

## Conformal Covariants<sup>\*</sup>)

By

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### § 0. Introduction

By a conformal covariant we mean a holomorphic differential operator which intertwines two holomorphic (“positive energy”) representations of the conformal group. In this article we determine all such.

Specifically, the representations are parametrized by triples  $\tau' = \tau' \left( -r, \frac{n+m}{2}, \frac{n-m}{2} \right)$  where  $n, m$  are non-negative integers and  $r$  is a half-integer ( $2r \in \mathbb{Z}$ ). The representation  $U_{\tau'}$  determined by  $\tau'$  has the form

$$(0.1) \quad (U_{\tau'}(g)f)(z) = \det(a+bz)^{\left(r+\frac{n-m}{2}\right)} \left( \bigotimes_s^n (a+bz)^{-1} \right) \\ \otimes \left( \bigotimes_s^m (d-(c+dz)(a+bz)^{-1}b)^{-1} \right) f((c+dz)(a+bz)^{-1})$$

on  $\left( \bigotimes_s^n \mathbb{C}^2 \right) \otimes \left( \bigotimes_s^m \mathbb{C}^2 \right)$ -valued holomorphic functions  $f$  on the generalized unit disk  $\mathcal{B} = \{z \in M(2, \mathbb{C}) \mid zz^* < 1\}$ , if  $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2, 2)$  ( $a, b, c$ , and  $d$  are  $2 \times 2$  complex matrices).

As was proved in [8], one may switch freely to holomorphic functions on the generalized upper half plane  $\mathcal{D} = \{z \in M(2, \mathbb{C}) \mid (z-z^*)/2i > 0\}$  and to an action of the conformal group  $G$ , identical to (0.1), but with  $a, b, c, d$  defined by another, isomorphic, real form of  $SU(2, 2)^{\mathbb{C}}$ , in the strong sense that if a differential operator  $D$  intertwines  $U_{\tau_1}$  and  $U_{\tau_2}$  on  $\mathcal{B}$ , then the *same* operator  $D$  intertwines the corresponding representations on  $\mathcal{D}$ . (Since  $\mathcal{B} \cap \mathcal{D} \neq \emptyset$ , it makes sense to compare constant coefficient differential operators, such as  $D$  above on  $\mathcal{B}$  and on  $\mathcal{D}$ .) To be precise; the realization of  $U_{\tau'}$  as acting on vector-valued holomorphic functions on  $\mathcal{D}$  is obtained by letting the  $U_{\tau'}$  in (0, 1) act on the element of  $SU(2, 2)^{\mathbb{C}}$  which corresponds to the Cayley trans-

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form  $z \rightarrow (1+iz)(1-iz)^{-1}$ . We can also remark along these lines that covariant differential operators are homogeneous and hence are also covariant with respect to change of sign on the variable. We may thus choose the following more usual version of  $U_{\tau'}$ :

$$(0.2) \quad (U_{\tau'}(g)f)(z) = \det(cz+d)^{r+\frac{n-m}{2}} \left( \otimes_s^n (cz+d)^{-1} \right) \otimes \left( \otimes_s^m (zc^*+d^*) \right) f((az+b)(cz+d)^{-1})$$

on  $\left( \otimes_s^n \mathbb{C}^2 \right) \otimes \left( \otimes_s^m \mathbb{C}^2 \right)$ -valued holomorphic functions on  $\mathcal{D}$  -obtained by means of conjugation by the Cayley transform, followed by conjugation by the element corresponding to conformal inversion  $z \rightarrow -z^{-1}$  on  $\mathcal{D}$ , followed by the map  $z \rightarrow -z$  (and a switch to an isomorphic  $SU(2, 2)$ ).

Dual to the space of  $K$ -finite vectors for  $U_{\tau'}$  is a highest weight module  $M(V_{\tau'})$  whose highest weight vector transforms under  $K$  ( $K$  is an appropriate maximal compact subgroup of  $G$ ) according to a representation  $\tau = \tau\left(r, \frac{n+m}{2}, \frac{n-m}{2}\right)$  dual to the  $\tau'$  of  $U_{\tau'}$ . The determination of the set of covariant differential operators is equivalent to the determination of the set of homomorphisms between such so-called generalized Verma modules [12], [8].

This is the road we shall take and we will not, in fact, return any more, in this article, to the representations  $U_{\tau'}$ .

We present here a complete description of the full set of homomorphisms between generalized Verma modules  $M(V_{\tau'})$  as above for the conformal group. Furthermore, we determine the subspace structure of the  $M(V_{\tau'})$ 's. This structure is often richer than what can directly be inferred from the homomorphisms.

The invariance of the solution subspace to a spin  $\left(\frac{n}{2}, \pm\right)$ -mass 0 wave equation under the conformal group seems to have been known to many physicists, at least for  $n=0, 1, 2$ , for several decades. The invariance of Maxwell's equations under  $G$  was first observed by Batemann [1] and Cunningham [6] though it may be argued that it is already contained in Lie's work [15].

More recently, Kostant [14] proved equivariance of the wave operator  $\square$  and in so doing, employed Verma module techniques. In Jakobsen-Vergne [10], the covariance of  $\square$  and of powers thereof as well as powers of the Dirac operator, was established. The covariance of wave operators on more general Lorentz manifolds has been established by Ørsted [21], [22], and later extended by Branson [4], [5] to differential forms, and by Paneitz [16] who found an

analogue of  $\square^2$  on an arbitrary pseudo-Riemannian manifold.

Returning to  $SU(2, 2)$ ; all situations  $U_{\tau_2} D = D U_{\tau_1}$  with  $U_{\tau_1}$  unitary on the set  $\{\varphi \mid D\varphi = 0\}$  were determined by Harris-Jakobsen [8]. A complete differential geometric treatment of Maxwell's equation and the Dirac equation was given by Paneitz and Segal [18], [19], and in a continuation of this work Paneitz [17] found, among other things, the composition series of all representations  $U_{\tau'}$  for which  $\tau'$  remains irreducible when restricted to the rotation subgroup of  $K$ . Viewing  $U_{\tau'}$  as a component of a degenerate series representation, he also paid attention to more general composition series, i.e. involving negative or mixed energy subspaces. Also, following a head-on investigation of Knapp-Stein intertwining operators around singularities, Petkova and Sotkov [20] have written down the composition series of a sizable subset of the representations above. Finally, we have recently learned that Enright and Shelton [7] have determined the composition series for generalized Verma modules for  $SU(p, q)$  of semi-regular highest weight and that, based on this, Boe and Collingwood [3] have found an abstract algorithm for determining the set of homomorphisms into generalized Verma modules of regular infinitesimal character.

The present investigations relies on the results on Bernstein-Gelfand-Gelfand [2], but besides that, it is quite straightforward. The key fact; Proposition 4, is proved in Chapter 1. It describes those covariant differential operators that contain a factor of  $\square^n$  for some  $n \geq 1$ . More generally, the idea behind most of what is going on is simply that the determinantal ideal in the set of polynomials in four variables corresponding to  $\square$ ;  $I_1(\square) = \mathcal{C}[z_1, z_2, z_3, z_4] \cdot (z_1^2 + z_2^2 + z_3^2 - z_4^2)$  or equivalently,  $I_2(\square) = \mathcal{C}[z_1, z_2, z_3, z_4] \cdot (z_1 z_4 - z_2 z_3)$ , is *prime*. Following this, there are two more chapters, entitled: 2. Conformal Covariants and 3. The Subspace Structure, respectively.

We wish to thank Tom Branson for providing us with some important examples.

### § 1. Fundamentals

The Lie algebra  $\mathfrak{g} = su(2, 2)$  is represented as in [9]. Let

$$(1.1) \quad k_{\mu}^{+} = \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & & 0 \end{array} \right], \quad k_{\mu}^{-} = \left[ \begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array} \right], \quad h_{\mu} = [k_{\mu}^{+}, k_{\mu}^{-}],$$

$$k_v^+ = \left[ \begin{array}{c|cc} 0 & & 0 \\ \hline & 0 & 0 \\ 0 & 1 & 0 \end{array} \right], \quad k_v^- = \left[ \begin{array}{c|cc} 0 & & 0 \\ \hline & 0 & 1 \\ 0 & 0 & 0 \end{array} \right], \quad \text{and} \quad h_v = [k_v^+, k_v^-].$$

Also let

$$(1.2) \quad \mathfrak{p}^- = \left\{ \left[ \begin{array}{cc|c} 0 & & 0 \\ \hline z_{11} & z_{12} & \\ z_{21} & z_{22} & 0 \end{array} \right] \mid z_{11}, z_{12}, z_{21}, \text{ and } z_{22} \in \mathbf{C} \right\},$$

$$\mathfrak{p}^+ = \left\{ \left[ \begin{array}{c|cc} 0 & z_{11} & z_{12} \\ \hline & z_{12} & z_{22} \\ 0 & & 0 \end{array} \right] \mid z_{11}, z_{12}, z_{21}, z_{22} \in \mathbf{C} \right\}, \quad \text{and}$$

$$K = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \mid u_1, u_2 \in U(2); \text{ and } \det(u_1 \cdot u_2) = 1 \right\}.$$

Elements of  $\mathfrak{p}^-$  corresponding to the appropriate matrices above are denoted  $z_{ij}$  whereas the analogous elements of  $\mathfrak{p}^+$  are denoted by  $z_{ij}^+$ . Further we let

$$(1.3.a) \quad h_{ij} = [z_{ij}^+, z_{ij}]; \quad i, j \in \{1, 2\}, \quad \text{and}$$

$$h = \frac{1}{2}[h_{11} + h_{22}].$$

Specifically, if  $D(a, b, c, d)$  denotes a  $4 \times 4$  diagonal matrix with entries from the top  $a, b, c$  and  $d$ , then

$$(1.3.b) \quad h_{ij} = D(\delta_{1,j}, \delta_{2,j}, -\delta_{i,1}, -\delta_{i,2}).$$

If  $\mathfrak{k}^{\mathbf{C}}$  denotes the complexification of the Lie algebra  $\mathfrak{k}$  of  $K$  then  $\mathfrak{k}^{\mathbf{C}} = su(2)^{\mathbf{C}} \oplus su(2)^{\mathbf{C}} \oplus \mathbf{C}$ , and  $k_{\mu}^+, k_{\mu}^-$  and  $h_{\mu}$  define one  $su(2)^{\mathbf{C}}$ ,  $k_v^+, k_v^-$ , and  $h_v$  the other, and  $h$  defines the center  $\mathbf{C}$ .

Any irreducible unitary representation  $\tau$  of  $K$  on a vector space  $V_{\tau}$  is uniquely determined by a vector  $v_{\tau} \in V_{\tau}$  which satisfies:

$$(1.4) \quad k_{\mu}^+ v_{\tau} = k_v^+ v_{\tau} = 0,$$

$$h_{\mu} v_{\tau} = n \cdot v_{\tau}, \quad h_v v_{\tau} = m \cdot v_{\tau}, \quad \text{and} \quad h v_{\tau} = r \cdot v_{\tau},$$

with  $n, m$ , and  $2r \in \mathbf{Z}$ , and  $m, n \geq 0$ . (We do not lose any generality by restricting  $2 \cdot r$  to  $\mathbf{Z}$ .)

We now define the generalized Verma module  $M(V_{\tau})$  by demanding that, in addition to (1.4),  $v_{\tau}$  satisfies

$$(1.5) \quad \mathfrak{p}^+ v_{\tau} = 0.$$

Thus,  $M(V_\tau)$  is the left  $\mathcal{U}(\mathfrak{g}^{\mathcal{C}})$ -module

$$(1.6) \quad M(V_\tau) = \mathcal{U}(\mathfrak{p}^-) \circ V_\tau.$$

In other words,  $M(V_\tau)$  is the linear span of all expressions  $z_{11}^a z_{12}^b z_{21}^c z_{22}^d \cdot v$  with  $(a, b, c, d) \in \mathbb{Z}_+^4$  and  $v \in V_\tau$ , and  $\mathfrak{g}, \mathfrak{g}^{\mathcal{C}}$ , or  $\mathcal{U}(\mathfrak{g}^{\mathcal{C}})$  acts from the left through the adjoint action together with (1.4)–(1.6). Note that the abelian algebras  $\mathfrak{p}^+$  and  $\mathfrak{p}^-$  are  $\mathfrak{k}^{\mathcal{C}}$ -modules and that  $[\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}^{\mathcal{C}}$ .

If  $\tau$  satisfies (1.4) we write  $\tau = \tau(n, m, r)$ . Any nontrivial homomorphism  $M(V_{\tau_1}) \rightarrow M(V_\tau)$  is then completely determined by a non-zero element  $q_{\tau_1} \in M(V_{\tau_1})$  satisfying:

$$(1.7) \quad \begin{aligned} \mathfrak{p}^+ q_{\tau_1} &= k_\mu^+ q_{\tau_1} = k_\nu^+ q_{\tau_1} = 0, \quad \text{and} \\ h_\mu q_{\tau_1} &= n_1 q_{\tau_1}, \quad h_\nu q_{\tau_1} = m_1 q_{\tau_1}, \quad \text{and} \quad h q_{\tau_1} = r_1 q_{\tau_1}. \end{aligned}$$

The elements  $h_{22}, h_\mu$ , and  $h_\nu$  form a basis of a maximal abelian subalgebra  $\mathfrak{h}$  of  $\mathfrak{k}^{\mathcal{C}}$ , and  $\mathfrak{h}$  is also a Cartan subalgebra of  $\mathfrak{g}^{\mathcal{C}}$ . Let  $e_1, e_2, e_3$  denote the usual basis vectors of  $\mathbb{C}^3$ . We identify  $\mathfrak{h}$  with  $\mathbb{C}^3$  by mapping  $h_{22}$  to  $e_1 - e_2$ ,  $h_\mu$  to  $e_2 + e_3$ , and  $h_\nu$  to  $e_2 - e_3$ . Furthermore, we also identify  $\mathfrak{h}^*$  with  $\mathbb{C}^3$ . Specifically, and defining the ordering at the same time, the positive roots of  $\mathfrak{h}$  in  $\mathfrak{g}^{\mathcal{C}}$  are taken to be  $e_2 \pm e_3$  (the compact) and  $e_1 \pm e_2, e_1 \pm e_3$  (the non-compact). We let

$$(1.8) \quad \begin{aligned} \beta &= e_1 - e_2, \quad \alpha_1 = e_1 - e_3, \quad \alpha_2 = e_1 + e_3, \quad r = e_1 + e_2, \\ \text{and} \quad \mu &= e_2 + e_3, \quad \nu = e_2 - e_3. \end{aligned}$$

A generalized Verma module is then determined by an element  $A = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{h}^*$  and the  $A$  corresponding to (1.4) is

$$(1.9) \quad A = \left( r, \frac{n+m}{2}, \frac{n-m}{2} \right).$$

For  $\sigma = (a, b, c) \in \mathfrak{h}^*$  we denote by  $S_\sigma$  the reflexion on  $\mathfrak{h}^*$  defined by

$$(1.10) \quad S_\sigma(A) = A - \langle \sigma, A \rangle \sigma$$

where

$$\langle \sigma, A \rangle = -\frac{2(a\lambda_1 + b\lambda_2 + c\lambda_3)}{a^2 + b^2 + c^2}.$$

Observe that  $\rho = \frac{1}{2}(\beta + \alpha_1 + \alpha_2 + r + \mu + \nu) = (2, 1, 0)$ .

It is a corollary to the celebrated Bernstein-Gelfand-Gelfand theorem (see [12, Proposition 1.5]) that one has the following.

**Proposition 1.** *If  $A_1 = \left(r_1, \frac{n_1+m_1}{2}, \frac{n_1-m_1}{2}\right)$  defines a subquotient of  $M(V_\tau)$  with  $\tau = \tau(n, m, r)$  corresponding to  $A$  as in (1.9), then  $A_1 + \rho$  is of the form  $S_\beta(A + \rho)$ ,  $S_{\alpha_1}(A + \rho)$ ,  $S_{\alpha_2}(A + \rho)$ ,  $S_\gamma(A + \rho)$ ,  $S_{\alpha_1}S_\beta(A + \rho)$ ,  $S_{\alpha_2}S_\beta(A + \rho)$ ,  $S_\gamma S_{\alpha_1}(A + \rho)$ ,  $S_\gamma S_{\alpha_2}(A + \rho)$ ,  $S_{\alpha_1}S_{\alpha_2}(A + \rho)$ ,  $S_{\alpha_1}S_{\alpha_2}S_\beta(A + \rho)$ ,  $S_\gamma S_{\alpha_1}S_{\alpha_2}(A + \rho)$ , or  $S_\gamma S_{\alpha_1}S_{\alpha_2}S_\beta(A + \rho)$ . Furthermore, if  $A_1 + \rho = S_{\gamma_i} \cdots S_{\gamma_1}(A + \rho)$  is one of these forms, then*

$$(1.11) \quad \forall j = 1, \dots, i: \langle \tau_j, S_{\gamma_{j-1}} \cdots S_{\gamma_1}(A + \rho) \rangle \in \{0, 1, 2, \dots\}.$$

*Remark 1.* It is straightforward to see that the above chains are the only possible (in relation to [12, Proposition 1.5]).

*Remark 2.* This proposition includes the situation of (1.7).

*Remark 3.* We have that  $S_{\alpha_1}S_{\alpha_2}S_\beta = S_\gamma S_{\alpha_1}S_{\alpha_2}$  and  $S_\gamma S_{\alpha_2}S_{\alpha_1}S_\beta = S_{\alpha_1}S_{\alpha_2}$ .

With  $A$  as in (1.9), let  $(z, x, y) = A + \rho$ . The two cases  $y \geq 0$  and  $y \leq 0$  are related by a simple exchange of  $\alpha_1$  and  $\alpha_2$ , and it is thus with no loss of generality that we from now on assume that  $y \geq 0$ .

Through a straightforward trial and error investigation the following result is easily obtained.

**Proposition 2.** *Let  $x$  and  $y$  be fixed and assume that  $y \geq 0$ . For  $z \in y + \mathbf{Z}$  as below, the given sequences of reflexions are the only possible which satisfy the requirements of Proposition 1:*

$$(1.12) \quad \begin{aligned} z \leq -x: & \text{ None} \\ -x < z \leq -y-1: & S_\gamma \quad (\text{if } x = y+1: \text{ void}) \\ -y < z < 1-y: & S_{\alpha_2} \\ 1-y \leq z \leq y-1: & S_{\alpha_2}, S_\gamma S_{\alpha_2} \quad (\text{if } y < 1: \text{ void}) \\ z = y: & S_{\alpha_2} \\ y+1 \leq z \leq x-1: & S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1}S_{\alpha_2}, S_\gamma S_{\alpha_1}S_{\alpha_2} \quad (\text{if } x = y+1: \text{ void}) \\ z = x: & S_{\alpha_1}S_{\alpha_2} \\ x+1 \leq z: & S_\beta, S_{\alpha_1}S_\beta, S_{\alpha_2}S_\beta, S_{\alpha_1}S_{\alpha_2}S_\beta, S_{\alpha_1}S_{\alpha_2} = S_\gamma S_{\alpha_1}S_{\alpha_2}S_\beta. \quad \square \end{aligned}$$

**Lemma 3.** *Inside  $\mathcal{U}(\mathfrak{p}^-)$  we have the following equations for non-negative integers  $\tilde{x}$ ,  $x'$ , and  $d$ :*

$$(1.13.a) \quad \begin{aligned} & z_{22}(\text{ad } k_{\tilde{\mu}})^{\tilde{x}}(\text{ad } k_{\tilde{\nu}})^{x'} z_{22}^d \\ &= (d - \tilde{x} + 1)(d - x' + 1)(d + 1)^{-2} (\text{ad } k_{\tilde{\mu}})^{\tilde{x}}(\text{ad } k_{\tilde{\nu}})^{x'} z_{22}^{d+1} \\ &\quad - d \cdot x \cdot x' \cdot (d + 1)^{-1} \det z (\text{ad } k_{\tilde{\mu}})^{\tilde{x}-1}(\text{ad } k_{\tilde{\nu}})^{x'-1} z_{22}^{d-1} \end{aligned}$$

$$(1.13.b) \quad \begin{aligned} & z_{12}(\text{ad } k_{\bar{\mu}})^{\tilde{x}}(\text{ad } k_{\bar{\nu}})^{x'} z_{22}^d \\ &= (d - \tilde{x} + 1)(d + 1)^{-2} (\text{ad } k_{\bar{\mu}})^{\tilde{x}} (\text{ad } k_{\bar{\nu}})^{x'+1} z_{22}^{d+1} \\ &\quad + d \cdot \tilde{x} \cdot (d + 1)^{-1} \det z (\text{ad } k_{\bar{\mu}})^{\tilde{x}-1} (\text{ad } k_{\bar{\nu}})^{x'} z_{22}^{d-1}, \end{aligned}$$

$$(1.13.c) \quad \begin{aligned} & z_{21}(\text{ad } k_{\bar{\mu}})^{\tilde{x}}(\text{ad } k_{\bar{\nu}})^{x'} z_{22}^d \\ &= -(d - x' + 1)(d + 1)^{-2} (\text{ad } k_{\bar{\mu}})^{\tilde{x}+1} (\text{ad } k_{\bar{\nu}})^{x'} z_{22}^{d+1} \\ &\quad - d \cdot x' \cdot (d + 1)^{-1} \det z (\text{ad } k_{\bar{\mu}})^{\tilde{x}} (\text{ad } k_{\bar{\nu}})^{x'-1} z_{22}^{d-1}, \end{aligned}$$

$$(1.13.d) \quad \begin{aligned} & z_{11}(\text{ad } k_{\bar{\mu}})^{\tilde{x}}(\text{ad } k_{\bar{\nu}})^{x'} z_{22}^d \\ &= -(d + 1)^{-2} (\text{ad } k_{\bar{\mu}})^{\tilde{x}+1} (\text{ad } k_{\bar{\nu}})^{x'+1} z_{22}^{d+1} \\ &\quad + d(d + 1)^{-1} \det z (\text{ad } k_{\bar{\mu}})^{\tilde{x}} (\text{ad } k_{\bar{\nu}})^{x'} z_{22}^{d-1}. \end{aligned}$$

*Proof.* Since  $z_{12} = [k_{\bar{\nu}}^-, z_{22}]$ ,  $z_{21} = -[k_{\bar{\mu}}^-, z_{22}]$ , and  $z_{11} = [k_{\bar{\nu}}^-, z_{21}]$ , it suffices to establish (1.13.a). With no loss of generality we assume that  $\tilde{x}, x' \leq d$ . We have that  $p = (\text{ad } k_{\bar{\mu}})^{\tilde{x}} (\text{ad } k_{\bar{\nu}})^{x'} z_{22}^d$  belongs to the  $\mathfrak{k}$ -irreducible subspace  $S$  of  $\otimes \mathfrak{p}^-$  whose highest weight vector is  $z_{22}^d$ .  $S = \otimes_s^d \mathfrak{p}^-$ . Thus  $z_{22} \circ p \in \mathfrak{p}^- \otimes S$ . More generally, the highest weight vectors of the  $\mathfrak{k}$ -module  $\mathcal{U}(\mathfrak{p}^-)$  are ( $\det z$  is defined in 1.16 below).

$$(1.14) \quad (\det z)^s z_{22}^{-i}; \quad i, s \in \mathbb{N} \cup \{0\}$$

and it follows easily that

$$(1.15) \quad \mathfrak{p}^- \otimes \otimes_s^d \mathfrak{p}^- = \otimes_s^{d+1} \mathfrak{p}^- + \det z \otimes_s^{d-1} \mathfrak{p}^-.$$

Hence  $z_{22} \circ p$  can be written as a sum according to (1.15), and it follows by looking at weights that the vectors involved are the ones given. The exact values of the coefficients are easily determined. It suffices to find just two equations involving these, and this can be done by looking at leading coefficients of e.g.  $z_{12}$  and  $z_{22}$ . We omit the details.  $\square$

The following proposition, which puts severe limitations on the  $\mathfrak{k}^{\mathcal{C}}$ -symmetry that  $q_{\tau_1}$  can possess, is crucial. Let

$$(1.16) \quad \det z = z_{11} z_{22} - z_{21} z_{12}.$$

**Proposition 4.** *Suppose that  $q_{\tau_1} \neq 0$  satisfies (1.7) and that furthermore*

$$(1.17) \quad q_{\tau_1} = (\det z)^s p_{\tau_1}$$

for some  $s \in \mathbb{N}$  and some  $p_{\tau_1} \in M(V_{\tau})$ . If  $n \neq m$ , then  $n \circ m = 0$  and  $n_1 \circ m_1 = 0$ . Furthermore, if  $(n, m) = (n, 0)$  then  $(n', m') = (0, m')$ , and, symmetrically.

*Proof.* Observe that  $k_\mu^+ p_{\tau_1} = k_\nu^+ p_{\tau_1} = 0$ . Furthermore we have that modulo  $\mathcal{U} \cdot \mathfrak{k}^+ = \mathcal{U}(\mathfrak{g}^{\mathcal{C}})k_\mu^+ + \mathcal{U}(\mathfrak{g}^{\mathcal{C}})k_\nu^+$ ,

$$(1.18.a) \quad z_{11}^+(\det z)^s = s(\det z)^{s-1}(z_{22}(h_{11}+2-s)) \pmod{\mathcal{U} \cdot \mathfrak{k}^+}$$

$$(1.18.b) \quad z_{12}^+(\det z)^s = -s(\det z)^{s-1}(z_{21}(h_{12}+2-s) - z_{22}k_\mu^-) \pmod{\mathcal{U} \cdot \mathfrak{k}^+}$$

$$(1.18.c) \quad z_{21}^+(\det z)^s = -s(\det z)^{s-1}(z_{12}(h_{21}+2-s) + z_{22}k_\nu^-) \pmod{\mathcal{U} \cdot \mathfrak{k}^+}$$

$$(1.18.d) \quad z_{22}^+(\det z)^s = s(\det z)^{s-1}(z_{11}(h_{22}+2-s) - z_{12}k_\mu^- + z_{21}k_\nu^-) \pmod{\mathcal{U} \cdot \mathfrak{k}^+}.$$

The exact form of  $(h_{ij}+2-s)$  follows from [9], since it is enough to establish it for  $n=m=0$ .

We may assume that  $p_{\tau_1}$  is homogeneous of degree  $d$ . Let  $\tilde{x}$  and  $x'$  be determined by

$$(1.19) \quad \begin{aligned} n_1 &= d+n-2\tilde{x} \\ m_1 &= d+m-2x'. \end{aligned}$$

As basis of  $V_\tau$  we choose  $\{(k_\mu^-)^i (k_\nu^-)^j v_\tau\}_{i,j=0, \dots, n}$ , and denote by  $p_0$  the coefficient of  $p_{\tau_1}$  with respect to  $v_\tau$ . Thus we have, for some  $N \geq 0$ ,

$$(1.20) \quad p_0 = \sum_{l=0}^N c_l \det z^l (\text{ad } k_\mu^-)^{\tilde{x}-l} (\text{ad } k_\nu^-)^{x'-l} z_{22}^{d-2l},$$

and  $c_0 \neq 0$ ;  $c_0=1$ , say.

Since the ideal generated by  $\det z$  inside  $\mathcal{C}[z_{11}, z_{12}, z_{21}, z_{22}]$  is *prime* it follows immediately from (1.18.a), by looking at  $v_\tau$ -coefficients, that

$$(1.21) \quad (h_{11}+2-s)p_0 \cdot v_\tau = 0.$$

Let us now assume that  $n_1 > 0$ . We observe that  $h_{12} = h_{11} - h_\mu$  and use (1.18.b) to look at  $v_\tau$ -coefficients in  $z_{12}^+(\det z)^s p_{\tau_1}$ . Specifically it follows that the  $v_\tau$ -coefficient of  $(z_{21}(h_{12}+2-s) - z_{22}k_\mu^-)p_{\tau_1}$  is proportional to  $\det z$ . Thus,  $n_1 z_{21} p_0 + z_{22}[k_\mu^-, p_0]$  is annihilated by  $(\text{ad } k_\mu^+)^{x+1}$  and  $(\text{ad } k_\nu^+)^{x'}$ . Hence, since  $[k_\nu^+, z_{21}] = 0$ ,

$$(1.22) \quad [n_1 z_{21} + z_{22} \text{ad } k_\mu^-] z_{21}^{\tilde{x}} z_{22}^{d-\tilde{x}} = 0,$$

and this implies, because  $[k_\mu^-, z_{22}] = -z_{21}$  and  $[k_\mu^-, z_{21}] = 0$ , that

$$(1.23) \quad n = \tilde{x}.$$

We now return to  $z_{11}^+$ . It follows from (1.7) that the terms of  $p_{\tau_1}$  involving  $v_\tau, k_\mu^- v_\tau,$  and  $k_\nu^- v_\tau$  are

$$(1.24) \quad \begin{aligned} p_{\tau_1}(v_\tau) + p_{\tau_1}(k_\mu^- v_\tau) + p_{\tau_1}(k_\nu^- v_\tau) &= ((\text{ad } k_\mu^-)^{\tilde{x}} (\text{ad } k_\nu^-)^{x'} z_{22}^d v_\tau \\ &\quad - \frac{\tilde{x}(d-\tilde{x}+1)}{n} (\text{ad } k_\mu^-)^{\tilde{x}-1} (\text{ad } k_\nu^-)^{x'} z_{22}^d k_\mu^- v_0 \\ &\quad - \frac{x'(d-x'+1)}{m} (\text{ad } k_\mu^-)^{\tilde{x}} (\text{ad } k_\nu^-)^{x'-1} z_{22}^d k_\nu^- v_0 \pmod{\det z}). \end{aligned}$$



We wish to evaluate the  $v_r$ -coefficient of  $z_{11}^+$  applied to (1.24). We write tacitly  $h_{11} \cdot v_r = w_{11} \cdot v_r$  though by (1.21),  $w_{11} = s - 2 + \tilde{x} + x'$ . We have the equations

$$(1.25) \quad \begin{aligned} [z_{11}^+, k_\mu^-] &= -z_{12}^+, & [z_{12}^+, k_\nu^-] &= z_{22}^+, \\ [z_{11}^+, k_\nu^-] &= z_{21}^+, & [z_{21}^+, z_{22}^+] &= k_\mu^+, & [z_{12}^+, z_{22}^+] &= -k_\nu^+, \\ [z_{11}^+, z_{21}^+] &= -k_\nu^+, & \text{and } [z_{11}^+, z_{12}^+] &= k_\mu^+. \end{aligned}$$

Thus,

$$(1.26) \quad \begin{aligned} & z_{11}^+ [(p_{\tau_1}(v_r) + p_{\tau_1}(k_\mu^- v_r) + p_{\tau_1}(k_\nu^- v_r)) \pmod{\det z}]_{v_r} \\ &= -\tilde{x} \cdot x' (\text{ad } k_\mu^-)^{\tilde{x}-1} (\text{ad } k_\nu^-)^{x'-1} z_{22}^{d-1} \cdot d(w_{11} - (d-1)) \\ &\quad - \tilde{x} \cdot x' (\text{ad } k_\mu^-)^{\tilde{x}-1} (\text{ad } k_\nu^-)^{x'-1} z_{22}^{d-1} \cdot d \cdot (d - (\tilde{x} - 1)) \\ &\quad - \tilde{x} \cdot x' (\text{ad } k_\mu^-)^{\tilde{x}-1} (\text{ad } k_\nu^-)^{x'-1} z_{22}^{d-1} \cdot d \cdot (d - (x' - 1)) \\ &= -\tilde{x} \cdot x' \cdot d(w_{11} + 3 + d - \tilde{x} - x') (\text{ad } k_\mu^-)^{\tilde{x}-1} (\text{ad } k_\nu^-)^{x'-1} z_{22}^{d-1}. \end{aligned}$$

By looking at weights it follows that the first interval in which a vector of the form (1.17) may exist is  $y + 1 \leq z \leq x - 1$ .

Let us first look at the situation corresponding to  $S_{\alpha_1} S_{\alpha_2}$ : It follows from (1.21) that any element  $p_{\tau_1}$  of the form (1.17), satisfying (1.7), and having weight  $S_{\alpha_1} S_{\alpha_2} (A + \rho) - \rho$  satisfies  $h_{11} q_{\tau_1} = -s - 2$ . Since  $A = (z, x, y)$  it follows that  $s = z + 1 - x$ , and this is never positive in the given interval. Let us then consider  $S_\gamma S_\alpha S_{\alpha_2}$ : We compute the  $v_r$ -coefficient of  $z_{11}^+ q_{\tau_1}$  by means of (1.18.a) and (1.20):

Observe that the computation of the  $v_r$ -coefficient corresponding to  $\det z^{l+s} z_{11}^+ (\text{ad } k_\mu^-)^{\tilde{x}-l} (\text{ad } k_\nu^-)^{x'-l} z_{22}^{d-2l}$  is given by (1.26) with  $(\tilde{x}, x', d)$  replaced by  $(\tilde{x} - l, x' - l, d - 2l)$ . The terms involving  $C_N$  (cf. (1.20)) are easily computed to be

$$(1.27) \quad \begin{aligned} & C_N \det z^{\bar{N}} (-1)(\tilde{x} - N)(x' - N)(d - 2N)(w_{11} + d - \tilde{x} - x' + 3) \\ &\quad \cdot (\text{ad } k_\mu^-)^{\tilde{x}-N-1} (\text{ad } k_\nu^-)^{x'-N-1} z_{22}^{d-2N-1} + \bar{N} \cdot C_N \det z^{N-1} \\ &\quad \cdot z_{22}(w_{11} + 2 + N - \tilde{x} - x') (\text{ad } k_\mu^-)^{\tilde{x}-N} (\text{ad } k_\nu^-)^{x'-N} z_{22}^{d-2N}, \end{aligned}$$

where  $\bar{N} = N + s$ .

The term containing  $\det z^N$  in the  $v_r$ -coefficient of  $z_{11}^+ q_{\tau_1}$  is given by (1.27); the contribution from the second summand can be found by means of (1.13.a). The result is easily computed to be

$$(1.28) \quad \begin{aligned} & C_N (\tilde{x} - N)(x' - N)(d - 2N)(d - 2N + 1)^{-1} (\bar{N} - 2z)(x + 1 + z - \bar{N}) \\ &\quad \cdot \det z^{\bar{N}} (\text{ad } k_\mu^-)^{\tilde{x}-N-1} (\text{ad } k_\nu^-)^{x'-N-1} z_{22}^{d-2N-1}. \end{aligned}$$

It is easy to see (cf. the diagrams of the following chapter) that  $1 \leq \bar{N} \leq x - y$ ,  $\tilde{x} = x + y - s$ ,  $x' = x - y - s$ , and  $d = x + z - 2s$ , hence all factors in (1.28) are non-zero.

In the case  $z = x$ , the only possible sequence of reflexions is  $S_{\alpha_1} S_{\alpha_2}$ . This sequence is ruled out by the computation below, by which it is also ruled out for  $z > x$ .

Consider finally  $z \geq x + 1$ : Two sequences may lead to a  $q_{\tau_1}$  of the form (1.17) and satisfying (1.7). The first is  $S_{\alpha_1} S_{\alpha_2} S_{\beta}$ : We have that  $S_{\alpha_1} S_{\alpha_2} S_{\beta}(z, x, y) - \rho = (-x - 2, z - 1, -y)$ , and it follows then from (1.21) that  $s = 1 + x - z \leq 0$ . The other possibility is  $S_{\gamma} S_{\alpha_1} S_{\alpha_2} S_{\beta} = S_{\alpha_1} S_{\alpha_2}$ : The  $v_{\tau}$ -coefficient of any  $q_{\tau_1} = q_{\gamma \alpha_1 \alpha_2 \beta}$  having highest weight  $S_{\alpha_1} S_{\alpha_2}(z, x, y) - \rho$  is

$$(1.29) \quad \det z^s \sum_{i=0}^m C_i \det z^i (\text{ad } k_{\bar{\mu}})^{n-i} (\text{ad } k_{\bar{\nu}})^{m-i} z_{22}^{n+m-2i}$$

with  $s = z - x + 1$ . By insisting that either  $z_{11}^+ q_{\tau_1} = 0$  or  $z_{22}^+ q_{\tau_1} = 0$ , it is easy to see that  $C_m \neq 0$ . Using (1.18.c) it is then straightforward to compute the leading term in  $\det z$  of the  $v_{\tau}$ -coefficient of  $z_{21}^+ q_{\tau_1}$ :

$$(1.30) \quad (z_{21}^+ q_{\tau_1})_{v_{\tau}} = (-1)^{n-m} (n-m)! C_m (m-n) (z-y) \det z^{s+m} z_{21}^{n-m-1} + \text{lower order terms in } \det z.$$

Thus,  $z_{21}^+ q_{\tau_1} \neq 0$ .  $\square$

We now proceed to prove that in the remaining cases we do have (1.7) satisfied for  $q_{\tau_1}$ 's of the form (1.17) (and  $r$  and  $r_1$  appropriate). Our proof follows the lines of the preceding investigation but we remark that it is possible to give an alternative proof of Proposition 5 (and probably also Proposition 6) along the lines of [11]: If the two pieces of the Dirac operator are denoted by  $\sigma$  and  $\tilde{\sigma}$  ( $\sigma = D^+$ ,  $\tilde{\sigma} = c(D)$  in the terminology of [11]), and if  $\square = d$ 'Alembertian, then the statement is equivalent to the assertion that  $\square^n \otimes_s^k \sigma$  as well as  $\square^n \otimes_s^k \tilde{\sigma}$  are intertwining differential operators for all non-negative integers  $n$  and  $k$ , and this fact can easily be obtained from [11] (or [10]).

**Proposition 5.** *Let  $A = (s + n/2 - 2, n/2, n/2)$  for  $s, n \in \mathbb{N}$ . Then (1.7) is satisfied precisely for a  $q_{\tau_1}$  of the form (1.17), and with  $\tau_1$  corresponding to  $A_1 = S_{\alpha_1} S_{\alpha_2}(A + \rho) - \rho = (-s + n/2 - 2, n/2, -n/2)$ . Similarly for  $\tilde{A} = (s + n/2 - 2, n/2, -n/2)$  and  $\tilde{A}_1 = (-s + n/2 - 2, n/2, n/2)$ .*

*Proof.* The two cases are clearly so similar that it suffices to prove one of them, say the first: Let then  $\tau \equiv A$  and let  $v_{\tau}$  be as in (1.4) and (1.5). Observe

that in the present situation, also  $k_{\nu}^{-}v_{\tau}=0$ . By looking at weights it follows that any  $q_{\tau_1}$  satisfying (1.7) must be of the form (1.17). In fact, it is easy to see that

$$(1.31) \quad q_{\tau_1} = (\det z)^s \sum_{i=0}^n \frac{1}{i!} z_{21}^{n-i} z_{22}^i (k_{\mu}^{-})^i v_{\tau}.$$

Since  $k_{\mu}^{-}q_{\tau_1}=0$ , and since  $A$  and  $A_1$  are set up appropriately, to establish (1.7) it suffices to prove that  $z_{21}^{+}q_{\tau_1}=0$ , and to this purpose we apply (1.18c). Observing that  $h_{22}v_{\tau}=(s-2)v_{\tau}$ ,  $h_{11}v_{\tau}=(s-2+n)v_{\tau}$ ,  $h_{21}v_{\tau}=(s-2+n)v_{\tau}$  and  $h_{12}v_{\tau}=(s-2)v_{\tau}$ , we get

$$(1.32) \quad z_{21}^{+}q_{\tau_1} = \det z^{s-1} \left( \det z \sum_{i=0}^n \frac{1}{i!} (n-i)sz_{21}^{n-i-1}z_{22}^i(k_{\mu}^{-})^i v_{\tau} - s \sum_{i=0}^n \frac{1}{i!} (-nz_{12}z_{21}^{n-i}z_{22}^i + z_{22}((n-i)z_{11}z_{21}^{n-i-1}z_{22}^i + iz_{21}^{n-i}z_{22}^{i-1}z_{12})) (k_{\mu}^{-})^i v_{\tau} \right),$$

and this is zero.  $\square$

**Proposition 6.** *Let  $A=(n-2+s, n, 0)$  for  $s, n \in \mathbb{N}$ . Then (1.7) is satisfied precisely for a  $q_{\tau_1}$  of the form (1.17), and with  $\tau_1$  corresponding to  $A_1=-\rho=(-n-s-2, n, 0)$ .*

*Proof.* The  $v_{\tau}$ -coefficient of  $q_{\tau_1}$  is given by

$$(1.33) \quad \sum_{i=0}^n c_i z_{22}^i \det z^{s-i} z_{11}^i$$

for some constants  $c_i$ , and  $z=n+s$ . The recursive relations imposed on the  $c_i$ 's by  $(z_{11}^{+}q_{\tau_1})_{v_{\tau}}=0$  are the same as those imposed by the analogous equation for  $z_{22}^{+}$ , namely

$$(1.34) \quad ic_i(-z+i-1-n) + c_{i-1}(z-i+1)(i-1-n) = 0$$

and these can of course be solved. Furthermore it is easy to see that the  $v_{\tau}$ -coefficients of  $z_{21}^{+}q_{\tau_1}$  and  $z_{12}^{+}q_{\tau_1}$  are zero. Thus there is a unique  $q_{\tau_1}$  for which the  $v_{\tau}$ -coefficient of any  $z^{+}q_{\tau_1}$  is zero. Since any non-zero highest weight vector for  $\mathfrak{k}^{\mathcal{C}}$  in  $M(V_{\tau})$  clearly has a non-zero  $v_{\tau}$ -coefficient, it follows that this  $q_{\tau_1}$  satisfies (1.7).  $\square$

*Remark.* For  $n=1,2$ , these homomorphisms correspond to the covariant differential operators recently determined by Branson [23].

### § 2. Conformal Covariants

The highest weight vectors of the  $\mathfrak{k}$ -module  $\mathcal{U}(\mathfrak{p}^-)$  are

$$(2.1) \quad (\det z)^s z_{22}^t; \quad t, s \in \mathbb{N} \cup \{0\}.$$

Through this one obtains a representation of  $U(2) \times U(2)$  given by the double diagram:

$$(2.2) \quad \begin{array}{|c|c|c|c|} \hline & \cdots & & \\ \hline & \cdots & & \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & \cdots & & \\ \hline & \cdots & & \\ \hline \end{array}$$

$\underbrace{\hspace{2cm}}_s \quad \underbrace{\hspace{2cm}}_t \qquad \underbrace{\hspace{2cm}}_s \quad \underbrace{\hspace{2cm}}_t$

This is not unique, and to make it correspond to (2.1) as a representation of  $K$  one should take the  $\otimes$ -product of (2.2) with the  $K$ -representation  $(u_1, u_2) \rightarrow (\det u_1)^{-(2s+t)}$  (cf. 1.2). In this sense, we are really only working with  $SU(2) \times SU(2)$  representations, but the form of (2.2) is convenient because the total number of boxes in one of the diagrams,  $2s+t$ , is equal to the degree  $d$  of (2.1) (and hence,  $\det z^s z_{22}^t \in \otimes^d \mathfrak{p}^-$ ).

In the same sense of non-uniqueness, a representation  $\tau$  gives rise to a double diagram

$$(2.3) \quad \begin{array}{|c|c|c|} \hline \cdots & & \\ \hline \end{array} \odot \begin{array}{|c|c|c|} \hline \cdots & & \\ \hline \end{array}$$

$\underbrace{\hspace{2cm}}_n \qquad \underbrace{\hspace{2cm}}_m$

We recall that  $\otimes$ -products of  $SU(n)$  representations can be handled completely by the Littlewood-Richardson rule (hereafter: the LR-rule) ([13]).

In the following we examine which of the sequences of reflexions in Proposition 2 in fact do correspond to a  $q_{\tau_1}$  as in (1.7). We do know that any single reflexion, as e.g.  $S_\gamma$  when  $-x < z \leq y-1$ , does correspond to a such ([12, Proposition 1.6]). The only problem at this level is that of multiplicities: Assume that  $-x < z \leq -y-1$ . Then  $A_1 = S_\gamma(A + \rho) - \rho = A - (z+x)\alpha_1$ . Thus,  $d = (z+x)$ ,  $n_1 = n-d$ , and  $m_1 = m-d$ , and thus the result is a double diagram

$$(2.4) \quad \begin{array}{|c|c|c|c|} \hline \cdots & & \cdots & \\ \hline 1 & \cdots & 1 & \\ \hline \end{array} \odot \begin{array}{|c|c|c|c|} \hline \cdots & & \cdots & \\ \hline 1 & \cdots & 1 & \\ \hline \end{array}$$

$\underbrace{\hspace{2cm}}_d \quad \underbrace{\hspace{2cm}}_{n-d} \qquad \underbrace{\hspace{2cm}}_d \quad \underbrace{\hspace{2cm}}_{m-d}$

and according to the LR-rule, this has multiplicity one (as implied by the indicated row of  $d$  1's).

An analogous argument gives that  $S_{\alpha_2}$  gives a  $q_{\tau_1}$  with multiplicity one when  $-y < z < 1-y$  as well as when  $1-y \leq z \leq y-1$ . Furthermore, in the last interval,  $S_\gamma$  corresponds to a unique non-trivial homomorphism into the module defined by  $S_{\alpha_2}(A + \rho) - \rho$ . However, as we shall see below,  $S_\gamma \circ S_{\alpha_2}$  does not correspond to an element of  $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$  and hence, the composite homomorphism is zero: We have that  $S_\gamma S_{\alpha_2}(A + \rho) - \rho = A - (z+y)\alpha_2 - (x-y)\gamma$ . It

follows that  $d=x+z$ ,  $n_1=n-(x+z)$ , and  $m_1=m+z-x+2y$ . This gives the following picture:

$$(2.5) \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline 1 & \dots & 1 & 1 & \dots & 1 & \dots & \dots & \dots & \dots \\ \hline \end{array}}^n \\ \underbrace{\hspace{2cm}}_{z+1} \quad \underbrace{\hspace{2cm}}_{x-1} \end{array} = \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots \\ \hline 1 & \dots & 1 & \dots & 1 & \dots \\ \hline \end{array}}^m \quad \overbrace{\begin{array}{|c|c|c|c|} \hline 1 & \dots & 1 & \dots \\ \hline 2 & \dots & 2 & \dots \\ \hline \end{array}}^{z-1} \end{array}$$

Observe that  $z+x \leq n=(x+y-1)$  and that  $x-y=m+1$ . According to the LR-rule, the diagram to the left can only occur in the  $\otimes$ -product

$$(2.6) \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline \dots & \dots & \dots & \dots \\ \hline \end{array}}^n \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline 1 & \dots & 1 & \dots \\ \hline \end{array}}^d \\ \underbrace{\hspace{2cm}} \end{array}$$

whereas the diagram to the right, due to the occurrence of at least one 2, which is implemented by the column of length 2 as indicated in (2.5), can only occur in  $\otimes$ -products of the form

$$(2.7) \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline \dots & \dots & \dots & \dots \\ \hline \end{array}}^m \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline 1 & \dots & 1 & 1 & \dots & 1 \\ \hline 2 & \dots & 2 & \dots & 2 & \dots \\ \hline \end{array}}^d \\ \underbrace{\hspace{2cm}}_s \end{array} \quad , s \geq 1$$

This contradicts (2.2) and thus (2.5) does not correspond to an element of  $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$ .

In the case  $z=y$  we have a single reflexion, so let us turn to the interval  $y+1 \leq z \leq x-1$ . Here it follows again that  $S_{\alpha_1}$  and  $S_{\alpha_2}$  give unique homomorphisms. Let  $q_{\alpha_1}$  and  $q_{\alpha_2}$  be the corresponding vectors as in (1.7). It is then easy to see that  $\mathcal{U}(\mathfrak{p}^-) \cdot q_{\alpha_1} \cap \mathcal{U}(\mathfrak{p}^-) \cdot q_{\alpha_2} \neq \{0\}$ . In fact, this follows by looking at multiplicities: The double diagram corresponding to  $S_{\alpha_1} S_{\alpha_2}$  is

$$(2.8) \quad \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^n \\ \underbrace{\hspace{2cm}}_{z+y} \quad \underbrace{\hspace{2cm}}_{z-y} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^m \\ \underbrace{\hspace{2cm}}_{z-y} \quad \underbrace{\hspace{2cm}}_{z+y} \end{array}$$

and the corresponding  $K$ -representation clearly occurs with multiplicity  $z-y+1$  in  $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$ . We have that  $d=2 \cdot z$  and it is easy to see from the LR-rule that (2.8) occurs with multiplicity 1 in the  $\otimes$ -product

$$(2.9) \quad \left( \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline \dots & \dots & \dots & \dots \\ \hline \end{array}}^n \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|} \hline \dots & \dots & \dots & \dots \\ \hline \end{array}}^m \\ \underbrace{\hspace{2cm}} \end{array} \right) \otimes \left( \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^s \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^{d-2s} \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^s \\ \underbrace{\hspace{2cm}} \end{array} \otimes \begin{array}{c} \overbrace{\begin{array}{|c|c|c|c|c|c|} \hline \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \end{array}}^{d-2s} \\ \underbrace{\hspace{2cm}} \end{array} \right)$$

for each  $s=0, 1, \dots, z-y$ . Thus, there is at least one  $q_{\tau_1}$  corresponding to  $S_{\alpha_1}S_{\alpha_2}$ . But if there were two linearly independent  $q_{\tau_1}$ 's then there would also be one of the form (1.17) and this is ruled out. Hence there is multiplicity one. Maintaining the assumption  $y+1 \leq z \leq x-1$ , we finally consider the double diagram corresponding to  $S_\gamma S_{\alpha_1} S_{\alpha_2}$ :

$$(2.10) \quad \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \hline \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \hline \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\ \hline \end{array}$$

$\underbrace{\hspace{10em}}_n \quad \underbrace{\hspace{10em}}_{z-y} \quad \underbrace{\hspace{10em}}_m \quad \underbrace{\hspace{10em}}_{z+y}$   
 $\underbrace{\hspace{2em}}_{z+y} \quad \underbrace{\hspace{2em}}_{x-z} \quad \underbrace{\hspace{2em}}_{z-y} \quad \underbrace{\hspace{2em}}_{x-z}$

Due to the fact that each diagram contains non-trivial columns, any  $q_{\tau_1}$  corresponding to (2.10) must be of the form (1.17) and is as such ruled out by the assumption that  $x > y+1$  since this implies that  $n \neq 0$  and  $m \neq 0$ .

Through analogous reasoning it follows that for  $z=x$ , unless  $x=y+1$  or  $y=0$ ,  $S_{\alpha_1}S_{\alpha_2}$  is ruled out by Proposition 4. Finally, when  $x+1 \leq z$ ,  $S_\beta$  gives a unique homomorphism,  $S_{\alpha_1}S_\beta$  and  $S_{\alpha_2}S_\beta$  do not correspond to elements of  $\mathcal{U}(\mathfrak{p}^-) \otimes V_\tau$ , and  $S_{\alpha_1}S_{\alpha_2}S_\beta$  as well as  $S_\gamma S_{\alpha_1}S_{\alpha_2}S_\beta$  correspond to cases in which the potential  $q_{\tau_1}$  is forced to be of the form (1.17) and hence are ruled out unless  $x=y+1$  or  $y=0$ . If  $x=y+1$  there is a unique homomorphism for  $S_\gamma S_{\alpha_1}S_{\alpha_2}S_\beta = S_{\alpha_1}S_{\alpha_2}$  given by Proposition 5 whereas Proposition 4 rules out  $S_{\alpha_1}S_{\alpha_2}S_\beta$ . Analogously, for  $y=0$ , Proposition 6 gives a homomorphism for  $S_{\alpha_1}S_{\alpha_2}$  whereas  $S_{\alpha_1}S_{\alpha_2}S_\beta$  again is ruled out.

All in all we have now established

**Theorem 7.** *Let  $x$  and  $y$  be fixed and assume that  $y \geq 0$ . For  $z \in y + \mathbb{Z}$  as below there is a non-trivial homomorphism into the generalized Verma module  $M(V_\tau)$  of highest weight  $\Lambda = (z, x, y) - \rho$  exactly for the given sequences of reflexions. There are no multiplicities:*

- i)  $x > y+1$ :
- $$(2.11) \quad \begin{array}{ll} z \leq -x: & \text{None} \\ -x < z \leq -y-1: & S_\gamma \\ -y < z \leq y & : S_{\alpha_2} \\ y+1 \leq z \leq x-1 & : S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1}S_{\alpha_2} \\ z = x: & \text{None} \\ x+1 \leq z & : S_\beta \end{array}$$

- ii)  $x = y+1$ :

$$\begin{aligned}
 z \leq 1-x & : \text{None} \\
 1-x < z \leq x-1 & : S_{\alpha_2} \\
 z = x & : S_{\alpha_1} S_{\alpha_2} \\
 x+1 \leq z & : S_{\beta}, S_{\alpha_1} S_{\alpha_2}.
 \end{aligned}$$

iii)  $y=0$ :

$$\begin{aligned}
 z \leq -x & : \text{None} \\
 -x < z \leq -1 & : S_{\gamma} \\
 1 \leq z \leq x-1 & : S_{\alpha_1}, S_{\alpha_2}, S_{\alpha_1} S_{\alpha_2} \\
 z = x & : S_{\alpha_1} S_{\alpha_2} \\
 x+1 \leq z & : S_{\beta}, S_{\alpha_1} S_{\alpha_2}. \quad \square
 \end{aligned}$$

*Remark.* As a part of the proof of Theorem 7 it was established that the homomorphism corresponding to  $S_{\alpha_1} S_{\alpha_2}$  when  $y+1 \leq z \leq x-1$  is the composite of the homomorphisms corresponding to  $S_{\alpha_1}$  and  $S_{\alpha_2}$ . It is furthermore a consequence of part i) that the homomorphism related to  $S_{\alpha_1} S_{\alpha_2}$  for  $z \geq x+1$  in part ii) and iii) does *not* factor through the homomorphism related to  $S_{\beta}$ .

### § 3. The Subspace Structure

Let  $A$  be an invariant subspace of  $M_{\tau}$  and assume that  $A \neq \{0\}$  and  $A \neq M(V_{\tau})$ . Then  $V_{\tau} \not\subseteq A$  and hence there is in  $A$  at least one element  $q$  of lowest degree, and  $d \geq 1$ . Since  $\mathfrak{k}$  preserves degree and  $\mathfrak{p}^+$  lowers it, we may assume that  $q$  satisfies (1.7). The space  $B = \mathcal{U}(\mathfrak{g}^c) \cdot q$  is then an invariant subspace contained in  $A$  and in case it is not irreducible we can repeat the above procedure. For the present investigation of the subspace structure of  $M(V_{\tau})$  it is, in view of the preceding determination of the full set of homomorphisms into  $M(V_{\tau})$ , more pertinent to investigate whether it may happen that  $A \setminus B \neq \{0\}$ . Should this occur, one is led, as above, to an element  $\check{q} \in A \setminus B$  of lowest degree which satisfies (1.7) *modulo*  $B$ . Thus, such a  $\check{q}$  does not necessarily satisfy (1.7) proper and hence needs not define a homomorphism into  $M(V_{\tau})$ .

The purpose of this chapter is to prove that this theoretical possibility, in the current situation *does* materialize for the representations  $A = (-1, n, 0)$ .

It is a consequence of the Bernstein-Gelfand-Gelfand theorem [2] that any  $\check{q}$  as above has a weight  $A_1$  as given by Proposition 1. We then begin our proof by looking anew at Proposition 2:

Since for  $|z| < y-1$ ,  $S_\gamma S_{\alpha_2}$  does not correspond to an element of  $M(V_\tau)$ , the first interval in which a phenomenon as above may occur is  $y+1 \leq z \leq x-1$ : Consider  $S_{\alpha_1} S_{\alpha_2}$ : We have seen that there is a unique homomorphism at this level; defined by the double diagram (2.8). Let us then suppose that there is at least one  $\check{q}$ , as above, of the same weight. We then have that  $z_{11}^+ \check{q}$  defines a highest weight vector for  $\mathfrak{k}^C$  and the corresponding double diagram is obtained by removing one box from each of the lowest rows in (2.8). Thus, the new lowest rows have lengths  $z+y-1$  and  $z-y-1$ , respectively, and this means that neither the diagram of the  $\mathfrak{k}$ -type  $q_{\alpha_1}$  that defines the homomorphism corresponding to  $S_{\alpha_1}$ , nor the diagram of the analogous  $q_{\alpha_2}$  is contained in the diagram of  $z_{11}^+ \check{q}$ . But then  $z_{11}^+ \check{q}$  cannot belong to  $\mathcal{U}(\mathfrak{g}^C)q_{\alpha_1} \cup \mathcal{U}(\mathfrak{g}^C)q_{\alpha_2}$ . Hence  $z_{11}^+ \check{q} = 0$ . Let  $q_{\alpha_1 \alpha_2}$  be the element of  $M(V_\tau)$  that defines the homomorphism for  $S_{\alpha_1} S_{\alpha_2}$ . Then also  $z_{11}^+ q_{\alpha_1 \alpha_2} = 0$ , and hence there is a  $\check{q}$  in the span of  $\check{q}$  and  $q_{\alpha_1 \alpha_2}$  which is of the form (1.17) and which is annihilated by  $z_{11}^+$ . Again it follows that (1.21) holds and also as in Chapter 1, it follows that  $((S_{\alpha_1} S_{\alpha_2}(z, x, y)) - \rho)(h_{11}) = ((-z, x, -y) - (2, 1, 0))(1, 1, 0) = -s-2$  for some  $s \in \mathbb{N}$ . But this equation becomes  $s = z-x+1$ , and in the given interval, such an  $s$  is less than or equal to zero.

It remains to investigate  $S_\gamma S_{\alpha_1} S_{\alpha_2}$ . As noted before, the shape of the diagram (2.10) implies that any element  $q_{\gamma \alpha_1 \alpha_2}$  of  $M(V_\tau)$  corresponding to this reflexion, contains a factor of  $\det z$ . If  $(z-y-1 \neq 0)$ , the following expression

$$(3.1) \quad \begin{aligned} & (z_{22}^+ + (z+y-1)^{-1} z_{12}^+ k_{\check{v}}^- - (z-y-1)^{-1} z_{21}^+ k_{\check{\mu}}^- \\ & \quad - (z+y-1)^{-1} (z-y-1)^{-1} z_{11}^+ k_{\check{\mu}}^- k_{\check{v}}^-) q_{\gamma \alpha_1 \alpha_2}, \end{aligned}$$

defines a highest weight vector for  $\mathfrak{k}^C$  corresponding to the removal of a box in each of the top rows in the diagram (2.10). Evidently, this expression is not in the ideal generated by  $q_{\alpha_1 \alpha_2}$  and hence it must be zero. By bringing the  $k_{\check{\mu}}^-$ 's and  $k_{\check{v}}^-$ 's to the other side of the  $z^+$ 's, we can use (1.18) to evaluate (3.1), but it suffices to observe that the term which does not contain any factor of  $\det z$ ;

$$(3.2) \quad \begin{aligned} & (az_{11} + a(z-y-1)^{-1} z_{12} k_{\check{\mu}}^- - a(z+y-1)^{-1} z_{21} k_{\check{v}}^- \\ & \quad - a(z-y-1)^{-1} (z+y-1)^{-1} z_{22} k_{\check{v}}^- k_{\check{\mu}}^-) (\det z)^{-1} q_{\gamma \alpha_1 \alpha_2}, \end{aligned}$$

for some  $a \in \mathbb{R}$ , has  $-a(z-y-1)^{-1} (z+y-1)^{-1} = (x+z) (z-y-1)^{-1} (z+y-1)^{-1}$ .

We now turn to the case  $z=y+1$ . (Still;  $z \leq x-1$ ): It follows that the  $v_\tau$ -coefficient of  $q_{\gamma \alpha_1 \alpha_2}$  is given by

$$(3.3) \quad (q_{\gamma \alpha_1 \alpha_2})_{v_\tau} = \det z (\text{ad } k_{\check{\mu}}^-)^n (\text{ad } k_{\check{v}}^-)^m z_{22}^n$$



where, as before,  $n=x+y-1$  and  $m=x-y-1$ . We have that

$$(3.4) \quad z_{11}^+ q_{\gamma_{\alpha_1 \alpha_2}}$$

is a highest weight vector for  $\mathfrak{k}^C$ . By Lemma 3 and (1.26), its  $v_\tau$ -coefficient is easily computed;

$$(3.4) \quad (z_{11}^+ q_{\gamma_{\alpha_1 \alpha_2}})_{v_\tau} = -m(n-m+1)(n+1)^{-2} (\text{ad } k_\mu^-)^n (\text{ad } k_\nu^-)^m z_{22}^{n+1} \\ + m(n+1)^{-1} n^2 (n+2)(m-n-1) \det z (\text{ad } k_\mu^-)^{n-1} \\ \cdot (\text{ad } k_\nu^-)^{m-1} z_{22}^{n-1}.$$

Recall that the  $\mathfrak{k}$ -type corresponding to (2.10) does not belong to  $\mathcal{U}(\mathfrak{p}^-) \cdot q_{\alpha_1 \alpha_2}$ . On the other hand, there must be a (unique)  $\mathfrak{k}$ -type  $\hat{q}$  in this ideal of the same weight as (3.4), and this  $\hat{q}$  must satisfy:

$$(3.5) \quad (z_{11} + z_{12} k_\mu^- - (n+1-m)^{-1} z_{21} k_\nu^- - (n+1-m) z_{22} k_\mu^- k_\nu^-) \hat{q} = 0.$$

We have that

$$(3.6) \quad (\hat{q})_{v_\tau} = (\text{ad } k_\mu^-)^n (\text{ad } k_\nu^-)^m z_{22}^{n+1} + \alpha \det z (\text{ad } k_\mu^-)^{n-1} (\text{ad } k_\nu^-)^{m-1} z_{22}^{n-1}$$

for some  $\alpha \in \mathbb{R}$ .

The  $\alpha$  is evaluated by means of Lemma 3, and it follows that  $\alpha = n^2(n+1) \cdot (n+2)$ . Thus,  $z_{11}^+ q_{\gamma_{\alpha_1 \alpha_2}} \in \mathcal{U}(\mathfrak{p}^-) \cdot q_{\alpha_1 \alpha_2}$ .

Assume now that  $n > m$ . Again we have an element;

$$(3.7) \quad (z_{21}^+ + (n-m)^{-1} z_{11}^+ k_\nu^-) q_{\gamma_{\alpha_1 \alpha_2}},$$

which is a highest weight vector for  $\mathfrak{k}^C$ . We compute:

$$(3.8) \quad [(z_{21}^+ + (n-m)^{-1} z_{11}^+ k_\nu^-) q_{\gamma_{\alpha_1 \alpha_2}}]_{v_\tau} \\ = -n^2(n-m+1)(n-1)(n+1)^{-1}(n-m)^{-1} \det z (\text{ad } k_\mu^-)^{n-1} (\text{ad } k_\nu^-)^m z_{22}^{n-1}.$$

At this level,  $n=1$  is excluded. It is then quite obvious, e.g. because the  $\alpha$  above is non-zero, that (3.8) cannot define an element in  $\mathcal{U}(\mathfrak{p}^-) q_{\alpha_1 \alpha_2}$ .

Finally,  $S_\gamma S_{\alpha_1} S_{\alpha_2}(1, x, 0) - \rho = (-x-2, 0, 0)$ . Thus, in case  $n=m$  and  $z=1$ , (3.4) is the only quantity that needs to be considered and hence, in this case we do get a quotient. Observe that due to the assumptions on the interval;  $x \geq 2$ .

Let us now continue our investigation with a look at  $z=x$ : If  $x > y+1$ , we have seen that there are no homomorphisms into  $M(V_\tau)$ , hence, evidently, there are no quotients either;  $M(V_\tau)$  is irreducible. If  $y=0$  there is a unique homomorphism given by Proposition 6. Since  $S_{\alpha_1} S_{\alpha_2}$  is the only possible sequence,

it follows that there are no other quotients. Similarly for  $x=y+1$ .

Finally, in case  $z \geq x+1$ , there is a unique non-trivial homomorphism  $M(V_{\tau_1}) \rightarrow M(V_\tau)$ , where  $\tau_1$  is defined by  $S_\beta(A+\rho) - \rho$ .  $S_{\alpha_1} S_\beta$  and  $S_{\alpha_2} S_\beta$  are ruled out as before. The remaining cases are  $S_{\alpha_1} S_{\alpha_2} S_\beta$  and  $S_\gamma S_{\alpha_1} S_{\alpha_2} S_\beta$ . They can be handled by counting multiplicities: Let  $\tilde{p}$  denote a highest weight vector in  $M(V_{\tau_1})$  ( $\tau_1$  as above), let  $v_{\tau_1}$  denote the highest weight in  $V_{\tau_1}$  ( $v_{\tau_1} = z_{22} v_\tau$ ) and let  $\tilde{p}_0$  denote the coefficient of  $\tilde{p}$  with respect to  $v_{\tau_1}$ . It then follows from Lemma 3 that if

$$(3.9) \quad \tilde{p}_0 = (\text{ad } k_\mu^-)^a (\text{ad } k_\nu^-)^b z_{22}^c \det z^b + \text{higher order terms in } \det z$$

and if the image of  $\tilde{p}$  under the homomorphism  $M(V_{\tau_1}) \rightarrow M(V_\tau)$  is zero, then

$$(3.10) \quad (n+1-a)(m+1-b) = 0.$$

Next, let  $\tau_2$  be the  $\mathfrak{k}$ -type defined by  $S_{\alpha_1} S_{\alpha_2} S_\beta(A+\rho) - \rho$ , and  $\tau_3$  the one defined by  $S_\gamma S_{\alpha_1} S_{\alpha_2} S_\beta(A+\rho) - \rho$ . Then, with  $m=x-y-1$ ,  $\tau_2$  has multiplicity  $m+1$  in  $M(V_\tau)$  and multiplicity  $m+2$  in  $M(V_{\tau_1})$ , whereas  $\tau_3$  has multiplicity  $m+1$  in  $M(V_\tau)$ , multiplicity 1 in  $M(V_{\tau_2})$ , and multiplicity  $m+1$  in  $M(V_{\tau_1})$ . (The multiplicity  $m+2$  of  $\tau_2$  in  $M(V_{\tau_1})$  is a reflexion of the fact that the homomorphism  $M(V_{\tau_2}) \rightarrow M(V_{\tau_1})$  maps into the kernel of  $M(V_{\tau_1}) \rightarrow M(V_\tau)$ , as follows from Theorem 7). It is then straightforward to see that  $S_{\alpha_1} S_{\alpha_2} S_\beta$  does not define a quotient. Moreover, unless  $x=y+1$  or  $y=0$ , neither does  $S_\gamma S_{\alpha_1} S_{\alpha_2} S_\beta$ , which again follows by counting multiplicities and by observing (via e.g. Lemma 3) that all representation of type  $\tau_3$  in  $M(V_{\tau_1})$  are mapped into non-zero elements of  $M(V_\tau)$ . In case  $x=y+1$  or  $y=0$  we have a homomorphism and it is easy to see that there can be no quotient besides the one thus obtained. Observe that in these last cases a very interesting phenomenon occurs: *The images of the homomorphisms do not overlap.* In fact; due to irreducibility the intersection must be trivial.

In the following,  $H$  denotes homomorphism and  $Q$  denotes quotient (i.e. not defined by a homomorphism). We can then state

**Theorem 8.** *The subspace structure of the generalized Verma module of highest weight  $(z, x, y) - \rho$  is given as follows. There are no multiplicities.*

i)  $x > y+1$ :

$$(3.11) \quad \begin{aligned} z \leq -x & & : \text{None} \\ -x < z \leq -y-1 & : S_\gamma(H) \\ -y < z \leq y & : S_{\alpha_2}(H) \end{aligned}$$

$$\begin{aligned}
 y+1 \leq z \leq x-1 &: S_{\alpha_1}(H), S_{\alpha_2}(H), S_{\alpha_1}S_{\alpha_2}(H) \\
 z = x &: \text{None} \\
 x+1 \leq z &: S_{\beta}(H).
 \end{aligned}$$

ii)  $x=y+1$ :

$$\begin{aligned}
 z \leq 1-x &: \text{None} \\
 1-x < z \leq x-1 &: S_{\alpha_2}(H) \\
 z = x &: S_{\alpha_1}S_{\alpha_2}(H) \\
 x+1 \leq z &: S_{\beta}(H), S_{\alpha_1}S_{\alpha_2}(H).
 \end{aligned}$$

iii)  $y=0$ :

$$\begin{aligned}
 z \leq -x &: \text{None} \\
 -x < z \leq -1 &: S_{\gamma}(H) \\
 z = 1 &: S_{\alpha_1}(H), S_{\alpha_2}(H), S_{\alpha_1}S_{\alpha_2}(H), S_{\gamma}S_{\alpha_1}S_{\alpha_2}(Q) \quad (x \geq 2) \\
 1 < z \leq x-1 &: S_{\alpha_1}(H), S_{\alpha_2}(H), S_{\alpha_1}S_{\alpha_2}(H) \\
 z = x &: S_{\alpha_1}S_{\alpha_2}(H) \\
 x+1 \leq z &: S_{\beta}(H), S_{\alpha_1}S_{\alpha_2}(H).
 \end{aligned}$$

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