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Cyclic Cohomology of $A[X]$ and $A[X, X^{-1}]$

By

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Abstract

Cyclic cohomology of A coefficient polynomial algebra $A[X]$ and Laurent polynomial algebra $A[X,X^{-1}]$ is studied by making use of a deformation of projective resolution and the machinery of A. Connes. The spectral sequence E_n^*, $n\geq 1$ behaves the same as algebraic K-theory.

§0. Introduction

In this paper, we give a formula of cyclic cohomology of $A[X]$ and $A[X, X^{-1}]$ in terms of the cyclic cohomology of A for an unital algebra A over $\mathbb{C}$. There is a theorem of algebraic K-theory that for regular and noether ring A,

(0.1) $$K_i(A[X])\simeq K_i(A)\,,$$

(0.2) $$K_i(A[X, X^{-1}])\simeq K_i(A)\oplus K_{i-1}(A)\,,$$

(see Quillen [5], Grayson [3]). We prove, in this paper, analogous formulae hold. Cyclic theory is supposed to be an algebraic homology or cohomology theory for algebra A and the right object must be the limit of the spectral sequence associated with the exact couple of A. Connes. We see $A[X]=\mathbb{C}[X]\otimes A$ and $A[X, X^{-1}]=\mathbb{C}[X, X^{-1}]\otimes A$. We can easily see that $H^{\mathrm{even}}(\mathbb{C}[X])\simeq\mathbb{C}$, $H^{\mathrm{odd}}(\mathbb{C}[X])\simeq 0$, and $H^{\mathrm{even}}(\mathbb{C}[X, X^{-1}])\simeq H^{\mathrm{odd}}(\mathbb{C}[X, X^{-1}])\simeq\mathbb{C}$. (More precisely, $E(^1_0[\mathbb{C}X])\simeq\mathbb{C}$, $E_1^n(\mathbb{C}[X])\simeq 0$, $n\geq 1$ and $E_1^n(\mathbb{C}[X, X^{-1}])\simeq\mathbb{C}$ for $n=0$, 1 and $E_1^n(\mathbb{C}[X, X^{-1}])\simeq 0$ for $n\geq 2$.) We also have the following natural maps:

(0.3) $$H_\lambda^n(A)\to H_\lambda^n(A[X])\,,\quad n\geq 0\,,$$

(0.4) $$H_\lambda^n(A)\to H_\lambda^n(A[X, X^{-1}])\,,\quad n\geq 0\,,$$

(0 5) $$H_\lambda^{n-1}(A)\to H_\lambda^n(A[X, X^{-1}])\,,\quad n\geq 1\,,$$

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where the maps (0.3) and (0.4) are given by the shuffle product with the 0-trace on $\mathbb{C}[X]$ and $\mathbb{C}[X, X^{-1}]$, respectively (see [1], [2]). In this case, the product is taken with 0-trace so that the shuffle product is the same as the cup product of A. Connes. The 0-trace on $\mathbb{C}[X]$ is the canonical one given by the evaluation of the constant term. The 0-trace on $\mathbb{C}[X, X^{-1}]$ is just the same. The maps (0.5) are given by the shuffle product (which is equal to $1/n$ times the cup product) with the 1-trace on $\mathbb{C}[X, X^{-1}]$ given by

$$\tau(a^0, a^1) = \sum_{l_0+l_1=0} l_1 a^0_{l_0} a^1_{l_1}, \tag{0.6}$$

where $a^i = \sum_{l_i \in \mathbb{Z}} a^i_{l_i} X^{l_i} \in \mathbb{C}[X, X^{-1}]$, $i=0,1$. By using (0.4) and (0.5), we have the natural maps

$$H^n_\lambda(A) \oplus H^{n-1}_\lambda(A) \to H^n_\lambda(A[X, X^{-1}]), \quad n \geq 1. \tag{0.7}$$

The same maps also induce maps of Hochschild cohomology. Our result is the following:

Theorem. *The maps induce isomorphisms of spectral sequences and*
(1) $E^*_n(A) \xrightarrow{\sim} E^*_n(A[X])$, $n \geq 1$,
(2) $E^*_n(A) \oplus E^{*-1}_n(A) \xrightarrow{\sim} E^*_n(A[X, X^{-1}])$, $n \geq 1$.

As an immediate corollary of this theorem, we can compute cyclic cohomology of rings like $\mathbb{C}[X_1, X_1^{-1}, \cdots, X_N, X_N^{-1}, Y_1, \cdots, Y_M]$, $N, M \in \mathbb{N}$.

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§1. Deformation of Projective Resolution

Let A be an unital algebra over $\mathbb{C}$ and let

$$0 \leftarrow A \overset{\varepsilon}{\leftarrow} B \overset{b_1}{\leftarrow} M_1 \overset{b_2}{\leftarrow} M_2 \leftarrow \cdots \tag{1.1}$$

be the canonical projective resolution with explicit homotopy maps $s: A \to B$ and $s_n: M_n \to M_{n+1}$, $n=0,1,2,\cdots$ (see [2]). We construct a projective resolution of $A[X, X^{-1}] \cong A \otimes \mathbb{C}[\mathbb{Z}]$. The case $A[X]$ will be discussed later. Actually the case $A[X] \cong A \otimes \mathbb{C}[\mathbb{N}]$ follows immediately from the discussion of $A[X, X^{-1}]$.

Let $V[X, \mathring{X}] = \mathbb{C}[X, X^{-1}] \otimes \mathbb{C}[\mathring{X}, \mathring{X}^{-1}]$ where $\mathbb{C}[\mathring{X}, \mathring{X}^{-1}] = \mathbb{C}[X, X^{-1}]^\circ$ is the opposite algebra of $\mathbb{C}[X,X^{-1}]$. We define $d\colon V[X,\mathring{X}]\to V[X,\mathring{X}]$ by the multiplication by$(X\otimes 1-1\otimes\mathring{X})$.

Definition 1.1. We put

$$0\leftarrow\tilde{A}\overset{\tilde{\varepsilon}}{\leftarrow}\tilde{B}\overset{\hat{b}_1}{\leftarrow}N_1\overset{\hat{b}_2}{\leftarrow}N_2\leftarrow\cdots \tag{1.2}$$

where $\tilde{A}=\mathbb{C}[X,X^{-1}]\otimes A$, $\tilde{B}=\tilde{A}\otimes\tilde{A}^0$ and

$$N_n=V[X,\mathring{X}]\otimes M_n\oplus V[X,\mathring{X}]\otimes M_{n-1}\,,\quad n\geq 1\,, \tag{1.3}$$

and the maps $\hat{b}_n\colon N_n\to N_{n-1}$ are defined by

$$\hat{b}_n(\omega,\tilde{\omega})=((1\otimes b_n)(\omega)+(-1)^n(d\otimes 1)(\tilde{\omega}),\,(1\otimes b_{n-1})(\tilde{\omega}))\,, \tag{1.4}$$

$\omega\in V[X,\mathring{X}]\otimes M_n$, $\tilde{\omega}\in V[X,\mathring{X}]\otimes M_{n-1}$ for $n\geq 2$ and $\hat{b}_1\colon N_1\to N_0\equiv\tilde{B}$ is defined by

$$\hat{b}_1(\omega,\tilde{\omega})=(1\otimes b_1)(\omega)-(d\otimes 1)(\tilde{\omega})\,, \tag{1.5}$$

$\omega\in V[X,\mathring{X}]\otimes M_1$, $\tilde{\omega}\in V[X,\mathring{X}]\otimes B=\tilde{B}$.

Lemma 1.2. *The complex* (1.2) *is a projective resolution of* $\tilde{A}=A[X,X^{-1}]$ *with a homotopy map* $\hat{s}_n\colon N_n\to N_{n+1}$ *given by*

$$\hat{S}_n(\omega,\tilde{\omega})=(((\mathcal{S}\circ E)\otimes s_n)(\omega),\,(-1)^{n+1}(\mathcal{S}_0\otimes 1)(\omega))\,, \tag{1.6}$$

$\omega\in V[X,\mathring{X}]\otimes M_n$, $\tilde{\omega}\in V[X,\mathring{X}]\otimes M_{n-1}$, *for* $n\geq 1$ *and*

$$\hat{S}_0(\omega)=(((\mathcal{S}\circ E)\otimes s_0)(\omega),\,-(\mathcal{S}_0\otimes 1)(\omega)) \tag{1.7}$$

$\omega\in V[X,\mathring{X}]\otimes M_0=V[X,\mathring{X}]\otimes B=\tilde{B}$, *where* $E\colon V[X,\mathring{X}]\to\mathbb{C}[X,X^{-1}]$, $\mathcal{S}\colon\mathbb{C}[X,X^{-1}]\to V[X,\mathring{X}]$ *and* $\mathcal{S}_0\colon V[X,\mathring{X}]\to V[X,\mathring{X}]$ *are given by*

$$E(\sum_{m,n}a_{m,n}X^m\otimes\mathring{X}^n)=\sum_{m,n}a_{m,n}X^{m+n},\quad a_{m,n}\in\mathbb{C} \tag{1.8}$$

$$\mathcal{S}(\sum_n a_nX^n)=\sum_n a_n(1\otimes\mathring{X}^n),\quad a_n\in\mathbb{C}\,, \tag{1.9}$$

$$\mathcal{S}_0(X^m\otimes\mathring{X}^n)=\Phi(X,\mathring{X},m)\,(1\otimes\mathring{X}^n)\,, \tag{1.10}$$

respectively, where

$$\Phi(X,\mathring{X},m)=\text{Sum}\,(l\colon m)X^l\otimes\mathring{X}^{m-1-l}\,, \tag{1.11}$$

and the notation Sum $(l\colon M)a_l$ *is defined by* $\sum_{l=0}^{M-1}a_l$ *for* $M>0$, *zero for* $M=0$, *and* $-(a_{-1}+\cdots+a_M)=-\sum_{l=-1}^{M}a_l$ *for* $M<0$.

Proof. By (1.3), N_n, $n\geq 0$, are algebraically free modules over $\tilde{B}=V[X,\mathring{X}]\otimes B$. By (1.4) and $\tilde{\varepsilon}=E\otimes\varepsilon$, $\hat{b}_{n-1}\hat{b}_n=0$, $n\geq 2$ and $\tilde{\varepsilon}\hat{b}_1=0$. So, we show that the maps $\hat{S}_n$ are homotopies of (1.2). For $n\geq 1$,

$$(1.12)\qquad \hat{b}_{n+1}\cdot\hat{S}_n(\omega,\tilde{\omega})$$
$$= \hat{b}_{n+1}((\mathcal{S}E\otimes S_n)(\omega), (-1)^{n+1}(\mathcal{S}_0\otimes 1)(\omega))$$
$$= ((\mathcal{S}E\otimes b_{n+1}S_n)(\omega)+(d\mathcal{S}_0\otimes 1)(\omega), (-1)^{n+1}(\mathcal{S}_0\otimes b_n)(\omega))$$

$$(1.13)\qquad \hat{S}_{n-1}\circ\hat{b}_n(\omega,\tilde{\omega})$$
$$= \hat{S}_{n-1}((1\otimes b_n)(\omega)+(-1)^n(d\otimes 1)(\tilde{\omega}), (1\otimes b_{n-1})(\tilde{\omega}))$$
$$= ((\mathcal{S}E\otimes S_{n-1}b_n)(\omega)+(-1)^n(\mathcal{S}Ed\otimes S_{n-1})(\omega),$$
$$(-1)^n(\mathcal{S}_0\otimes b_n)(\omega)+(\mathcal{S}_0 d\otimes 1)(\tilde{\omega}))$$
$$= ((\mathcal{S}E\otimes S_{n-1}b_n)(\omega), (-1)^n(\mathcal{S}_0\otimes b_n)(\omega)+\tilde{\omega}),$$

$(\omega,\tilde{\omega})\in N_n$, where we use $Ed=0$ and $\mathcal{S}_0 d=1$. Hence, by using $\mathcal{S}E+d\mathcal{S}_0=1$, $b_{n+1}S_n+S_{n-1}b_n=1$, we obtain $\hat{b}_{n+1}\hat{S}_n+\hat{S}_{n-1}\hat{b}_n=1$. For $n=0$,

$$(1.14)\qquad \hat{b}_1\circ\hat{S}_0(\omega) = \hat{b}_1((\mathcal{S}E\otimes S_0)(\omega), -(\mathcal{S}_0\otimes 1)(\omega))$$
$$= (\mathcal{S}E\otimes b_1S_0)(\omega)+(d\mathcal{S}_0\otimes 1)(\omega)$$

$$(1.15)\qquad \tilde{S}\circ\tilde{\varepsilon}(\omega) = \tilde{S}((E\otimes\varepsilon)(\omega))$$
$$= (\mathcal{S}E\otimes S\varepsilon)(\omega),$$

$\omega\in\tilde{B}$. Hence we obtain $\hat{b}_1\circ\hat{S}_0+\tilde{S}\circ\tilde{\varepsilon}=1$ by using $b_1S_0+S\varepsilon=1$ and $\mathcal{S}E+d\mathcal{S}_0=1$.

Q.E.D.

We next construct quasi-isomorphisms between the canonical projective resolution and the projective resolution (1.2). We see that A is a subalgebra of $\tilde{A}=\boldsymbol{C}[X,X^{-1}]\otimes A$, naturally. Let

$$(1.16)\qquad 0\leftarrow \tilde{A}\overset{\tilde{\varepsilon}}{\leftarrow}\tilde{B}\overset{\tilde{b}_1}{\leftarrow}\tilde{M}_1\overset{\tilde{b}_2}{\leftarrow}\tilde{M}_2\leftarrow\cdots$$

be the canonical projective resolution of $\tilde{A}=\boldsymbol{C}[X,X^{-1}]\otimes A$.

Definition 1.3. Let h_n: $V[X,\overset{\circ}{X}]\otimes M_n\to\tilde{M}_n$, $n\geq 1$ be the $\tilde{B}$-linear maps determined by

$$(1.17)\quad h_n(1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^n) = 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^n,\ a^j\in A,\ j=1,\cdots,n,$$

and let $\tilde{h}_n$: $V[X,\overset{\circ}{X}]\otimes M_{n-1}\to\tilde{M}_n$, $n\geq 1$ be the $\tilde{B}$-linear maps determined by

$$(1.18)\qquad \tilde{h}_n(1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{n-1})$$
$$=\sum_{l=0}^{n-1}(-1)^l 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1},$$
$$a^j\in A,\ j=1,\cdots,n-1.$$

We now define H_n: $N_n\to\tilde{M}_n$, $n\geq 1$ by

$$(1.19)\qquad H_n(\omega,\tilde{\omega}) = h_n(\omega)+(-1)^n\tilde{h}_n(\tilde{\omega}),\ (\omega,\tilde{\omega})\in N_n.$$

Lemma 1.4. *The following diagrams commute*:

$$
\begin{array}{ccc}
\tilde{M}_{n-1} & \xleftarrow{\ \tilde{b}_n\ } & \tilde{M}_n \\
\uparrow H_{n-1} & & \uparrow H_n \\
N_{n-1} & \xleftarrow{\ \hat{b}_n\ } & N_n
\end{array}, \quad n\geq 1 . \tag{1.20}
$$

Proof. First, we compute for $(\omega,0)\in N_n$. Let $\omega=1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^n$. Then by (1.17) and (1.18),

$$H_n(\omega, 0) = 1_V\otimes 1_B\otimes a^1\otimes\cdots\otimes a^n , \tag{1.21}$$

where we use $1_{\tilde{B}}=1_V\otimes 1_B$, $V=V[X,\mathring{X}]$. It follows

$$
\begin{aligned}
\tilde{b}_n\circ H_n(\omega, 0) &= 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^n \\
&\quad+\sum_{j=1}^{n-1}(-1)^j 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^n \\
&\quad+(-1)^n 1_V\otimes(1\otimes\mathring{a}^n)\otimes a^1\otimes\cdots\otimes a^{n-1} .
\end{aligned} \tag{1.22}
$$

On the other hand,

$$
\begin{aligned}
(1\otimes b_n)(\omega) &= 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^n \\
&\quad+\sum_{j=1}^{n-1}(-1)^j 1_V\otimes 1_B\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^n \\
&\quad+(-1)^n 1_V\otimes(1\otimes\mathring{a}^n)\otimes a^1\otimes\cdots\otimes a^{n-1} .
\end{aligned} \tag{1.23}
$$

Hence, the diagram (1.20) commutes for $(\omega,0)\in N_n$.

Next, we compute for $(0,\tilde{\omega})\in N_n$. Let $\tilde{\omega}=1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{n-1}$. Then by (1.18) and (1.19),

$$
\begin{aligned}
&H_n(0, \tilde{\omega}) \\
&\quad=(-1)^n\sum_{l=0}^{n-1}(-1)^l 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} .
\end{aligned} \tag{1.24}
$$

Suppose $1\leq l\leq n-2$. Then

$$
\begin{aligned}
&\tilde{b}_n(1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1}) \\
&\quad= 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
&\qquad+\sum_{j=1}^{l-1}(-1)^j 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
&\qquad+(-1)^l 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{l-1}\otimes a^l X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
&\qquad+(-1)^{l+1} 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X a^{l+1}\otimes a^{l+2}\otimes\cdots\otimes a^{n-1} \\
&\qquad+\sum_{j=l+1}^{n-2}(-1)^{j+1} 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1} \\
&\qquad+(-1)^n 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-2} ,
\end{aligned} \tag{1.25}
$$

and for $l=0$, $l=n-1$,

$$(1.26)\quad \tilde{b}_n(1_{\tilde{B}}\otimes X\otimes a^1\otimes\cdots\otimes a^{n-1})$$
$$= (X\otimes 1)\otimes 1_B\otimes a^1\otimes\cdots\otimes a^{n-1} - 1_{\tilde{B}}\otimes Xa^1\otimes a^2\otimes\cdots\otimes a^{n-1}$$
$$+\sum_{j=1}^{n-2}(-1)^{j-1}1_{\tilde{B}}\otimes X\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1}$$
$$+(-1)^n 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes X\otimes a^1\otimes\cdots\otimes a^{n-2},$$

$$(1.27)\quad \tilde{b}_n(1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{n-1}\otimes X)$$
$$= 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^{n-1}\otimes X$$
$$+\sum_{j=1}^{n-2}(-1)^j 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1}\otimes X$$
$$+(-1)^{n-1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{n-2}\otimes a^{n-1}X$$
$$+(-1)^n(1\otimes\mathring{X})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^{n-1}.$$

By putting (1.25), (1.26), (1.27) into (1.24), we obtain

$$(1.28)\quad \tilde{b}_n\circ H_n(0,\tilde{\omega})$$
$$= (-1)^n(X\otimes 1)\otimes 1_B\otimes a^1\otimes\cdots\otimes a^{n-1}$$
$$-(-1)^n 1_{\tilde{B}}\otimes Xa^1\otimes a^2\otimes\cdots\otimes a^{n-1}$$
$$+(-1)^n\sum_{j=1}^{n-2}(-1)^{j-1}1_{\tilde{B}}\otimes X\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1}$$
$$+1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes X\otimes a^1\otimes\cdots\otimes a^{n-2}$$
$$+(-1)^n\sum_{l=1}^{n-2}(-1)^l 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1}$$
$$+(-1)^n\sum_{l=1}^{n-2}\sum_{j=1}^{l-1}(-1)^{l+j}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^l \otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1}$$
$$+(-1)^n\sum_{l=1}^{n-2}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{l-1}\otimes a^l X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1}$$
$$-(-1)^n\sum_{l=1}^{n-2}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes Xa^{l+1}\otimes a^{l+2}\otimes\cdots\otimes a^{n-1}$$
$$+(-1)^n\sum_{l=1}^{n-2}\sum_{j=l+1}^{n-2}(-1)^{l+j+1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1} \otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1}$$
$$+\sum_{l=1}^{n-2}(-1)^l 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-2}$$
$$-1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^{n-1}\otimes X$$
$$+\sum_{j=1}^{n-2}(-1)^{j+1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^{n-1}\otimes X$$
$$+(-1)^n 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^{n-2}\otimes a^{n-1}X$$
$$+(-1)^{n+1}(1\otimes\mathring{X})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^{n-1}.$$

It is seen that the second, seventh, eighth and thirteenth terms cancel and then (1.28) is equal to:

$$
\begin{aligned}
(1.29)\quad & (-1)^n(X\otimes 1-1\otimes \mathring{X})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^{n-1} \\
& +(-1)^n\sum_{l=1}^{n-1}(-1)^l 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
& +(-1)^n\sum_{l=1}^{n-1}\sum_{j=1}^{l-1}(-1)^{l+j}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^ja^{j+1}\otimes\cdots\otimes a^l \\
& \qquad\qquad \otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
& +(-1)^n\sum_{l=0}^{n-2}\sum_{j=l+1}^{n-2}(-1)^{l+j+1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots \\
& \qquad\qquad \otimes a^ja^{j+1}\otimes\cdots\otimes a^{n-1} \\
& +\sum_{l=0}^{n-2}(-1)^l 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-2}\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(1.30)\quad (1\otimes b_{n-1})(\tilde{\omega}) = {} & 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^{n-1} \\
& +\sum_{j=1}^{n-2}(-1)^j 1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^ja^{j+1}\otimes\cdots\otimes a^{n-1} \\
& +(-1)^{n-1}1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^{n-2}\,.
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
(1.31)\quad & \tilde{h}_{n-1}\circ(1\otimes b_{n-1})(\tilde{\omega}) \\
& =\sum_{l=0}^{n-2}(-1)^l 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^{l+1}\otimes X\otimes a^{l+2}\otimes\cdots\otimes a^{n-1} \\
& \quad +\sum_{j=1}^{n-2}\sum_{l=0}^{j-1}(-1)^{j+l}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots \\
& \qquad\qquad \otimes a^ja^{j+1}\otimes\cdots\otimes a^{n-1} \\
& \quad +\sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1}(-1)^{j+l+1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^ja^{j+1}\otimes\cdots\otimes a^l\otimes X \\
& \qquad\qquad \otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
& \quad +(-1)^{n-1}\sum_{l=0}^{n-2}(-1)^l 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^l\otimes X \\
& \qquad\qquad \otimes a^{l+1}\otimes\cdots\otimes a^{n-2}\,.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
(1.32)\quad & (-1)^{n-1}\tilde{h}_{n-1}\circ(1\otimes b_{n-1})(\tilde{\omega}) \\
& =(-1)^n\sum_{l=1}^{n-1}(-1)^l 1_V\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
& \quad +(-1)^n\sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1}(-1)^{j+l}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^ja^{j+1}\otimes\cdots \\
& \qquad\qquad \otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-1} \\
& \quad +(-1)^n\sum_{j=1}^{n-2}\sum_{l=0}^{j-1}(-1)^{j+l+1}1_{\tilde{B}}\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots \\
& \qquad\qquad \otimes a^ja^{j+1}\otimes\cdots\otimes a^{n-1}
\end{aligned}
$$

$$+\sum_{l=0}^{n-2}(-1)^l 1_V\otimes(1\otimes\mathring{a}^{n-1})\otimes a^1\otimes\cdots\otimes a^l\otimes X\otimes a^{l+1}\otimes\cdots\otimes a^{n-2}.$$

By viewing $\sum_{j=1}^{n-2}\sum_{l=j+1}^{n-1}=\sum_{l=1}^{n-1}\sum_{j=1}^{l-1}$ and $\sum_{j=1}^{n-2}\sum_{l=0}^{j-1}=\sum_{l=0}^{n-2}\sum_{j=l+1}^{n-2}$ in (1.32), and by using (1.28), (1.29), we obtain

$$\tilde{b}_n\circ H_n(0,\tilde{\omega})=(-1)^n(d\otimes 1)(\tilde{\omega})+(-1)^{n-1}\tilde{h}_{n-1}\circ(1\otimes b_{n-1})(\tilde{\omega})\,. \tag{1.33}$$

Hence, we obtain the commutativity of the diagram (1.20) due to (1.4) and (1.19). Q.E.D.

Definition 1.5. Let $k_n\colon \tilde{M}_n\to V[X,\mathring{X}]\otimes M_n$, $n\geq 1$ be the $\tilde{B}$-linear maps determined by

$$\begin{aligned}&k_n(1_{\tilde{B}}\otimes a^1X^{l_1}\otimes\cdots\otimes a^nX^{l_n})\\ &\quad=(1\otimes\mathring{X}^{l_1+\cdots+l_n})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^n\,,\\ &\qquad a^j\in A,\ l_j\in\mathbf{Z},\ j=1,\cdots,n\,,\end{aligned}\tag{1.34}$$

and let $\tilde{k}_n\colon \tilde{M}_n\to V[X,\mathring{X}]\otimes M_{n-1}$, $n\geq 1$ be the $\tilde{B}$-linear maps determined by

$$\begin{aligned}&\tilde{k}_n(1_{\tilde{B}}\otimes a^1X^{l_1}\otimes\cdots\otimes a^nX^{l_n})\\ &\quad=\Phi(X,\mathring{X},l_1)(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^n\,,\\ &\qquad a^j\in A,\ l_j\in\mathbf{Z},\ j=1,\cdots,n\,.\end{aligned}\tag{1.35}$$

We now define $\tilde{K}_n\colon \tilde{M}_n\to N_n$, $n\geq 1$ by

$$\tilde{K}_n(\omega)=(k_n(\omega),(-1)^n\tilde{k}_n(\omega)),\ \omega\in\tilde{M}_n\,. \tag{1.36}$$

Lemma 1.6. (1) $k_{n-1}\circ\tilde{b}_n=(1\otimes b_n)\circ k_n+(d\otimes 1)\circ\tilde{k}$, $n\geq 2$,

(2) $\tilde{b}_1=(1\otimes b_1)\circ k_1+(d\otimes 1)\circ\tilde{k}_1$,

(3) $\tilde{k}_{n-1}\circ\tilde{b}_n=-(1\otimes b_{n-1})\circ\tilde{k}_n$, $n\geq 2$.

Proof. (1) Let $\omega=1_{\tilde{B}}\otimes a^1X^{l_1}\otimes\cdots\otimes a^nX^{l_n}\in\tilde{M}_n$, $a^j\in A$, $l_j\in\mathbf{Z}$, $j=1,\cdots,n$. Then

$$\begin{aligned}\tilde{b}_n(\omega)&=(X^{l_1}\otimes 1)\otimes(a^1\otimes 1)\otimes a^2X^{l_2}\otimes\cdots\otimes a^nX^{l_n}\\ &\quad+\sum_{j=1}^{n-1}(-1)^j 1_{\tilde{B}}\otimes a^1X^{l_1}\otimes\cdots\otimes a^jX^{l_j}a^{j+1}X^{l_{j+1}}\otimes\cdots\otimes a^nX^{l_n}\\ &\quad+(-1)^n(1\otimes\mathring{X}^{l_n})\otimes(1\otimes\mathring{a}^n)\otimes a^1X^{l_1}\otimes\cdots\otimes a^{n-1}X^{l_{n-1}}.\end{aligned}\tag{1.37}$$

So, by using (1.34),

$$\begin{aligned}k_{n-1}\circ\tilde{b}_n(\omega)&=(X^{l_1}\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^n\\ &\quad+\sum_{j=1}^{n-1}(-1)^j(1\otimes\mathring{X}^{l_1+\cdots+l_n})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^ja^{j+1}\otimes\cdots\otimes a^n\\ &\quad+(-1)^n(1\otimes\mathring{X}^{l_1+\cdots+l_n})\otimes(1\otimes\mathring{a}^n)\otimes a^1\otimes\cdots\otimes a^{n-1}\,.\end{aligned}\tag{1.38}$$

On the other hand, by (1.34),

$$
\begin{aligned}
(1.39)\quad & (1\otimes b_n)\circ k_n(\omega) \\
&= (1\otimes \mathring{X}^{l_1+\cdots+l_n})\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots\otimes a^n \\
&\quad + \sum_{j=1}^{n-1}(-1)^j(1\otimes\mathring{X}^{l_1+\cdots+l_n})\otimes 1_B\otimes a^1\otimes\cdots\otimes a^j a^{j+1}\otimes\cdots\otimes a^n \\
&\quad +(-1)^n(1\otimes\mathring{X}^{l_1+\cdots+l_n})\otimes(1\otimes\mathring{a}^n)\otimes a^1\otimes\cdots\otimes a^{n-1}\,.
\end{aligned}
$$

Hence, by (1.35) and $d(\Phi(X,\mathring{X},m))=X^m\otimes 1-1\otimes\mathring{X}^m$, we obtain the assertion.

(2) Let $\omega=1_{\tilde{B}}\otimes aX^n\in\tilde{M}_1$, $a\in A$, $n\in\mathbb{Z}$. Then

$$
\begin{aligned}
(1.40)\quad & (1\otimes b_1)\circ k_1(\omega)+(d\otimes 1)\circ\tilde{k}_1(\omega) \\
&= (1\otimes b_1)\,((1\otimes\mathring{X}^n)\otimes 1_B\otimes a)+(X^n\otimes 1-1\otimes\mathring{X}^n)\otimes(a\otimes 1) \\
&= (1\otimes\mathring{X}^n)\otimes(a\otimes 1-1\otimes\mathring{a})+(X^n\otimes 1-1\otimes\mathring{X}^n)\otimes(a\otimes 1) \\
&= (X^n\otimes 1)\otimes(a\otimes 1)-(1\otimes\mathring{X}^n)\otimes(1\otimes\mathring{a}) \\
&= \tilde{b}_1(\omega)\,.
\end{aligned}
$$

(3) Let $\omega=1_B\otimes a^1X^{l_1}\otimes\cdots\otimes a^nX^{l_n}\in\tilde{M}_n$, $a^j\in A$, $l_j\in\mathbb{Z}$, $j=1,\cdots,n$. Then by (1.35) and (1.37)

$$
\begin{aligned}
(1.41)\quad & \tilde{k}_{n-1}\circ\tilde{b}_n(\omega) \\
&= \Phi(X,\mathring{X},l_2)\,(X^{l_1}\otimes\mathring{X}^{l_3+\cdots+l_n})\otimes(a^1a^2\otimes 1)\otimes a^3\otimes\cdots\otimes a^n \\
&\quad -\Phi(X,\mathring{X},l_1+l_2)\,(1\otimes\mathring{X}^{l_3+\cdots+l_n})\otimes(a^1a^2\otimes 1)\otimes a^3\otimes\cdots\otimes a^n \\
&\quad +\sum_{j=2}^{n-1}(-1)^j\Phi(X,\mathring{X},l_1)\,(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots \\
&\qquad\qquad \otimes a^ja^{j+1}\otimes\cdots\otimes a^n \\
&\quad +(-1)^n\Phi(X,\mathring{X},l_1)\,(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes\mathring{a}^n)\otimes a^2\otimes\cdots\otimes a^{n-1} \\
&= -\Phi(X,\mathring{X},l_1)\,(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1a^2\otimes 1)\otimes a^3\otimes\cdots\otimes a^n \\
&\quad -\sum_{j=1}^{n-2}(-1)^j\Phi(X,\mathring{X},l_1)\,(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes 1)\otimes a^2\otimes\cdots \\
&\qquad\qquad \otimes a^{j+1}a^{j+2}\otimes\cdots\otimes a^n \\
&\quad -(-1)^{n-1}\Phi(X,\mathring{X},l_1)\,(1\otimes\mathring{X}^{l_2+\cdots+l_n})\otimes(a^1\otimes\mathring{a}^n)\otimes a^2\otimes\cdots\otimes a^{n-1}\,,
\end{aligned}
$$

where we use $\Phi(X,\mathring{X},l_1+l_2)=(X^{l_1}\otimes 1)\,\Phi(X,\mathring{X},l_2)+(1\otimes\mathring{X}^{l_2})\,\Phi(X,\mathring{X},l_1)$ for the second equality. Hence, in view of (1.35), we obtain the formula. Q.E.D.

Corollary 1.7. *The following diagrams commute*:

$$
(1.42)\qquad
\begin{array}{ccc}
\tilde{M}_{n-1} & \xleftarrow{\ \tilde{b}_n\ } & \tilde{M}_n \\
\downarrow{\scriptstyle \tilde{K}_{n-1}} & & \downarrow{\scriptstyle \tilde{K}_n} \\
N_{n-1} & \xleftarrow{\ \hat{b}_n\ } & N_n
\end{array}
,\quad n\geq 1\,.
$$

We want the fact that the resolution (1.2) is a retraction of the canonical projective resolution. We need a little correction to construct a retraction K_n of H_n, $n\geq 1$.

Definition 1.8. Let $C_n: N_n \to N_n$, $n\geq 1$, be the $\tilde{B}$-linear maps determined by

$$C_n(\omega, \tilde{\omega}) = (\omega - \Psi_n(\tilde{\omega}), \tilde{\omega}), \quad (\omega, \tilde{\omega}) \in N_n, \tag{1.43}$$

$$\Psi_n(\tilde{\omega}) = (-1)^n \sum_{l=0}^{n-1} (-1)^l (1\otimes \mathring{X}) \otimes 1_B \otimes b^1 \otimes \cdots \otimes b^l \otimes 1 \otimes b^{l+1} \otimes \cdots \otimes b^{n-1} \tag{1.44}$$

where $\tilde{\omega} = 1_{\tilde{B}} \otimes b^1 \otimes \cdots \otimes b^{n-1} \in V[X, \mathring{X}] \otimes M_{n-1}$.

Lemma 1.9. $(1\otimes b_n)\Psi_n = \Psi_{n-1}(1\otimes b_{n-1})$, $n\geq 2$.

Proof. Both sides don't include any operations involving $\mathring{X}$, so we omit $1\otimes \mathring{X}$ in (1.44). Then the proof is just the same as that of the last half of Lemma 1.4 replacing X by 1. Q.E.D.

Corollary 1.10. *The following diagrams commute:*

$$\begin{array}{ccc} N_{n-1} & \xleftarrow{\hat{b}_n} & N_n \\ \downarrow C_{n-1} & & \downarrow C_n \\ N_{n-1} & \xleftarrow{\hat{b}_n} & N_n \end{array}, \quad n\geq 1. \tag{1.45}$$

Proof. The case $n>1$ is by Lemma 1.9 and by the definitions of C_n and $\hat{b}_n$. The case $n=1$ follows from $(1\otimes b_1)\Psi_1=0$. Q.E.D.

Proposition 1.11. *We put* $K_n = C_n \tilde{K}_n: \tilde{M}_n \to N_n$, $n\geq 1$.

(1) *The following diagrams commute:*

$$\begin{array}{ccc} \tilde{M}_{n-1} & \xleftarrow{\tilde{b}_n} & \tilde{M}_n \\ \downarrow K_{n-1} & & \downarrow K_n \\ N_{n-1} & \xleftarrow{\hat{b}_n} & N_n \end{array}, \quad n\geq 1, \tag{1.46}$$

(2) $K_n \circ H_n = 1$ *on* N_n, $n\geq 1$.

Proof. (1) follows from Corollaries 1.7 and 1.10.

(2) Let $\omega = 1_{\tilde{B}} \otimes a^1 \otimes \cdots \otimes a^n$, $\tilde{\omega} = 1_{\tilde{B}} \otimes b^1 \otimes \cdots \otimes b^{n-1}$, $a^1, \cdots, a^n, b^1, \cdots, b^{n-1} \in A$. Then

$$\begin{aligned} H_n(\omega, \tilde{\omega}) = {} & 1_{\tilde{B}} \otimes a^1 \otimes \cdots \otimes a^n \\ & + (-1)^n \sum_{l=0}^{n-1} (-1)^l 1_{\tilde{B}} \otimes b^1 \otimes \cdots \otimes b^l \otimes X \otimes b^{l+1} \otimes \cdots \otimes b^{n-1}. \end{aligned} \tag{1.47}$$

Hence,

$$\tilde{K}_n \circ H_n(\omega, \tilde{\omega}) = (\omega + \Psi_n(\tilde{\omega}), \omega). \tag{1.48}$$

So, we obtain $K_n \circ H_n = 1$. Q.E.D.

Remark 1.12. By Lemma 1.4, and Proposition 1.11 together with the explicit homotopy $\tilde{S}_j$, $j \geq 1$ of the canonical projective resolution, we also obtain homotopy maps of our resolution (1.2). However, it doesn't coincide with the homotopy maps (1.6) (homotopy map is not unique).

§2. Splitting of E_1-terms

By making use of the quasi-isomorphisms H_n and K_n, $n \geq 1$, we compute $E_1^*(\tilde{A})$, in this section. We identify $\tilde{A}^*$ with $\prod_{\mathbf{Z}} A^*$ (algebraic dual), where $\tilde{A} = \mathbf{C}[X, X^{-1}] \otimes A$. By using the similar identification, also using the quasi-isomorphisms H_n and K_n, $n \geq 1$, we obtain the following diagrams:

(2.1)
$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{C}}(\tilde{A}_n, \mathbf{C}) & \xrightarrow{\tilde{b}^*_{n+1}} & \mathrm{Hom}_{\mathbf{C}}(\tilde{A}_{n+1}, \mathbf{C}) \\
\wr\| & & \wr\| \\
\mathrm{Hom}_{\tilde{B}}(\tilde{M}_n, \tilde{A}^*) & \xrightarrow{b^*_{n+1}} & \mathrm{Hom}_{\tilde{B}}(\tilde{M}_{n+1}, \tilde{A}^*) \\
K_n^* \uparrow \downarrow H_n^* & & K_{n+1}^* \uparrow \downarrow H_{n+1}^* \\
\mathrm{Hom}_{\tilde{B}}(N_n, \tilde{A}^*) & \xrightarrow{\hat{b}^*_{n+1}} & \mathrm{Hom}_{\tilde{B}}(N_{n+1}, \tilde{A}^*) \\
\wr\| & & \wr\| \\
\prod_{\mathbf{Z}} [\mathrm{Hom}_B(M_n, A^*)] \oplus \prod_{\mathbf{Z}} [\mathrm{Hom}_B(M_{n-1}, A^*)] & & \prod_{\mathbf{Z}} [\mathrm{Hom}_B(M_{n+1}, A^*)] \oplus \prod_{\mathbf{Z}} [\mathrm{Hom}_B(M_n, A^*)] \\
\wr\| & & \wr\| \\
\prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_n, \mathbf{C})] \oplus \prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_{n-1}, \mathbf{C})] & \xrightarrow{\hat{b}^*_{n+1}} & \prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_{n+1}, \mathbf{C})] \oplus \prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_n, \mathbf{C})]
\end{array}$$

We identify an element $\phi \in \prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_n, \mathbf{C})] \oplus \prod_{\mathbf{Z}} [\mathrm{Hom}_{\mathbf{C}}(A_{n-1}, \mathbf{C})]$ by a pair of $\mathbf{Z}$-indexed sequences $\phi = (\phi^0, \phi^1)$, where $\phi^0_m \in \mathrm{Hom}(A_n, \mathbf{C})$, $\phi^1_m \in \mathrm{Hom}(A_{n-1}, \mathbf{C})$, $m \in \mathbf{Z}$. We will write de Rham differential in terms of this expression.

Lemma 2.1. (1) $H^0(\tilde{A}, \tilde{A}^*) \cong \prod_{\mathbf{Z}} [H^0(A, A^*)]$.

(2) $H^n(\tilde{A}, \tilde{A}^*) \cong (\prod_{\mathbf{Z}} [H^n(A, A^*)]) \oplus (\prod_{\mathbf{Z}} [H^{n-1}(A, A^*)])$, $n \geq 1$.

Proof. By (1.4), let $\hat{b}^*_{n+1}(\phi^0, \phi^1) = (\varphi^0, \varphi^1)$. Then, $\varphi^0_m = \phi^0_m \circ b_{n+1}$, $\varphi^1_m = \phi^1_m \circ b_n + (-1)^n (\phi^0_{m+1} - \phi^0_{m+1}) = \phi^1_m \circ b_n$. Hence we obtain the assertion. Q.E.D.

We now compute the de Rham differential of the exact couple obtained by the Hochschild and the cyclic cohomology of $\tilde{A}$, in terms of the splitting given by Lemma 2.1. For the purpose of reducing complicated formulas, we prepare some lemmas.

Lemma 2.2. *Let* $\phi\in Z^{n+1}(A,A^*)\subset\mathrm{Hom}_{\mathbf{C}}(A_{n+1},\mathbf{C})$ *and define* $\varphi\in\mathrm{Hom}_{\mathbf{C}}(A_n,\mathbf{C})$ *by*

$$\varphi(b^0, b^1, \cdots, b^n)=\sum_{l=0}^{n}(-1)^l\phi(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^n)\,, \tag{2.2}$$

$b^j\in A$, $0\leq j\leq n$. *Then* φ *is a Hochschild coboundary.*

Proof. For $0\leq l\leq n$, we have

$$\begin{aligned}
&b_{n+2}(b^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^n)\\
&=\sum_{j=0}^{l-1}(-1)^j(b^0, b^1, \cdots, b^jb^{j+1}, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^n)\\
&\quad+(-1)^l(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^n)\\
&\quad+(-1)^{l+1}(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^n)\\
&\quad+(-1)^{l+2}(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^n)\\
&\quad+\sum_{j=l+1}^{n-1}(-1)^j(b^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^jb^{j+1}, \cdots, b^n)\\
&\quad+(-1)^{n+2}(b^nb^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^{n-1})\,.
\end{aligned}\tag{2.3}$$

where $b_{n+2}\colon A_{n+2}\to A_{n+1}$ is the Hochschild boundary operator, see Connes [2] or Loday-Quillen [4]. Hence, by using $\phi\circ b_{n+2}=0$, we obtain

$$\begin{aligned}
&-\sum_{l=0}^{n}(-1)^l\phi(b^0, \cdots, b^l, 1, b^{l+1}, \cdots, b^n)\\
&=\sum_{l=1}^{n}\sum_{j=0}^{l-1}(-1)^j\phi(b^0, \cdots, b^jb^{j+1}, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^n)\\
&\quad+\sum_{l=0}^{n-2}\sum_{j=l+1}^{n-1}(-1)^j\phi(b^0, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^jb^{j+1}, \cdots, b^n)\\
&\quad+(-1)^n\sum_{l=0}^{n-1}\phi(b^nb^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^{n-1})\\
&=\sum_{l=0}^{n-1}\sum_{j=0}^{l}(-1)^j\phi(b^0, \cdots, b^jb^{j+1}, \cdots, b^{l+1}, 1, 1, b^{l+2}, \cdots, b^n)\\
&\quad+\sum_{l=0}^{n-2}\sum_{j=l+1}^{n-1}(-1)^j\phi(b^0, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^jb^{j+1}, \cdots, b^n)\\
&\quad+\sum_{l=0}^{n-1}(-1)^n\phi(b^nb^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^{n-1})\\
&=\sum_{l=0}^{n-2}\Big\{\sum_{j=0}^{l}(-1)^j\phi(b^0, \cdots, b^jb^{j+1}, \cdots, b^{l+1}, 1, 1, b^{l+2}, \cdots, b^n)\\
&\quad+\sum_{j=l+1}^{n-1}(-1)^j\phi(b^0, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^jb^{j+1}, \cdots, b^n)\\
&\qquad+(-1)^n\phi(b^nb^0, b^1, \cdots, b^l, 1, 1, b^{l+1}, \cdots, b^{n-1})\Big\}
\end{aligned}\tag{2.4}$$

$$+\sum_{j=0}^{n-1}(-1)^j\phi(b^0,\cdots,b^jb^{j+1},\cdots,b^n,1,1)$$
$$+(-1)^n\phi(b^nb^0,b^1,\cdots,b^{n-1},1,1)$$
$$=\sum_{l=0}^{n-1}\phi^{(l+1)}\circ b_n(b^0,\cdots,b^n)\,,$$

where we put

$$\phi^{(l)}(a^0,\cdots,a^n)=\phi(a^0,\cdots,a^{l-1},1,1,a^{l+1},\cdots,a^n)\,, \tag{2.5}$$

$a^j\in A$, $0\leq j\leq n$, and $1\leq l\leq n$. Q.E.D.

Corollary 2.3. *The map $C_n: N_n\to N_n$, $n\geq 1$ defined by Definition 1.8 induces an identity map of the Hochschild cohomology $\prod_{Z}[H^n(A,A^*)]\oplus\prod_{Z}[H^{n-1}(A,A^*)]$. Furthermore, the maps $\tilde{K}_n$ and K_n, $n\geq 1$ induce the same isomorphisms of Hochschild cohomology.*

Lemma 2.4. *Let $\phi\in Z^n(A,A^*)\subset\mathrm{Hom}_C(A_n,C)$ and define $\varphi\in\mathrm{Hom}_C(A_n,C)$ by*

$$\varphi(a^0,a^1,\cdots,a^n)=\phi(a^na^0,a^1,\cdots,a^{n-1},1)\,, \tag{2.6}$$

$a^j\in A$, $0\leq j\leq n$. *Then φ is a Hochschild coboundary.*

Proof. We have

$$b_{n+1}(a^0,a^1,\cdots,a^n,1)=\sum_{j=0}^{n-1}(-1)^j(a^0,\cdots,a^ja^{j+1},\cdots,a^n,1)\,. \tag{2.7}$$

Hence by using $\phi\circ b_{n+1}=0$, we obtain

$$\sum_{j=0}^{n-1}(-1)^j\phi(a^0,\cdots,a^ja^{j+1},\cdots,a^n,1)=0\,. \tag{2.8}$$

So, if we put $\tilde{\phi}(a^0,\cdots,a^{n-1})=\phi(a^0,\cdots,a^{n-1},1)$, then

$$\tilde{\phi}\circ b_n(a^0,\cdots,a^n)=\sum_{j=0}^{n-1}(-1)^j\phi(a^0,\cdots,a^ja^{j+1},\cdots,a^n,1) \tag{2.9}$$
$$+(-1)^n\phi(a^na^0,a^1,\cdots,a^{n-1},1)$$
$$=(-1)^n\varphi(a^0,\cdots,a^n)\,.$$

Q.E.D.

Lemma 2.5. *Let $[\varphi]=([\varphi^0],[\varphi^1])\in\prod_{Z}[H^{n+1}(A,A^*)]\oplus\prod_{Z}[H^n(A,A^*)]$, and $\tilde{D}_{n+1}: H^{n+1}(\tilde{A},\tilde{A}^*)\to H^n(\tilde{A},\tilde{A}^*)$, $n\geq 0$. Then $\tilde{D}_{n+1}([\varphi])=[\phi]$ is induced by*

$$\phi_m^0=\varphi_m^0\circ\tilde{B}_n+m(-1)^n\varphi_{m-1}^1\,, \tag{2.10}$$

$$\phi_m^1=\varphi_m^1\circ\tilde{B}_{n-1}\,, \tag{2.11}$$

where $\tilde{B}_n: A_n\to A_{n+1}$ is the map defined by

$$(2.12)\quad \tilde{B}_n(a^0, \cdots, a^n) = \sum_{j=0}^{n} (-1)^{n(j+1)}(a^{n-j+1}, \cdots, a^{n-j}, 1) + \sum_{j=0}^{n} (-1)^{nj}(1, a^{n-j+1}, \cdots, a^{n-j}), \quad (a^0, \cdots, a^n) \in A_n$$

which induces the de Rham differential of the Hochschild cohomology of A.

Proof. By (1.17), (1.18) and (1.19), the isomorphism $H^*(\tilde{A}, \tilde{A}^*) \simeq \prod_{Z}[H^*(A, A^*)] \oplus \prod_{Z}[H^{*-1}(A, A^*)]$ is induced by

$$(2.13)\quad \phi_m^0(a^0, \cdots, a^n) = \phi(a^0X^m, a^1, \cdots, a^n)$$

$$(2.14)\quad \phi_m^1(b^0, \cdots, b^{n-1}) = (-1)^n \sum_{l=0}^{n-1} (-1)^l \phi(b^0X^m, b^1, \cdots, b^l, X, b^{l+1}, \cdots, b^{n-1}),$$

where $[\phi] \in H^n(\tilde{A}, \tilde{A}^*)$, and by (1.34), (1.35), (1.36) and Corollary 2.3, the isomorphism of opposite direction is induced by

$$(2.15)\quad \phi(a^0X^{l_0}, \cdots, a^nX^{l_n}) = \phi^0_{l_0+\cdots+l_n}(a^0, \cdots, a^n) + (-1)^n l_1 \phi^1_{l_0+\cdots+l_n-1}(a^0a^1, a^2, \cdots, a^n),$$

where $([\phi^0], [\phi^1]) \in \prod_{Z}[H^n(A, A^*)] \oplus \prod_{Z}[H^{n-1}(A, A^*)]$, $n \geq 1$.

Let $\tilde{D}_{n+1}([\varphi]) = [\phi]$. Then by (2.12), (2.13) and (2.15), we obtain

$$(2.16)\quad \phi_m^0(a^0, \cdots, a^n) = \sum_{j=0}^{n} (-1)^{n(j+1)} \varphi_m^0(a^{n-j+1}, \cdots, a^{n-j}, 1) + \sum_{j=0}^{n} (-1)^{nj} \varphi_m^0(1, a^{n-j+1}, \cdots, a^{n-j}) + (-1)^n m \varphi^1_{m-1}(a^na^0, a^1, \cdots, a^{n-1}, 1) + (-1)^n m \varphi^1_{m-1}(a^0, \cdots, a^n), \quad (a^0, \cdots, a^n) \in A_n.$$

The third term is a coboundary by Lemma 2.4, hence we obtain (2.10).

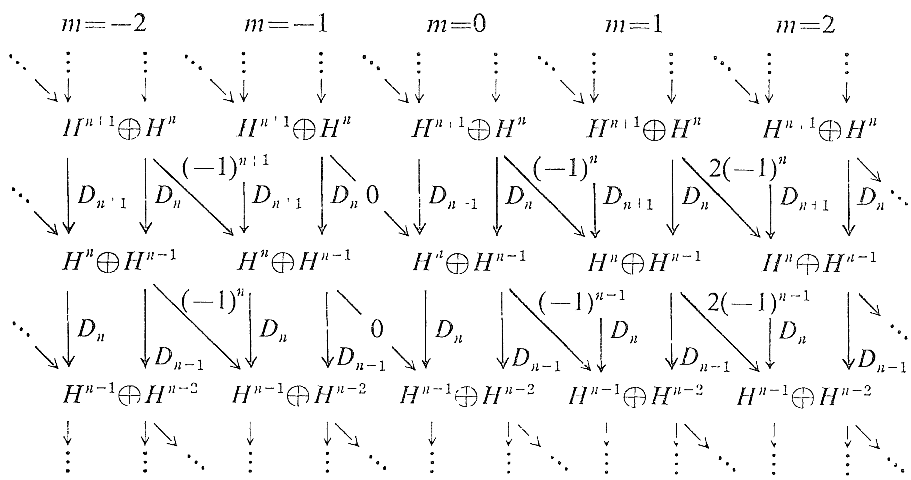
Next, by (2.12), (2.14) and (2.15), we obtain

$$\begin{aligned}(2.17)\quad \phi_m^1(b^0, \cdots, b^{n-1}) &= (-1)^n \sum_{l=0}^{n-1} (-1)^l \varphi^0_{m+1} \circ \tilde{B}_n(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^{n-1}) \\ &+ (-1)^n \sum_{l=0}^{n-1} (-1)^l (-1)^{n+1} m \varphi_m^1(b^nb^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^{n-1}, 1) \\ &+ (-1)^n \sum_{l=0}^{n-1} (-1)^l (-1)^{n+1} m \varphi_m^1(b^0, b^1, \cdots, b^l, 1, b^{l+1}, \cdots, b^{n-1}) \\ &+ (-1)^n \sum_{l=0}^{n-1} (-1)^l (-1)^{n+1} (-1)^{n(n-l+2)} \varphi_m(b^l, b^{l+1}, \cdots, b^{n-1}, b^0, \cdots, b^{l-1}, 1) \\ &+ (-1)^n \sum_{l=0}^{n-1} (-1)^l (-1)^{n+1} (-1)^{n(n-l)} \varphi_m(1, b^{l+1}, \cdots, b^{n-1}, b^0, \cdots, b^l),\end{aligned}$$

where the first term on the right hand side is the contribution coming from the first term on the right hand side of (2.15), which is coboundary by Lemma 2.2. The second and third terms are the contribution of the b^0X^m part in (2.14) to the second term on the right hand side of (2.15), which are again coboundary, by Lemmas 2.2 and 2.4 (for example, by putting $\Psi(b^0, b^1, \cdots, b^n)=\varphi^1_m(b^nb^0, b^1, \cdots, b^{n-1}, 1)$). The fourth and fifth terms of the right hand side of (2.17) are the contribution of X between b^l and b^{l+1} in (2.14) to the second term on the right hand side of (2.15) which are now equal to $\varphi^1_m\circ\tilde{B}_{n-1}(b^0, \cdots, b^{n-1})$. Q.E.D.

Here, we mention that in all the discussions so far, we can replace $\mathbb{C}[X, X^{-1}]$ by $\mathbb{C}[X]$ and $\mathbb{Z}$ by $\mathbb{N}=\{0,1,2,\cdots\}$. Hence, we can also discuss the case $\mathbb{C}[X]\otimes A$ by using the computations which we have already done.

First, we discuss the case $\tilde{A}=\mathbb{C}[X,X^{-1}]\otimes A$. By (2.10) and (2.11) of Lemma 2.5, we obtain the following complex computing $E_1^*(\tilde{A})$ as follows:



where $H^*=H^*(A, A^*)$ and D_n: $H^n\to H^{n-1}$ is the de Rham differential. The case $\tilde{A}=\mathbb{C}[X]\otimes A$ corresponds to the right half of the above diagram. In view of this diagram, we immediately obtain the following.

Proposition 2.6. (1) *Let* $\tilde{A}=\mathbb{C}[X,X^{-1}]\otimes A$. *Then* $E_1^*(\tilde{A})\cong E_1^*(A)\oplus E_n^{*-1}(A)$.
(2) *Let* $\tilde{A}=\mathbb{C}[X]\otimes A$. *Then* $E_1^*(\tilde{A})\cong E_1^*(A)$.

§3. Splitting of Spectral Sequence

In this section, we prove that for $\tilde{A}=\mathbb{C}[X, X^{-1}]\otimes A$, the spectral sequence for $\tilde{A}$ is a two direct sum of the spectral sequence of A with one of which degree shifted by -1.

By (2.15) we have two families of Hochschild cohomology maps $\tilde{\Psi}^n: H^n(A,A^*) \to H^n(\tilde{A},\tilde{A}^*)$, $n\geq 0$ and $\Psi^n: H^{n-1}(A,A^*)\to H^n(\tilde{A},\tilde{A}^*)$, $n\geq 1$ induced by

$$(3.1)\quad \tilde{\Psi}^n(\phi)(a^0,\cdots,a^n)=\sum_{l_0+\cdots+l_n=0}\phi(a^0_{l_0},a^1_{l_1},\cdots,a^n_{l_n}),\ [\phi]\in H^n(A,A^*)\,,$$

$$(3.2)\quad \Psi^n(\phi)(a^0,\cdots,a^n)=(-1)^n\sum_{l_0+\cdots+l_n=0} l_1\phi(a^0_{l_0}a^1_{l_1},a^2_{l_2},\cdots,a^n_{l_n}),\ [\phi]\in H^{n-1}(A,A^*)\,,$$

where $a^j=\sum_{l_j\in \boldsymbol{Z}} a^j_{l_j}X^{l_j}\in \boldsymbol{C}[X,X^{-1}]\otimes A=\tilde{A}$, $j=0,\cdots,n$.

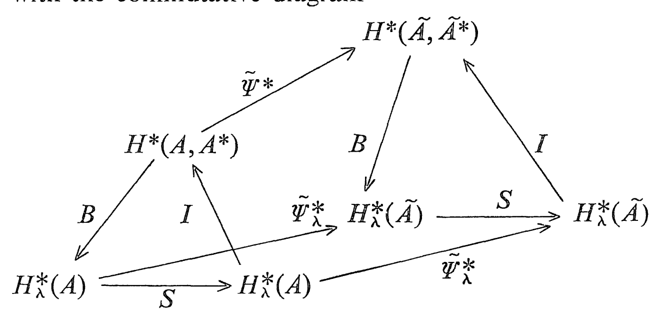
We also have two families of cyclic cohomology maps $\tilde{\Psi}^n_\lambda: H^n_\lambda(A)\to H^n_\lambda(\tilde{A})$, $n\geq 0$ and $\Psi^n_\lambda: H^{n-1}_\lambda(A)\to H^n_\lambda(\tilde{A})$, $n\geq 1$ where $\tilde{\Psi}^n_\lambda=\tilde{\Psi}^n$ and Ψ^n_λ is induced by

$$(3.3)\quad \Psi^n_\lambda(\phi)(a^0,\cdots,a^n)$$
$$=\frac{1}{n}(-1)^n\sum_{l_0+\cdots+l_n=0}\sum_{i=0}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n})\,,$$

where $a^j=\sum_{l_j\in \boldsymbol{Z}} a^j_{l_j}X^{l_j}\in\tilde{A}$, $j=0,\cdots,n$.

Remark 3.1. The case $\tilde{A}=\boldsymbol{C}[X]\otimes A$ is very simple. We only have $\tilde{\Psi}^n$ and $\tilde{\Psi}^n_\lambda$, $n\geq 0$ with the commutative diagram

(3.4)



Note that in this case, the expression of $\tilde{\Psi}^n=\tilde{\Psi}^n_\lambda$ is quite simple. There is only one summand in the right hand side of (3.1) which is the term with $l_0=l_1=\cdots=l_n=0$. Further, by Proposition 2.6(2), this morphism gives isomorphisms on E^*_1. So, by diagram chase using the five lemmas, we can see that this morphism actually induces an isomorphism of spectral sequence so that the limits coincide (see Remark 3.11).

We now go back to the case $\tilde{A}=\boldsymbol{C}[X,X^{-1}]\otimes A$. By (3.1), we can see that the diagram (3.4) still commutes for the case $\tilde{A}=\boldsymbol{C}[X,X^{-1}]\otimes A$. Next, we show that the diagrams:

$$(3.5)\qquad \begin{array}{ccc} H^{n-1}_\lambda(A) & \xrightarrow{\Psi^n_\lambda} & H^n_\lambda(\tilde{A}) \\ \downarrow I & & \downarrow I \\ H^{n-1}(A,A^*) & \xrightarrow{\Psi^n} & H^n(\tilde{A},\tilde{A}^*) \end{array}\,,\qquad n\geq 1$$

commute.

Lemma 3.2. *Let $\phi\in Z^{n-1}(A,A^*)$. We put*

$$\phi_k(a^0,\cdots,a^n)=\sum_{l_0+\cdots+l_n=0} l_k\sum_{j=0}^{k-1}(-1)^j\phi(a^0_{l_0},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n}), \tag{3.6}$$

$1\le k\le n$, where $a^j=\sum_{l_j\in Z} a^j_{l_j}X^{l_j}\in\tilde{A}$. Then $\phi_{k+1}-\phi_k\in B^n(\tilde{A},\tilde{A}^*)$, $1\le k\le n-1$.

Proof. We put

$$\varphi_k(a^0,\cdots,a^{n-1})=\sum_{l_0+\cdots+l_{n-1}=0} l_k\phi(a^0_{l_0},\cdots,a^{n-1}_{k_{n-1}}). \tag{3.7}$$

Then

$$\begin{aligned}
&\varphi_k\circ b_n(a^0,\cdots,a^n) \\
&=\sum_{l_0+\cdots+l_n=0}\{l_{k+1}\sum_{j=0}^{k-1}(-1)^j\phi(a^0_{l_0},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^k_{l_k},\cdots,a^n_{l_n}) \\
&\quad+(l_k+l_{k+1})(-1)^k\phi(a^0_{l_0},\cdots,a^k_{l_k}a^{k+1}_{l_{k+1}},\cdots,a^n_{l_n}) \\
&\quad+l_k\sum_{j=k+1}^{n-1}(-1)^j\phi(a^0_{l_0},\cdots,a^k_{l_k},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n}) \\
&\quad+l_k(-1)^n\phi(a^n_{l_n}a^0_{l_0},a^1_{l_1},\cdots,a^{n-1}_{l_{n-1}})\} \\
&=\sum_{l_0+\cdots+l_n=0}\{l_{k+1}\sum_{j=0}^{k}(-1)^j\phi(a^0_{l_0},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n}) \\
&\quad-l_k\sum_{j=0}^{k-1}(-1)^j\phi(a^0_{l_0},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n})\}.
\end{aligned} \tag{3.8}$$

Q.E.D.

Corollary 3.3. *The diagrams* (3.5) *commute.*

Proof. We show that (3.2) and (3.3) are Hochschild equivalent. By using (3.6),

$$\begin{aligned}
&\sum_{l_0+\cdots+l_n=0}\sum_{i=0}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n}) \\
&=\sum_{l_0+\cdots+l_n=0}\{(l_1+l_2+\cdots+l_n)\phi(a^0_{l_0}a^1_{l_1},a^2_{l_2},\cdots,a^n_{l_n}) \\
&\quad+(l_2+\cdots+l_n)\phi(a^0_{l_0},a^1_{l_1}a^2_{l_2},a^3_{l_3},\cdots,a^n_{l_n})+\cdots \\
&\quad+(-1)^{n-1}l_n\phi(a^0_{l_0},\cdots,a^{n-1}_{l_{n-1}}a^n_{l_n})\} \\
&=\sum_{k=1}^{n}\phi_k(a^0,\cdots,a^n).
\end{aligned} \tag{3.9}$$

By Lemma 3.2, this is Hochschild equivalent to $n\phi_1(a^0,\cdots,a^n)$ and we also have

$$\phi_1(a^0,\cdots,a^n)=\sum_{l_0+\cdots+l_n=0} l_1\phi(a^0_{l_0}a^1_{l_1},a^2_{l_2},\cdots,a^n_{l_n}). \tag{3.10}$$

Hence, we obtain the assertion. Q.E.D.

The next step is to show that the diagrams

$$(3.11)\qquad \begin{array}{ccc} H^n(A,A^*) & \xrightarrow{\Psi^{n+1}} & H^{n+1}(\tilde{A},\tilde{A}^*) \\ \downarrow B & & \downarrow B \\ H_\lambda^{n-1}(\tilde{A}) & \xrightarrow{\Psi_\lambda^n} & H_\lambda^n(\tilde{A}) \end{array}\quad , \qquad n\geq 1$$

commute. (It is easy to see that $H^0(A,A^*)\to H^1(\tilde{A},\tilde{A}^*)\to H_\lambda^0(\tilde{A})$ is a zero map.)

Let $\phi\in Z^n(A,A^*)$ and put $\Psi^{n+1}(\phi)=\tilde{\Psi}$. By (2.12) and (3.2), we obtain

$$(3.12)\quad B(\tilde{\Psi})(a^0,\cdots,a^n)$$
$$=\sum_{j=0}^{n}(-1)^{n(j+1)}(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}l_{n-j+2}\phi(a^{n-j+1}_{l_{n-j+1}}a^{n-j+2}_{l_{n-l+2}},a^{n-j+3}_{l_{n-j+3}},\cdots,a^{n-j}_{l_{n-j}},1)$$
$$+\sum_{j=0}^{n}(-1)^{nj}(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}l_{n-j+1}\phi(a^{n-j+1}_{l_{n-j+1}},\cdots,a^{n-j}_{l_{n-j}})\,.$$

Lemma 3.4. *Let $\phi\in Z^n(A,A^*)$ and put*

$$(3.13)\quad \varphi_j(a^0,\cdots,a^n)$$
$$=\sum_{l_0+\cdots+l_n=0}\{\sum_{i=0}^{j}(-1)^i l_{j+1}\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n},1)$$
$$-\sum_{i=0}^{j-1}(-1)^i l_j\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^j_{l_j},\cdots,a^n_{l_n},1)\}\;,$$

$1\leq j\leq n-1$. *Then $\varphi_j\circ N_n\in B_\lambda^n(\tilde{A})$.*

Proof. We put

$$(3.14)\qquad \tilde{\varphi}_j(a^0,\cdots,a^{n-1})=\sum_{l_0+\cdots+l_{n-1}=0}l_j\phi(a^0_{l_0},\cdots,a^{n-1}_{l_{n-1}},1)\,.$$

Then $\tilde{\varphi}_j\circ N_{n-1}\in C_\lambda^{n-1}(\tilde{A})$ so that $-\tilde{\varphi}_j\circ N_{n-1}\circ b_n=\tilde{\varphi}_j\circ\tilde{b}_n\circ N_n\in Z_\lambda^n(\tilde{A})$ due to $-N_{n-1}\circ b_n=\tilde{b}_nN_n$ (see Connes [2] or Loday-Quillen [4]). By (3.14) and the definition of $\tilde{b}_n$ (see Loday-Quillen [4]), we obtain

$$(3.15)\quad \tilde{\varphi}_j\circ\tilde{b}_n(a^0,\cdots,a^n)$$
$$=\sum_{l_0+\cdots+l_n=0}\{\sum_{i=0}^{j-1}(-1)^i l_{j+1}\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n},1)$$
$$+(-1)^j(l_j+l_{j+1})\phi(a^0_{l_0},\cdots,a^j_{l_j}a^{j+1}_{l_{j+1}},\cdots,a^n_{l_n},1)$$
$$+\sum_{i=j+1}^{n-1}(-1)^i l_j\phi(a^0_{l_0},\cdots,a^j_{l_j},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n},1)\}\,.$$

By using $\phi\in Z^n(A,A^*)$, we have

$$(3.16)\qquad \sum_{i=1}^{n-1}(-1)^i\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n},1)=0\,.$$

Hence putting (3.16) into the right hand side of (3.15), we obtain $\tilde{\varphi}_j\circ\tilde{b}_n=\varphi_j$ and obtain the assertion. Q.E.D.

By Lemma 3.4, we obtain $(\varphi_1+\cdots+\varphi_{n-1})N_n\in B_\lambda^n(\tilde{A})$. Now we compute

$$(3.17)\quad (\varphi_1+\cdots+\varphi_{n-1})(a^0,\cdots,a^n)$$
$$=\sum_{l_0+\cdots+l_n=0}\{\sum_{i=0}^{n-1}(-1)^i l_n\varphi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n},1)-l_1\phi(a^0_{l_0}a^1_{l_1},a^2_{l_2},\cdots,a^n_{l_n},1)\}$$
$$=-\sum_{l_0+\cdots+l_n=0}l_1\phi(a^0_{l_0}a^1_{l_1},\cdots,a^n_{l_n},1),$$

where we use (3.16). We can easily see that the first sum of the right hand side of (3.12) is equal to $-(\varphi_1+\cdots+\varphi_{n-1})\circ N_n$ by using (3.17). Hence we can omit the first sum of the right hand side of (3.12) for the proof of the commutativity of (3.11).

We next compute $\Psi^n_\lambda(B(\phi))\in Z^n_\lambda(\tilde{A})$. We put

$$(3.18)\qquad \Phi_1(a^0,\cdots,a^{n-1})=\sum_{j=0}^{n-1}(-1)^{(n-1)(j+1)}\phi(a^{n-j},\cdots,a^{n-j-1},1),$$

$$(3.19)\qquad \Phi_2(a^0,\cdots,a^{n-1})=\sum_{j=0}^{n-1}(-1)^{(n-1)j}\phi(1,a^{n-j},\cdots,a^{n-j-1}).$$

By (2.12), $B(\phi)=\Phi_1+\Phi_2$. On the other hand, if we put

$$(3.20)\quad \tilde{\psi}_j(a^0,\cdots,a^n)$$
$$=\sum_{l_0+\cdots+l_n=0}\sum_{i=0}^{j-1}(-1)^i l_j\phi(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^j_{l_j},\cdots,a^n_{l_n},1),$$

$1\leq j\leq n-1$. Then $\tilde{\psi}_j\circ N_n\in B^n_\lambda(\tilde{A})$.

Lemma 3.5. $n\Psi^n_\lambda(\Phi_n)=-(\tilde{\psi}_1+\cdots+\tilde{\psi}_{n-1})\circ N_n$.

Proof. By (3.3)

$$(3.21)\quad n\Psi^n_\lambda(\Phi_1)(a^0,\cdots,a^n)$$
$$=(-1)^n\sum_{l_0+\cdots+l_n=0}\sum_{i=0}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i\Phi_1(a^0_{l_0},\cdots,a^i_{l_i}a^{i+1}_{l_{i+1}},\cdots,a^n_{l_n}).$$

In the proof of this lemma, we write a^i instead of $a^i_{l_i}$ to avoid the complicated formula. We have no way of confusion. By (3.18),

$$(3.22)\quad \Phi_1(a^0,\cdots,a^ia^{i+1},\cdots,a^n)$$
$$=\sum_{j=0}^{n-i-1}(-1)^{(n-1)(j+1)}\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1)$$
$$=\sum_{j=n-i}^{n-1}(-1)^{(n-1)(j+1)}\phi(a^{n-j},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j-1},1).$$

Since $\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}=\sum_{j=0}^{n-1}\sum_{i=0}^{n-j-1}$ and $\sum_{i=0}^{n-1}\sum_{j=n-i}^{n-1}=\sum_{j=1}^{n-1}\sum_{i=n-j}^{n-1}$, we obtain

$$(3.23)\qquad \sum_{i=0}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i\Phi_1(a^0,\cdots,a^ia^{i+1},\cdots,a^n)$$
$$=\sum_{j=0}^{n-1}\sum_{i=0}^{n-i-1}(l_{i+1}+\cdots+l_n)(-1)^i(-1)^{(n-1)(j+1)}$$

$$\times\Phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1)$$
$$+\sum_{j=1}^{n-1}\sum_{i=n-j}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i(-1)^{(n-1)(j+1)}$$
$$\times\phi(a^{n-j},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j-1},1)$$
$$=\sum_{j=0}^{n-1}\sum_{i=0}^{n-j-2}(l_{i+1}+\cdots+l_n)(-1)^i(-1)^{(n-1)(j+1)}$$
$$\times\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1)$$
$$+\sum_{j=0}^{n-1}(l_{n-j}+\cdots+l_n)(-1)^{n-j-1}(-1)^{(n-1)(j+1)}$$
$$\times\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^{n-j-1}a^{n-j},1)$$
$$+\sum_{j=2}^{n}\sum_{i=n-j+1}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i(-1)^{(n-1)j}$$
$$\times\phi(a^{n-j+1},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j},1),$$

where we replace j by $j-1$ in the second double sum to obtain the second quality. By using $\phi\in Z^n(A,A^*)$, we obtain

$$(3.24)\qquad (-1)^{n-j-1}\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^{n-j-1}a^{n-j},1)$$
$$=\sum_{i=n-j+1}^{n-1}(-1)^{n+i}\phi(a^{n-j+1},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j},1)$$
$$+\phi(a^{n-j+1},\cdots,a^na^0,\cdots,a^{n-j},1)$$
$$+\sum_{i=0}^{n-j-2}(-1)^{i+1}\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1).$$

Hence (3.23) is equal to

$$(3.25)\qquad \sum_{j=0}^{n-1}\sum_{i=0}^{n-j-2}(-1)^{(n-1)(j+1)}(-1)^i(l_{i+1}+\cdots+l_{n-j-1})$$
$$\times\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1)$$
$$+\sum_{j=1}^{n-1}(-1)^{(n-1)(j+1)}(l_{n-j}+\cdots+l_n)\phi(a^{n-j+1},\cdots,a^na^0,\cdots,a^{n-j},1)$$
$$+\sum_{j=2}^{n}\sum_{i=n-j+1}^{n-1}(-1)^{(n-1)j}(-1)^{i+1}(l_{n-j}+\cdots+l_i)$$
$$\times\phi(a^{n-j+1},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j},1),$$

which we put $\sum_{j=0}^{n}F_j(a^0,\cdots,a^n)$. For $j=0$, we get

$$(3.26)\qquad F_0(a^0,\cdots,a^n)$$
$$=(-1)^{n-1}\sum_{i=0}^{n-2}(-1)^i(l_{i+1}+\cdots+l_{n-1})\phi(a^0,\cdots,a^ia^{i+1},\cdots,a^n,1),$$

and hence, in view of (3.20) and (3.21),

$$(3.27)\qquad (-1)^n\sum_{l_0+\cdots+l_n=0}F_0(a^0,\cdots,a^n)=-(\tilde{\psi}_1+\cdots+\tilde{\psi}_{n-1})(a^0,\cdots,a^n).$$

So, the remaining part of the proof is to show

$$(3.28)\qquad \sum_{l_0+\cdots+l_n=0} F_{j+1}(a^0,\cdots,a^n)=\sum_{l_0+\cdots+l_n=0}(-1)^n F_j(a^n,a^0,\cdots,a^{n-1}).$$

For generic j (i.e. $2\leq j\leq n-1$)

$$\begin{aligned}(3.29)\qquad & F_j(a^0,\cdots,a^n)\\ &=\sum_{i=0}^{n-j-2}(-1)^{(n-1)(j+1)}(-1)^i(l_{i+1}+\cdots+l_{n-j-1})\\ &\qquad\times\phi(a^{n-j+1},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j},1)\\ &\quad+(-1)^{(n-1)(j+1)}(l_{n-j}+\cdots+l_n)\phi(a^{n-j+1},\cdots,a^na^0,\cdots,a^{n-j},1)\\ &\quad+\sum_{i=n-j+1}^{n-1}(-1)^{(n-1)j}(-1)^{i+1}(l_{n-j}+\cdots+l_i)\\ &\qquad\times\phi(a^{n-j+1},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j},1).\end{aligned}$$

So, we obtain

$$\begin{aligned}(3.30)\qquad & (-1)^nF_j(a^n,a^0,\cdots,a^{n-1})\\ &=(-1)^n\sum_{i=1}^{n-j-2}(-1)^{(n-1)(j+1)}(-1)^i(l_i+\cdots+l_{n-j-2})\\ &\qquad\times\phi(a^{n-j-2},\cdots,a^{n-1},a^n,a^0,\cdots,a^{i-1}a^i,\cdots,a^{n-j-1},1)\\ &\quad+(-1)^n(-1)^{(n-1)(j+1)}(l_0+\cdots+l_{n-j-2})\\ &\qquad\times\phi(a^{n-j-2},\cdots,a^na^0,\cdots,a^{n-j-1},1)\\ &\quad+(-1)^n(-1)^{(n-1)(j+1)}(l_{n-j-1}+\cdots+l_{n-1})\\ &\qquad\times\phi(a^{n-j},\cdots,a^{n-1}a^n,a^0,\cdots,a^{n-j-1},1)\\ &\quad+(-1)^n\sum_{i=n-j+1}^{n-1}(-1)^{(n-1)j}(-1)^{i+1}(l_{n-j-1}+\cdots+l_{i-1})\\ &\qquad\times\phi(a^{n-j},\cdots,a^{i-1}a^i,\cdots,a^{n-1},a^n,a^0,\cdots,a^{n-j-1},1).\end{aligned}$$

By replacing $i-1$ by i, the first term is equal to

$$\begin{aligned}(3.31)\qquad & (-1)^n\sum_{i=0}^{n-j-3}(-1)^{(n-1)(j+1)}(-1)^{i+1}(l_{i+1}+\cdots+l_{n-j-2})\\ &\qquad\times\phi(a^{n-j-2},\cdots,a^n,a^0,\cdots,a^ia^{i+1},\cdots,a^{n-j-1},1).\end{aligned}$$

By using $l_0+\cdots+l_n=0$, the second term is equal to

$$\begin{aligned}(3.32)\qquad & (-1)^{n-1}(-1)^{(n-1)(j+1)}(l_{n-j-1}+\cdots+l_n)\\ &\qquad\times\phi(a^{n-j-2},\cdots,a^na^0,\cdots,a^{n-j-1},1).\end{aligned}$$

The sum of the third and the fourth terms is

$$\begin{aligned}(3.33)\qquad & (-1)^n\sum_{i=n-j+1}^{n}(-1)^{(n-1)j}(-1)^{i+1}(l_{n-j-1}+\cdots+l_{i-1})\\ &\qquad\times\phi(a^{n-j},\cdots,a^{i-1}a^i,\cdots,a^n,a^0,\cdots,a^{n-j-1},1)\\ &=(-1)^n\sum_{i=n-j}^{n-1}(-1)^{(n-1)j}(-1)^i(l_{n-j-1}+\cdots+l_i)\\ &\qquad\times\phi(a^{n-j},\cdots,a^ia^{i+1},\cdots,a^n,a^0,\cdots,a^{n-j-1},1),\end{aligned}$$

where we replace $i-1$ by i for the equality. So, by looking at (3.31), (3.32), (3.33) and (3.29), we obtain (3.28) for generic j. The non-generic j is rather easy to check. Q.E.D.

To avoid the complexity of our computation, we use Corollary 3.3 which guarantees that $\Psi^{n+1}(\phi)$ and $\Psi_\lambda^{n+1}(\phi)$ are Hochschild equivalent. Since B is a cohomology map, $B(\Psi^{n+1}(\phi))$ and $B(\Psi_\lambda^{n+1}(\phi))$ are cyclic equivalent, i.e. the difference belongs to $B_\lambda^n(\tilde{A})$. We now compute $B(\Psi_\lambda^{n+1}(\phi))$. Let $\Psi=\Psi_\lambda^{n+1}(\phi)$ and we put

$$B_r(\Psi)(a^0, \cdots, a^n) = \sum_{j=0}^{n} (-1)^{n(j+1)} \Psi(a^{n-j+1}, \cdots, a^{n-j}, 1), \tag{3.34}$$

$$B_l(\Psi)(a^0, \cdots, a^n) = \sum_{j=0}^{n} (-1)^{nj} \Psi(1, a^{n-j+1}, \cdots, a^{n-j}). \tag{3.35}$$

By (2.12), $B(\Psi)=B_r(\Psi)+B_l(\Psi)$. Since

$$\begin{aligned} &\Psi(b^0, \cdots, b^n, 1) \\ &= \frac{1}{n+1}(-1)^{n+1} \sum_{m_0+\cdots+m_n=0} \sum_{i=0}^{n-1} (-1)^i (m_{i+1}+\cdots+m_n) \\ &\qquad \times \phi(b^0_{m_0}, \cdots, b^i_{m_i} b^{i+1}_{m_{i+1}}, \cdots, b^n_{m_n}, 1), \end{aligned} \tag{3.36}$$

we obtain

$$\begin{aligned} B_r(\Psi) &= -\frac{1}{n+1}(\tilde{\psi}_1+\cdots+\tilde{\psi}_n)\circ N_n \\ &= -\frac{1}{n+1}(\tilde{\psi}_1+\cdots+\tilde{\psi}_{n-1})\circ N_n \end{aligned} \tag{3.37}$$

by looking at the proof of Lemma 3.5 (especially (3.26) and (3.27)). Hence, it is a cyclic coboundary.

We next compute $B_l(\Psi)$. By (3.3), we get

$$\begin{aligned} &\Psi_\lambda^{n+1}(\phi)(1, b^1, \cdots, b^{n+1}) \\ &= \frac{1}{n+1}(-1)^{n+1} \sum_{m_1+\cdots+m_{n+1}=0} \sum_{i=1}^{n} (m_{i+1}+\cdots+m_{n+1})(-1)^i \\ &\qquad \times \phi(1, b^1_{m_1}, \cdots, b^i_{m_i} b^{i+1}_{m_{i+1}}, \cdots, b^{n+1}_{m_{n+1}}). \end{aligned} \tag{3.38}$$

Hence,

$$\begin{aligned} &\Psi_\lambda^{n+1}(\phi)(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^{n-j}) \\ &= \frac{1}{n+1}(-1)^{n+1} \sum_{l_0+\cdots+l_n=0} \{\sum_{i=1}^{j-1} (l_{n-j+i+1}+\cdots+l_n+l_0+\cdots+l_{n-j})(-1)^i \\ &\qquad \times \phi(1, a^{n-j+1}, \cdots, a^{n-j+i} a^{n-j+i+1}, a^n, a^0, \cdots, a^{n-j}) \\ &\quad +(l_0+\cdots+l_{n-j})(-1)^j \phi(1, a^{n-j+1}, \cdots, a^n a^0, \cdots, a^{n-j}) \end{aligned} \tag{3.39}$$

$$+\sum_{i=j+1}^{n}(l_{i-j}+\cdots+l_{n-j})(-1)^{i}$$
$$\times\phi(1, a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{i-j-1}a^{i-j}, \cdots, a^{n-j})\},$$

where we use simplified notation as before. By using $\phi\in Z^{n}(A, A^{*})$, we obtain

$$(3.40)\qquad \begin{aligned}&(-1)^{j}\phi(1, a^{n-j+1}, \cdots, a^{n}a^{0}, \cdots, a^{n-j})\\ &=-\phi(a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j})\\ &\quad-\sum_{i=1}^{j-1}(-1)^{i}\phi(1, a^{n-j+1}, \cdots, a^{n-j+i}a^{n-j+i+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j})\\ &\quad-\sum_{i=j+1}^{n}(-1)^{i}\phi(1, a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{i-j-1}a^{i-j}, \cdots, a^{n-j})\\ &\quad-(-1)^{n+1}\phi(a^{n-j}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j}).\end{aligned}$$

Hence, by (3.39), $B_{l}(\Psi)(a^{0}, \cdots, a^{n})$ is equal to

$$(3.41)\qquad \begin{aligned}&\frac{1}{n+1}(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}\{\sum_{j=2}^{n}\sum_{i=1}^{j-1}(-1)^{nj}(l_{n-j+i+1}+\cdots+l_{n})(-1)^{i}\\ &\qquad\times\phi(1, a^{n-j+1}, \cdots, a^{n-j+i}a^{n-j+i+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j})\\ &\quad+\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{nj}(l_{0}+\cdots+l_{i-j-1})(-1)^{i+1}\\ &\qquad\times\phi(1, a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{i-j-1}a^{i-j}, \cdots. a^{n-j})\}\\ &\quad-\frac{1}{n+1}(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}\{\sum_{j=0}^{n}(-1)^{nj}(l_{0}+\cdots+l_{n-j})\\ &\qquad\times\phi(a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j})\\ &\quad+(-1)^{n+1}\sum_{j=0}^{n}(-1)^{nj}(l_{0}+\cdots+l_{n-j})\phi(a^{n-j}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j-1})\}.\end{aligned}$$

The second sum of the summand of $\sum_{l_0+\cdots+l_n=0}$ of the second term of (3.41) is computed as follows:

$$(3.42)\qquad \begin{aligned}&(-1)^{n+1}\sum_{j=0}^{n}(-1)^{nj}(l_{0}+\cdots+l_{n-j})\phi(a^{n-j}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j-1})\\ &=(-1)^{n^2+n+1}l_{0}\phi(a^{0}, \cdots, a^{n})\\ &\quad+(-1)^{n+1}\sum_{j=0}^{n-1}(-1)^{nj}(l_{0}+\cdots+l_{n-j})\phi(a^{n-j}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j-1})\\ &=-l_{0}\phi(a^{0}, \cdots, a^{n})\\ &\quad+(-1)^{n+1}\sum_{j=1}^{n}(-1)^{n(j-1)}(l_{0}+\cdots+l_{n-j+1})\phi(a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j}),\end{aligned}$$

where we replace $j+1$ by j to obtain the second equality. By using $\sum_{j=0}^{n}(-1)^{nj}\cdot(l_{0}+\cdots+l_{n-j})=\sum_{j=1}^{n}(-1)^{nj}(l_{0}+\cdots+l_{n-j})$ due to $l_{0}+\cdots+l_{n}=0$, the summand of $\sum_{l_0+\cdots+l_n=0}$ of the second term of (3.41) is equal to

$$(3.43)\qquad -\sum_{j=1}^{n}(-1)^{nj}l_{n-j+1}\phi(a^{n-j+1}, \cdots, a^{n}, a^{0}, \cdots, a^{n-j})-l_{0}\phi(a^{0}, \cdots, a^{n})$$

$$= -\sum_{j=0}^{n}(-1)^{nj} l_{n-j+1}\phi(a^{n-j+1}, \cdots, a^{n-j}) .$$

So, the second term of (3.41) is equal to

$$(3.44)\qquad \frac{1}{n+1}(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}\sum_{j=0}^{n}(-1)^{nj} l_{n-j+1}\phi(a^{n-j+1}, \cdots, a^{n-j}) ,$$

which is cyclic cohomologous to $\frac{1}{n+1}B(\tilde{\Psi})(a^0, \cdots, a^n)$ due to (3.12) and (3.17).

In view of (3.41), we define $\Xi \in Z_\lambda^n(\tilde{A})$ by

$$(3.45)\qquad \begin{aligned} &\Xi(a^0, \cdots, a^n) \\ &= (-1)^{n+1}\sum_{l_0+\cdots+l_n=0}\{\sum_{j=2}^{n}\sum_{i=1}^{j-1}(-1)^{nj}(l_{n-j+i+1}+\cdots+l_n)(-1)^i \\ &\qquad \times\phi(1, a^{n-j+1}, \cdots, a^{n-j+i}a^{n-j+i+1}, \cdots, a^n, a^0, \cdots, a^{n-j}) \\ &\quad +\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{nj}(l_0+\cdots+l_{i-j-1})(-1)^{i+1} \\ &\qquad \times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^{i-j-1}a^{i-j}, \cdots, a^{n-j})\} . \end{aligned}$$

Lemma 3.6. *Ξ is cyclic cohomologous to $nB(\tilde{\Psi})$.*

Proof. Due to the discussion after the proof of Lemma 3.5, $B(\tilde{\Psi})$ is cyclic cohomologous to $B(\Psi)$ which is again cyclic cohomologous to $B_l(\Psi)$ by (3.37). By (3.41), $B_l(\Psi)=\frac{1}{n+1}\Xi+(3.44)$, which is cyclic cohomologous to $\frac{1}{n+1}\Xi+\frac{1}{n+1}B(\tilde{\Psi})$ as we have already discussed. So, combining all these matters, $\left(1-\frac{1}{n+1}\right)B(\tilde{\Psi})$ is cyclic cohomologous to $\frac{1}{n+1}\Xi$. Multiplying $(n+1)$ on both sides, we obtain the assertion. Q.E.D.

Lemma 3.7. $n\Psi_\lambda^n(\phi_2)=\Xi$.

Proof. Replacing $n-j+1$ by i, we obtain

$$(3.46)\qquad \begin{aligned} &\sum_{j=2}^{n}\sum_{i=1}^{j-1}(-1)^{nj}(l_{n-j+i+1}+\cdots+l_n)(-1)^i \\ &\qquad \times\phi(1, a^{n-j+1}, \cdots, a^{n-j+i}a^{n-j+i+1}, \cdots, a^n, a^0, \cdots, a^{n-j}) \\ &= \sum_{j=2}^{n}\sum_{i=n-j+1}^{n-1}(-1)^{nj+n+j}(l_{i+1}+\cdots+l_n)(-1)^j \\ &\qquad \times\phi(1, a^{n-j+1}, \cdots, a^i a^{i+1}, \cdots, a^n, a^0, \cdots, a^{n-j}) . \end{aligned}$$

Using $l_0+\cdots+l_n=0$ and replace $i-j-1$ by i, we obtain

$$(3.47)\qquad \begin{aligned} &\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{nj}(l_0+\cdots+l_{i-j-1})(-1)^{i+1} \\ &\qquad \times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^{i-j-1}a^i, \cdots, a^{n-j}) \end{aligned}$$

$$
\begin{aligned}
&=\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}(-1)^{nj}(l_{i-j}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^{i-j-1}a^i, \cdots, a^{n-j}) \\
&=\sum_{j=0}^{n-1}\sum_{i=0}^{n-j-1}(-1)^{nj+j+1}(l_{i+1}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^ia^{i+1}, \cdots, a^{n-j})\,.
\end{aligned}
$$

By putting (3.46) and (3.47), using $\sum_{j=2}^{n}\sum_{i=n-j+1}^{n-1}=\sum_{i=1}^{n-1}\sum_{j=n-i+1}^{n}$ and $\sum_{j=0}^{n-1}\sum_{i=0}^{n-j-1}=\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}$, into the right hand side of (3.45), wc obtain

$$
\begin{aligned}
(3.48)\qquad &\Xi(a^0, \cdots, a^n) \\
&=(-1)^{n+1}\sum_{l_0+\cdots+l_n=0}\{\sum_{i=1}^{n-1}\sum_{j=n-i+1}^{n}(-1)^{nj+n+j}(l_{i+1}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^ia^{i+1}, \cdots, a^n, a^0, \cdots, a^{n-j}) \\
&\quad+\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}(-1)^{nj+j+1}(l_{i+1}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^ia^{i+1}, \cdots, a^{n-j})\}\,.
\end{aligned}
$$

Next, we compute $n\Psi^n_\lambda(\Phi_2)$. By (3.19),

$$
\begin{aligned}
(3.49)\qquad &\Phi_2(a^0, \cdots, a^ia^{i+1}, \cdots, a^n) \\
&=\sum_{j=0}^{n-i-1}(-1)^{(n-1)j}\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^ia^{i+1}, \cdots, a^{n-j}) \\
&\quad+\sum_{j=n-i}^{n-1}(-1)^{(n-1)j}\phi(1, a^{n-j}, \cdots, a^ia^{i+1}, \cdots, a^n, a^0, \cdots, a^{n-j-1}) \\
&=\sum_{j=0}^{n-i-1}(-1)^{(n-1)j}\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^ia^{i+1}, \cdots, a^{n-j}) \\
&\quad+\sum_{j=n-i+1}^{n}(-1)^{(n-1)(j-1)}\phi(1, a^{n-j+1}, \cdots, a^ia^{i+1}, \cdots, a^n, a^0, \cdots, a^{n-j})\,,
\end{aligned}
$$

where we replace $j+1$ by j in the second sum to obtain the second equality. Hence, by (3.3),

$$
\begin{aligned}
(3.50)\qquad &n\Psi^n_\lambda(\Phi_2)(a^0, \cdots, a^n) \\
&=(-1)^n\sum_{l_0+\cdots+l_n=0}\sum_{i=0}^{n-1}(l_{i+1}+\cdots+l_n)(-1)^i\Phi_2(a^0, \cdots, a^ia^{i+1}, \cdots, a^n) \\
&=(-1)^n\sum_{l_0+\cdots+l_n=0}\{\sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}(-1)^{(n-1)j}(l_{i+1}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^n, a^0, \cdots, a^ia^{i+1}, \cdots, a^{n-j}) \\
&\quad+\sum_{i=1}^{n-1}\sum_{j=n-i+1}^{n}(-1)^{(n-1)(j-1)}(l_{i+1}+\cdots+l_n)(-1)^i \\
&\qquad\times\phi(1, a^{n-j+1}, \cdots, a^ia^{i+1}, \cdots, a^n, a^0, \cdots, a^{n-j})\}\,.
\end{aligned}
$$

which coincides with (3 48). Q.E.D.

Corollary 3.8. *The diagrams* (3.11) *commute.*

Proof. Let $\phi \in Z^n(A, A^*)$. By Lemma 3.6, $B(\Psi^{n+1}(\phi)) = B(\tilde{\Psi})$ is cyclic cohomologous to $\frac{1}{n}\Xi$ which is equal to $\Psi_\lambda^n(\Phi_2)$ by Lemma 3.7. By Lemma 3.5, $\Psi_\lambda^n(\Phi_2)$ is cyclic cohomologous to $\Psi_\lambda^n(\Phi_1) + \Psi_\lambda^n(\Phi_2) = \Psi_\lambda^n(\Phi) = \Psi_\lambda^n(B(\phi))$. This proves the commutativity of the diagram. Q.E.D.

Lemma 3.9. *The diagrams*

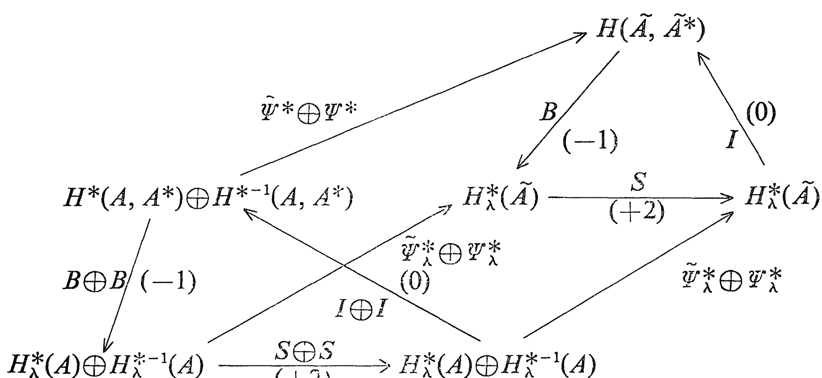
(3.51)
$$\begin{array}{ccc} H_\lambda^{n-1}(A) & \xrightarrow{\Psi_\lambda^n} & H_\lambda^n(\tilde{A}) \\ \downarrow S & & \downarrow S \\ H_\lambda^{n+1}(A) & \xrightarrow{\Psi_\lambda^{n+2}} & H_\lambda^{n+2}(\tilde{A}) \end{array}, \quad n \geq 1$$

commute, where S is assumed to be given by the shuffle product by the generator $\sigma \in H_\lambda^2(\boldsymbol{C})$ (*which differs from the S-operator using cup product by constant multiple*).

Proof. This is a consequence of the associativity of shuffle product (equivalently, cup product) and graded commutativity of the products (note that the degree of the generator σ of $H_\lambda^2(\boldsymbol{C})$ is two). Q.E.D.

By the commutative diagrams (3.4), (3.5), (3.11) and Lemma 3.9, we obtain the following commutative diagram (map between exact couples):

(3.52)



Theorem 3.10. *The maps* $(\tilde{\Psi}^* \oplus \Psi^*, \tilde{\Psi}_\lambda^* \oplus \Psi_\lambda^*)$ *of exact couples induces isomorphisms of spectral sequences*

(3.53)
$$E_n^*(A) \oplus E_n^{*-1}(A) \xrightarrow{\sim} E_n^*(A[X, X^{-1}]), \quad n \geq 1,$$

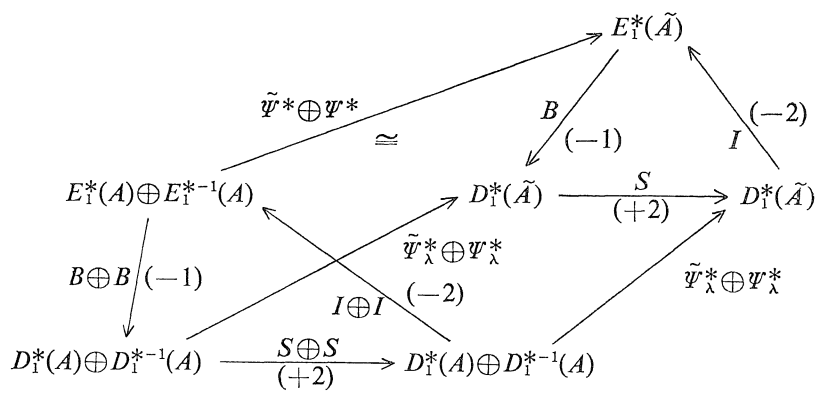
where we put $E_n^{-1}(A) = 0$.

Remark 3.11. The same, but much simpler, mechanisms works to prove that the maps $(\tilde{\Psi}^*, \tilde{\Psi}^*_\lambda)$ of exact couples induces isomorphisms of spectral sequences.

(3.54) $$E^*_n(A) \xrightarrow{\sim} E^*_n(A[X]), \quad n \geq 1 .$$

Proof of Theorem 3.10. By Proposition 2.6, the maps $\tilde{\Psi}^* \oplus \Psi^* : H^n(A, A^*) \oplus H^{n-1}(A, A^*) \to H^n(\tilde{A}, \tilde{A})$ are quasi-isomorphisms for the computation of de Rham terms. Further, through this quasi-isomorphism, the de Rham differential splits into each component by Lemma 2.5. So, the maps of exact couples induce maps of E_1-terms:

(3.55)



By using the fact that the maps $E^*_1(A) \oplus E^{*-1}_1(A) \to E^*_1(\tilde{A})$ are isomorphisms, we can easily conclude that the maps $D^*_1(A) \oplus D^{*-1}_1(A) \to D^*_1(\tilde{A})$ are isomorphisms by the repeated use of 5-lemmas from lower degree. Actually, the starting of the exact sequence is as follows:

(3.56)
$$\begin{array}{ccccccc}
\cdots \to E^3_1(\tilde{A}) & \longrightarrow & D^2_1(\tilde{A}) & \longrightarrow & D^4_1(\tilde{A}) & \longrightarrow & E^2_1(\tilde{A}) \\
\uparrow \wr & & \uparrow \wr & & \uparrow & & \uparrow \wr \\
\cdots \to E^3_1(A) \oplus E^2_1(A) & \to & D^2_1(A) \oplus D^1_1(A) & \to & D^4_1(A) \oplus D^3_1(A) & \to & E^2_1(A) \oplus E^1_1(A) \\
 & & \wr\| & & & & \\
 & & D^2_1(A) & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \| & & & & & & \\
\longrightarrow & & D^1_1(\tilde{A}) & \longrightarrow & D^3_1(\tilde{A}) & \xrightarrow{\sim} & E^1_1(\tilde{A}) & \longrightarrow & D^0_1(\tilde{A}) = 0 \\
 & & \uparrow & & \uparrow \wr & & \uparrow \wr & & \uparrow \\
\to & & D^1_1(A) \oplus D^0_1(A) & \to & D^3_1(A) \oplus D^2_1(A) & \xrightarrow{\sim} & E^1_1(A) \oplus E^0_1(A) & \to & D^0_1(A) = 0 \\
 & & \| & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & D_1^2(\tilde{A}) & \xrightarrow{\sim} & E_1^0(\tilde{A}) & \to & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \\ 0 & \to & D_1^2(A)\oplus D_1^1(A) & \xrightarrow{\sim} & E_1^0(A) & \to & 0 \\ & & \wr\| & & & & \\ & & D_1^2(A) & & & & \end{array}$$

From the above diagram, we immediately obtain that the maps $D_1^2(A)\to D_1^2(\tilde{A})$ and $D_1^3(A)\oplus D_1^2(A)\to D_1^3(\tilde{A})$ are isomorphisms. By using these, 5-lemma applies to show that the map $D_1^4(A)\oplus D_1^3(A)\to D_1^4(\tilde{A})$ is an isomorphism. The repeated use of this type of argument implies that the induced maps $D_1^*(A)\oplus D_1^{*-1}(A)\to D_1^*(A)$ are isomorphisms. So the maps of the exact couples actually induce isomorphisms of higher derived couples and this proves the theorem.

Q.E.D.

§4. An Example

In this section, we give explicit generators of cyclic cohomology of $A=\boldsymbol{C}[X_1,X_1^{-1},\cdots,X_N,X_N^{-1},Y_1,\cdots,Y_M]$, $N, M\in\boldsymbol{N}$. By using our theorem (Theorem 3.10+Remark 3.11), we obtain $E_n^k(A)\simeq\boldsymbol{C}^{\binom{N}{k}}$ $n\geq 1$, where we used the basic fact that $E_n^0(\boldsymbol{C})\simeq\boldsymbol{C}, E_n^k(\boldsymbol{C})\simeq 0, k\geq 1$ for $n\geq 1$. Hence we can easily see that $H^{\text{even}}(A)\simeq H^{\text{odd}}(A)\simeq\boldsymbol{C}^{2^{N-1}}$. Let $a^0,\cdots,a^k\in A$ be of the following form:

$$a^j=\sum_{l_1^j\in\boldsymbol{Z}}\cdots\sum_{l_N^j\in\boldsymbol{Z}}\ \sum_{m_1^j\in\boldsymbol{N}}\cdots\sum_{m_M^j\in\boldsymbol{N}} a^j_{l_1^j,\cdots,l_N^j,m_1^j,\cdots,m_M^j} \times X_1^{l_1^j}\cdots X_N^{l_N^j}Y_1^{m_1^j}\cdots Y_M^{m_M^j},\quad 0\leq j\leq k. \tag{4.1}$$

We put

$$\begin{aligned}(4.2)\qquad &\phi_{e_{p_1}\wedge\cdots\wedge e_{p_k}}(a^0,\cdots,a^k)\\ &=\frac{(-1)^k}{k!}\sum_{l_1}\cdots\sum_{l_n}\Big(\sum_{\sigma\in\mathcal{S}(k)}\operatorname{sgn}(\sigma)l_{p_1}^{\sigma(1)}\cdots l_{p_k}^{\sigma(k)}\Big)\\ &\qquad\times a^0_{l_1^0,\cdots,l_N^0,0,\cdots,0}\cdots a^k_{l_1^k,\cdots,l_N^k,0,\cdots,0},\end{aligned}$$

where $\sum_{l_q}$ means $\sum_{l_q^0+\cdots+l_q^k=0}$, $1\leq q\leq N$, and $e_{p_1}\wedge\cdots\wedge e_{p_k}$ is viewed as an element of $\Lambda^k\boldsymbol{C}^N$ under the standing assumption that $e_1,\cdots,e_N$ is the fixed basis of $\boldsymbol{C}^N$. Then, it is seen that $\phi_{e_{p_1}\wedge\cdots\wedge e_{p_k}}\in Z_\lambda^k(A)$ and furthermore, they generate E_1^*. We will omit the proof in detail. But we shall give a rough sketch. Getting rid of $Y_1,\cdots,Y_M$ is easy. Only the evaluation of degree zero coefficient survive. For the part $X_1,\cdots,X_N$, we use induction. First we assume that the formula holds for $X_2,\cdots,X_N$, and use $\tilde{\Psi}_\lambda^*$, Ψ_λ^*; we obtain the desired generator. Actually, the maps $\tilde{\Psi}_\lambda^*$ leave the degree of the cyclic cocycle as before, and the maps Ψ_λ^* raise the degree by one.

There is another way of looking at these generators. On the algebra, there is a canonical normalized trace τ given by the evaluation of degree zero coefficient. We also have derivations $\delta_1, \cdots, \delta_N$ corresponding to the variables $X_1, \cdots, X_N$ which are given by

$$\delta_j(X_1^{l_1}\cdots X_N^{l_N}Y_1^{m_1}\cdots Y_M^{m_M}) = l_j X_1^{l_1}\cdots X_N^{l_N}Y_1^{m_1}\cdots Y_M^{m_M}, \tag{4.3}$$

$1\leq j\leq N$. Then it is seen that the trace is δ_j-invariant for $1\leq j\leq N$ and we can construct 2^N-numbers of cyclic cocycles. These cyclic cocycles differ from (4.2) by constant multiple $\frac{(-1)^k}{k!}$.

§5. Discussions

The first point is that all our discussion works well not only for $\mathbb{C}$ but also for any field k of characteristic zero.

The next point is that all our discussion works well also for the topological case. It means that if A is a unital topological ring, then our discussion works if we replace $A[X]$ by $\mathcal{S}(\mathbb{N})\hat{\otimes}A$ and $A[X,X^{-1}]$ by $\mathcal{S}(\mathbb{Z})\hat{\otimes}A$. In this case, our formula for $\mathcal{S}(\mathbb{Z})\hat{\otimes}A$ is interpreted as a special case of Künneth formula with one of the variables $C^\infty(\mathbb{T}^1)$ and our example in Section 4 gives the homology of $\mathbb{T}^N$.

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Note added: While this paper was being typed, R. Staffeldt let the author know of the preprint

Kassel, C., *Cyclic homology, comodules and mixed complexes*, in which a Künneth type exact sequence for cyclic homology is proved and the formulas which are analogous to ours are proved. The author is grateful to R. Staffeldt for his notice.

The author also received the preprint

Burghelea, D., *Künneth formula in cyclic homology*, in which the same kind of result as above was obtained.