi: \mathcal{F}_1 \to \mathcal{F}_2$ is *defined over* $(\mathcal{G}, \tau)$, if the following equality holds: for any $g \in \mathcal{G}$ and $f \in \mathcal{F}_1$,

$$(2.1.13) \qquad \phi(gf) = \tau(g)\phi(f) \quad \text{or} \quad \phi(gf) = \phi(f)\tau(g)$$

respectively.

Let σ be the $\boldsymbol{C}$-linear anti-isomorphism of the universal enveloping algebra $U(\mathcal{L})$ defined by

$$(2.1.14) \qquad \sigma(e_n) = e_{-n} \quad (n \in \boldsymbol{Z}) \quad \text{and} \quad \sigma(e_0') = e_0',$$

and define the $\boldsymbol{C}$-linear isomorphism

$$(2.1.15) \qquad \sigma: M(h, c) \to M^{\dagger}(h, c)$$

defined over $(U(\mathcal{L}), \sigma)$ by

$$\sigma(|h, c\rangle) = \langle c, h|.$$

Then σ preserves weights for $\mathfrak{h}$ and for any M,

$$\sigma(|M, h, c\rangle) = \langle c, h, M|.$$

Define the bilinear form on $M(h, c)$

$$(2.1.16) \qquad (\ |\ ): M(h, c) \times M(h, c) \to \boldsymbol{C}$$

by

$$(|v\rangle \,|\, |v'\rangle) = \langle \sigma(|v\rangle) | v'\rangle .$$

Consider the restriction of the vacuum expectation values to the degree d subspaces:

$$(2.1.17) \qquad \langle\ |\ \rangle_{h,c}^{d}: M_d^{\dagger}(h, c) \times M_d(h, c) \to \boldsymbol{C} .$$

Take the basis (2.1.11) in $M_d(h, c)$ and $M_d^{\dagger}(h, c)$, then the matrices of the bilinear forms $(2.1.16)_d$ and (2.1.17) coincide with each other, and denote them by $A_d(h, c)$, that is

$$(2.1.18) \quad A_d(h, c)_N{}^M = \langle c, h, M | N, h, c\rangle_{h,c}^{d} = (|M, h, c\rangle \,|\, |N, h, c\rangle)_d .$$

It is easily seen that the matrix $A_d(h, c)$ is symmetric.

The determinant of the matrix $A_d(h, c)$ plays a very crucial role in the irreducibility problem of Verma modules, and is calculated by V.G. Kac as a function of parameters $(h, c)\in\mathfrak{h}^*$.

Proposition 2.1 (V.G. Kac [1979], F.L. Feigin-D.B. Fuks [1982]).

$$\det{}^2 A_d(h, c) = \text{const.} \prod_{k=1}^{d} \prod_{j|k} \Phi_{j,k/j}(h, c)^{p(d-k)}, \tag{2.1.19}$$

where

$$\Phi_{k_1,k_2}(h, c) := \prod_{j=1}^{2} \{h+\frac{1}{24}(k_j^2-1)(c-13)+\frac{1}{2}(k_1k_2-1)\}+\frac{1}{16}(k_1^2-k_2^2)^2. \tag{2.1.20}$$

Here we remark the following symmetry of the function (2.1.20):

$$\Phi_{k_1,k_2}(h, c) = \Phi_{-k_1,-k_2} = \Phi_{k_2,k_1} = \Phi_{-k_2,-k_1} \tag{2.1.21}$$

for any (h, c), moreover we can decompose it to the product of linear factors:

Proposition 2.2.

$$\begin{aligned}\Phi_{k_1,k_2}(h, c) &= \frac{1}{4}(w+\frac{s}{2}k_1-\frac{1}{s}k_2)(w-\frac{s}{2}k_1+\frac{1}{s}k_2)\\ &\times(w+\frac{s}{2}k_2-\frac{1}{s}k_1)(w-\frac{s}{2}k_2+\frac{1}{s}k_1)\end{aligned} \tag{2.1.22}$$

where

$$\begin{cases} c = c(\lambda) := -12\lambda^2+1 = \left(\frac{3}{\alpha}-2\right)(3\alpha-2), \\ h = h(w, \lambda) := \frac{1}{2}(w^2-\lambda^2) = \frac{w^2}{2}-\frac{(\alpha-1)^2}{4\alpha}, \\ \alpha = \alpha(s) := \frac{s^2}{2} \quad \text{and} \quad \lambda = \lambda(s) := \frac{1}{s}-\frac{s}{2}. \end{cases} \tag{2.1.23}$$

Note. The meaning of the parametrization (2.1.23) will be explained below.

Note. $\boldsymbol{C}^*\ni s\mapsto\lambda=\frac{1}{s}-\frac{s}{2}\in\boldsymbol{C}$ is a double covering with branching points $s=\pm\sqrt{2}i$.

The mapping

$$\pi:\ \boldsymbol{C}^2\ni(w, \lambda)\mapsto(h, c)\in\boldsymbol{C}^2 \tag{2.1.24}$$

defined by (2.1.23) is a 4-fold branched covering.

Remark. In this article, we use the same notations for vectors of Verma modules $M(h, c)$ and Fock spaces $\mathcal{F}(w, \lambda)$, for example $|h, c\rangle$ and $|w, \lambda\rangle$. But we consider that the readers aren't confused, because we always use the letters

h, c for parameters of Verma modules and w, λ for Fock spaces.

2.2) Relations of Fock Spaces and Verma Modules

For each $(w, \lambda)\in \boldsymbol{C}^2$, let $h=h(w, \lambda)$ and $c=c(\lambda)$ as are given in (2.1.23). Due to the universality of the Verma modules and the formulae (1.2.3), we get the unique left $\mathcal{L}$-module map

$$\pi_{w,\lambda}\colon\ M(h(w, \lambda), c(\lambda)) \to \mathcal{F}(w, \lambda) \tag{2.2.1}$$

which sends the vacuum $|h(w, \lambda), c(\lambda)\rangle$ to the vacuum $|w, \lambda\rangle$, and the unique right $\mathcal{L}$-module map

$$\pi^{\dagger}_{w,\lambda}\colon\ M^{\dagger}(h, c) \to \mathcal{F}^{\dagger}(w, \lambda) \tag{2.2.2}$$

which sends the vacuum to the vacuum. They preserve degrees and can be explicitly represented as

$$\begin{aligned}\pi_{w,\lambda}(|M, h, c\rangle) &= \pi(e_+(M))|w, \lambda\rangle \\ &= \sum_{||N||=d} C_d(w, \lambda)_N{}^M|N, w, \lambda\rangle,\end{aligned} \tag{2.2.3}$$

$$\begin{aligned}\pi^{\dagger}_{w,\lambda}(\langle c, h, M|) &= \langle\lambda, w|\pi(e_-(M)) \\ &= \sum_{||N||=d} C^{\dagger}_d(w, \lambda)_M{}^N\langle\lambda, w, N|\end{aligned} \tag{2.2.4}$$

for a multi-index M with $||M||=d$, where $C_d(w, \lambda)_N{}^M$ and $C^{\dagger}_d(w, \lambda)_N{}^M$ are polynomials of w and λ.

Then we can see that the vacuum expectation values are compatible in the following sense.

Proposition 2.3. *For any $(w, \lambda)\in \boldsymbol{C}^2$, the following diagram is commutative.*

$$\begin{array}{ccccc} \langle\ |\ \rangle\colon & M^{\dagger}(h(w, \lambda), c(\lambda)) \times & M(h(w, \lambda), c(\lambda)) & \to & \boldsymbol{C} \\ & \downarrow \pi^{\dagger}_{w,\lambda} & \downarrow \pi_{w,\lambda} & & \downarrow \mathrm{id} \\ \langle\ |\ \rangle\colon & \mathcal{F}^{\dagger}(w, \lambda) \times & \mathcal{F}(w, \lambda) & \longrightarrow & \boldsymbol{C} \end{array} \tag{2.2.5}$$

Let σ be the $\boldsymbol{C}$-linear anti-automorphism of the algebra $\hat{\mathcal{A}}$ defined by

$$\sigma(p_n) = p_{-n} \quad (n\in \boldsymbol{Z}) \quad \text{and} \quad \sigma(\Lambda) = -\Lambda\,. \tag{2.2.6}$$

Then by the formulae (1.2.3) and (2,.1.14), the following diagram commutes:

$$\begin{array}{cccc} \sigma\colon & U(\mathcal{L}) & \longrightarrow & U(\mathcal{L}) \\ & \downarrow \pi & & \downarrow \pi \\ \sigma\colon & \hat{\mathcal{A}} & \longrightarrow & \hat{\mathcal{A}} \end{array} \tag{2.2.7}$$

Define the $\boldsymbol{C}$-linear isomorphism

$$\sigma\colon \mathcal{F}_d(w, -\lambda) \to \mathcal{F}_d^\dagger(w, \lambda) \tag{2.2.8}$$

defined over $(\hat{\mathcal{A}}, \sigma)$ which sends the vacuum vector $|w, -\lambda\rangle$ to the vacuum $\langle\lambda, w|$. Then the following diagram commutes:

$$\begin{array}{ccc} \sigma\colon M_d(h, c) & \longrightarrow & M_d^\dagger(h, c) \\ \downarrow \pi_{w,-\lambda} & & \downarrow \pi^\dagger_{w,\lambda} \\ \sigma\colon \mathcal{F}_d(w, -\lambda) & \longrightarrow & \mathcal{F}_d^\dagger(w, \lambda) \end{array} \tag{2.2.9}$$

Hence the matrices of the mappings $\pi_{w,-\lambda}$ and $\pi^\dagger_{w,\lambda}$ are related as

$$C_d^\dagger(w, \lambda)_M{}^N = C_d(w, -\lambda)_N{}^M\,. \tag{2.2.10}$$

Then by the above commutative diagrams, we get the decomposition of the Kac's matrix $A_d(h, c)$.

Proposition 2.4. *For each* $(w, \lambda)\in\boldsymbol{C}^2$ *and any* $d\geqq 0$,

$$A_d(h(w, \lambda), c(\lambda)) = {}^tC_d(w, \lambda)D_dC_d(w, -\lambda) \tag{2.2.11}$$

that is,

$$A_d(h, c)_N{}^M = \sum_{\|N'\|=\|M'\|=d} C_d(w, \lambda)_{N'}{}^N D_{d,N'}{}^{M'} C_d(w, \lambda)_{M'}{}^M\,. \tag{2.2.12}$$

Note again that D_d is a nonsingular diagonal matrix whose diagonal components are given in (1.1.16).

2.3) Singular Vectors

In this paragraph we summarize results about the Verma modules and the Fock space representations.

At first we give the definition of singular vectors of the Verma module. A vector v of $M(h, c)$ is called *singular*, if

$$e_{-n}v = 0 \tag{2.3.1}$$

for any $n\geqq 1$. Denote by $\mathcal{S}_d(h, c)$ the set of singular vectors of degree d in $M(h, c)$.

Recall that for any $(w, \lambda)\in\boldsymbol{C}^2$, $(h, c)\in\mathfrak{h}^*$ and $d\geqq 0$,

$$\begin{aligned} \dim M_d(h, c) &= \dim M_d^\dagger(h, c) \\ &= \dim \mathcal{F}_d(w, \lambda) = \dim \mathcal{F}_d^\dagger(w, \lambda) = p(d)\,. \end{aligned} \tag{2.3.2}$$

Then standard arguments lead us to the following proposition.

Proposition 2.5. *For each* $(w, \lambda)\in \boldsymbol{C}^2$, *let*

$$h = h(w, \lambda) = \frac{1}{2}(w^2-\lambda^2) \quad and \quad c = c(\lambda) = 1-12\lambda^2 . \tag{2.3.3}$$

Then the following conditions are equivalent.

(1) *The Verma module* $M(h(w, \lambda), c(\lambda))$ *is irreducible.*

(2) *There is no singular vector in* $M(h, c)$ *of positive degree.*

(3) $\det A_d(h, c)\neq 0$ *for any* $d\geqq 0$.

(4) *The mappings* $\pi_{w,\lambda}: M(h, c)\to \mathcal{F}(w, \lambda)$ *and* $\pi^{\dagger}_{w,\lambda}: M^{\dagger}(h, c)\to \mathcal{F}^{\dagger}(w, \lambda)$ *are isomorphisms.*

(5) $\det C_d(w, \lambda)\neq 0$ *and* $\det C_d(w, -\lambda)\neq 0$ *for any* $d\geqq 0$.

(6) *There are no singular vectors in* $\mathcal{F}(w, \lambda)$ *and* $\mathcal{F}(w, -\lambda)$ *of positive degree.*

(7) *The Fock space representation* $\mathcal{F}(w, \lambda)$ *is irreducible.*

Moreover we get the following proposition which is essentially proved by F.L. Feigin and D.B. Fuks [1982].

Proposition 2.6. (i) *For any* $(w, \lambda)\in \boldsymbol{C}^2$ *and* $d\geqq 1$,

$$\dim \mathcal{S}_d(w, \lambda)\leqq 1 . \tag{2.3.4}$$

(ii) *The set*

$$\Sigma_{\mathcal{F}}(d) = \{(w, \lambda)\in \boldsymbol{C}^2;\ \dim \mathcal{S}_d(w, \lambda) = 1\} \tag{2.3.5}$$

is an algebraic set of $\boldsymbol{C}^2$.

Proof. Fix an integer $d\geqq 1$. The space $\mathcal{S}_d=\mathcal{S}_d(w, \lambda)$ of singular vectors of degree d is the kernel of the linear mapping L^-:

$$L^- = \sum_{n\geqq 1} L_{-n}\colon \mathcal{F}_d = \mathcal{F}_d(w, \lambda) \to \overline{\mathcal{F}}_{d-1} = \sum_{k=0}^{d-1} \mathcal{F}_k(w, \lambda) . \tag{2.3.6}$$

For a multi-index $M=(m_1, m_2, \cdots)$, put

$$|M\rangle = \begin{cases} |M, w, \lambda\rangle/n^M n! & \text{if } m_j\geqq 0 \text{ for any } j\geqq 1 \\ 0 & \text{if } m_j<0 \text{ for some } j\geqq 1 \end{cases} \tag{2.3.7}$$

Then we get

$$p_n|M\rangle=n(m_n+1)|M+\delta_n\rangle \quad\text{and}\quad p_{-n}|M\rangle=|M-\delta_n\rangle \qquad (n\geqq 1), \tag{2.3.8}$$

where δ_j is the multi-index whose i-th component equals to δ_{ij} (Kronecker's delta).

Introduce the order among these vectors as the lexicographic order for corresponding multi-indices $M=(m_1, m_2, \cdots)$, that is, the vector $|M\rangle$ is called higher than $|N\rangle$ (denoted as $|M\rangle > |N\rangle$), if there exists $j \geqq 1$ such that

$$m_i = n_i \quad (i<j) \qquad \text{and} \qquad m_j > n_j .$$

By this filtration in the space $\mathcal{F}(w, \lambda)$, decompose $\mathcal{F}_d$ and $\overline{\mathcal{F}}_{d-1}$ as

$$(2.3.9) \quad \begin{cases} \mathcal{F}_d = \sum_{j=0}^{d} \mathcal{F}_d^j ; \ \mathcal{F}_d^j = \text{span of } \{|M\rangle ; \ \|M\| = d, \ m_1 = d-j\}, \\ \overline{\mathcal{F}}_{d-1} = \sum_{j=0}^{d-1} \overline{\mathcal{F}}_{d-1}^j ; \ \overline{\mathcal{F}}_{d-1}^j = \text{span of } \{|N\rangle ; \ \|N\| \leqq d-1, \ n_1 = d-1-j\}, \end{cases}$$

then we get

$$(2.3.10) \quad \begin{cases} \dim \mathcal{F}_d^0 = q(0) = 1 ; \ \dim \mathcal{F}_d^1 = q(1) = 0 ; \\ \dim \mathcal{F}_d^j = q(j) \leqq q(0) + \cdots + q(j-2) = \dim \overline{\mathcal{F}}_{d-1}^{j-2} \quad (d \geqq j \geqq 2) , \end{cases}$$

where

$$(2.3.11) \qquad q(j) = \sharp\{(m_1, m_2, \cdots) ; \ m_i \geqq 0, \sum_{i \geqq 2} i m_i = j\} .$$

By the remark (2.3.10), it is sufficient for the statement (i) to show that the restriction of the operator L^- to the subspace $\sum_{j=2}^{d} \mathcal{F}_d^j$ is injective.

By the formula (2.3.8), we get the explicit formula for the operator L^- as

$$(2.3.12) \quad L^-|M\rangle = \sum_{\|N\| \leqq d-1} E_N{}^M |N\rangle$$
$$= \sum_{n=1}^{d} \{(w+n\lambda)|M-\delta_n\rangle + \frac{1}{2} \sum_{j=1}^{n-1} |M-\delta_j-\delta_{n-j}\rangle + \sum_{j \geqq 1} j(m_j+1)|M+\delta_j-\delta_{j+n}\rangle\} .$$

Note that all of $E_N{}^M$ are polynomials of w and λ, so the second statement (ii) is obvious. By this expression (2.3.12) of the operator L^-, we get

$$L^-(\mathcal{F}_d^k) \subset \sum_{j=k-2}^{k+1} \overline{\mathcal{F}}_{d-1}^j$$

for any k, in particular

$$(2.3.13) \qquad E_N{}^M = 0 \qquad \text{for} \quad |N\rangle \in \overline{\mathcal{F}}_{d-1}^k \quad \text{and} \quad |M\rangle \in \mathcal{F}_d^j \quad (j \geqq k+3) .$$

For $|M\rangle \in \mathcal{F}_d$, let $|N_M\rangle$ be the highest vector among the vectors $|N\rangle$ with $E_N{}^M \neq 0$, and we call $|N_M\rangle$ the peak of the column corresponding to the vector $|M\rangle$. Then for $|M\rangle \in \mathcal{F}_d$ with $m_1 \leqq d-2$, $E_{N_M}{}^M$ belongs to the submatrix $E(k)$ for $0 \leqq k \leqq d-2$, where

(2.3.14) $$E(k) = (E_N{}^M)_{|N\rangle \in \mathcal{F}^k_{d-1},\ |M\rangle \in \mathcal{F}^{k+2}_d} .$$

Hence by the remark (2.3.13), it is sufficient for the statement (i) to show that the matrix $E(k)$ is of full rank for $0 \leqq k \leqq d-2$.

The operator $L^-(k)$

$$L^-(k):\ \mathcal{F}^{k+2}_d \to \mathcal{F}^k_{d-1}$$

corresponding to the submatrix $E(k)$ is given as

(2.3.15) $$L^-(k)|M\rangle = (d-k-1)\sum_{n=1}^{d-1} |M+\delta_1-\delta_{n+1}\rangle .$$

For any $M=(m_1, m_2, \cdots)$, we get the peak corresponding to the vector $|M\rangle$ explicitly as

(2.3.16) $$N_M = M+\delta_1-\delta_g \qquad (g = \max\{n;\ m_n \neq 0\}) .$$

Note that for any $|N\rangle \in \mathcal{F}^k_{d-1}$, there is only one vector $|M\rangle \in \mathcal{F}^{k+2}_d$ such that $E_N{}^M \neq 0$, that is, $M=N-\delta_1+\delta_l$ with $l=d+1-||N||$. Hence the columns corresponding to all vectors $|M\rangle \in \mathcal{F}^{k+2}_d$ are linearly independent, that is, the matrix $E(k)$ is of full rank for $0 \leqq k \leqq d-2$. *q e.d.*

2.4) Determinant Formula for Fock Spaces

In this paragraph we show the determinant formula for the canonical $\mathcal{L}$-module mapping

(2.2.1) $$\pi_{w,\lambda}:\ M(h, c) \to \mathcal{F}(w, \lambda) .$$

Here we recall the relations among the complex parameters h, c, w, λ, s and α:

(2.1.23) $$\begin{cases} c = c(\lambda) := -12\lambda^2+1 = \left(\dfrac{3}{\alpha}-2\right)(3\alpha-2) , \\ h = h(w, \lambda) := \dfrac{1}{2}(w^2-\lambda^2) = \dfrac{w^2}{2} - \dfrac{(\alpha-1)^2}{4\alpha} , \\ \alpha = \alpha(s) := \dfrac{s^2}{2} \quad \text{and} \quad \lambda = \lambda(s) := \dfrac{1}{s} - \dfrac{s}{2} , \end{cases}$$

For each $\lambda \in \boldsymbol{C}$, let $s_\pm$ be the roots of the equation $\lambda=\lambda(s)$, and denote

(2.4.1) $$w_{a,b} = w_{a,b}(s_+) = \frac{a}{2}s_+ + \frac{b}{s_+} = \frac{a}{2}s_+ + \frac{b}{2}s_- .$$

Then we get

Proposition 2.7.

$$(2.4.2)\qquad \begin{cases} \det C_d(w, \lambda) = \text{const.} \prod_{k=1}^{d} \prod_{a|k} \left(w + \frac{a}{2}s_+ + \frac{k}{2a}s_-\right)^{p(d-k)} \\ \det C_d^\dagger(w, \lambda) = \text{const.} \prod_{k=1}^{d} \prod_{a|k} \left(w - \frac{a}{2}s_+ - \frac{k}{2a}s_-\right)^{p(d-k)} \end{cases}$$

As a corollary, we get

Proposition 2.8.

(1) *The $\mathcal{L}$-module mapping $\pi_{w,\lambda}$: $M(h(w, \lambda), c(\lambda)) \to \mathcal{F}(w, \lambda)$ is isomorphic, if and only if the equation*

$$(2.4.3)\qquad w + \frac{a}{2}s_+ + \frac{b}{2}s_- = 0$$

has no integral solutions $(a, b) \in \boldsymbol{Z}^2$ with $a \geqq 1$ and $b \geqq 1$.

(2) *The $\mathcal{L}$-module mapping $\pi_{w,\lambda}^\dagger$: $M^\dagger(h(w, \lambda), c(\lambda)) \to \mathcal{F}^\dagger(w, \lambda)$ is isomorphic, if and only if the equation* (2.4.3) *has no integral solutions $(a, b) \in \boldsymbol{Z}^2$ with $a \leqq -1$ and $b \leqq -1$.*

(3) *$\mathcal{F}(w, \lambda)$ is irreducible as an $\mathcal{L}$-module, if and only if the equation* (2.4.3) *has no integral solutions $(a, b) \in \boldsymbol{Z}^2$ with $ab \geqq 1$.*

Now we summarize the fact about the existence of nontrivial intertwining operators between Fock spaces in the form used in the proof of Propositions 2.1 and 2.7. This is immediately obtained from Theorem 3.8 and Proposition 4.4. (Note that the proofs of these statements are not dependent on the results obtained in this section.)

Proposition 2.9. *Let a, b be positive integers, and $\lambda \in \boldsymbol{C}$. Assume that $\alpha(s) \notin \boldsymbol{Q}$. Put $w = w_{a,b}(s_+)$ and $w' = w_{-a,b}(s_+)$. Then there exist intertwining operators*

$$O(a, b) \colon \mathcal{F}(w', \lambda) \to \mathcal{F}(w, \lambda)$$

and

$$O(a, -b) \colon \mathcal{F}(-w, \lambda) \to \mathcal{F}(-w', \lambda)$$

such that i) *$O(a, b)|w', \lambda\rangle$ is a nonzero singular vector of degree ab; and* ii) *there exists a vector $|v\rangle \in \mathcal{F}_{ab}(-w, \lambda)$ such that $O(a, -b)|v\rangle = |-w', \lambda\rangle$.*

2.5) Proofs of Determinant Formulas

In this paragraph, we prove Theorem 2.1 and Proposition 2.7, and use the notations in §§2.4.

Proof of Proposition 2.1. At first, we note that the degree of det $A_d(h, c)$ as a polynomial of h is equal to $\sum_{j=1}^{d} jp_j(d)$, where $p_j(d)$ is the number of partitions of the positive integer d by just j positive integers. Hence the degrees of both side of the formula (2.1.19) coincide with each other, due to the following combinatorial identity:

$$\sum_{j=1}^{d} jp_j(d) = \sum_{j=1}^{d} d(j)p(d-j)\,, \tag{2.5.1}$$

where $d(j)$ is the number of positive divisors of the integer j, i.e.

$$d(j) = \sharp\{k\in \mathbf{Z}_+;\, k\,|\, j\}.$$

Hence we must only prove that $\det^2 A_d(h, c)$ can be divided by the left hand side of the formula (2.1.19).

For each $c\in \mathbf{C}$, choose λ with $c=c(\lambda)$ and $s=s_+=s_+(\lambda)$ with $\lambda=\lambda(s)$ smoothly. Then the roots of $c=c(\lambda)$ are $\pm\lambda$, and $s_-=-\frac{2}{s}$, $s_+(-\lambda)=-s$ and $s_-(-\lambda)=\frac{2}{s}$.

Now by the induction on d, we prove the formula (2.1.19) and the following assertion:

$(\sharp)_d$ There exists a nonzero singular vector in $M_d(h(w_{a,b}(s), c(\lambda(s)))$ for any pair $(a, b)\in \mathbf{Z}_+^2$ with $d=ab$, generically w.r.t. s.

At first, let $d=1$. Then $\det A_1(h, c)=\langle c, h\,|\,e_{-1}e_1\,|\,h, c\rangle=h$ and $\Phi_{1,1}(h, c)=h^2$. And $w_{1,1}(s)=-\lambda$, so $h(w_{1,1}(s), \lambda)=0$. Hence $(\sharp)_1$ holds for all s, since $e_1\,|\,0, c\rangle\in M_1(0, c)$ is a singular vector.

Now assume that the formula $(2.1.19)_j$ and the assertion $(\sharp)_j$ hold for all $1\leqq j<d$.

If $|\,v\rangle\in M_j(h, c)$ is a nonzero singular vector, then $U(\mathfrak{n}_+)\,|\,v\rangle$ is contained in $I(h, c)$, where $I(h, c)$ is the kernel of the bilinear form (2.1.16) $(\ |\ )$: $M(h, c) \times M(h, c)\to \mathbf{C}$. Since Verma modules are $U(\mathfrak{n}_+)$-free, the dimension of $(U(\mathfrak{n}_+)\,|\,v\rangle)\cap M_k(h, c)$ is equal to $p(k-j)$ for any $k\geqq i$.

Using notation (2.4.1) for $w_{a,b}$, we can rewrite $\Phi_{a,b}$ as

$$4\Phi_{a,b} = (w+w_{a,b})\,(w-w_{a,b})\,(w+w_{b,a})\,(w-w_{b,a})\,. \tag{2.5.2}$$

So, by the remark above and the induction hypothesis $(\sharp)_j$ $(0\leqq j<d)$, the polynomial $\det^2 A_d(h, c)$ is divided by

$$\prod_{k=1}^{d-1}\prod_{j|k}\Phi_{j,k/j}(h, c)^{p(d-k)}.$$

Let $(a, b)\in \mathbf{Z}_{+}^{2}$ with $d=ab$. By Proposition 2.9, there exist singular vectors $|v\rangle\in \mathcal{F}_{ab}(w_{a,b}(s), \lambda(s))$ and $|v'\rangle\in \mathcal{F}_{ab}(w_{a,b}(-s), \lambda(-s))=\mathcal{F}_{ab}(-w_{a,b}(\lambda(s)), -\lambda(s))$.

Hence by Proposition 2.3 and the commutativity (2.2.9), the bilinear form $(\ |\ )$ on $M_d(h_0, c_0)\times M_d(h_0, c_0)$ is degenerate, that is, $\det A_d(h_0, c_0)=0$, where $h_0=h(w_{a,b}(s))$ or $h_0=-w_{a,b}(s)$, and $c_0=c(\lambda(s))=c(-\lambda(s))$.

On the other hand, $\det A_j(h_0, c_0)\neq 0$ for each $j\ (1\leqq j<d)$ and generic s, since $\det^2 A_j(h, c)$ is the product of the form (2.5.2) for $ab\leqq j(<d)$ as a polynomial in (h, c).

By the above 2 facts, we get a singular vector in $M_d(h_0, c_0)$ for generic s. Thus $\det A_d(h, c)$ is divided by $(w-w_{a,b}(s))\,(w+w_{a,b}(s))$, so $\det^2 A_d(h, c)$ is divided by $\Phi_{a,b}(h, c)$. *q.e.d.*

Proof of Proposition 2.7. Fix $\lambda\in \mathbf{C}$ and denote by $s_{\pm}$ the roots of the equation $\lambda=\lambda(s)$.

Assume that $\alpha=\alpha(s_+)\in \mathbf{C}\backslash \mathbf{Q}$ (note that $\alpha(s_-)=\alpha(s_+)^{-1}$). Then the equation (2.4.3) has at most one integral solution $(a, b)\in \mathbf{Z}^2$.

Fix integers $a, b\geqq 1$ and put

$$\text{(2.5.3)}\qquad \begin{cases} w_0=w_{a,b}(s);\ c=c(\lambda)\,. \\ h=h(w_0, \lambda);\ h'=h+ab\,. \end{cases}$$

Let $I(h, c)$ be the kernel of the bilinear form (2.1.16), then any submodule of $M(h, c)$ is contained in $I(h, c)$. Then by Propositions 2.1 and 2.2, the Verma module $M(h', c)$ is irreducible, and $\dim I_{ab}(h, c)=1$, where $I_{ab}(h, c)=I(h, c)\cap M_{ab}(h, c)$. By the similar proposition for Verma modules as Proposition 2.6 (i), the space $I_{ab}(h, c)$ is spanned by singular vector $|v\rangle$. Denote by $M'(h, c)$ the $\mathcal{L}$-submodule generated by this $|v\rangle$. Then we get the intertwining operator

$$\Phi_{a,b}\colon M(h', c)\to M(h, c)$$

of degree ab, which sends the vacuum vector $|h', c\rangle$ to the singular vector $|v\rangle$. This mapping $\Phi_{a,b}$ is injective by the irreducibility of $M(h', c)$. Then by Proposition 2.1,

$$\dim M_d(h, c)/I_d(h, c)=\dim M_d(h, c)/\Phi_{a,d}(M_{d-ab}(h', c))$$

for any $d\geqq 0$. Hence

$$\Phi_{a,b}(M(h', c)) = M'(h, c) = I(h, c)\,,$$

and this is the unique proper submodule of $M(h, c)$.

By Proposition 2.9, we get a nontrivial intertwining operator

$$O(s_+;\, a,\, -b)\colon\ \mathcal{F}(-w_0, \lambda) \to \mathcal{F}(-w_0', \lambda)$$

of degree $-ab<0$, where $w_0'=w_{-a,b}(s)$.

Then the $\mathcal{L}$-module mapping $\pi_{-w_0,\lambda}\colon M(h, c)\to\mathcal{F}(-w_0, \lambda)$ cannot be isomorphic, since the $\mathcal{L}$-module mapping $O(s_+;\, a,\, -b)\circ\pi_{-w_0,\lambda}$ is a zero-mapping (Note that any Verma module is generated by its vacuum vector). Hence by the uniqueness of the proper submodule of $M(h, c)$, we get

$$\ker(\pi_{-w_0,\lambda}) = M'(h, c) = \Phi_{a,b}(M(h', c))\,,$$

and

$$\dim(\ker(\pi_{-w_0,\lambda})\cap M_d(h, c)) = \dim M_{d-ab}(h', c) = p(d-ab)$$

for any $d\geqq ab$. So the polynomial $\det C_d(w, \lambda)$ of w is divisible by the power of linear factor

$$(w+w_0)^{p(d-ab)} = \left(w+\frac{a}{2}s_+ + \frac{b}{2}s_-\right)^{p(d-ab)}.$$

Note that

$$\left(w+\frac{a}{2}s_+ + \frac{b}{2}s_-\right)\left(w+\frac{b}{2}s_+ + \frac{a}{2}s_-\right) = w^2+ab\lambda^2-(a+b)\lambda w-\frac{(a-b)^2}{2}\,.$$

By the same argument, it is seen that $\det C_d(w, -\lambda)$ is divisible by $\left(w-\frac{a}{2}s_+ - \frac{b}{2}s_-\right)^{p(d-ab)}$. Thus we get the formula (2.4.2) due to Propositions 2.1, 2.2, 2.4 and the formula (2.2.10). *q.e.d.*

§3. Vertex Operators and Construction of Intertwining Operators

3.1) Operators on Fock Spaces

At first we introduce weakly defined operators on Fock spaces.

For each $(w, \lambda)\in\mathbf{C}^2$, the Fock space $\mathcal{F}=\mathcal{F}(w, \lambda)$ is endowed with a topology by the filtration

$$\mathcal{F} = G_0(\mathcal{F})\supseteqq G_1(\mathcal{F})\supseteqq G_2(\mathcal{F})\supseteqq\cdots\,, \tag{3.1.1}$$

where G_d's are defined analogously as (1.1.4). Then the completed Fock space $\hat{\mathcal{F}}(w, \lambda)$ is defined by

$$\hat{\mathcal{F}}(w, \lambda) = \varprojlim_n \mathcal{F}(w, \lambda)/G_n(\mathcal{F}(w, \lambda)) . \tag{3.1.2}$$

Then $\hat{\mathcal{F}}(w, \lambda)$ is a Hausdorff complete topological vector space with the dense subspace $\mathcal{F}(w, \lambda)$. The left action of $\mathcal{A}$ on $\mathcal{F}(w, \lambda)$ can be extended uniquely to a continuous left action on $\hat{\mathcal{F}}(w, \lambda)$, and the vacuum expectation value

$$\langle \mid \rangle \colon \mathcal{F}^{\dagger}(w, \lambda) \times \mathcal{F}(w, \lambda) \to \boldsymbol{C} \tag{1.1.14}$$

can be extended uniquely to a continuous bilinear map

$$\langle \mid \rangle \colon \mathcal{F}^{\dagger}(w, \lambda) \times \hat{\mathcal{F}}(w, \lambda) \to \boldsymbol{C} , \tag{3.1.3}$$

where the topology in $\mathcal{F}^{\dagger}(w, \lambda)$ is considered as discrete. Consider the map

$$\Phi \colon \hat{\mathcal{F}}(w, \lambda) \to \mathrm{Hom}_{\boldsymbol{C}} (\mathcal{F}^{\dagger}(w, \lambda), \boldsymbol{C}) \tag{3.1.4}$$

defined by

$$\Phi(u)(v) = \langle v \mid u \rangle \tag{3.1.5}$$

where $u \in \hat{\mathcal{F}}(w, \lambda)$ and $v \in \mathcal{F}^{\dagger}(w, \lambda)$. Then this map Φ is a topological linear isomorphism between topological vector spaces $\hat{\mathcal{F}}(w, \lambda)$ and $\mathrm{Hom}_{\boldsymbol{C}}(\mathcal{F}^{\dagger}(w, \lambda), \boldsymbol{C})$.

Fix two pairs of indices (w_1, λ_1) and $(w_2, \lambda_2) \in \boldsymbol{C}^2$, and consider a linear map

$$O \colon \mathcal{F}(w_1, \lambda_1) \to \hat{\mathcal{F}}(w_2, \lambda_2) , \tag{3.1.6}$$

which we call an *operator*.

If an operator $O \colon \mathcal{F}(w_1, \lambda_1) \to \hat{\mathcal{F}}(w_2, \lambda_2)$ depends on complex variables $\zeta_1, \cdots, \zeta_a$, and any matrix element $\langle v \mid O(\zeta) u \rangle$ of the operator $O(\zeta)$ is a holomorphic function for $\langle v \mid \in \mathcal{F}^{\dagger}(w_2, \lambda_2)$ and $\mid u \rangle \in \mathcal{F}(w_1, \lambda_1)$, then we call the operator $O(\zeta)$ is *holomorphic* on $\zeta_1, \cdots, \zeta_a$.

For two operators

$$O_1 \colon \mathcal{F}(w_1, \lambda_1) \to \hat{\mathcal{F}}(w_2, \lambda_2) \quad \text{and} \quad O_2 \colon \mathcal{F}(w_2, \lambda_2) \to \hat{\mathcal{F}}(w_3, \lambda_3) , \tag{3.1.7}$$

their composition product can not be defined in general. So remark the following fact (it means that we consider operators always in the weak sense).

Lemma 3.1. *There is a one to one correspondence between the following objects.*

(1) *Operators*

$$O \colon \mathcal{F}(w_1, \lambda_1) \to \hat{\mathcal{F}}(w_2, \lambda_2) , \tag{3.1.6}$$

(2) *Bilinear maps*

$$\hat{O}:\ \mathcal{F}^{\dagger}(w_2, \lambda_2)\times\mathcal{F}(w_1, \lambda_1)\rightarrow \boldsymbol{C}\,. \tag{3.1.8}$$

The correspondence is defined by

$$\langle u|\hat{O}|v\rangle = \langle u|Ov\rangle \tag{3.1.9}$$

for $\langle u|\in\mathcal{F}^{\dagger}(w_2, \lambda_2)$ *and* $|v\rangle\in F(w_1,\lambda_1)$.

For each integer $d\geqq 0$, let $\{|u_{d,1}\rangle, \cdots, |u_{d,p(d)}\rangle\}$ be a basis of $\mathcal{F}_d(w_2, \lambda_2)$ and $\{\langle u_{d,1}|, \cdots, \langle u_{d,p(d)}|\}$ be its dual basis of $\mathcal{F}_d^{\dagger}(w_2, \lambda_2)$. Then the expression

$$\sum_{j=1}^{p(d)} |u_{d,j}\rangle\langle u_{d,j}|$$

can be considered as the identity operator of $\mathcal{F}_d(w_2, \lambda_2)$.

If the series

$$\sum_{d=0}^{\infty}\left|\sum_{j=1}^{p(d)}\langle v|\hat{O}_2|u_{d,j}\rangle\langle u_{d,j}|\hat{O}_1|u\rangle\right| \tag{3.1.10}$$

is convergent for any $\langle v|\in\mathcal{F}^{\dagger}(w_3, \lambda_3)$ and $|u\rangle\in\mathcal{F}(w_1, \lambda_1)$, then the formula

$$\langle v|\hat{O}_3|u\rangle = \sum_{d=0}^{\infty}\sum_{j=1}^{p(d)}\langle v|\hat{O}_2|u_{d,j}\rangle\langle u_{d,j}|\hat{O}_1|u\rangle$$

defines a $\boldsymbol{C}$-bilinear mapping

$$\hat{O}_3:\ \mathcal{F}^{\dagger}(w_3, \lambda_3)\times\mathcal{F}(w_1, \lambda_1)\rightarrow \boldsymbol{C}\,, \tag{3.1.11}$$

hence by Lemma 3.1, this mapping $\hat{O}_3$ defines an operator

$$O_3:\ \mathcal{F}(w_1, \lambda_1)\rightarrow \hat{\mathcal{F}}(w_3, \lambda_3)\,, \tag{3.1.12}$$

and this operator O_3 is called the *composition* of these operators O_1 and O_2, and is denoted by $O_3 = O_2\circ O_1$.

3.2) Definition of Vertex Operator

Fix $(w, \lambda)\in\boldsymbol{C}^2$. For variables $s\in\boldsymbol{C}$ and $\zeta\in\boldsymbol{C}^*$, consider the operators

$$\zeta^{sp_0}:\ \mathcal{F}(w, \lambda)\rightarrow\mathcal{F}(w, \lambda)\,,\quad \mathcal{F}^{\dagger}(w, \lambda)\rightarrow\mathcal{F}^{\dagger}(w, \lambda) \tag{3.2.1}$$

$$E_+(s, \zeta):\ \mathcal{F}(w, \lambda)\rightarrow\hat{\mathcal{F}}(w, \lambda)\,,\quad \mathcal{F}^{\dagger}(w, \lambda)\rightarrow\mathcal{F}^{\dagger}(w, \lambda) \tag{3.2.2}$$

$$E_-(s, \zeta):\ \mathcal{F}(w, \lambda)\rightarrow\mathcal{F}(w, \lambda)\,,\quad \mathcal{F}^{\dagger}(w, \lambda)\rightarrow\hat{\mathcal{F}}^{\dagger}(w, \lambda) \tag{3.2.3}$$

defined by

$$
(3.2.4)\qquad \begin{cases} \zeta^{sp_0} = \exp(sp_0 \log \zeta)\,, \\ E_+(s, \zeta) = \exp\left(s \sum_{n\geqq 1} \zeta^n \frac{p_n}{n}\right) \quad \text{and} \\ E_-(s, \zeta) = \exp\left(-s \sum_{n\geqq 1} \zeta^{-n} \frac{p_{-n}}{n}\right) \end{cases}
$$

respectively, and the translation operator T_s is uniquely defined as a $\boldsymbol{C}$-linear mapping

$$
(3.2.5)\qquad \begin{cases} T_s\colon \mathcal{F}(w, \lambda) \to \mathcal{F}(w+s, \lambda) \\ T_s\colon \mathcal{F}^\dagger(w, \lambda) \to \mathcal{F}^\dagger(w-s, \lambda) \end{cases}
$$

under the following conditions:

$$
(3.2.6)\qquad \begin{cases} T_s|w, \lambda\rangle = |w+s, \lambda\rangle;\ \langle\lambda, w|T_s = \langle\lambda, w-s|\,, \\ T_s p_n = p_n T_s \quad (n\neq 0);\quad T_s A = A T_s\,. \end{cases}
$$

Now define the vertex operator

$$
(3.2.7)\qquad X(s, \zeta)\colon\ \mathcal{F}(w, \lambda) \to \hat{\mathcal{F}}(w+s, \lambda)
$$

as

$$
(3.2.8)\qquad X(s, \zeta) = E_+(s, \zeta)E_-(s, \zeta)T_s\zeta^{s^2/2+sp_0}\,.
$$

By the definitions (3.2.4, 6), we get

$$
(3.2.9)\qquad \begin{cases} p_0T_s - T_sp_0 = sT_s\,, \\ \langle v|T_su\rangle = \langle vT_s|u\rangle \quad (v\in\mathcal{F}^\dagger(w+s, \lambda),\ u\in\mathcal{F}(w, \lambda)); \end{cases}
$$

$$
(3.2.10)\qquad \zeta^{sp_0}T_s = \zeta^{s^2}T_s\zeta^{sp_0}\,,\quad T_sE_+ = E_+T_s\,,\quad T_sE_- = E_-T_s\,.
$$

Note. The vertex operator

$$
X(s, \zeta) = \zeta^{sp_0-s^2/2}T_sE_+(s, \zeta)E_-(s, \zeta)
$$

operates also as a linear map

$$
X(s, \zeta)\colon\ \mathcal{F}^\dagger(w, \lambda) \to \hat{\mathcal{F}}^\dagger(w-s, \lambda)\,.
$$

By the commutation relations (1.1.1), we get

$$
(3.2.11)\qquad \begin{cases} [p_n, E_+(s, \zeta)] = \begin{cases} s\zeta^{-n}E_+(s, \zeta) & (n\leqq -1) \\ 0 & (n\geqq 0) \end{cases} \\ [p_n, E_-(s, \zeta)] = \begin{cases} s\zeta^{-n}E_-(s, \zeta) & (n\geqq 1) \\ 0 & (n\leqq 0)\,, \end{cases} \end{cases}
$$

so for $s, t\in\boldsymbol{C}$ and $\zeta_1, \zeta_2\in\boldsymbol{C}^*$, the composition $E_-(s, \zeta_1)\ E_+(t, \zeta_2)$ can be defined as formal Laurent series of ζ_1 and ζ_2 and is expressed as

$$(3.2.12)\quad E_-(s,\zeta_1)E_+(t,\zeta_2)=\exp\{-st\sum_{n=1}^{\infty}\frac{1}{n}(\zeta_2/\zeta_1)^n\}E_+(t,\zeta_2)E_-(s,\zeta_1)\,.$$

Hence

Proposition 3.2. *For s, t, ζ_1, $\zeta_2\in\boldsymbol{C}$ with $|\zeta_1|>|\zeta_2|>0$, the composition* (3.2.12) *can be defined in the sense of* (3.1.11) *and the composition*

$$(3.2.13)\qquad E_-(s,\zeta_1)E_+(t,\zeta_2)=(1-\zeta_2/\zeta_1)^{st}E_+(t,\zeta_2)E_-(s,\zeta_1)\,,$$

is holomorphic in the domain $\{(\zeta_1,\zeta_2)\in\boldsymbol{C}^2;\ |\zeta_1|>|\zeta_2|>0\}$ as an operator-valued function of ζ_1 and ζ_2, where in the right hand side of (3.2.13) *we take the principal branch of the multi-valued function $(1-\zeta_2/\zeta_1)^{st}$.*

For an integer $a\geqq 1$, define the submanifold M_a of $(\boldsymbol{C}^*)^a$ as

$$(3.2.14)\qquad M_a=\{\zeta=(\zeta_1,\cdots,\zeta_a)\in(\boldsymbol{C}^*)^a;\ \zeta_i\neq\zeta_j\qquad(i\neq j)\}\,,$$

and the multi-valued holomorphic function $F(\alpha;\zeta)$ on M_a parametrized by $\alpha\in\boldsymbol{C}$ as

$$(3.2.15)\qquad F(\alpha;\zeta_1,\cdots,\zeta_a)=\prod_{1\leqq i<j\leqq a}(\zeta_i-\zeta_j)^{2\alpha}\prod_{1\leqq i\leqq a}\zeta_i^{-(a-1)\alpha}$$

Then the function F is symmetric in $\zeta_1,\cdots,\zeta_a$ and is invariant under the $\boldsymbol{C}^*$-action of the form

$$(3.2.16)\qquad (\zeta_1,\zeta_2,\cdots,\zeta_a)\mapsto(k\zeta_1,k\zeta_2,\cdots,k\zeta_a)\qquad(k\in\boldsymbol{C}^*)\,.$$

For $s\in\boldsymbol{C}$, define the operator-valued functions

$$E_+(s;\zeta_1,\cdots,\zeta_a):\ \mathcal{F}(w,\lambda)\to\hat{\mathcal{F}}(w,\lambda),\ \mathcal{F}^\dagger(w,\lambda)\to\mathcal{F}^\dagger(w,\lambda)\,,$$
$$E_-(s;\zeta_1,\cdots,\zeta_a):\ \mathcal{F}(w,\lambda)\to\mathcal{F}(w,\lambda),\ \mathcal{F}^\dagger(w,\lambda)\to\hat{\mathcal{F}}^\dagger(w,\lambda)$$

for $(\zeta_1,\cdots,\zeta_a)\in(\boldsymbol{C}^*)^a$ and

$$Z(s;\zeta_1,\cdots,\zeta_a):\ \mathcal{F}(w,\lambda)\to\hat{\mathcal{F}}(w+as,\lambda),\ \ \mathcal{F}^\dagger(w,\lambda)\to\hat{\mathcal{F}}^\dagger(w-as,\lambda)$$

for $(\zeta_1,\cdots,\zeta_a)\in M_a$ as

$$(3.2.17)\qquad E_\pm(s;\zeta_1,\cdots,\zeta_a)=\exp\left(\pm s\sum_{n\geqq 1}(\zeta_1^{\pm n}+\cdots+\zeta_a^{\pm n})\frac{p_{\mp n}}{n}\right)$$

and

$$(3.2.18)\qquad Z(s;\zeta_1,\cdots,\zeta_a)$$
$$=F\left(\frac{s^2}{2};\zeta_1,\cdots,\zeta_a\right)E_+(s;\zeta_1,\cdots,\zeta_a)E_-(s;\zeta_1,\cdots,\zeta_a)T_s^{\,a}\,.$$

Then these operators are holomorphic on $\zeta=(\zeta_1, \cdots, \zeta_a)$. By using iteratedly Proposition 3.2, we get

Proposition 3.3 (**Short Distance Expansion**).

For any integer $a\geqq 1$ *and any* $s, \zeta_1, \cdots, \zeta_a \in \boldsymbol{C}$ *with* $|\zeta_1|>|\zeta_2|>\cdots>|\zeta_a|>0$, *the compositions* $E_\pm(s, \zeta_1)\cdots E_\pm(s, \zeta_a)$ *and* $X(s, \zeta_1)\cdots X(s, \zeta_a)$ *are defined in the sense of* (3.1.12), *and*

$$E_\pm(s, \zeta_1) \cdots E_\pm(s, \zeta_a) = E_\pm(s; \zeta_1, \cdots, \zeta_a)\,, \quad \text{and} \tag{3.2.19}$$

$$X(s, \zeta_1) \cdots X(s, \zeta_a) = Z(s; \zeta_1, \cdots, \zeta_a)\,(\zeta_1 \cdots \zeta_a)^{sp_0+(a/2)s^2}\,, \tag{3.2.20}$$

where in the right hand side of (3.2.20), *we take the principal branch of the multi-valued function*

$$F\left(\frac{s^2}{2}; \zeta_1, \cdots, \zeta_a\right)(\zeta_1 \cdots \zeta_a)^{sp_0+(a/2)s^2}.$$

Note. $E_+(s; \zeta_1, \cdots, \zeta_a)$ and $E_-(s; \zeta_1, \cdots, \zeta_a)$ are single-valued and symmetric in $\zeta_1, \cdots, \zeta_a \in \boldsymbol{C}^*$, so $Z(s; \zeta_1, \cdots, \zeta_a)$ is also symmetric.

3.3) Conformal Covariance

From the formulae (3.2.10, 11) and the definition (3.2.8) of the vertex operator, we get

$$\begin{cases} [A, X(s, \zeta)] = 0 \quad \text{and} \\ [p_n, X(s, \zeta)] = s\zeta^{-n}X(s, \zeta) \qquad (n\in \boldsymbol{Z})\,. \end{cases} \tag{3.3.1}$$

By these formulae, the expressions (1.2.3) of the operators L_n and Proposition 3.3, we get the following important relations.

Proposition 3.4 (**Conformal Covariance**). *For any* $n\in \boldsymbol{Z}$

$$[L_n, X(s, \zeta)] = \zeta^{-n}\left\{\zeta\frac{d}{d\zeta}-n\left(sA+\frac{s^2}{2}\right)\right\}X(s, \zeta)\,, \tag{3.3.2}$$

for $s\in \boldsymbol{C}$ *and* $\zeta\in \boldsymbol{C}^*$, *and*

$$\begin{aligned}[L_n, Z(s; \zeta_1, \cdots, \zeta_a)] \\ = \sum_{j=1}^{a} \zeta_j^{-n}\left[\zeta_j\frac{\partial}{\partial\zeta_j}+\left\{sp_0-\frac{a}{2}s^2-n\left(sA+\frac{s^2}{2}\right)\right\}\right]Z(s; \zeta)\,,\end{aligned} \tag{3.3.3}$$

for $s\in \boldsymbol{C}$ *and* $\zeta\in M_a$.

Proof. For example,

(3.3.4) $$\zeta\frac{d}{d\zeta}X(s,\zeta) = s\Big(\sum_{j\geqq 0}\zeta^j p_j-\frac{s}{2}\Big)X(s,\zeta)+s\sum_{j\geqq 1}\zeta^{-j+sp_0-s^2/2}E_+p_{-j}E_-T_s = [L_0, X(s,\zeta)]\,.$$

3.4) Intertwining Operators

Let $\mathcal{S}_\alpha^*$ be a local system with coefficients in $\boldsymbol{C}$, associated to the monodromy group of the multi-valued function $F(\alpha;\zeta_1,\zeta_2,\cdots,\zeta_a)$ on the manifold M_a (for example, see P. Deligne [1970]). Denote by $\mathcal{S}_\alpha$ the dual of the local system $\mathcal{S}_\alpha^*$ over M_a.

Take an element Γ of the homology group $H_a(M_a;\mathcal{S}_\alpha)$, and define the operator

(3.4.1) $$O(s,\Gamma;\,l_1,\cdots,l_a)=\int_\Gamma d\zeta_1 d\zeta_2\cdots d\zeta_a\zeta_1^{-l_1-1}\cdots\zeta_a^{-l_a-1}Z(s;\zeta)$$

for a complex number s and integers $l_1,\cdots,l_a$.

Here we remark the following fact which will be proved in §3.5.

Proposition 3.6. *The integral*

(3.4.2) $$\int_\Gamma \zeta_1^{-l_1-1}\cdots\zeta_a^{-l_a-1}F(s;\zeta_1,\zeta_2,\cdots,\zeta_a)d\zeta_1\cdots d\zeta_a$$

vanishes unless $l_1+l_2+\cdots+l_a=0$.

Then we get

Proposition 3.7. *The operator* (3.4.1) *is homogeneous of degree* $l_1+l_2+\cdots+l_a$*, and can be considered as the linear mapping*

(3.4.3) $$O(s,\Gamma;\,l_1,\cdots,l_a)\colon\ \mathcal{F}(w,\lambda)\to\mathcal{F}(w+as,\lambda)\,.$$

Moreover

(3.4.4) $$[L_{-n},O(s,\Gamma;\,l_1,\cdots,l_a)] =\sum_{j=1}^{a}\Big\{l_j-n+sp_0-\frac{as^2}{2}+n\Big(sA+\frac{s^2}{2}\Big)\Big\}O(s,\Gamma;\,l_1,\cdots,l_j-n,\cdots,l_a)\,.$$

Proof. From Proposition 3.6 and the definition (3.2.18) of $Z(s;\zeta)$, we get the homogeneity of the operator (3.4.1) Hence for any vector $|v\rangle\in\mathcal{F}(w,\lambda)$, its image lies in $\mathcal{F}(w+as,\lambda)$, because the summation is finite in the expression

$O(s, \Gamma; l_1, \cdots, l_a)|v\rangle$. The relation (3.4.4) follows from Proposition 3.5 and the integration by parts. *q.e d.*

Put $b=l_1=\cdots=l_a$, and let the coefficients in the right hand side of (3.4.4) be equal to zero for any $n\in\boldsymbol{Z}$, that is,

$$-1+sA+\frac{s^2}{2}=0 \quad \text{and} \quad b+sp_0-\frac{as^2}{2}=0\,. \tag{3.4.5}$$

Note that $A=\lambda\mathrm{id}$ and $p_0=(w+as)\mathrm{id}$ on the space $\mathscr{F}(w+as, \lambda)$. Hence we get that if the parameters satisfy the equalities

$$\frac{s^2}{2}+\lambda s-1=0 \quad \text{and} \quad w=-\frac{a}{2}s-\frac{b}{s}\,, \tag{3.4.6}$$

then the operators of Fock space representations commute with the operator

$$O(s, \Gamma; a, b)=O(s, \Gamma; \underbrace{b, \cdots, b}_{a})\colon \mathscr{F}(w, \lambda)\to\mathscr{F}(w+as, \lambda)\,, \tag{3.4.7}$$

that is,

$$[L_{-n}, O(s, \Gamma; a, b)]=0 \qquad \text{for any} \quad n\in\boldsymbol{Z}\,, \tag{3.4.8}$$

Due to Proposition 3.7, the operator $O(s, \Gamma; a, b)$ is homogeneous of degree ab.

Remark. For $a=2$, the operator $O(s, \Gamma; 2, b)$ in (3.4.7) coincides with the operator $O(s, \Gamma; b, b)$ in (3.4.3). We hope that this notational confusion does not bother the readers, since the distinction is clear in the context.

Summarizing above facts, we get one of our main results.

Theorem 3.8. *For each $s\in\boldsymbol{C}^*$ and integers $a\geqq 1$ and b, take a cycle $\Gamma\in H_a(M_a; \mathcal{S}_\alpha)$ $(\alpha=s^2/2)$ and put*

$$\lambda=\lambda(s)=\frac{1}{s}-\frac{s}{2} \quad \textit{and} \quad w=a\frac{s}{2}-\frac{b}{s}\,. \tag{3.4.9}$$

Then the operator

$$O(s, \Gamma; a, b)\colon \mathscr{F}(w-as, \lambda)\to\mathscr{F}(w, \lambda) \tag{3.4.10}$$

is an intertwining operator of degree ab.

Remark. In the following we will show the existence of a cycle Γ of $H_a(M_a; \mathcal{S}_\alpha)$ for which the intertwining operator $O(s, \Gamma; a, b)$ is nontrivial.

Remark. When $\alpha=\frac{s^2}{2}$ is an integer, the local system $\mathcal{S}_\alpha$ is trivial. We can take a cycle $\Gamma\in H_a(M_a; \mathcal{S}_\alpha)=H_a(M_a; \boldsymbol{C})$ as the cycle which represents the residue around $\zeta_1=0, \cdots, \zeta_a=0$. For this cycle Γ, the operator $O(s, \Gamma; a, b)$ gives a nontrivial operator.

3.5) Proof of Proposition 3.6

By §3.6 we can assume that $a\geqq 2$. Define the manifold

(3.5.1) $$Y_{a-1} = \{(k_1, \cdots, k_{a-1})\in(\boldsymbol{C}^*)^{a-1};\ k_i\neq k_j\ (i\neq j),\ k_i\neq 1\},$$

and consider the $\boldsymbol{C}^*$-bundle

(3.5.2) $$\eta_a:\ M_a\ni(\zeta_1, \cdots, \zeta_a)\mapsto(k_1, \cdots, k_{a-1})\in Y_{a-1}$$

defined by

(3.5.3) $$k_i = \zeta_{i+1}/\zeta_1 \qquad (i = 1, \cdots, a-1)\,.$$

Then this bundle is trivial, in fact, the mapping

(3.5.4) $$\begin{array}{ccc} Y_{a-1}\times\boldsymbol{C}^* & \longrightarrow & M_a \\ \Cup & & \Cup \\ ((k_1, \cdots, k_{a-1}), \zeta) & \mapsto & (\zeta, k_1\zeta, \cdots, k_{a-1}\zeta) \end{array}$$

is an isomorphism of $\boldsymbol{C}^*$-bundles. By this coordinate transformation, the function $F(\alpha; \zeta_1, \cdots, \zeta_a)$ changes into the function

(3.5.5) $$\begin{aligned} G(\alpha; k_1, k_2, \cdots, k_{a-1}, \zeta) &= G(\alpha; k_1, k_2, \cdots, k_{a-1}) \\ &= \prod_{1\leqq i<j\leqq a-1}(k_i-k_j)^{2\alpha}\prod_{1\leqq j\leqq a-1}(1-k_j)^{2\alpha}k_j^{-(a-1)\alpha}\,, \end{aligned}$$

which is independent of the fiber variable ζ, and can be considered as a multi-valued holomorphic function on Y_{a-1} for each α. Hence the local system $\mathcal{S}_\alpha$ is decomposed as the product of the constant local system $\boldsymbol{C}$ on $\boldsymbol{C}^*$ and the local system $\mathcal{S}'_\alpha$ on Y_{a-1} analogously defined by the function $G(s; k_1, \cdots, k_{a-1})$. Then we get

Lemma 3.9. *For any* $\alpha\in\boldsymbol{C}$,

(3.5.6) $$H_j(Y_{a-1}\colon \mathcal{S}'_\alpha) = 0 \qquad (j\geqq a)\,.$$

The proof of this lemma will be given in §4.1. By this lemma and Künneth's theorem, we may assume that a cycle $\Gamma\in H_a(M_a, \mathcal{S}_\alpha)$ is taken as a product of a cycle $\Gamma_1\in H_1(\boldsymbol{C}^*; \boldsymbol{C})$ and a cycle $\Gamma_2\in H_{a-1}(Y_{a-1}; \mathcal{S}'_\alpha)$. Hence the integral (3.4.2) changes into the integral

$$(3.5.7)\qquad \int_{\Gamma_1} \zeta^{-(l_1+\cdots+l_a)}\frac{d\zeta}{\zeta}\int_{\Gamma_2} k_1^{-l_2-1}\cdots k_{a-1}^{-l_a-1}G(\alpha; k)\,dk_1\cdots dk_{a-1}\,,$$

whose first factor is nothing else but the residue.

3.6) Case Where $a=1$

Let $a=1$, then $M_1=\boldsymbol{C}^*$, $F(\alpha; \zeta)=1$ and the local system $\mathcal{S}_\alpha$ is constant for any $\alpha\in\boldsymbol{C}^*$. Hence

$$(3.6.1)\qquad H_1(M_1; \mathcal{S}_\alpha) = H_1(\boldsymbol{C}^*; \boldsymbol{C})\cong\boldsymbol{C}e_1\,,$$

where e_1 is the positively oriented unit circle on the plane $\boldsymbol{C}$. So in this case, Proposition 3.6 is obvious.

Take a cycle $\Gamma=\dfrac{1}{2\pi\sqrt{-1}}e_1$ of $H_1(M_1; \mathcal{S}_\alpha)$, then we get

$$(3.6.2)\qquad \begin{aligned} O(s, \Gamma; 1, b) &= \int_0^1 e^{-2\pi bt\sqrt{-1}}E_+(s; \zeta)E_-(s; \zeta)T_s dt \quad (\zeta = e^{-2\pi b\sqrt{-1}t}) \\ &= \Phi_b(s)T_s\,, \end{aligned}$$

where $\Phi_b(s)$ is the homogeneous component of degree b of the operator $E_+(s; \zeta)E_-(s; \zeta)$.

Here we remark the well-known facts (see for example, D.E. Littlewood [1958]):

$$(3.6.3)\qquad \exp\,(\sum_{l\geqq 1}\zeta^l x_l) = \sum_{l\geqq 0}P_l(x)\zeta^l\,,$$

where $P_l(x)$ is the character polynomial of the irreducible representation corresponding to the Young diagram $Y_l=(l)=\underbrace{\square\square\cdots\square}_{l}$ of the group $\mathrm{GL}(N, \boldsymbol{C})$ where N is sufficiently large, and

$$(3.6.4)\qquad x_l = \frac{1}{l}\operatorname{tr} g^l \qquad (g\in \mathrm{GL}(N, \boldsymbol{C}))\,.$$

In particular, the polynomial $P_l(x)$ is homogeneous of degree l, where $\deg x_l=l$. For example,

$$(3.6.5)\qquad \begin{aligned} &P_0(x) = 1\,, \qquad P_1(x) = x_1\,, \\ &P_2(x) = x_2+x_1^2/2\,, \quad P_3(x) = x_3+x_1x_2+x_1^3/6\,, \\ &P_4(x) = x_4+x_1x_3+x_2^2/2+x_1^2x_2/2+x_1^4/24\,, \qquad \text{etc.} \end{aligned}$$

So we get

(3.6.6) $$\Phi_b(s) = \sum_{\substack{l-l'=b \\ l,l'\geqq 0}} P_l(sp_1, \cdots, sp_n/n, \cdots)P_{l'}(\cdots, -sp_{-n}/n, \cdots).$$

Let $b=0$. Then the operator

(3.6.7) $$O(s, \Gamma; 1, 0): \mathcal{F}\left(-\frac{s}{2}, \lambda(s)\right) \to \mathcal{F}\left(\frac{s}{2}, \lambda(s)\right)$$

is nontrivial, since

(3.6.8) $$O(s, \Gamma; 1, 0)\left|-\frac{s}{2}, \lambda(s)\right\rangle = \left|\frac{s}{2}, \lambda(s)\right\rangle.$$

Let $b>0$. Then the image of the vacuum vector

(3.6.9) $$O(s, \Gamma; 1, b)|w-s, \lambda\rangle = P_b(\cdots, sp_n/n, \cdots)|w,\lambda\rangle$$

is a nonzero vector of degree b of the Fock space $\mathcal{F}(w, \lambda)$, where

(3.6.10) $$w = \frac{s}{2} - \frac{b}{s} \quad \text{and} \quad \lambda = \lambda(s) = \frac{1}{s} - \frac{s}{2},$$

that is, the operator $O(s, \Gamma; 1, b)$ is nontrivial and its image is a proper submodule of $\mathcal{F}(w, \lambda)$.

Let $b<0$. Then by the regularity of the vacuum expectation value (1.1.14), we get a vector $|v\rangle \in \mathcal{F}(w, \lambda)$ of degree $-b$ such that

(3.6.11) $$P_{-b}\left(\cdots, -\frac{s}{n}p_{-n}, \cdots\right)|v\rangle = |w, \lambda\rangle.$$

Hence the vector $T_s^{-1}|v\rangle \in \mathcal{F}(w-s, \lambda)$ is mapped to the vacuum by the intertwining operator $O(s, \Gamma; 1, b)$, that is,

(3.6.12). $$O(s, \Gamma; 1, b)T_s^{-1}|v\rangle = |w, \lambda\rangle.$$

3.7) Case Where $a=2$

Let $a=2$. Then the base space Y_1 of the $\mathbf{C}^*$-bundle η_2 (3.5.2) is

(3.7.1) $$Y_1 = \{k \in \mathbf{C}^*;\ k \neq 1\} = \mathbf{C} - \{0, 1\},$$

and the function (3.5.5) is simply

(3.7.2) $$G(\alpha; k) = (1-k)^{2\alpha}k^{-\alpha}.$$

In this case, Lemma 3.9 is obvious.

For a pair of integers $m=(m_1, m_2)$ with $m_1 \geqq m_2 \geqq 0$, consider the polynomial

$$(3.7.3)\qquad M(m)=M(m;\ t_1,t_2)=\begin{cases} t_1^{m_1}t_2^{m_2} & \text{if } m_1=m_2 \\ t_1^{m_1}t_2^{m_2}+t_1^{m_2}t_2^{m_1} & \text{if } m_1\neq m_2 \end{cases}$$

and expand the exponential function

$$(3.7.4)\qquad \exp\Big(\sum_{n=1}^{\infty}(t_1^n+t_2^n)x_n\Big)=\sum_{d\geqq 0}\sum_{|m|=d}M(m)N(m),$$

where $m=(m_1, m_2)$, $|m|:=m_1+m_2$ and $N(m)=N(m;\ x_1, x_2, \cdots)\in \boldsymbol{C}[x_1, x_2, \cdots]$ with $\deg N(m)=|m|$. Then the polynomials $N(m)$ are linearly independent.

Take a cycle $\Gamma=\Gamma_1\times\Gamma_2\in H_2(M_2;\ \mathcal{S}_\alpha)\cong H_1(\boldsymbol{C}^*;\ \boldsymbol{C})\otimes H_1(Y_1;\ \mathcal{S}'_\alpha)$, then the intertwining operator $O(s, \Gamma;\ 2, b)$ is

$$(3.7.5)\qquad O(s, \Gamma;\ 2, b)=\int_\Gamma d\zeta_1 d\zeta_2 \zeta_1^{-b-1}\zeta_2^{-b-1}Z(s;\ \zeta_1, \zeta_2)$$

$$=\sum_{\substack{d_1\geqq 0\\ d_2\geqq 0}}\sum_{\substack{|m|=d_1\\ |n|=d_2}}\int_\Gamma d\zeta_1 d\zeta_2 \zeta_1^{-b-1}\zeta_2^{-b-1}F\Big(\frac{s^2}{2};\ \zeta_1, \zeta_2\Big)$$

$$\times M(m;\ \zeta_1, \zeta_2)M(n;\ \zeta_1^{-1}, \zeta_2^{-1})N\Big(m;\ \cdots, \frac{sp_n}{n}, \cdots\Big)N\Big(n;\ \cdots, \frac{sp_{-n}}{-n}, \cdots\Big)T_{2s}$$

but the summation on d_1 and d_2 here is in fact only for the set $\{(d_1, d_2)\in\boldsymbol{Z}^2;\ d_1, d_2\geqq 0,\ d_1-d_2=2b\}$. So we must calculate the integral of the form

$$(3.7.6)\qquad \int_\Gamma d\zeta_1 d\zeta_2 \zeta_1^{-l_1-1}\zeta_2^{-l_2-1}F\Big(\frac{s^2}{2};\ \zeta_1, \zeta_2\Big)$$

$$=\int_{\Gamma_1} d\zeta\zeta^{-l_1-l_2-1}\int_{\Gamma_2} dk k^{-l_2-1}G\Big(\frac{s^2}{2};\ k\Big),$$

hence of the form

$$(3.7\ 7)\qquad \int_{\Gamma_2}(1-k)^{2\alpha}k^\beta dk\,,$$

where β is taken as $\beta+\alpha$ is an integer.

To construct a cycle $\Gamma_2\in H_1(Y_1;\ \mathcal{S}'_\alpha)$, we divide the three cases i) $2\alpha\notin\boldsymbol{Z}$, ii) 2α is an odd integer and iii) 2α is aneven integer. Note that these cases correspond to the conditions (4.2.5).

Case i) Let $2\alpha\notin\boldsymbol{Z}$, and take a cycle $\Gamma_2\in H_1(Y_1\colon \mathcal{S}'_\alpha)$ as

$$(3.7.8)\qquad \Gamma_2=\Gamma^\varepsilon=-e_0^\varepsilon/(e^{2\pi\alpha\sqrt{-1}}-1)+(\varepsilon, 1-\varepsilon)+e_1^\varepsilon/(e^{-4\pi\alpha\sqrt{-1}}-1)\,,$$

where e_i^ε is the standard circle with the center i and originating at the point

$i+(-1)^i\varepsilon$ ($i=0$ or 1) for some small $\varepsilon>0$. Note that $\dim H_1(Y_1; \mathcal{S}'_\alpha)=1$.

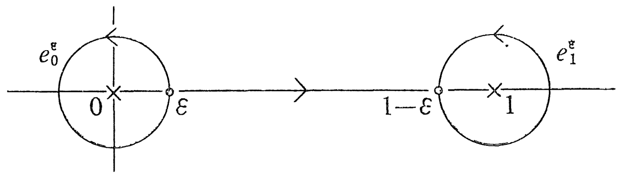


Figure 3.1

Take α and β as $\mathrm{Re}\, 2\alpha$, $\mathrm{Re}\,\beta>-1$, then the integral (3.7.7) equals to the beta function

(3.7.9)
$$B(2\alpha+1, \beta+1) = \int_0^1 (1-k)^{2\alpha} k^\beta dk = \frac{\Gamma(2\alpha+1)\Gamma(\beta+1)}{\Gamma(2\alpha+\beta+2)},$$

and for other values of α and β, the value of the integral (3.7.7) is obtained by the analytic continuation of the beta function (3.7.9). Hence, the integral (3.7.7) is meromorphic as a function of α, and its poles and zeroes are all simple and situated at most at half-integers $\left(\text{that is, } \alpha\in\frac{1}{2}\mathbb{Z}\right)$, because $\beta+\alpha\in\mathbb{Z}$.

Case ii) Let $2\alpha\in\mathbb{Z}\backslash 2\mathbb{Z}$, and take a cycle $\Gamma_2\in H_1(Y_1; \mathcal{S}'_\alpha)$ as

(3.7.10)
$$\Gamma_2 = \Gamma^\varepsilon = me_0^\varepsilon/2\pi\sqrt{-1}\,.$$

Note that $\dim H_1(Y_1; \mathcal{S}'_\alpha)=1$. Then we get that the integral (3.7.7) equals to $\delta_{2\alpha+1,0}$.

Case iii) Let $\alpha\in\mathbb{Z}$, and take a cycle $\Gamma_2\in H_1(Y_1; \mathcal{S}'_\alpha)$ as

(3.7.11)
$$\Gamma_2 = \Gamma^\varepsilon = me_0^\varepsilon/2\pi\sqrt{-1}+ne_1^\varepsilon/2\pi\sqrt{-1}\,.$$

Note that the local system $\mathcal{S}'_\alpha$ is trivial and $\dim H_1(Y_1; \mathcal{S}'_\alpha)=2$. Then we get that the integral (3.7.7) equals to $m\delta_{\beta+1,0}+n\delta_{2\alpha+1,0}$, hence to $m\delta_{\beta+1,0}$.

§4. Nontriviality of Intertwining Operators (Generic Case)

4.1) Vanishing of Homology

In this paragraph, we prove Lemma 3.9 in a more general setting.

For an integer $m\geqq 1$ and complex numbers α, β and γ, consider the manifold Y_m and the holomorphic and multi-valued function G on Y_m defined by

$$Y_m = \{(k_1, \cdots, k_m) \in (C^*)^m;\ k_i \neq 0,1 \quad \text{and} \quad k_i \neq k_j\ (i \neq j)\}, \tag{4.1.1}$$

$$G(\alpha, \beta, \gamma; k_1, \cdots, k_m) = \prod_{1 \leqq i < j \leqq m} (k_i - k_j)^{2\alpha} \prod_{j=1}^{m} k_j^{\beta}(1-k_j)^{\gamma}. \tag{4.1.2}$$

Then define the local system $\mathcal{S}^*_{\alpha,\beta,\gamma}$ on the manifold Y_m by the monodromy of the function (4.1.2), and denote by $\mathcal{S}_{\alpha,\beta,\gamma}$ the dual local system of $\mathcal{S}^*_{\alpha,\beta,\gamma}$ (note that $\mathcal{S}_{\alpha,-m\alpha,2\alpha} = \mathcal{S}'_\alpha$; see §3.4~5).

Proposition 4.1. *For each integer $m \geqq 1$ and complex numbers α, β, γ*

$$H_j(Y_m; \mathcal{S}_{\alpha,\beta,\gamma}) = 0 \qquad (j \geqq m+1). \tag{4.1.3}$$

Proof. For integers $p, q \geqq 1$, define the manifold

$$F_{p,q} = \{(k_1, \cdots, k_p) \in Y_p;\ k_i \neq 1, 2, \cdots, q\ (1 \leqq i \leqq p)\}, \tag{4.1.4}$$

and consider the local trivial fibering

$$\begin{array}{ccc} \pi: F_{p,q} & \longrightarrow & B_q \\ \Cup & & \Cup \\ (k_1, \cdots, k_p) & \mapsto & k_p \end{array} \tag{4.1.5}$$

where the base space B_q is the region obtained by omitting $q+1$ points from the complex plane C:

$$B_q = C - \{0, 1, 2, \cdots, q\}. \tag{4.1.6}$$

Then the fiber over the point $q+1 \in B_q$ is just the manifold $F_{p-1,q+1}$.

Fix an integer $m \geqq 1$. Then we get a sequence of locally trivial fiberings

$$F_{p,m-p+1} \to F_{p+1,m-p} \xrightarrow{\pi_p} B_{m-p} \tag{4.1.7}$$

for $1 \leqq p \leqq m-1$. Note that $F_{m,1} = Y_m$ and $F_{1,m} = B_m$.

Denote by the same symbol $\mathcal{S}_{\alpha,\beta,\gamma}$ the restriction of the local system $\mathcal{S}_{\alpha,\beta,\gamma}$ on the fiber $F_{p-1,m-p+2}$ at each stage. Then we can show by the induction on p $(1 \leqq p \leqq m)$ that

$$H_j(F_{p,m-p+1}; \mathcal{S}_{\alpha,\beta,\gamma}) = 0 \qquad (j \geqq p+1). \tag{4.1.8}$$

In fact, the first step

$$H_j(F_{1,m}; \mathcal{S}_{\alpha,\beta,\gamma}) = 0 \qquad (j \geqq 2) \tag{4.1.9}$$

is obvious, and it is well-known that there is a Leray spectral sequence $\{E^n_{i,j}\}$ of the fibering (4.1.7) such that

(4.1.10) $$E^2_{i,j} = H_i(B_{m-p};\ \mathcal{H}_j(F_{p,m-p+1};\ \mathcal{S}_{\alpha,\beta,\gamma})) \Rightarrow H_*(F_{p+1,m-p};\ \mathcal{S}_{\alpha,\beta,\gamma})\,.$$

Note that the Leray sheaf $\mathcal{H}_j(F_{p,m-p+1}; \mathcal{S}_{\alpha,\beta,\gamma})$ on B_{m-p} is also locally constant. Then by the assumption of the induction, we get

(4.1.11) $$H_i(B_{m-p};\ \mathcal{H}_j(F_{p,m-p+1};\ \mathcal{S}_{\alpha,\beta,\gamma})) = 0\,,$$

if $i \geqq 2$ or $j \geqq p$. Thus the proof is completed.

Define the set $\Omega(m) \subset \mathbb{C}^3$ as

(4.1.12) $$\Omega(m) = \{(\alpha, \beta, \gamma) \in \mathbb{C}^3;\ d(d+1)\alpha \notin \mathbb{Z}\,,\quad d(d-1)\alpha + d\beta \notin \mathbb{Z}\,,\quad d(d-1)\alpha + d\gamma \in \mathbb{Z}\ (1 \leqq d \leqq m)\}\,,$$

for which we can construct good cycles (see Proposition 4.2). Here we give some conjectures about lower homologies.

Conjecture 4.A. *For* $(\alpha, \beta, \gamma) \in \Omega(m)$,

(4.1.13) $$H_j(Y_m;\ \mathcal{S}_{\alpha,\beta,\gamma}) = 0 \qquad (j \neq m)\,.$$

The symmetric group Σ_m of m letters acts freely on the manifold Y_m as permutations of the coordinates $k_1, \cdots, k_m$. The function $G(\alpha, \beta, \gamma; k_1, \cdots, k_m)$ is symmetric in $k_1, \cdots, k_m$, that is, Σ_m-invariant, hence it can be considered as a function $\bar{G}(\alpha, \beta, \gamma; k_1, \cdots, k_m)$ on the quotient manifold $W_m = Y_m/\Sigma_m$. Then $\bar{G}$ is holomorphic and multi-valued on the manifold W_m. Denote by $\mathcal{S}_{\alpha,\beta,\gamma}$ the local system on W_m defined by the multi-valued function $\bar{G}$ analogously as $\mathcal{S}_{\alpha,\beta,\gamma}$. Then we get that $G(\alpha, \beta, \gamma; k) = \pi^*\bar{G}(\alpha, \beta, \gamma; k)$ and $\mathcal{S}_{\alpha,\beta,\gamma} = \pi^*\bar{\mathcal{S}}_{\alpha,\beta,\gamma}$, where π is the projection:

$$\pi:\ Y_m \to W_m = Y_m/\Sigma_m\,.$$

Conjecture 4.B. *For each* $(\alpha, \beta, \gamma) \in \Omega(m)$,

(4.1.14) $$\dim H_j(W_m;\ \bar{\mathcal{S}}_{\alpha,\beta,\gamma}) = \begin{cases} 1 & \text{for } j = m \\ 0 & \text{for } j \neq m\,. \end{cases}$$

Remark. In the trivial coefficient case, the homology groups $H_j(Y_m; \mathbb{C})$ were calculated by F.R. Cohen [1976], and by his results it can be shown that the Euler characteristic is given as

(4.1.15) $$\sum_{j=0}^{m} (-1)^j \dim H_j(Y_m;\ \mathcal{S}_{\alpha,\beta,\gamma}) = (-1)^m m!\,.$$

By this formula, it can be shown that Conjecture 4.A implies Conjecture 4.B.

4.2) Cycles and Selberg's Integrals

Fix an integer $m\geqq 1$. Let $\Gamma(\alpha, \beta, \gamma)$ be a cycle of $H_m(Y_m; \mathcal{S}_{\alpha,\beta,\gamma})$ given for each (α, β, γ) of an open set $\Omega\subset \boldsymbol{C}^3$. Then we call $\Gamma(\alpha, \beta, \gamma)$ holomorphic on Ω, if for any holomorphic function $g(k_1, \cdots, k_m)$ defined on Y_m, the integral

$$(4.2.1) \qquad \int_{\Gamma(\alpha,\beta,\gamma)} G(\alpha, \beta, \gamma; k_1, \cdots, k_m)g(k_1, \cdots, k_m)dk_1, \cdots, dk_m$$

is a holomoiphic function of $(\alpha, \beta, \gamma)\in\Omega$.

The set $\Omega(m)$ defined in (4.1.12) is connected, dense and open in $\boldsymbol{C}^3$. Then we get

Proposition 4.2. *There exist cycles $\Gamma(\alpha, \beta, \gamma)\in H_m(Y_m; \mathcal{S}_{\alpha,\beta,\gamma})$ defined on $\Omega(m)$ such that*

1) *$\Gamma(\alpha, \beta, \gamma)$ is holomorphic on $\Omega(m)$.*
2) *If $(\alpha, \beta, \gamma)\in\Omega(m)$ and $(l_1, \cdots, l_m)\in\boldsymbol{Z}^m$ satisfy the inequalities*

$$(4.2.2) \qquad \mathrm{Re}\,\alpha>0\,, \quad \mathrm{Re}\,\gamma>0 \quad \text{and} \quad \mathrm{Re}\,\beta>-\min_j l_j\,,$$

then the equality of integrals

$$(4.2.3) \qquad \int_{\Gamma(\alpha,\beta,\gamma)} G(\alpha, \beta, \gamma; k_1, \cdots, k_m)k_1^{l_1}\cdots k_m^{l_m}dk_1\cdots dk_m = \int_{\Delta(m)} G(\alpha, \beta, \gamma; k_1, \cdots, k_m)k_1^{l_1}\cdots k_m^{l_m}dk_1\cdots dk_m$$

holds, where $\Delta(m)$ is the open simplex in $\boldsymbol{R}^m$ defined by

$$(4.2.4) \qquad \Delta(m) = \{(k_1, \cdots, k_m)\in\boldsymbol{R}^m;\ 1>k_1>\cdots>k_m>0\}\,,$$

and the right hand side of the equality (4.2.3) *is considered as an improper integral which is absolutely convergent.*

The proof of this proposition will be given in §5.

Remark. For each integer $a\geqq 1$, define the set

$$(4.2.5) \qquad \Omega_a = \{\alpha\in\boldsymbol{C};\ d(d+1)\alpha\notin\boldsymbol{Z},\ d(a-d)\alpha\notin\boldsymbol{Z}\ \ (1\leqq d\leqq a-1)\}\,,$$

then for any $\alpha\in\Omega_{m+1}$, the triple $(\alpha, -m\alpha, 2\alpha)$ belongs to the set $\Omega(m)$.

The integrals of these types were considered by many people, such as A. Selberg [1944], F.J. Dyson [1962], K. Aomoto [1984] etc. The following proposition is due to A. Selberg.

Proposition 4.3 (A. Selberg [1944], I.G. Macdonald [1982]).
Let $\alpha, \beta, \gamma \in \boldsymbol{C}$ satisfy the inequalities

$$\text{(4.2.6)} \qquad \operatorname{Re}\beta > -1\,, \quad \operatorname{Re}\gamma > -1\,, \quad \operatorname{Re}\alpha > -\min\left\{\frac{1}{m}\,, \frac{\operatorname{Re}\beta+1}{m-1}\,, \frac{\operatorname{Re}\gamma+1}{m-1}\right\},$$

then the improper integral (4.2.7) *converges absolutely and is explicitly expressed as*

$$\text{(4.2.7)} \qquad \int_{\Delta(m)} \prod_{1\leqq i<j\leqq m} (k_i-k_j)^{2\alpha} \prod_{j=1}^{m} k_j^{\beta}(1-k_j)^{\gamma} dk_1\cdots dk_m$$
$$= \frac{1}{m!} \prod_{j=1}^{m} \frac{\Gamma(j\alpha+1)\Gamma((j-1)\alpha+\beta+1)\Gamma((j-1)\alpha+\gamma+1)}{\Gamma(\alpha+1)\Gamma((m+j-2)\alpha+\beta+\gamma+2)}\,.$$

Note. In the case that $m=1$, the integral (4.2.7) is the beta function

$$\text{(4.2.8)} \qquad \int_0^1 k^{\beta}(1-k)^{\gamma} dk = \frac{\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)}$$

(see §3.7).

Problem 4.C. *For $(\alpha, \beta, \gamma) \in \boldsymbol{C}^3$, $(l_1, \cdots, l_m) \in \boldsymbol{Z}^m$ and a symmetric function $f \in \boldsymbol{C}[k_1, \cdots, k_m, k_1^{-1}, \cdots, k_m^{-1}]^{\Sigma_m}$, compute the integral*

$$\text{(4.2.9)} \qquad \int_{\Delta(m)} \prod_{1\leqq i<j\leqq m} (k_i-k_j)^{2\alpha} \prod_{j=1}^{m} k_j^{\beta}(1-k_j)^{\gamma} f(k_1, \cdots, k_m) dk_1\cdots dk_m\,.$$

4.3) Nontriviality of Intertwining Operators

Take integers $a \geqq 1$ and b, and a complex number s with $\alpha = s^2/2 \in \dot{\Omega}_a$ (see (4.2.5)). Put $m=a-1$, $w=\frac{a}{2}s-\frac{b}{s}$ and $\lambda=\frac{1}{s}-\frac{s}{2}$. Take a cycle $\Gamma_2 = \Gamma(\alpha, -m\alpha, 2\alpha)$ of $H_m(Y_m; \mathcal{S}_{\alpha,-m\alpha,2\alpha})$ as Proposition 4.2, and let $\Gamma=\Gamma_1\times\Gamma_2$, where Γ_1 is a generator of $H_1(\boldsymbol{C}^*; \boldsymbol{C})$. In this situation, as a corollary of Proposition 4.3, we get

Proposition 4.4. *The intertwining operator*

$$\text{(4.3.1)} \qquad O(s, \Gamma; a, b)\colon \mathcal{F}(w-as, \lambda) \to \mathcal{F}(w, \lambda)$$

is nontrivial in the sense that

1) *for $b \geqq 0$, the image $O(s, \Gamma; a, b)|w-as, \lambda\rangle$ is a nonzero singular vector of degree ab.*

2) *for $b<0$, there exists a vector $|v\rangle$ of degree $-ab$ whose image is the vacuum vector $|w, \lambda\rangle$, that is, $O(s, \Gamma; a, b)|v\rangle = |w, \lambda\rangle$.*

Before the proof of this proposition, we prepare some facts about symmetric functions (cf. I.G. Macdonald's book [1979]).

Fix an integer $a \geqq 1$. Consider the polynomial algebra $\boldsymbol{C}[t_1, \cdots, t_a]$, and the subalgebra

$$A = \boldsymbol{C}[t_1, \cdots, t_a]^{\Sigma_a} \tag{4.3.2}$$

of symmetric polynomials, where the symmetric group Σ_a acts on $\boldsymbol{C}[t_1, \cdots, t_a]$ by permutations of indices. And consider the polynomial algebra

$$V = \boldsymbol{C}[x_1, x_2, \cdots] \tag{4.3.3}$$

of an infinite number of variables $x_1, x_2, \cdots$. These algebras are made graded algebras by defining degrees as

$$\deg t_j = 1 \ (1 \leqq j \leqq a) \quad \text{and} \quad \deg x_n = n \ (n = 1, 2, \cdots). \tag{4.3.4}$$

and

$$A = \sum_{d \geqq 0} A_d \quad \text{and} \quad V = \sum_{d \geqq 0} V_d \tag{4.3.5}$$

are their homogeneous decompositions.

For each integer $d \geqq 0$, consider the set

$$P_{a,d} = \{m = (m_1, \cdots, m_a) \in \boldsymbol{Z}^a;\ m_1 \geqq \cdots \geqq m_a \geqq 0,\ |m| := \sum_{j=1}^{a} m_j = d\}, \tag{4.3.6}$$

and let $p_{a,d} = \# P_{a,d}$ the number of partitions of the integer d by at most a positive integers, then

$$p_{a,d} \leqq p_{a+1,d} \leqq \cdots \leqq p_d = p_{d,d} = p_{d+1,d} = \cdots. \tag{4.3.7}$$

Note that

$$p_{a,d} = \dim A_d \quad \text{and} \quad p_d = \dim V_d. \tag{4.3.8}$$

Now choose a family of elements $M(m) = M(m; t_1, \cdots, t_a) \in A$ parametrized by $m \in \bigcup_{d \geqq 0} P_{a,d}$ with the property:

(4.3.9) the set $\{M(m);\ m \in P_{a,d}\}$ is a basis of A_d for each $d \geqq 0$.

Then the following lemma is well-known.

Lemma 4.5. *Consider the expansion*

$$\exp\Big(\sum_{n=1}^{\infty} (t_1^n + \cdots + t_a^n) x_n\Big) = \sum_{d \geqq 0} \sum_{m \in P_{a,d}} M(m) N(m) \in \sum_{d \geqq 0} A_d \otimes V_d, \tag{4.3.10}$$

then the elements $N(m)=N(m; x_1, x_2, \cdots)$ *for* $m\in P_{a,d}$ *are linearly independent in the space* V_d *for* $d\geqq 0$.

Now for $m\in P_{a,d}$, specify the element $M(m)\in A_d$ as

$$M(m)=M(m; t_1, \cdots, t_a)=\sum_{\sigma\in\Sigma_a/\Sigma_a(m)} t_{\sigma(1)}^{m_1}\cdots t_{\sigma(a)}^{m_a}, \tag{4.3.11}$$

where $\Sigma_a(m)$ is the subgroup of Σ_a defined by

$$\Sigma_a(m)=\{\sigma\in\Sigma_a;\ t_{\sigma(1)}^{m_1}\cdots t_{\sigma(a)}^{m_a}=t_1^{m_1}\cdots t_a^{m_a}\}, \tag{4.3.12}$$

and denote by $N(m)\in V_d$ the corresponding element given by Lemma 4.5.

Proof of Proposition 4.4. At first consider the case where $b\geqq 0$. Since the elements $N(m)$ for $m\in P_{a,d}$ are linearly independent in the space V_d, there exists an element $\langle v|\in\mathcal{F}_{ab}^{\dagger}(w, \lambda)$ such that

$$\langle v|N(m; sp_1, \frac{sp_2}{2}, \frac{sp_3}{3}, \cdots)|w, \lambda\rangle=\begin{cases}1 & \text{if } m=(b, \cdots, b)\in P_{a,ab}\\ 0 & \text{otherwise}\end{cases} \tag{4.3.13}$$

by the nondegeneracy of the vacuum expectation values (1.1.14). Then

$$\begin{aligned}(4.3.14)\quad &\langle v|E_+(s; \zeta_1, \cdots, \zeta_a)E_-(s; \zeta_1, \cdots \zeta_a)|w, \lambda\rangle\\ &=\sum_{m\in P_{a,ab}} M(m; \zeta_1, \cdots, \zeta_a)\langle v|N(m; sp_1, \frac{sp_2}{2}, \cdots)|w, \lambda\rangle\\ &=\zeta_1^b\cdots\zeta_a^b.\end{aligned}$$

Hence for the given $\Gamma=\Gamma_1\times\Gamma_2\in H_a(M_a; \mathcal{S}_\alpha)$, we get

$$\begin{aligned}(4.3.15)\quad &\langle v|O(s, \Gamma; a, b)|w-as, \lambda\rangle\\ &=\int_\Gamma d\zeta_1\cdots d\zeta_a\zeta_1^{-b-1}\cdots\zeta_a^{-b-1}F(\alpha; \zeta_1, \cdots, \zeta_a)\\ &\qquad\times\langle v|E_+(s; \zeta_1, \cdots, \zeta_a)E_-(s; \zeta_1, \cdots, \zeta_a)|w, \lambda\rangle\\ &=\int_\Gamma F(\alpha; \zeta_1, \cdots, \zeta_a)\zeta_1^{-1}\cdots\zeta_a^{-1}d\zeta_1\cdots d\zeta_a\\ &=\int_{\Gamma_2} G(\alpha; k_1, \cdots, k_{a-1})dk_1\cdots dk_{a-1}\\ &=\frac{1}{(a-1)!}\prod_{j=1}^{m}\frac{\Gamma(j\alpha+1)\Gamma((j-a)\alpha+1)\Gamma((j+1)\alpha+1)}{\Gamma(\alpha+1)\Gamma(j\alpha+2)}.\end{aligned}$$

This does not vanish by the condition for α.

For the case where $b<0$, take an element $|v\rangle\in\mathcal{F}_{ab}(w-as, \lambda)$ such that

$$\begin{aligned}(4.3.16)\quad &\langle\lambda, w-as|N(m; -sp_{-1}, \frac{sp_{-2}}{-2}, \cdots)|v\rangle\\ &=\begin{cases}1 & \text{if } m=(-b, \cdots, -b)\in P_{a,-ab}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

then the remaining of the proof is similar as above.

§5. Construction of Cycles

This section is devoted to the proof of Proposition 4.2. In §3.7, we already discussed and prove it in the case that $m=1$. So fix an integer $m\geqq 2$ throughout this section.

In the first few paragraphs, we prepare some concepts and make some observations about the geometries of the manifold Y_m and the simplex $\Delta(m)$.

5.1) Faces of Simplex $\bar{\Delta}(m)$

We use the same symbols $k_1, \cdots, k_m$ for coordinates of the spaces $\boldsymbol{R}^m \subset \boldsymbol{C}^m$, and the conventions that $k_0=1$ and $k_{m+1}=0$. Recall that

$$\Delta(m) = \{(k_1, \cdots, k_m) \in \boldsymbol{R}^m;\ 1>k_1>\cdots>k_m>0\} .$$

In this subsection, we introduce the parametrizations $\mathcal{K}$ of hyperplanes of $\boldsymbol{R}^m$ and $\boldsymbol{C}^m$, and $\mathcal{J}$ of faces of the closed simplex $\bar{\Delta}(m)$, which correspond to the singularities of the function

$$G(\alpha, \beta, \gamma; k_1, \cdots, k_m) = \prod_{1\leqq i<j\leqq m} (k_i-k_j)^{2\alpha} \prod_{j=1}^{m} k_j^{\beta}(1-k_j)^{\gamma} .$$

The parametrizations $\mathcal{K}$ and $\mathcal{J}$ are the families of subsets of $\boldsymbol{N}(m):= \{0, 1, 2, \cdots, m+1\}$ defined as

$$(5.1.1)\quad \begin{cases} \mathcal{K}:= \{K=(i,j);\ 0\leqq i<j\leqq m+1,\ K\neq(0, m+1)\} \subset \boldsymbol{N}(m)\times\boldsymbol{N}(m) , \\ \mathcal{J}:= \{J=[i,j]=(i, i+1, \cdots, j);\ 0\leqq i<j\leqq m+1,\ J\neq[0, m+1]\} . \end{cases}$$

Define the depth of their elements as

$$(5.1.2)\qquad d((i,j)) = d([i,j]) = j-i ,$$

then $\mathcal{K}$ and $\mathcal{J}$ are the disjoint union of subsets of the same depth:

$$(5.1.3)\qquad \mathcal{K} = \bigcup_{1\leqq d\leqq m} \mathcal{K}(d);\quad \mathcal{J} = \bigcup_{1\leqq d\leqq m} \mathcal{J}(d);$$

where

$$\begin{cases} \mathcal{K}(d):= \{K\in\mathcal{K};\ d(K)=d\} = \{(i, i+d);\ 0\leqq i\leqq m-d+1\} , \\ \mathcal{J}(d):= \{J\in\mathcal{J};\ d(J)=d\} = \{[i, i+d];\ 0\leqq i\leqq m-d+1\} , \end{cases}$$

and note that $\#\mathcal{K}(d)=\#\mathcal{J}(d)=m+2-d$ for $1\leqq d\leqq m$.

For an element $K=(i,j)\in\mathcal{K}$, define hyperplanes $D(K)$ of $\boldsymbol{C}^m$ and $D(K)_{\boldsymbol{R}}$ of $\boldsymbol{R}^m$ by

$$(5.1.4)\qquad \begin{cases} D(K) := \{(k_1, \cdots k_m)\in \mathbb{C}^m;\ k_i = k_j\} \\ D(K)_{\mathbb{R}} := D(K)\cap \mathbb{R}^m . \end{cases}$$

For each $J=[i,j]\in\mathcal{J}$, define the connected complex submanifold $L(J)$ of $\mathbb{C}^m$ of codimension $d=d(J)$ and its real section $L(J)_{\mathbb{R}}$ by

$$(5.1.5)\qquad \begin{cases} L(J) := \bigcap_{\substack{K\in\mathcal{K}\\ J\subset K}} D(K) = \{k\in\mathbb{C}^m;\ k_i = k_{i+1} = \cdots = k_{i+d}\} , \\ L(J)_{\mathbb{R}} := L(J)\cap\mathbb{R}^m = \bigcap_{\substack{K\in\mathcal{K}\\ J\subset K}} D(K)_{\mathbb{R}} . \end{cases}$$

Then for $K\in\mathcal{K}$ and $I, J\in\mathcal{J}$,

$$(5.1.6)\qquad \begin{cases} D(K)\supset L(J), \text{ if and only if } \quad K\subset J , \\ L(I)\supset L(J), \text{ if and only if } \quad I\subset J , \end{cases}$$

where the inclusions among elements of $\mathcal{K}$ and $\mathcal{J}$ are considered as subsets of $\mathbb{N}(m)$.

We identify $\mathcal{J}(1)=\mathcal{K}(1)$, since for $d=1$,

$$(5.1.7)\qquad L([i, i+1]) = D((i, i+1)) \qquad (0\leqq i\leqq m) .$$

Now we introduce *primitive* faces of closed simplex $\bar{\Delta}(m)$ by the following (5.1.8), with which any face of $\bar{\Delta}(m)$ is represented as in the formula (5.1.11).

For $J=[i,j]\in\mathcal{J}$, let

$$(5.1.8)\qquad \begin{cases} \bar{\Delta}(J) := L(J)\cap\bar{\Delta}(m) = \{(k_1, \cdots, k_m)\in\bar{\Delta}(m);\ k_i = k_{i+1} = \cdots = k_j\} \\ \Delta(J) := \mathrm{int}\,\bar{\Delta}(J) , \end{cases}$$

then

$$(5.1.9)\qquad \bar{\Delta}(J) = D(J)\cap\bar{\Delta}(m)$$

and $\Delta(J)$ is a face of $\bar{\Delta}(m)$ of codimension $d(J)$. And for each $1\leqq d'<d(J)$,

$$\bar{\Delta}(J) = \bigcap_{\substack{I\in\mathcal{J}(d')\\ I\subset J}} \bar{\Delta}(I) ,$$

in particular,

$$(5.1.10)\qquad \bar{\Delta}(J) = \bar{\Delta}([i,j]) = \bar{\Delta}([i,j-1])\cap\bar{\Delta}([i+1,j]) .$$

Note that all codimension 1 faces of $\bar{\Delta}(m)$ are primitive, that is, they are expressed as $\Delta(J)$ for $J\in\mathcal{J}(1)$.

The faces of the closed simplex $\bar{\Delta}(m)$ are parametrized by disjoint unions of some elements of $\mathcal{J}$. More precisely, take $J_1=[i_1, j_1], \cdots, J_r=[i_r, j_r]\in\mathcal{J}$ with the conditions

$$j_l<i_{l+1} \quad (1\leqq l\leqq r-1) ;$$

(note that $[0, m+1]\notin\mathcal{J}$) and let

$$\text{(5.1.11)} \qquad \begin{cases} \bar{\varDelta}(J_1, \cdots, J_r):=\bigcap_{l=1}^{r}\bar{\varDelta}(J_l) \\ \varDelta(J_1, \cdots, J_r):=\operatorname{int}\bar{\varDelta}(J_1, \cdots, J_r), \end{cases}$$

then $\varDelta(J_1, \cdots, J_l)$ is a face of $\bar{\varDelta}(m)$ of codimension $\sum_{l=1}^{r}(j_l-i_l)$, and any face of $\bar{\varDelta}(m)$ is of this form. By (5.1.9),

$$\text{(5.1.12)} \qquad \bar{\varDelta}(J_1, \cdots, J_r):=\bigcap_{l=1}^{r}\bigcap_{\substack{K\subset J_l\\ K\in\mathcal{K}(1)}}\bar{\varDelta}(K).$$

It is easily seen that any primitive face $\varDelta(J)$ is an $(m-d(J))$-simplex, and has coordinates similar as $\varDelta(m)$ such that primitive faces of $\varDelta(J)$ are similarly given by these coordinates:

Lemma 5.1. *For each $J=[i, j]\in\mathcal{J}(d)$ with $2\leqq d\leqq m$, there exist coordinates $(u, v)=(u_1, \cdots, u_d, v_1, \cdots, v_{m-d})$ of $\boldsymbol{C}^m$ such that*

$$\text{(5.1.13)} \qquad \begin{cases} \varDelta(m)=\{(u, v)\in\boldsymbol{R}^m;\ 1>u_1>\cdots>u_d>0,\ 1>v_1>\cdots>v_{m-d}>0\} \\ \varDelta(J)=\{(u, v)\in\boldsymbol{R}^m;\ u=0(u_1=\cdots=u_d=0),\ 1>v_1>\cdots>v_{m-d}>0\} \\ \bar{\varDelta}(J)=\{(u, v)\in\bar{\varDelta}(m);\ u=0\}\subset L(J)=\{(u, v)\in\boldsymbol{C}^m;\ u=0\} \end{cases}$$

and for any $I\subset J$ (here we write I as $I=[i+i_1, i+i_2]$ (if $i>0$) or $I=[j-i_2, j-i_1]$ (if $i=0$))

$$\text{(5.1.14)} \qquad \begin{cases} \bar{\varDelta}(I)=\{(u, v)\in\bar{\varDelta}(m);\ u_{i_1+1}=u_{i_1+2}=\cdots=u_{i_2+1}=0\} \\ L(I)=\{(u, v)\in\boldsymbol{C}^m;\ u_{i_1+1}=u_{i_1+2}=\cdots=u_{i_2+1}=0\}, \end{cases}$$

where we use the convention that $u_{d+1}=0$.

5.2) Exponent of Analytic Form

Consider a pair $(M, \mathcal{N})$ of an m-dimensional complex manifold M and a finite collection $\mathcal{N}$ of connected and closed complex submanifolds of codimension 1 of the manifold M. Then we call a family $\mathcal{N}$ be *normal crossing* at a point $p\in M$, if there exist local coordinates $(U; z)$ near the point p such that

i) $z(p)=(z_1(p), \cdots z_m(p))=(0, \cdots, 0)$ (in this situation, we call these coordinates $z=(z_1, \cdots, z_m)$ are given *around* the point p);

ii) for some $0 \leqq d \leqq m$

$$U \cap (\bigcup_{D \in \mathfrak{N}} D) = \{z \in U;\ z_1 = 0, \text{ or } \cdots, \text{ or } z_d = 0\}\ . \tag{5.2.1}$$

A family $\mathfrak{N}$ is called *normal crossing*, if $\mathfrak{N}$ is normal crossing at any point of M.

We call that a triple $(M, \mathfrak{N}, \Theta)$ satisfies the *condition* (E), if it satisfies the following conditions i)–iv):

i) M is an m-dimensional complex manifold and $\mathfrak{N}$ is a finite collection of connected and closed complex submanifolds of codimension 1 of the manifold M.

ii) $\mathfrak{N}$ is a finite collection of connected and closed complex submanifolds of codimension 1 of the manifold M. Any pair (D, D') from $\mathfrak{N}$ is normal crossing.

iii) Θ is a multi-valued analytic m-form on the manifold M and is holomorphic and does not vanish on $M \backslash N$, where N is the codimension 1 subvariety $N = \bigcup_{D \in \mathfrak{N}} D$;

iv) for each point $p \in M$, there exist local coordinates $(U; z)$ around p such that for some d and $\alpha_j \in \boldsymbol{C}$ $(1 \leqq j \leqq d)$,

$$\Theta|_U = \prod_{j=1}^{d} f_j^{\alpha_j} f(z)\, dz_1 \cdots dz_m\,, \tag{5.2.2}$$

where $f(z)$ is a nonvanishing holomorphic function on U and $f_j = 0$ is the local equation of some element $D_j \in \mathfrak{N}$ through the point p, that is,

$$D_j \cap U = \{f_j = 0\} \quad \text{and} \quad (df_j)_q \neq 0 \quad (q \in U)\,. \tag{5.2.3}$$

Let a triple $(M, \mathfrak{N}, \Theta)$ satisfy the condition (E), and take a codimension 1 submanifold $D \in \mathfrak{N}$. Choose a point $p \in D$ and local coordinate $(U; z)$ around p such that

i) $U \cap D' = \emptyset$ for any $D' \in \mathfrak{N} \backslash \{D\}$; $U \cap D = \{z_1 = 0\}$;

ii) the form Θ is written on U as

$$\Theta = z_1^e g(z)\, dz_1 \cdots dz_m \tag{5.2.4}$$

for some complex number e, where $g(z)$ is a nonvanishing holomorphic function on U.

Then this number e in (5.2.4) proves to be independent of the choice of a point p and local coordinates $(U; z)$, hence we denote this number by $e = e(D, \Theta)$, and call $e(D, \Theta)$ the *exponent* of the m-form Θ along a codimension 1 submanifold $D\in\mathfrak{N}$.

Now return to our situation of Proposition 4.2.

At first, we recall that the manifold Y_m and the simplex $\Delta(m)$ are of the form:

$$\text{(5.2.5)} \qquad \begin{cases} Y_m = \{(k_1, \cdots, k_m)\in\boldsymbol{C}^m;\ k_i\neq k_j\ (0\leqq i<j\leqq m+1)\}\ . \\ \Delta(m) = \{(k_1, \cdots, k_m)\in\boldsymbol{R}^m;\ 1>k_1>\cdots>k_m>0\}\ . \end{cases}$$

Consider the collections $\mathfrak{N}$ of closed, connected codimension 1 submanifolds of $\boldsymbol{C}^m$ and the subvariety N defined by

$$\text{(5.2.6)} \qquad \begin{cases} \mathfrak{N}:=\{D(K);\ K\in\mathcal{K}\} \\ N:=\bigcup_{D\in\mathfrak{N}} D=\bigcup_{K\in\mathcal{K}} D(K)\,, \end{cases}$$

then we get

$$\text{(5.2.7)} \qquad Y_m=\boldsymbol{C}^m\backslash N\,.$$

For each $(\alpha, \beta, \gamma)\in\boldsymbol{C}^3$, consider the analytic m-form $\Theta(\alpha, \beta, \gamma)$ on $\boldsymbol{C}^m$ defined by

$$\text{(5.2.8)} \qquad \Theta:=\Theta(\alpha, \beta, \gamma)=G(\alpha, \beta, \gamma;\ k_1, \cdots, k_m)dk_1\cdots dk_m$$

where

$$G(\alpha, \beta, \gamma;\ k_1, \cdots, k_m)=\prod_{1\leqq j\leqq m} k_j^{\beta}(1-k_j)^{\gamma}\prod_{1\leqq i<j\leqq m}(k_i-k_j)^{2\alpha}\,,$$

then $\Theta(\alpha, \beta, \gamma)$ is multi-valued on $\boldsymbol{C}^m$ and is holomorphic on Y_m.

It is easily checked that this triple $(\boldsymbol{C}^m, \mathfrak{N}, \Theta)$ satisfies the condition (E), and the exponents of Θ are given as follows: for $K=(i, j)\in\mathcal{K}$

$$\text{(5.2.9)} \qquad e(D(K), \Theta)=\begin{cases} \beta & \text{if } j=m+1 \\ \gamma & \text{if } i=0 \\ 2\alpha & \text{otherwise}\,. \end{cases}$$

However, the family $\mathfrak{N}$ is not normal crossing, so it is difficult to construct a desired cycle in $H_m(\boldsymbol{C}^m\backslash N;\ \mathcal{S}_{\alpha,\beta,\gamma})$. In order to avoid this difficulty, in the next paragraph we will desingularize $L(J)$ $(J\in\mathcal{J}(2)\cup\cdots\cup\mathcal{J}(m))$ such that any divisor of singularities of the m-form intersects with the closed cell $\bar{\Delta}$ in some 1-codimensional face, and the family of these divisors is normal crossing at any point of $\bar{\Delta}$.

5.3) Blowing Up

In this paragraph, we will construct a complex m-dimensional manifold

M_∞, and a proper holomorphic mapping

$$\pi_\infty\colon M_\infty \to M_0 := \mathbb{C}^m \tag{5.3.1}$$

which has the following properties:

(I) The restriction

$$\pi_\infty\colon M_\infty \backslash N_\infty \to \mathbb{C}^m \backslash N = Y_m \tag{5.3.2}$$

is biholomorphically homeomorphic, where $N_\infty = \pi_\infty^{-1}(N)$.

(II) For $D \in \mathcal{N}$, $\pi_\infty^{-1}(D)$ is a connected and closed submanifold of M_∞ of codimension 1.

(III) The family $\mathcal{N}_\infty = \{\pi_\infty^{-1}(D);\ D \in \mathcal{N}\}$ is normal crossing at each point of the closed cell $\bar{\Delta}_\infty$, where

$$\Delta_\infty = \pi_\infty^{-1}(\Delta(m)) \subset M_\infty \backslash N_\infty\,. \tag{5.3.3}$$

By the induction on $d (d=1, 2, \cdots, m-1)$, we will construct this mapping $\pi_\infty\colon M_\infty \to \mathbb{C}^m$ as a composition of a sequence of mappings:

$$\pi_\infty\colon M_\infty = M_{m-1} \xrightarrow{\pi_{m-1}} M_{m-2} \xrightarrow{\pi_{m-2}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = \mathbb{C}^m\,. \tag{5.3.4}$$

The induction data $\{M_d, \mathcal{N}_d, \mathcal{L}_d, \Delta_d\}_{(0 \leq d \leq m-1)}$ and $\{\pi_d\}_{(1 \leq d \leq m-1)}$ are given as follows:

(1) M_d is an m-dimensional complex manifold.

(2) $\mathcal{N}_d$ is a collection of $(m-1)$-dimensional connected and closed submanifolds of M_d, parametrized as

$$\mathcal{N}_d = \mathcal{N}_d(1) \cup \mathcal{N}_d(2)\,, \qquad \begin{cases} \mathcal{N}_d(1) = \{D_d(K);\ K \in \mathcal{K}(2) \cup \cdots \cup \mathcal{K}(m)\} \\ \mathcal{N}_d(2) = \{D_d(J);\ J \in \mathcal{J}(1) \cup \bigcup_{j=0}^{d-1} \mathcal{J}(m-j)\}\,. \end{cases} \tag{5.3.5}$$

(3) $\mathcal{L}_d = \bigcup_{l=2}^{m-d} \mathcal{L}_d(l)$, where $\mathcal{L}_d(l)$ is a collection of $(m-l)$-dimensional submanifolds of M_d, parametrized as

$$\mathcal{L}_d(l) = \{L_d(J);\ J \in \mathcal{J}(l)\}\,,$$

such that any two d-dimensional submanifolds from $\mathcal{L}_d(m-d)$ are mutually disjoint.

(4) $\pi_{d+1}\colon M_{d+1} \to M_d$ is the blowing up of M_d along the d-dimensional submanifold $L_d \subset M_d$, where $L_d = \bigcup_{J \in \mathcal{J}(m-d)} dL_d(J)$.

(5) $\varDelta_d = \pi_{d-1}^{-1}(\varDelta_{d-1})$ is an m-dimensional cell in $M_d \backslash N_d$, which is biholomorphically homeomorphic to $\varDelta_0 = \varDelta(m)$, where $N_d = \bigcup_{D \in \mathfrak{N}_d} D$.

For the stage that $d=0$, we may add the subscript 0 to corresponding objects in §5.1. We must only note that $\mathcal{L}_0(m)$ consists of two distinct points $L_0([0, m]) = (1, \cdots, 1) \in \boldsymbol{C}^m$ and $L_0([1, m+1]) = (0, \cdots, 0)$, and $L_0 = L_0([0, m]) \cup L_0([1, m+1]) \subset \boldsymbol{C}^m$.

After the blowing up π_{d+1}, we define objects at the $(d+1)$-stage as follows: for $K \in \bigcup_{j=2}^{m} \mathcal{K}(j)$ and $J \in \mathcal{J}(1) \cup \bigcup_{l=m-d+1}^{m} \mathcal{J}(l)$, let

$$(5.3.6) \qquad \begin{cases} D_{d+1}(K) := \text{the closure of } \pi_{d+1}^{-1}(D_d(K) \backslash (D_d(K) \cap L_d)), \\ D_{d+1}(J) := \text{the closure of } \pi_{d+1}^{-1}(D_d(J) \backslash (D_d(J) \cap L_d)), \end{cases}$$

for $J \in \mathcal{J}(m-d)$

$$(5.3.7) \qquad D_{d+1}(J) = P\nu(L_d(J)),$$

and for $J \in \bigcup_{l=2}^{m-d-1} \mathcal{J}(l)$,

$$(5.3.8) \qquad L_{d+1}(J) := \text{the closure of } \pi_{d+1}^{-1}(L_d(J) - (L_d(J) \cap L_d)),$$

where $P\nu(L_d(J))$ is the projective normal bundle of $L_d(J)$ in M_d. (Note that submanifolds $L_d(J)$ $(J \in \mathcal{J}(m-d))$ are mutually disjoint, so we can blow up individually.)

Then it is easily seen that these objects satisfy the assumption (1)–(4) of the induction except that any two submanifolds from $\mathcal{L}_{d+1}(m-d-1)$ are mutually disjoint.

For $I \in \bigcup_{l=2}^{m-d-1} \mathcal{J}(l)$ and $J \in \mathcal{J}(1) \cup \bigcup_{l=m-d}^{m} \mathcal{J}(l)$, let

$$(5.3.9) \qquad \bar{\varDelta}_{d+1}(I) = L_{d+1}(I) \cap \bar{\varDelta}_{d+1}; \quad \bar{\varDelta}_{d+1}(J) = D_{d+1}(J) \cap \bar{\varDelta}_{d+1}.$$

By the construction, we can see that

$$(5.3.10) \qquad \mathfrak{N}_{d+1}(2) = \{D \in \mathfrak{N}_{d+1};\ D \cap \bar{\varDelta}_{d+1} \neq \emptyset\}$$

and any codimension 1 face of the closed m-cell $\bar{\varDelta}_{d+1}$ is expressed as $\bar{\varDelta}_{d+1} \cap D$ for some $D \in \mathfrak{N}_{d+1}(2)$. Moreover, the family $\mathfrak{N}_{d+1}(2)$ is normal crossing at any point of $\bar{\varDelta}_{d+1}$, where $\varDelta_{d+1} = \pi_{d+1}^{-1}(\varDelta_d)$.

We illustrate for the case that $d=0$. Let $J=[1, m+1]$, then the blowing up of $M_0 = \boldsymbol{C}^m$ along the point $L_0(J) = (0, \cdots, 0)$ is expressed by the coordinate change:

(5.3.11) $$k_1 = w_1,\ k_2 = w_1 w_2,\ \cdots,\ k_m = w_1 w_m$$

in a neighborhood U of $\bar{\Delta}_1 \cap L_1(J)$. Then $\{w_2, \cdots w_m\}$ gives affine coordinates of $P\nu(L_0(J)) = P^{m-1}$ in U, and

(5.3.12) $$\bar{\Delta}_1(J) = \{(w_1, \cdots, w_m) \in \bar{\Delta}_1 \cap U;\ w_1 = 0,\ 1 > w_2 > w_3 > \cdots > w_m > 0\}\ .$$

For $I = [i, j] \subsetneqq J$,

$$\bar{\Delta}_1(J) \cap U = \begin{cases} \{(w_1, \cdots, w_m) \in \bar{\Delta}_1 \cap U;\ 1 = w_2 = w_3 = \cdots = w_j\} & \text{if } i = 1 \\ \{(w_1, \cdots, w_m) \in \bar{\Delta}_1 \cap U;\ w_i = w_{i+1} = \cdots = w_j\} & \text{if } i \neq 1\ . \end{cases}$$

In particular,

$$\begin{cases} L_1([1, m]) \cap U = \{(w_1, \cdots, w_m) \in U;\ 1 = w_2 = w_3 = \cdots = w_m\} \\ L_1([2, m+1]) \cap U = \{(w_1, \cdots, w_m) \in U;\ w_2 = w_3 = \cdots = w_m = 0\}\ . \end{cases}$$

Hence

$$L_1([1, m]) \cap L_1([2, m+1]) = \emptyset\ ,$$

since the left hand side is included in

$$\pi_1^{-1}(L_0([1, m]) \cap L_0([2, m+1]) = \pi_1^{-1}(L_0([1, m+1])) = L_1([1, m+1])\ ,$$

so in U. It is also clear that $\mathcal{N}_1(2)$ is normal crossing in U.

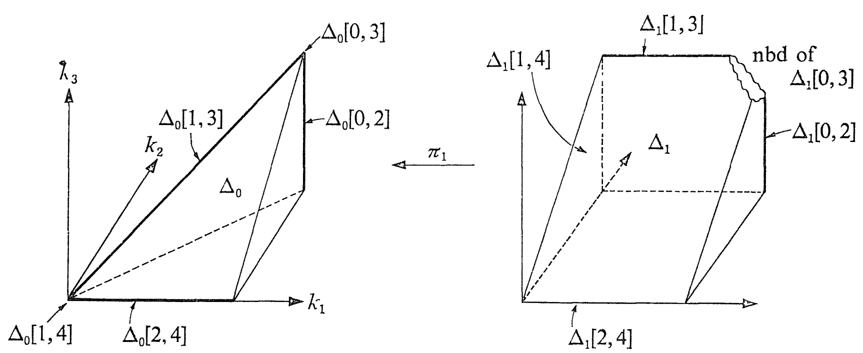


Figure 5.1 $m=3$, $J=[1, 4]$

The blowing up along the point $(1, \cdots 1) = L_0([0, m])$ is similarly described. Other primitive faces have the same structure as in the $d=0$ stage, except that we can consider one variable freely (w_1 in the above case, see Lemma 5.1).

Thus we can show that for any $J = [i, i+d]$ $(d \geqq 2)$, $L_{m-d}(J)$ doesn't intersect with other $L_{m-d}(I)$ with $d = d(I)$, and after the blowing up of M_{m-d} along

$L_{m-d}(J)$,

$$L_{m-d+1}([i, i+d-1]) \cap L_{m-d+1}([i+1, i+d]) = \emptyset .$$

More in detail, similarly as Lemma 5.1, we can introduce coordinates $\{u_1, \cdots, u_d, v_1, \cdots v_{m-d}\}$ in a neighborhood U of $\bar{\Delta}_{m-d} \cap L_{m-d}(J)$ such that

$$(5.3.13) \quad \begin{cases} \Delta_{m-d} \cap U = \{(u, v) \in \mathbf{R}^m \cap U;\ 1 > u_1 > \cdots > u_d > 0,\ 1 > v_i > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqq i \leqq m-d)\} \\ \Delta_{m-d}(J) \cap U = \{(u, v) \in \mathbf{R}^m \cap U;\ u_1 = \cdots = u_d = 0,\ 1 > v_1 > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqq i \leqq m-d)\} , \end{cases}$$

and for I_1 and $I_2 \in \mathcal{J}(d-1)$ such that $I_1 \cup I_2 = J$,

$$\begin{cases} \bar{\Delta}_{m-d}(I_1) \cap U = \{(u, v) \in \bar{\Delta}_{m-d} \cap U;\ 1 > u_1 = u_2 = \cdots = u_d > 0,\ 1 > v_i > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqq i \leqq m-d)\} \\ \bar{\Delta}_{m-d}(I_2) \cap U = \{(u, v) \in \bar{\Delta}_{m-d} \cap U;\ 1 > u_1 > 0,\ u_2 = \cdots = u_d = 0,\ 1 > v_i > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1 \leqq i \leqq m-d)\} . \end{cases}$$

The blowing up π_{m-d+1} of M_{m-d} along $L_{m-d}(J)$ is expressed by the coordinate change

$$(5.3.14) \qquad u_1 = w_1,\ u_2 = w_1 w_2,\ \cdots,\ u_d = w_1 w_d$$

in a neighborhood U' of $\bar{\Delta}_{m-d+1} \cap L_{m-d+1}(J)$, so

$$\begin{cases} L_{m-d+1}(I_1) \cap U' = \{(w, v) \in U';\ 1 = w_2 = \cdots = w_d\} \\ L_{m-d+1}(I_2) \cap U' = \{(w, v) \in U';\ w_2 = \cdots = w_d = 0\} , \end{cases}$$

hence $L_{m-d+1}(I_1) \cap L_{m-d+1}(I_2) = \emptyset$.

Now the induction procedure is justified, so we get the proper holomorphic mapping

$$\pi_\infty \colon M_\infty = M_{m-1} \xrightarrow{\pi_{m-1}} M_{m-2} \xrightarrow{\pi_{m-2}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = \mathbf{C}^m$$

which satisfies the three conditions (I)–(III) in the first part of this subsection. Since every blowing up above is defined over $\mathbf{R}$, we can define real sections of M_∞ and $D_\infty(J) = D_{m-1}(J)$ $(J \in \mathcal{J})$ as

$$M_{\infty,\mathbf{R}} = \pi_\infty^{-1}(\mathbf{R}^m);\ D_{\infty,\mathbf{R}}(J) = D_\infty(J) \cap M_{\infty,\mathbf{R}} .$$

Note that $\mathcal{L}_{m-1} = \emptyset$. Replace the subscript $m-1$ with ∞.

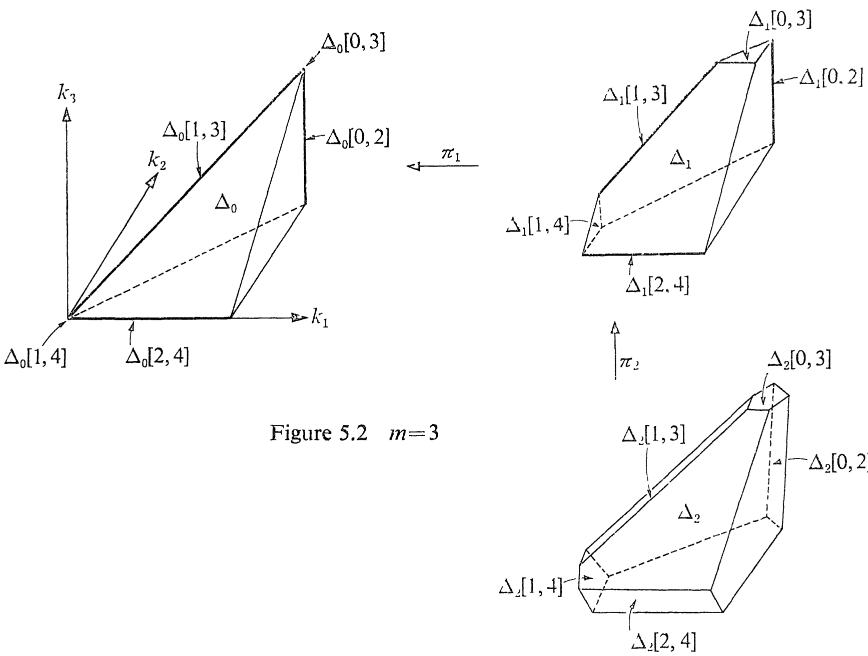


Figure 5.2 $m=3$

The family $\mathcal{N}_\infty(2)_{\boldsymbol{R}}=\{D_\infty(J)_{\boldsymbol{R}};J\in\mathcal{J}\}$ of connected, closed, 1-codimensional and real analytic manifolds of $M_{\infty,\boldsymbol{R}}$ is normal crossing at each point of $M_{\infty,\boldsymbol{R}}$ at least near $\bar{\Delta}_\infty$. And Δ_∞ is one of connected components of $M_{\infty,\boldsymbol{R}}\setminus\bigcup_{J\in\mathcal{J}}D_{\infty,\boldsymbol{R}}$.

For $0\leqq d\leqq m$, let

(5.3.15) $E(d)=\{p\in\bar{\Delta}_\infty;$ there exist just d elements of $\mathcal{N}_\infty$ containing $p\}$ then

$$\bar{\Delta}_\infty=\bigcap_{d=0}^{m}E(d)\,. \tag{5.3.16}$$

Then the subanalytic set $\bar{\Delta}_\infty$ in $M_{\infty,\boldsymbol{R}}$ has the structure of a stratified set:

$$E(0)=\Delta_\infty;\ E(1)=\bigcup_{J\in\mathcal{J}}\Delta_\infty(J)\,, \tag{5.3.17}$$

and $\bar{\Delta}_\infty(J)$ is homeomorphic to an $(m-1)$-dimensional disk. Each connected component of $E(d)$ is the interior of the intersection of some elements of $\{\bar{\Delta}_\infty(J);J\in\mathcal{J}\}$, and is a real codimension d submanifold of $M_{\infty,\boldsymbol{R}}$ which is homeomorphic to an $(m-d)$-dimensional cell.

Now define the analytic m-form Θ_∞ on M_∞ by

$$\Theta_\infty=\Theta_\infty(\alpha,\beta,\gamma)=\pi_\infty^*(\Theta(\alpha,\beta,\gamma))\,, \tag{5.3.18}$$

then the triple $(M_\infty,\mathcal{S}_\infty,\Theta_\infty)$ satisfies the condition (E), and the exponents are computed as follows:

Proposition 5.2. *For* $J=[i,j]\in\mathcal{J}(d)$ $(1\leqq d\leqq m)$,

$$(5.3.19)\qquad e(D_\infty(J),\Theta_\infty)=\begin{cases} d(d-1)\alpha+d\gamma+d-1 & \text{if } i=0\\ d(d-1)\alpha+d\beta+d-1 & \text{if } j=m+1\\ d(d+1)\alpha+d-1 & \text{otherwise}.\end{cases}$$

Proof. For $J\in\mathcal{J}(1)$, the blowing up π_∞ gives no essential changes for coordinates near a generic point of $D_\infty(J)$, so the formula (5.2.9) is the desired.

For any $J\in\mathcal{J}(d)$ $(2\leqq d\leqq m)$, take a generic point p of $D_\infty(J)$, then coordinates near p are changed from ones near $\pi_\infty(p)$ essentially (i.e. singularly) only by the blowing up π_{m-d+1} (see (5.3.13, 14)).

More precisely, we can introduce coordinates $\{w_1,\cdots,w_d,x_1,\cdots,x_{m-d}\}$ around p in a neighborhood $U\subset\mathbf{C}^m$ and $\{u_1,\cdots,u_d,v_1,\cdots,v_{m-d}\}$ around $\pi_\infty(p)$ in $\pi_\infty(U)\subset M_\infty$ such that 1) the mapping π_∞: $U\to\pi_\infty(U)$ is given as

$$u_1=w_1,\ u_2=w_1w_2,\ \cdots,\ u_d=w_1w_d,\ v_i=x_i\ (1\leqq i\leqq m-d),$$

and 2) the m-form Θ_0 is expressed as

$$\Theta_0=\prod_{1\leqq i\leqq d}u_i^\nu\prod_{1\leqq i<j\leqq d}(u_i-u_j)^{2\alpha}f(u_1,\cdots,u_d,v_1,\cdots,v_{m-d})du_1\cdots du_d dv_1\cdots dv_{m-d},$$

where $f(u,v)$ does not vanish on $\pi_\infty(U)$ and

$$\nu=\gamma\ (\text{if } i=0),\ \nu=\beta\ (\text{if } j=m+1),\ \nu=2\alpha\ (\text{otherwise}).$$

Hence

$$\Theta_m=w_1^e g(w_1,\cdots,w_d,x_1,\cdots,x_{m-d})\,dw_1\cdots dw_d dx_1\cdots dx_{m-d},$$

where $g(w,x)\neq 0$ on U and

$$e=d\nu+\frac{d(d-1)}{2}\times 2\alpha+d-1.$$

5.4) Proof of Proposition 4.2 (Construction of Cycles)

Here we use the notations in the preceding paragraphs. Fix an integer $m\geqq 2$ and an element (α,β,γ) of the set $\Omega\subset\mathbf{C}^3$ defined by

$$(5.4.1)\quad \Omega=\Omega(m)=\{(\alpha,\beta,\gamma)\in\mathbf{C}^3;\ e(J;\alpha,\beta,\gamma)\notin\mathbf{Z}\text{ for any }J\in\mathcal{J}\}$$
$$=\{(\alpha,\beta,\gamma)\in\mathbf{C}^3;\ d(d+1)\alpha\notin\mathbf{Z},\ d(d-1)\alpha+d\beta\notin\mathbf{Z},\ d(d-1)\alpha+d\gamma\notin\mathbf{Z}\quad(1\leqq d\leqq m)\}.$$

Since $\pi_\infty: M_\infty - N_\infty \to M_0 - N_0 = Y_m$ is homeomorphic, we get an isomorphism between the homology groups

(5.4.2) $\pi_{\infty *}: H_m(M_\infty - N_\infty; \mathcal{S}_{\Theta_\infty}) \to H_m(M_0 - N_0; \mathcal{S}_{\Theta_0}) = H_m(Y_m; \mathcal{S}_{\alpha,\beta,\gamma})$

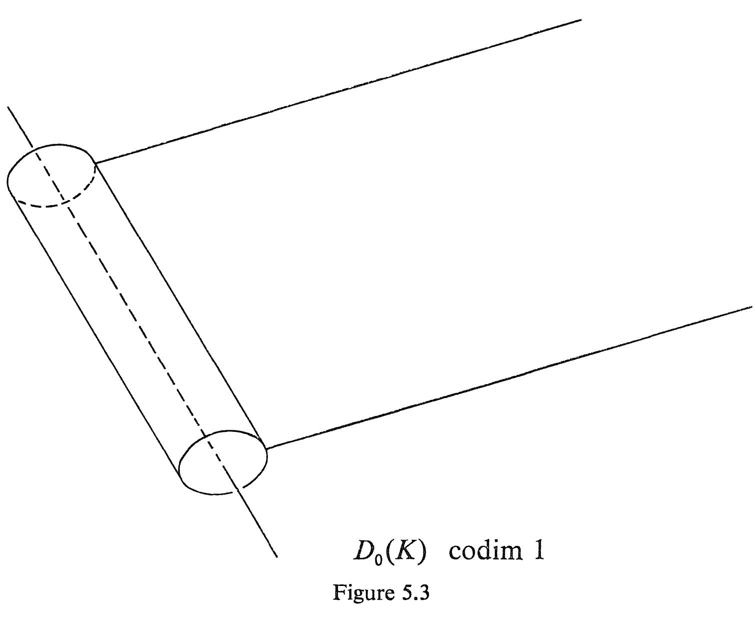
where $\mathcal{S}_{\Theta_0}$ and $\mathcal{S}_{\Theta_\infty}$ are the local system on the manifolds $M_\infty - N_\infty$ and $Y_m = M_0 - N_0$ determined by the m-forms Θ_∞ and Θ_0 similarly as $\mathcal{S}_\alpha$. Hence we will construct a cycle $\Gamma_\infty(\alpha, \beta, \gamma)$ and project Γ_∞ to get the desired cycle $\Gamma(\alpha, \beta, \gamma)$of $H_m(Y_m; \mathcal{S}_{\alpha,\beta,\gamma})$.

Recall that $\Delta_\infty(J)$ for $J \in \mathcal{J}$ is an open subset in $D_\infty(J)_{\mathbf{R}}$ and the family $\mathcal{N}_{\infty,\mathbf{R}} = \{D_\infty(J)_{\mathbf{R}}; J \in \mathcal{J}\}$ is normal crossing at any point of $\bar{\Delta}_\infty$.

Introduce a riemannian metric in M_∞ and fix a small $\varepsilon > 0$, and consider the ε-neighborhood $U_\varepsilon(d)$ of $E(d)$ in M_∞ and the real section $U_\varepsilon(d)_{\mathbf{R}} = U_\varepsilon(d) \cap M_{\infty,\mathbf{R}}$.

We construct chains $c(d)$ and $e(d) \in C_{m-d}(M_\infty - N_\infty; \mathcal{S}_{\Theta_\infty})$ inductively on $d = 0, 1, \cdots, m$ such that

$$\text{(5.4.3)} \qquad \begin{cases} c(d) = c(d-1) + e(d)\,, \\ \text{support } (\partial c(d)) \subset \bigcup_{j=d+1}^{m} U_\varepsilon(d)\,, \\ \text{support } (e(d)) \subset U_\varepsilon(d)\,. \end{cases}$$



$D_0(K)$ codim 1

Figure 5.3

At first we introduce the orientation in $\Delta_0 = \Delta(m) \subset M_{0,\mathbf{R}} = \mathbf{R}^m = \{(k_1, \cdots, k_m)\}$

by the ordering of this coordinate system, and introduce the orientation on Δ_∞ by the homeomorphism $\pi_\infty: \Delta_\infty \to \Delta_0$.

Put

$$c(0) = \Delta_\infty - (\bigcup_{j=1}^{m} U_{\varepsilon/2}(d)_{\boldsymbol{R}}) \cap \Delta_\infty \tag{5.4.4}$$

with the orientation on Δ_∞, then

$$\partial c(0) \subset \Delta_\infty - \bigcup_{j=1}^{m} U_\varepsilon(d) . \tag{5.4.5}$$

Take a point $p \in E(1)$, then there exists an element $J \in \mathcal{J}$ such that $p \in \Delta_\infty(J)$ and near the point p we have a local coordinate system $(U; z)$ such that

$$\begin{cases} U = \{z = (z_1, \cdots, z_m) \in \boldsymbol{C}^m;\ |z_i| < 5\ \ (1 \leqq i < m)\} , \\ U_{\boldsymbol{R}}: = U \cap M_{\infty,\boldsymbol{R}} = \{z \in U;\ \mathrm{Im} z_i = 0\ \ (1 \leqq i \leqq m)\} , \\ U \cap \Delta_\infty = \{z \in U_{\boldsymbol{R}};\ z_1 > 0\} , \\ U \cap \Delta_\infty(J) = \{z \in U_{\boldsymbol{R}};\ z_1 = 0\} , \\ U \cap U_\varepsilon(1) = \{z \in U;\ |z_1| < \varepsilon\} . \end{cases} \tag{5.4.6}$$

Now the exponent $e = e(D_\infty(J), \Theta_\infty(\alpha, \beta, \gamma))$ along $D_\infty(J)$ is not an integer, since $(\alpha, \beta, \gamma) \in \Omega$. So we can suppose the orientation ω_∞ coincides with the orientation of $U \cap M_{\infty,\boldsymbol{R}}$ induced by the ordering of this coordinate $(z_1, \cdots, z_m)$.

On this coordinate neighborhood U,

$$c(0) = \{z \in U \cap M_{\infty,\boldsymbol{R}};\ z_1 \geqq \frac{\varepsilon}{2}\} \tag{5.4.7}$$

with this orientation. Define a chain $e(1)$ as

$$e(1)|_U = \frac{1}{e^{-2\pi i e} - 1} S^1(\frac{\varepsilon}{2}) \times \{z' = (z_2, \cdots, z_m);\ \mathrm{Im} z_j = 0\ (2 \leqq j \leqq m)\} , \tag{5.4.8}$$

where

$$S^1(\frac{\varepsilon}{2}) = \{z_1 \in \boldsymbol{C};\ |z_1| = \frac{\varepsilon}{2}\}$$

with a positive orientation.

By patching together this $e(1)|_U$ at each point $p \in E(1) \backslash (\bigcup_{j=2}^{m} U_{\varepsilon/2}(j) \cap M_{\infty,\boldsymbol{R}})$, we can get a chain $e(1)$ such that $c(1) = c(0) + e(1)$ and $e(1)$ satisfy the conditions (5.4.6).

In the second stage $d=2$, we can construct $e(2)$ analogously. In fact, take a point $p \in E(2)$ then there exist local coordinates $(U; z)$ such that

$$(5.4.9)\quad \begin{cases} U=\{z=(z_1,\cdots,z_m)\in\mathbf{C}^m;\ |z_i|<5\ \ (1\leqq i\leqq m)\}\ , \\ U_{\mathbf{R}}=U\cap M_{\infty,\mathbf{R}}=\{z\in U;\ \mathrm{Im}\,z_i=0\ \ (1\leqq i\leqq m)\}\ , \\ U\cap\Delta_\infty=\{z\in U_{\mathbf{R}};\ z_1>0,\ z_2>0\}\ , \\ U\cap E(1)=\{z\in U_{\mathbf{R}};\ z_1=0,\ z_2>0\}\cup\{z\in U_{\mathbf{R}};\ z_1>0,\ z_2=0\}\ , \\ U\cap U_\varepsilon(1)=\{z\in U;\ |z_1|<\varepsilon,\ |\mathrm{Im}\,z_j|<\varepsilon\ \ (2\leqq j\leqq m),\ \mathrm{Re}\,z_2>-\varepsilon\} \\ \qquad\qquad\cup\{z\in U;\ |z_2|<\varepsilon,\ |\mathrm{Im}\,z_j|<\varepsilon\ \ (j\neq 2),\ \mathrm{Re}\,z_1>-\varepsilon\}\ , \\ U\cap U_\varepsilon(2)=\{z\in U;\ |z_1|<\varepsilon,\ |z_2|<\varepsilon,\ |\mathrm{Im}\,z_j|<\varepsilon\ \ (3\leqq j\leqq m)\}\ , \\ U\cap E(2)=\{z\in U_{\mathbf{R}};\ z_1=z_2=0\}\ , \end{cases}$$

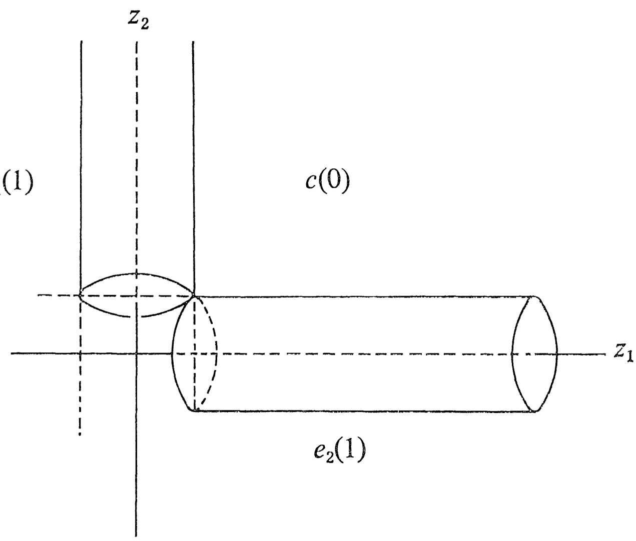


Figure 5.4

then

$$(5.4.10)\quad \begin{cases} c(0)=\{z\in U\cap M_{\infty,\mathbf{R}};\ z_1\geqq\frac{\varepsilon}{2},\ z_2\geqq\frac{\varepsilon}{2}\}\ , \\ e(1)|_U=e_1(1)|_U+e_2(1)|_U\ , \\ e_1(1)|_U=\dfrac{1}{e^{-2\pi i\alpha_1}-1}S_1^1(\frac{\varepsilon}{2})\times\{(z_2,\cdots,z_m);\ \mathrm{Im}\,z_j=0,\ \frac{\varepsilon}{2}\leqq z_j\leqq 5\}\ , \\ e_2(1)|_U=\dfrac{1}{e^{-2\pi i\alpha_2}-1}\{z_1;\ \mathrm{Im}\,z_1=0,\ \frac{\varepsilon}{2}\leqq z_1\leqq 5\}\times S_2^1(\frac{\varepsilon}{2}) \\ \qquad\qquad\times\{(z_3,\cdots,z_m);\ \mathrm{Im}\,z_j=0,\ \frac{\varepsilon}{2}\leqq z_j\leqq 5\}\ , \end{cases}$$

where

$$S_l^1(\frac{\varepsilon}{2})=\{z_l\in\mathbf{C};\ |z_l|=\frac{\varepsilon}{2}\}\qquad(l=1,2)$$

with a positive orientation, and α_1 and α_2 are the exponents along the sub-manifolds $\{z_1=0\}$ and $\{z_2=0\}$ respectively.

Then define a chain $e(2)$ as

$$(5.4.11)\qquad e(2)|_U = \frac{1}{(e^{-2\pi i\alpha_1}-1)(e^{-2\pi i\alpha_2}-1)} S_1^1(\frac{\varepsilon}{2})\times S_2^1(\frac{\varepsilon}{2}) \times\{(z_3,\cdots,z_m);\ \mathrm{Im}\, z_j=0,\ \frac{\varepsilon}{2}\leqq z_j\leqq 5\}\ ,$$

and by patching together this $e(2)|_U$ at each point $p\in E(2)\backslash(\bigcup_{j=3}^m U_{\varepsilon/2}(j)_{\boldsymbol{R}}\cap \Delta_\infty)$, we can get a chain $e(2)$ such that $c(2)=c(0)+e(1)+e(2)$ and $e(2)$ satisfy the conditions (5.4.6).

By induction on d, we get the chain $c=c(m)$ and this chain is the desired cycle $\Gamma_\infty(\alpha,\beta,\gamma)\in H_m(M_\infty\backslash N_\infty, \mathcal{S}_{\Theta_\infty(\alpha,\beta,\gamma)})$.

The key points to the inductive steps are

(1) the family $\mathfrak{N}_\infty$ is normal crossing at any point of $\bar{\Delta}_\infty$;

(2) along each $D\in\mathfrak{N}_\infty$, the exponent of Θ_∞ is not an integer.

The work is tedious but not difficult. In order to make the patching of locally defined chains $e(d)|_U$, we use the technique of controlled tubular neighborhood system of the stratified set $\bar{\Delta}_\infty=\bigcup_{d=0}^m E(d)$ due to Thom-Mather (see J. Mather [1970]).

Appendix. Fermi-Bose Correspondence

A.0) In this appendix, we show the way how the Fock space representations $\mathcal{F}(w,\lambda)$ of the Virasoro algebra $\mathcal{L}$ can be constructed by using charged Fermi operators.

When we were studing the work of F.L. Feigin and D.B. Fuks ([1982]), we arrived at our Fock space representations $\mathcal{F}(w,\lambda)$. In that paper, they constructed representations of $\mathcal{L}$ depending on two parameters, by using exterior algebras. Here we reconstruct representations of this type by using charged Fermi operators, then due to the Fermi-Bose correspondence developped in the paper of E. Date et al. [1983], we express these representations by Bose operators. This is the way how we found our Fock space representations $\mathcal{F}(w,\lambda)$.

A.1) Charged Fermi Operators and Representations of Virasoro Algebra

Consider the associative algebra $\mathfrak{B}$ over $\boldsymbol{C}$, generated by ψ_n and $\psi_n^\dagger$ $(n\in\boldsymbol{Z})$

with the following Fermi commutation relations:

(A.1.1) $$\begin{cases} [\psi_m, \psi_n]_+ = [\psi_m^\dagger, \psi_n^\dagger]_+ = 0 \\ [\psi_m, \psi_n^\dagger]_+ = \delta_{m,n}, \end{cases}$$

where

(A.1.2) $$[A, B]_+ = AB + BA .$$

Moreover the algebra $\mathfrak{B}$ has the $\boldsymbol{Z}$-graded algebra structure

(A.1.3) $$\mathfrak{B} = \sum_{n \in \boldsymbol{Z}} \mathfrak{B}_n ,$$

by defining the degrees of ψ_n and $\psi_n^\dagger$ as

(A.1.4) $$\deg \psi_n = 1 \quad \text{and} \quad \deg \psi_n^\dagger = -1$$

for any $n \in \boldsymbol{Z}$, and then we call these degrees the *charge*.

Consider the vector spaces V and $V^\dagger$ defined by

(A.1.5) $$V = \sum_{n \in \boldsymbol{Z}} \boldsymbol{C} \psi_n \quad \text{and} \quad V^\dagger = \sum_{n \in \boldsymbol{Z}} \boldsymbol{C} \psi_n^\dagger ,$$

and put

(A.1.6) $$W = V \oplus V^\dagger .$$

By the paring $\langle \psi_m, \psi_n^\dagger \rangle = \delta_{n,m}$, the vector spaces V and $V^\dagger$ are dual with each other, and the sets $\{\psi_n, n \in \boldsymbol{Z}\}$ and $\{\psi_n^\dagger, n \in \boldsymbol{Z}\}$ constitute the dual bases.

Fix another polarization of the space W defined as $W = W_+ \oplus W_-$, where

(A.1.7) $$\begin{cases} W_+ = \sum_{n \geq 0} \boldsymbol{C} \psi_n + \sum_{n < 0} \boldsymbol{C} \psi_n^\dagger , \\ W_- = \sum_{n < 0} \boldsymbol{C} \psi_n + \sum_{n \geq 0} \boldsymbol{C} \psi_n^\dagger . \end{cases}$$

Since any two elements in W_+(or W_-) anti-commute with each other respectively, we get the isomorphism

(A.1.8) $$\mathfrak{B} \cong \Lambda(W_+) \otimes \Lambda(W_-) = \Lambda(W_+ \oplus W_-) = \Lambda(W)$$

as left $\Lambda(W_+)$- and right $\Lambda(W_-)$-modules, where $\Lambda(W)$ is the exterior algebra of W. Hence we can define the normal product

(A.1.9) $$: \; : \; \Lambda(W) \to \mathfrak{B} ,$$

as the uniquely defined $\boldsymbol{C}$-linear isomorphism under the conditions that (1) $:1: = 1$ and (2) $: \; :$ is a left $\Lambda(W_+)$- and right $\Lambda(W_-)$-module mapping.

Consider the left $\mathfrak{B}$-module $\mathcal{H}$ with the cyclic vector $|0\rangle$ satisfying

(A.1.10) $$W_-|0\rangle = 0,$$

and also the right $\mathfrak{B}$-module $\mathcal{H}^\dagger$ with the cyclic vector $\langle 0|$ satisfying

(A.1.11) $$\langle 0|W_+ = 0.$$

Then the module $\mathcal{H}$ is a free $\Lambda(W_+)$-module and the module $\mathcal{H}^\dagger$ is a free $\Lambda(W_-)$-module, that is, the two mappings

(A.1.12) $$\Lambda(W_+)\ni a \mapsto a|0\rangle \in \mathcal{H} = \mathfrak{B}/\mathfrak{B}W_- \quad \text{and}$$

(A.1.13) $$\Lambda(W_-)\ni b \mapsto \langle 0|b \in \mathcal{H}^\dagger = W_+\mathfrak{B}\backslash\mathfrak{B}$$

are $\boldsymbol{C}$-linear isomorphisms.

By these isomorphisms, the grading in $\Lambda(W_+)$ and $\Lambda(W_-)$ can be transferred to $\mathcal{H}$ and $\mathcal{H}^\dagger$, and the degrees in $\mathcal{H}$ and $\mathcal{H}^\dagger$ are also called *charges*: for homogeneous elements $a\in\Lambda(W_+)$ and $b\in\Lambda(W_-)$,

(A.1.14) $$\deg a|0\rangle = \deg a \quad \text{and} \quad \deg\langle 0|b = -\deg b;$$

(A.1.15) $$\mathcal{H} = \sum_{l\in\boldsymbol{Z}} \mathcal{H}_l \quad \text{and} \quad \mathcal{H}^\dagger = \sum_{l\in\boldsymbol{Z}} \mathcal{H}_l^\dagger .$$

The homogeneous components $\mathcal{H}_l$ and $\mathcal{H}_l^\dagger$ of the decompositions (A.1.15) are called the charge l sectors.

The vacuum expectation value

(A.1.16) $$\langle\,|\,\rangle \colon \mathcal{H}^\dagger\times\mathcal{H} \to \boldsymbol{C}$$

is uniquely defined by the following conditions:

i) $\langle\,|\,\rangle$ is $\boldsymbol{C}$-bilinear; ii) $\langle 0|0\rangle=1$;

iii) $\langle va|u\rangle = \langle v|au\rangle$ for any $\langle v|\in\mathcal{H}^\dagger$, $|u\rangle\in\mathcal{H}$ and $a\in\mathfrak{B}$.

Then it is easily seen that the restriction of $\langle\,|\,\rangle$ to $\mathcal{H}_l^\dagger\times\mathcal{H}_{l'}$ vanishes if $l\neq l'$, and is nondegenerate if $l=l'$.

Define the operators H_n and U_n for $n\in\boldsymbol{Z}$ by the formulae:

(A.1.17) $$\begin{cases} H_n = \sum_{j\in\boldsymbol{Z}} :\psi_j\psi_{j+n}^\dagger: \\ U_n = \sum_{j\in\boldsymbol{Z}} j:\psi_j\psi_{j+n}^\dagger: . \end{cases}$$

Then these operators are well-defined as operators on the spaces $\mathcal{H}$ and $\mathcal{H}^\dagger$.

Identify the vector space V with the space $\boldsymbol{C}[z, z^{-1}]$ of Laurent polynomials of z, by setting

(A.1.18) $$\psi_n = z^n \quad (n\in\boldsymbol{Z}).$$

For each $(w, \lambda)\in \boldsymbol{C}^2$, define the representation $\rho_{(w,\lambda)}$ of the Virasoro algebra $\mathcal{L}$ on the space $V=\boldsymbol{C}[z, z^{-1}]$ by the formulae:

$$\text{(A.1.19)} \qquad \begin{cases} \rho_{(w,\lambda)}(e_n) = z^{n+1}\dfrac{d}{dz}+\{w+\dfrac{1}{2}-(\lambda-\dfrac{1}{2})n\}z^n & (n\in \boldsymbol{Z}) \\ \rho_{(w,\lambda)}(e_0') = 0\,. \end{cases}$$

And define the representation $\rho^\dagger$ of $\mathcal{L}$ on the dual space $V^\dagger$, skew-adjoint to the representation ρ, that is,

$$\text{(A.1.20)} \quad \langle \rho^\dagger_{(w,\lambda)}(e)\phi^\dagger | \psi\rangle + \langle \phi^\dagger | \rho_{(w,\lambda)}(e)\psi\rangle = 0 \quad (e\in\mathcal{L},\ \phi^\dagger\in V^\dagger,\ \psi\in V)\,.$$

Then

$$\begin{cases} \rho_{(w,\lambda)}(e_n)\psi_m = (m+w+\dfrac{1}{2}-(\lambda-\dfrac{1}{2})n)\psi_{m+n} \\ \rho^\dagger_{(w,\lambda)}(e_n)\psi^\dagger_m = -(m+w+\dfrac{1}{2}-(\lambda+\dfrac{1}{2})n)\psi^\dagger_{m-n}\,, \end{cases}$$

for any $n, m\in\boldsymbol{Z}$.

Remark. Consider the quotient Lie algebra $\mathcal{L}'=\mathcal{L}/\boldsymbol{C}e_0'$, then the space V can be considered as an $\mathcal{L}'$-module, that is,

$$\text{(A.1.21)} \qquad e_n = z^{n+1}\frac{d}{dz} \quad (n\in\boldsymbol{Z})\,.$$

Easily we can show that

$$\text{(A.1.22)} \qquad H^1(\mathcal{L}';\ V)\cong \boldsymbol{C}^2$$

and a representative of a cycle $(w, \lambda)\in\boldsymbol{C}^2$ can be taken as

$$\text{(A.1.23)} \qquad (w, \lambda)\,(e_n) = (w+\frac{1}{2}-(\lambda-\frac{1}{2})n)z^n \quad (n\in\boldsymbol{Z})\,.$$

For $(w, \lambda)\in\boldsymbol{C}^2$, define the linear mapping

$$\text{(A.1.24)} \qquad \hat{\rho}_{(w,\lambda)}\colon \mathcal{L}\to \mathrm{Hom}_{\boldsymbol{C}}(\mathcal{H}, \mathcal{H})$$

by the following formulae:

$$\text{(A.1.25)} \quad \begin{cases} \hat{\rho}_{(w,\lambda)}(e_n) = U_{-n}+\{w+\dfrac{1}{2}-(\lambda+\dfrac{1}{2})n\}H_{-n}+\dfrac{1}{2}\delta_{n,0}(w^2-\lambda^2)\mathrm{id} \ (n\in\boldsymbol{Z}) \\ \hat{\rho}_{(w,\lambda)}(e_0') = (-12\lambda^2+1)\,\mathrm{id}\,. \end{cases}$$

By easy but long calculations, we get

Proposition A.1.

1) $\hat{\rho}_{(w,\lambda)}$ *is a representation of the Virasoro algebra* $\mathcal{L}$ *in the space* $\mathcal{H}$.

2) *For each* $m\in\boldsymbol{Z}$ *and* $e\in\mathcal{L}$, $\hat{\rho}_{(w,\lambda)}(e)$ *preserves the charge m sector* $\mathcal{H}_m$, *hence* $\hat{\rho}_{(w,\lambda)}$ *induces a representation of* $\mathcal{L}$ *in the space* $\mathcal{H}_m$.

3) *For each* $e\in\mathcal{L}$, $\phi^\dagger\in V^\dagger$ *and* $\psi\in V$,

$$\text{(A.1.26)}\qquad \begin{cases} [\hat{\rho}_{(w,\lambda)}(e),\ \psi] = \rho_{(w,\lambda)}(e)\,(\psi) \\ [\hat{\rho}_{(w,\lambda)}(e),\ \phi^\dagger] = \rho^\dagger_{(w,\lambda)}(e)\,(\phi^\dagger), \end{cases}$$

where the left hand side of (A.1.26) *is considered in the operator algebra.*

Remark. Consider the generating function defined by

$$\text{(A.1.27)}\qquad E(\zeta;\, w,\, \lambda) := \sum_{n\in\boldsymbol{Z}} \zeta^n \hat{\rho}_{(w,\lambda)}(e_n)\,,$$

then by (A.2.20–21)

$$\text{(A.1.28)}\quad E(\zeta;\, w,\, \lambda) = :(\zeta\frac{d}{d\zeta}\psi(\zeta))\psi^\dagger(\zeta): + (w+\frac{1}{2}):\psi(\zeta)\psi^\dagger(\lambda): + (\lambda+\frac{1}{2})\zeta\frac{d}{d\zeta}:\psi(\zeta)\psi^\dagger(\zeta): + \frac{1}{2}(w^2-\lambda^2)\mathrm{id}\,,$$

where we set

$$\text{(A.1.29)}\qquad \psi(\zeta) = \sum_{n\in\boldsymbol{Z}} \psi_n\zeta^n \quad\text{and}\quad \psi^\dagger(\zeta) = \sum_{n\in\boldsymbol{Z}} \psi^\dagger_n\zeta^{-n}\,.$$

A.2) Fermi-Bose Correspondence (after E. Date et al. [1983])

Proposition A.2 (E. Date et al. [1983]).

1) *For any n,* $m\in\boldsymbol{Z}$

$$\text{(A.2.1)}\qquad [H_n,\, H_m] = n\,\delta_{n+m,0}\, id\,.$$

2) *For any vector* $|v\rangle\in\mathcal{H}$, *there exists a positive integer m such that* $H_n|v\rangle=0$ *for any* $n\geqq m$.

3) *For any* $n\geqq 1$ *and any vector* $|v\rangle\in\mathcal{H}$, *there exists a positive integer m such that* $(H_n)^m|v\rangle=0$.

4) *For any* $l\in\boldsymbol{Z}$

$$\text{(A.2.2)}\qquad \begin{cases} \mathcal{H}_l = \{|v\rangle\in\mathcal{H};\ H_0|v\rangle = l|v\rangle\} \quad \text{and} \\ \mathcal{H}^\dagger_l = \{\langle v|\in\mathcal{H}^\dagger;\ \langle v|H_0 = \langle v|l\}\,. \end{cases}$$

Introduce an infinite set of variables $x=(x_1,\, x_2,\, \cdots)$ and set

$$\text{(A.2.4)}\qquad H(x) = \sum_{n\geqq 1} x_n H_n \quad\text{and}$$

(A.2.4) $$e^{H(x)}=\sum_{n\geq 0}\frac{1}{n!}(H(x))^n\,.$$

By Proposition A.2, the actions of the operators $H(x)$ and $e^{H(x)}$ on the spaces $\mathcal{H}$ and $\mathcal{H}^\dagger$ are well-defined, and

(A.2.5) $$H(x)|0\rangle=0\,,\ \text{hence}\ e^{H(x)}|0\rangle=|0\rangle\,,$$

but here note that $\langle 0|H(x)\neq 0$.

Define a state of charge n as

(A.2.6) $$\langle n|=\begin{cases}\langle 0|\psi_{-1}\cdots\psi_n & (n<0)\\ \langle 0| & (n=0)\\ \langle 0|\psi_0^\dagger\cdots\psi_{n-1}^\dagger & (n>0)\end{cases}$$

Consider the $\boldsymbol{C}$-linear mapping

(A.2.7) $$\Phi\colon\mathcal{H}\to\mathcal{V}=\boldsymbol{C}[u,u^{-1},x_1,x_2,\cdots]$$

defined by

$$\Phi(a|0\rangle)=\sum_{m\in\boldsymbol{Z}}\langle m|e^{H(x)}a|0\rangle u^m\qquad(a\in\mathfrak{B})\,.$$

Proposition A.3 (E. Date et al. [1983]).

(1) *The mapping Φ is a graded $\boldsymbol{C}$-linear isomorphism, where the grading in the space $\mathcal{V}=\sum_{m\in\boldsymbol{Z}}\mathcal{V}_m$ is given as*

(A.2.8) $$\deg u=1\ \ \textit{and}\ \ \deg x_n=0\quad(n\geq 1)\,,$$

hence

(A.2.9) $$\Phi(\mathcal{H}_m)=\mathcal{V}_m=u^m\boldsymbol{C}[x_1,x_2,\cdots]\quad(m\in\boldsymbol{Z})\,.$$

(2) *The isomorphism Φ gives the action of the operators H_n on the space $\mathcal{V}=\boldsymbol{C}[u,u^{-1},x_1,x_2,\cdots]$ as*

(A.2.10) $$\begin{cases}H_n=\partial_n & (n>0)\\ H_0=u\partial_u & \\ H_{-n}=n\,x_n & (n>0)\end{cases}$$

where $\partial_n=\partial/\partial x_n$ $(n\geq 1)$ and $\partial_u=\partial/\partial u$.

By the same method to prove the Proposition A.3 (2), we get

Proposition A.4. *The isomorphism Φ gives the action of the operators U_n on the space $\mathcal{V}=\boldsymbol{C}[u,u^{-1},x_1,x_2,\cdots]$ as*

(A.2.11)
$$\begin{cases} U_n = (u\partial_u - \frac{n+1}{2})\partial_n + \frac{1}{2}\sum_{j=1}^{n-1}\partial_j\partial_{n-j} + \sum_{j\geqq 1} jx_j\partial_{n+j}\,, \\ U_0 = \frac{1}{2}(u\partial_u)^2 - \frac{1}{2}u\partial_u + \sum_{j\geqq 1} jx_j\partial_j\,, \\ U_{-n} = (u\partial_u + \frac{n-1}{2})nx_n + \frac{1}{2}\sum_{j=1}^{n-1} j(n-j)x_jx_{n-j} + \sum_{j\geqq 1}(n+j)x_{n+j}\partial_j \end{cases}$$

for $n\geqq 1$.

Sketch of Proof. For a complex number $\zeta\in\boldsymbol{C}^*$ we introduce the operators defined by

(A.2.12)
$$\psi(\zeta) = \sum_{n\in\boldsymbol{Z}}\psi_n\zeta^n \quad\text{and}\quad \psi^\dagger(\zeta) = \sum_{n\in\boldsymbol{Z}}\psi_n^\dagger\zeta^{-n}\,,$$

then $\psi(\zeta)$ and $\psi^\dagger(\zeta)$ are holomorphic operator-valued functions in the sense that

(A.2.13)
$$\langle u|\psi(\zeta)|v\rangle \quad\text{and}\quad \langle u|\psi^\dagger(\zeta)|v\rangle$$

are holomorphic functions of ζ for any $\langle u|\in\mathcal{H}^\dagger$ and $|v\rangle\in\mathcal{H}$. Then for $(\zeta_1, \zeta_2)\in(\boldsymbol{C}^*)^2$, the normal product of generating functions

(A.2.14)
$$:\psi(\zeta_1)\psi^\dagger(\zeta_2): = \sum_{l,m\in\boldsymbol{Z}}\zeta_1^l\zeta_2^{-m}:\psi_l\psi_m^\dagger:$$

are also a holomorphic operator-valued function on $(\boldsymbol{C}^*)^2$. Then we get

(A.2.15)
$$:\psi(\zeta)\psi^\dagger(\zeta): = \sum_{m\in\boldsymbol{Z}}\zeta^{-m}(\sum_{l\in\boldsymbol{Z}}:\psi_l\psi_{l+m}^\dagger:) = \sum_{m\in\boldsymbol{Z}}\zeta^{-m}H_m \quad\text{and}$$

(A.2.16)
$$[\zeta_1\frac{d}{d\zeta_1}:\psi(\zeta_1)\psi^\dagger(\zeta_2):]_{\zeta=\zeta_1=\zeta_2} = \sum_{m\in\boldsymbol{Z}}\zeta^{-m}\{\sum_{l\in\boldsymbol{Z}} l:\psi_l\psi_{l+m}^\dagger:\} = \sum_{m\in\boldsymbol{Z}}\zeta^{-m}U_m\,.$$

Thus we must calculate the operator form on the space $\mathcal{V}$ of the operator $:\psi(\zeta)\psi^\dagger(\zeta):$. Consider the product

(A.2.17)
$$\psi(\zeta_1)\psi^\dagger(\zeta_2) = \sum_{l,m\in\boldsymbol{Z}}\zeta_1^l\zeta_2^{-m}\psi_l\psi_m^\dagger$$

as an operator-valued Laurent series on ζ_1 and ζ_2. Then as Laurent series, we get

(A.2.18)
$$\begin{aligned}\psi(\zeta_1)\psi^\dagger(\zeta_2) &= \sum_{l,m\in\boldsymbol{Z}}\zeta_1^l\zeta_2^{-m}:\psi_l\psi_m^\dagger: + \sum_{l<0}(\zeta_1/\zeta_2)^l[\psi_l, \psi_l^\dagger]_+ \\ &= \sum_{l,m\in\boldsymbol{Z}}\zeta_1^l\zeta_2^{-m}:\psi_l\psi_m^\dagger: + \sum_{l>0}(\zeta_2/\zeta_1)^l\,.\end{aligned}$$

Here we note that in the domain $|\zeta_1|>|\zeta_2|>0$, the functions

(A.2.19) $$\sum_{l\geq 1}(\zeta_2/\zeta_1)^l = \frac{\zeta_2}{\zeta_1-\zeta_2}$$

and $:\psi(\zeta_1)\psi^{\dagger}(\zeta_2):$ are holomorphic, hence $\psi(\zeta_1)\psi^{\dagger}(\zeta_2)$ is also holomorphic in this domain, and satisfies the equality

(A.2.20) $$:\psi(\zeta_1)\psi^{\dagger}(\zeta_2): = \psi(\zeta_1)\psi^{\dagger}(\zeta_2) - \frac{\zeta_2}{\zeta_1-\zeta_2}.$$

On the other hand, E. Date et al. [1983] calculated the operator form of $\psi(\zeta_1)\psi^{\dagger}(\zeta_2)$ on the charge m sector $\mathcal{V}_m$, so they obtained

(A.2.21) $$:\psi(\zeta_1)\psi^{\dagger}(\zeta_2): = \frac{\zeta_2}{\zeta_1-\zeta_2}\{(\zeta_1/\zeta_2)^m X(\zeta_1, \zeta_2) - 1\}$$

where

(A.2.22) $$X(\zeta_1, \zeta_2) = e^{\xi(x,\zeta_1)-\xi(x,\zeta_2)}e^{-\xi(\tilde{\partial},\zeta_1)+\xi(\tilde{\partial},\zeta_2)},$$

and

(A.2.23) $$\xi(x, \zeta) = \sum_{n\geq 1} x_n\zeta^n \quad\text{and}\quad \xi(\tilde{\partial}, \zeta) = \sum_{n\geq 1}\frac{1}{n}\zeta^{-n}\partial_n.$$

On the other hand, consider the operators A and B defined by

(A.2.24) $$\begin{cases} \xi(x, \zeta_1)-\xi(x, \zeta_2) = (\zeta_1-\zeta_2)\sum_{n\geq 1}\dfrac{\zeta_1^n-\zeta_2^n}{\zeta_1-\zeta_2}x_n = (\zeta_1-\zeta_2)A, \\ \xi(\tilde{\partial}, \zeta_2)-\xi(\tilde{\partial}, \zeta_1) = (\zeta_1-\zeta_2)\sum_{n\geq 1}\dfrac{\zeta_2^{-n}-\zeta_1^{-n}}{\zeta_1-\zeta_2}\dfrac{\partial_n}{n} = (\zeta_1-\zeta_2)B, \end{cases}$$

then A and B are operator-valued holomorphic functions of $(\zeta_1, \zeta_2)\in(\boldsymbol{C}^*)^2$, and note that these are holomorphic even at $(\zeta, \zeta)\neq(0, 0)$. So we get

(A.2.25) $$A(\zeta, \zeta) = \sum_{n\geq 1} nx_n\zeta^{n-1}, \qquad B(\zeta, \zeta) = \sum_{n\geq 1}\partial_n\zeta^{-n-1},$$

and

(A.2.26) $$X(\zeta_1, \zeta_2) = \sum_{k\geq 0} X_k(\zeta_1, \zeta_2)(\zeta_1-\zeta_2)^k,$$

where

(A.2.27) $$X_k(\zeta_1, \zeta_2) = \sum_{\substack{i,j\geq 0\\ i+j=k}}\frac{1}{i!\,j!}A^iB^j.$$

For each $m\in\boldsymbol{Z}$, we can expand the function $(\zeta_1/\zeta_2)^m$ as

(A.2.28) $$(\zeta_1/\zeta_2)^m = \left(1+\frac{\zeta_1-\zeta_2}{\zeta_2}\right)^m = \sum_{j\geq 0}\binom{m}{j}\left(\frac{\zeta_1-\zeta_2}{\zeta_2}\right)^j.$$

Hence we get

(A.2.29) $$:\psi(\zeta_1)\psi^{\dagger}(\zeta_2): = \frac{\zeta_2}{\zeta_1-\zeta_2}\{(\zeta_1/\zeta_2)^m X(\zeta_1, \zeta_2)-1\}$$
$$= \sum_{h\geqq 0}\{\sum_{\substack{j+k=h+1\\ j\geqq 0, k\geqq 1}} \binom{m}{j} \zeta_2^{-j+1}X_k(\zeta_1, \zeta_2)\}(\zeta_1-\zeta_2)^h .$$

Note here that the function $\sum_{j+k=h+1} \binom{m}{j} \zeta_2^{-j+1}X_k(\zeta_1, \zeta_2)$ are holomorphic at $\zeta_1=\zeta_2\neq 0$. So we get

(A.2.30) $$[\zeta_1\frac{d}{d\zeta_1} :\psi(\zeta_1)\psi^{\dagger}(\zeta_2):]_{\zeta=\zeta_1=\zeta_2}$$
$$= m\zeta X_1(\zeta, \zeta)+\zeta^2X_2(\zeta, \zeta)+\zeta^2[\frac{\partial}{\partial\zeta_1}X_1(\zeta_1, \zeta_2)]_{\zeta=\zeta_1=\zeta_2} .$$

Hence by (A.2.16), we get the formulae (A.2.11) for the operators U_n.

A.3) By Proposition A.4, we get the explicit form of the operators (A.1. 25) on the space $\mathcal{V}=\boldsymbol{C}[u, u^{-1}, x_1, x_2, \cdots]$.

Proposition A.5. *On the space* $\mathcal{V}$,

(A.3.1) $$\hat{\rho}_{(w,\lambda)}(e_n) = (w+u\partial_u-n\lambda)nx_n+\frac{1}{2}\sum_{j=1}^{n-1}j(n-j)x_jx_{n-j}+\sum_{j\geqq 1}(n+j)x_{n+j}\partial_j$$

(A.3.2) $$\hat{\rho}_{(w,\lambda)}(e_{-n}) = (w+u\partial_u+n\lambda)\partial_n+\frac{1}{2}\sum_{j=1}^{n-1}\partial_j\partial_{n-j}+\sum_{j\geqq 1}jx_j\partial_{n+j}$$

(A.3.3) $$\hat{\rho}_{(w,\lambda)}(e_0) = \frac{1}{2}\{(w+u\partial_u)^2-\lambda^2\}\,\mathrm{id}+\sum_{j\geqq 1}jx_j\partial_j$$

(A.3.4) $$\hat{\rho}_{(w,\lambda)}(e_0') = (1-1\ 2\lambda^2)\mathrm{id} .$$

If we restrict the representation $\hat{\rho}_{(w,\lambda)}$ on the charge zero sector $\mathcal{V}_0=\boldsymbol{C}[x_1, x_2, \cdots]$, then this representation is nothing else but the Fock space represectation $\pi_{w,\lambda}$ on $\mathcal{F}(w, \lambda)$ of the Virasoro algebra $\mathcal{L}$ (see (1.1.28, 1.2.3, 10)). Here we identify

(A.3.5) $$\mathcal{F}(w, \lambda)\cong\boldsymbol{C}[x_1, x_2, \cdots]$$

and

(A.3.6) $$p_n = nx_n,\ p_{-n} = \partial_n\ (n\geqq 1),\ p_0 = w\,\mathrm{id},\ \Lambda = \lambda\,\mathrm{id} .$$

References

Aomoto, K. [1984], Configurations and invariant theory of Gauss-Manin systems, in Group Representations and Systems of Differential Equations, *Advanced Studies in pure Mathematics*, **4**, Kinokuniya (1984), 165–179.

Belavin, A.A., Polyakov, A.M. and Zamolodchikov, A.B. [1984], Infinite conformal symmetry of critical fluctuations in two dimensions, *J. of Stat. Phys.*, **34**-5/6 (1984), 763–774.

Cohen, F.R. [1976], The homology of C_{n+1}-space, $n\geqq 0$, in The Homology of Iterated Loop Spaces, *Lecture Notes in Math.*, **533** Springer (1976), 207–351.

Date, E., Jimbo, M., Kashiwara, M. and Miwa, T. [1983], Transformation groups for soliton equations, in *Proc. of RIMS symposium* (ed. M. Jimbo & T. Miwa), World Scientific, Singapore (1983), 39–120.

Deligne, P. [1970], Equations Differentielles a Points Singuliers Reguliers, *Lect. Notes in Math.*. **163**, Springer 1970.

Dotsenko, Vl.S. and Fateev, V.A. [1984], Conformal algebra and multipoint correlation functions in 2D statistical models, *Nuclear Physics*, **B240** [FS12] (1984), 312–348.

Dyson, F.J. [1962], Statistical theory of the energy levels of complex systems I, *J. Math. Phys.*, **3** (1962), 140–156.

Feigin, F.L. and Fuks, D.B. [1982], Skew-symmetric invariant differential operators on a straight line and Virasoro algebra, *Func. Anal. & its Appl.*, **16**-**2** (1982), 47–63 (in Russian).

Feigin, F.L. and Fuks, D.B. [1983], Verma modules over Virasoro algebra, *Func. Anal. & its Appl.*, **17**-**3** (1983), 91–92 (in Russian).

Kac, V.G. [1979], Contravariant form for infinite dimensional Lie algebras and superalgebras, *Lect. Notes in Physics*, **94** (1979), 441–445.

Kac, V.G. and Kazhdan, D.A. [1979], Structure of representations with highest weight of infinite dimensional Lie algebras, *Advances in Math.*, **34** (1979), 97–108.

Littlewood, D.E. [1950], *The Theory of Group Characters and Matrix Representations*, 2nd ed., Clarendon Press, Oxford 1950.

Macdonald, I.G. [1979], *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford 1979.

Macdonald, I.G. [1982], Some conjectures for root systems, *SIAM J. Math. Anal.*, **13**-**6** (1982), 988–1007.

Mandelstam, S. [1974], Dual-Resonance Models, *Physics Reports*, **13**-**6** (1974), 259–353.

Mather, J. [1970], Notes on topological stability, mineographed notes, Harvard University 1970.

Selberg, A. [1944], Bemerkninger om et Mulitiplet Integral, *Norsk Mat. Tidsskrift*, **26** (1944), 71–78.