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Radonification Problem for Cylindrical Measures on Tensor Products of Banach Spaces

By

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Abstract

An operator $w_1 \otimes w_2$ is said to be p-Radonifying if it maps every cylindrical measure of type p, defined on the tensor product $E \otimes F$ of two Banach spaces, into a Radon probability of order p on the completion of some normed product $G \otimes_\alpha H$. In this paper we prove that $w_1 \otimes w_2$ is p-Radonifying, $1<p<\infty$, if and only if it is $\tilde{p}$-summing.

§1. Introduction

The Radonification problem for cylindrical measures on Banach spaces has been studied by A. Badrikian, S. Chevet, B. Maurey, Y. Okazaki, L. Schwartz and others, cf. e.g. [1], [11], [12] and in the references stated there. In the Schwartz's approach to this problem one try to find all operators w: $E \to G$ which map every cylindrical measure on E of type p into a Radon probability on G of order p. Such operators are called p-Radonifying. The main result is: for $1<p<\infty$, w is p-Radonifying if and only if it is p-summing. For $0 \leqslant p \leqslant 1$, the situation is more complex.

B. Maurey considered in [5] a class of F-cylindrical probabilities on $E \otimes F$, which lies somewhere between cylindrical measures on $E \otimes F$ and probabilities on some completion of this space (nearer to the first ones). He tries to find (p, F)-Radonifying operators W: $E \otimes F \to G \otimes F$ of the form $W = w \otimes 1_F$, which map every F-cylindrical probability on $E \otimes F$ of type (p, F) into a Radon probability on some completion $G \hat{\otimes}_\alpha F$ of the space $G \otimes F$. It turns out that (p, F)-summing operators are (p, F)-Radonifying for $1<p<\infty$, under some additional assumptions on the norm α and on the space F (cf. [5], Exposé II, Théoréme 2). As an example, it is shown that if w: $E \to G$ is p-summing, then $w \otimes 1_F$:

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$E\otimes F\to G\hat{\otimes}_{\varepsilon}F$ is (p, F)-summing. Also, if w is p-left-nuclear, then $w\otimes 1_F$: $E\otimes F\to G\hat{\otimes}_{d_p}F$ is (p, F)-summing (cf. [2], Proposition 5 and [3]. Theorem 3).

In this paper we give an analogous result for the class of cylindrical measures on $E\otimes F$ of type p. It will be shown that $\tilde{p}$-summing operators of the form $w_1\otimes w_2$ are p-Radonifying, for $1<p<\infty$.

Let $1<p<\infty$. By classical methods it is easy to obtain that an operator of the form $1_E\otimes w$: $E\otimes F\to E\otimes H$, where w: $F\to H$ is p-summing, maps every cylindrical measure on $E\otimes F$ of type p into an H-cylindrical probability on $E\otimes H$ of type (p, H). Thus, Maurey's result shows that the tensor product $w_1\otimes w_2$ of two p-summing operators is p-Radonifying, from $E\otimes F$ into $G\hat{\otimes}_{\varepsilon}H$, under additional assumption of reflexivity of the space H.

The direct approach which we use in this paper gives something more. Namely, [2] Theorem 3 shows that the product $w_1\otimes w_2$ of two p-summing operators is $\tilde{p}$-summing from $E\otimes F$ into $G\hat{\otimes}_{\alpha}H$, whenever α satisfies $\alpha\leqslant /d_p$ or $\alpha\leqslant g_p\backslash$. Thus, Theorem 6.3 states that such operator is p-Radonifying from $E\otimes F$ in $G\hat{\otimes}_{\alpha}H$, $\alpha\leqslant /d_p$ or $\alpha\leqslant g_p\backslash$, without assumption on the reflexivity of the space H.

§2 is preparatory. In §3 we define cylindrical measures on $E\otimes F$ and establish the connection between them and probabilities on some completion $E\hat{\otimes}_{\alpha}F$. Other necessary notions (type, convergence, Fourier transform, image of cylindrical measure by an operator of the form $w_1\otimes w_2$, etc.) are introduced in §4 and §5. The main theorem, announced before, is proved in §6.

§2. Notation and some Preliminary Results

Throughout this paper E, F, G, H will denote real Banach spaces, E', F', G', H' their topological duals. $\mathcal{L}(E, G)$ stands for the space of all continuous linear operators: $E\to G$. By $[u, x']\in F$ we denote the action of an element $u\in E\otimes F$ on vectors in E'. The element $u\in E\otimes F$ induces a finite-dimensional linear operator $\hat{u}$: $E'\to F$ by $\hat{u}x' := [u, x']$. $\langle\circ, \circ\rangle$ will always denote the canonical pairing, in various settings, e.g., for $x'\in E'$, $y'\in F'$ and $u\in E\otimes F$ it holds $\langle u, x'\otimes y'\rangle = \langle [u, x'], y'\rangle = \langle \hat{u}x', y'\rangle$.

Let $\{x_j\}$ be a sequence in E. By $N_p(x_j)$ we denote the number, finite or not

$$N_p(x_j) := \begin{cases} \{\sum_j \|x_j\|^p\}^{1/p} & , 1\leqslant p<\infty \\ \sup_j \|x_j\| & , p=\infty \end{cases}$$

and by

$$M_p(x_j) := \sup \{N_p(\langle x_j, x'\rangle), \|x'\| \leqslant 1\}$$

For $\{u_j\} \subset E \otimes F$ (and similarly for $\{v_j\} \subset \mathcal{L}(E, G)$) we denote

$$Q_p(u_j) := \sup \{N_p(\langle u_j, x' \otimes y'\rangle), \|x'\| \leqslant 1, \|y'\| \leqslant 1\}$$
$$S_p(u_j) := \sup \{N_p([u_j, x']), \|x'\| \leqslant 1\}$$

Linear operators $w: E \to G$, $W: E \otimes F \to G$ for which it exists a constant $C \geqslant 0$ such that

$$N_p(wx_j) \leqslant C \; M_p(x_j) \tag{2.1}$$

$$N_p(W(u_j)) \leqslant C \; S_p(u_j) \tag{2.2}$$

$$N_p(W(u_j)) \leqslant C \; Q_p(u_j) \tag{2.3}$$

for all finite sets $\{x_1, \cdots, x_n\}$ in E or $\{u_1, \cdots, u_n\}$ in $E \otimes F$, are called p-summing, (p, F)-summing and $\tilde{p}$-summing (respectively). The infimum of all constants C in (2.1)–(2.3) is denoted by $\pi_p(w)$, $\pi_{p,F}(W)$, $\tilde{\pi}_p(W)$, respectively.

It is known that $\tilde{p}$-summing operators: $E \otimes F \to G$ are (p, F)-summing, and also a p-summing from $E \otimes_\alpha F$ into G, for arbitrary reasonable norm α (cf. [2], Proposition 1) and hence continuous from $E \otimes_\varepsilon F$ into G, with the norm

$$\|W\| \leqslant \tilde{\pi}_p(W) \tag{2.4}$$

(ε denotes the least reasonable crossnorm).

§3. Cylindrical Measures on $E \otimes F$ and Radon Probabilities on $E \hat{\otimes}_\alpha F$

By $FC(E)$ we denote the family of all closed subspaces in E of the finite codimension. The canonical projection $E \to E/N$, $N \in FC(E)$ is denoted by π_N, the projections $E/N_1 \to E/N_2$, $N_1 \subset N_2$ by $\pi_{N_2 N_1}$. A cylindrical measure on E is a projective system $\{\lambda_N, \pi_N, N \in FC(E)\}$ of Radon probabilities on finite-dimensional quotients of the space E; for $N_1 \subset N_2$ it holds $\lambda_{N_2} = \pi_{N_2 N_1}(\lambda_{N_1})$. It is well known that such system defines a finitely additive measure λ on the algebra of cylindrical sets in E, by

$$\lambda(B) := \lambda_N(B_N)$$

where $B = \pi_N^{-1}(B_N)$, and B_N is a Borel set in E/N. We denote $\lambda = (\lambda_N)$.

For convenience, we denote $\pi_{N \otimes M} := \pi_N \otimes \pi_M$: $E \otimes F \to (E/N) \otimes (F/M)$, $N \in FC(E)$, $M \in FC(F)$, and by $\pi_{N_2 N_1 \otimes M_2 M_1} := \pi_{N_2 N_1} \otimes \pi_{M_2 M_1}$: $(E/N_1) \otimes (F/M_1) \to (E/N_2) \otimes (F/M_2)$, $N_1 \subset N_2$, $M_1 \subset M_2$.

The following is obvious:

Proposition 3.1. *If $N_1 \subset N_2$, $M_1 \subset M_2$ are closed subspaces of the finite codimension, then the following diagram commutes*:

$$\begin{array}{ccc} & \nearrow^{\pi_{N_1 \otimes M_1}} & (E/N_1)\otimes(F/M_1) \\ E\otimes F & & \downarrow \pi_{N_2 N_1 \otimes M_2 M_1} \\ & \searrow_{\pi_{N_2 \otimes M_2}} & (E/N_2)\otimes(F/M_2) \end{array} \tag{3.1}$$

Definition. A *cylindrical measure* λ on $E\otimes F$ is a projective system $\{\lambda_{N\otimes M}, \pi_{N\otimes M}, N\in FC(E), M\in FC(F)\}$ of the Radon probabilities on finite dimensional spaces $(E/N)\otimes(F/M)$.

$\mathcal{M}^c(E\otimes F)$ stands for the space of all cylindrical measures on $E\otimes F$.

The cylindrical algebra on a vector space X depends only on the dual pair (X, X'), and remains the same if the original topology on X is replaced by another which gives the same dual. More generally, if (X, Y) is a pair of vector spaces in duality, such that Y separates points in X, then the cylindrical algebra on X depends only on the space Y.

$(E\otimes F, E'\otimes F')$ is a pair of vector spaces in separated duality. Hence, we can define the cylindrical algebra on $E\otimes F$ not introducing any topology on $E\otimes F$. A cylinder is a set of the form

$$C = \{u\in E\otimes F: (\langle u, u'_j\rangle)_{1\leq j\leq n}\in B\} \tag{3.2}$$

where $n\in N$, $u'_1, \cdots, u'_n\in E'\otimes F'$ and B is a set in the Borel algebra $\mathcal{B}(\boldsymbol{R}^n)$.

It is not quite obvious that cylindrical measure can measure the cylinders! Namely, sets C of the form (3.2) need not be of the form $\{u\in E\otimes F: \pi_{N\otimes M}(u)\in B\}$ for some $N\in FC(E)$, $M\in FC(F)$, so we must prove that $\lambda(C)$ is (well) defined. For the sequel, it will be sufficient to prove

Proposition 3.2. *Let λ be a cylindrical measure on $E\otimes F$. For $u'\in E'\otimes F'$ and $B\in\mathcal{B}(\boldsymbol{R})$, the measure*

$$u'(\lambda)(B) := \lambda\{u\in E\otimes F: \langle u, u'\rangle\in B\}$$

is well defined.

Proof. Take a representation $u' = \sum_{j=1}^{n} \xi'_j \otimes \eta'_j$ where n is minimal with this property. Then, $\{\xi'_1, \cdots, \xi'_n\}$ and also $\{\eta'_1, \cdots, \eta'_n\}$ are linearly independent (cf. [10], Lemma 1.2). Denote $N^0 := \text{span}\{\xi_1', \cdots, \xi_n'\} \subset E'$, $M^0 := \text{span}\{\eta_1', \cdots, \eta_n'\} \subset F'$, and let $N := (N^0)^0 \subset E$, $M := (M^0)^0 \subset F$ be their polars. By Auerbach lemma, there exists a basis $\{x_1', \cdots, x_n'\}$ for N^0 and a basis $\{\bar{x}_1, \cdots, \bar{x}_n\}$ for $E/N \simeq (N^0)'$ such that $\langle x_i', \bar{x}_j \rangle = \delta_{ij}$ (cf. [6], p. 22). Similarly, there exists $\{y_1', \cdots, y_n'\} \subset M^0$ and $\{\bar{y}_1, \cdots, \bar{y}_n\} \subset F/M$ with the same properties. Then it holds $\pi_N = \sum_{j=1}^n x_j' \otimes \bar{x}_j$, $\pi_M = \sum_{k=1}^n y_k' \otimes \bar{y}_k$, and u' has a representation $u' = \sum_{j,k=1}^n t_{jk} x_j' \otimes y_k'$.

Define now ρ: $(E/N) \otimes (F/M) \to \mathbb{R}$ by $\rho(\bar{x}_j \otimes \bar{y}_k) := t_{jk}$. ρ is continuous linear mapping, and it holds

$$\begin{aligned}(\rho \circ \pi_{N \otimes M})(u) &= \rho\Big(\sum_{j,k} \langle u, x_j' \otimes y_k' \rangle \bar{x}_j \otimes \bar{y}_k\Big) \\ &= \sum_{j,k} t_{jk} \langle u, x_j' \otimes y_k' \rangle = \langle u, u' \rangle\end{aligned}$$

for every $u \in E \otimes F$. Thus, $u'(\lambda) = (\rho \circ \pi_{N \otimes M})(\lambda) = \rho(\lambda_{N \otimes M})$ is well defined probability on $\mathbb{R}$.

In the classical situation, every probability (normed σ-additive measure on the Borel σ-algebra $\mathcal{B}(E)$) on a separable Banach space defines a cylindrical measure (finitely additive measure on the smaller cylindrical algebra) in the unique way. Moreover, for a given cylindrical measure λ on the cylindrical algebra $\mathcal{A}(E)$, there exists at most one probability μ such that $\mu | \mathcal{A}(E) = \lambda$. The necessary and sufficient condition for the existence of μ is the σ-additivity of λ. The reason for this uniqueness lies in the fact that $\mathcal{B}(E)$ is the σ-algebra generated by $\mathcal{A}(E)$. Even in the non-separable case, a cylindrical measure defines at most one Radon probability on E ([11], p. 174), the tightness property of Radon probabilities becomes now essential.

The connection between cylindrical measures on $E \otimes F$ and Radon probabilities on some completion $E \hat{\otimes}_\alpha F$ of this space is more complex. If we wish to obtain an one-to-one correspondence between them, we need some additional conditions on the norm α. Namely, the cylindrical algebra on $E \otimes F$ is far away from the cylindrical algebra on $E \hat{\otimes}_\alpha F$, the first one is considerably smaller.

Let us suppose that α satisfies

$$|\langle u, x' \otimes y' \rangle| \leqslant \alpha(u) \|x'\| \, \|y'\| \tag{3.3}$$

for all $x' \in E'$, $y' \in F'$ and $u \in E \otimes F$. If ψ denotes the canonical embedding $E \otimes_\varepsilon F \to \mathcal{L}(E', F)$, then (3.3) ensures that ψ is continuous, with the norm $\leqslant 1$. Hence, it can be extended by continuity to the completion, $\hat{\psi}$: $E \hat{\otimes}_\varepsilon F \to \mathcal{L}(E', F)$.

Let μ be a Radon probability on $E\hat{\otimes}_\alpha F$. It is natural to put $\mu_{N\otimes M}:=\pi_{N\otimes M}(\mu)$ in order to obtain a cylindrical measure on $E\otimes F$, but first we must be sure that the operator $\pi_{N\otimes M}\colon E\otimes F\to(E/N)\otimes(F/M)$ has a continuous prolongation to the space $E\hat{\otimes}_\alpha F$.

We can identify an element $\pi_{N\otimes M}(u)\in(E/N)\otimes(F/M)$ with the operator

$$(\pi_{N\otimes M}(u))^\wedge=\pi_M\circ\hat{u}\circ{}^t\pi_N\colon (E/N)'\to F/M$$

Therefore, we can write $\pi_{N\otimes M}=\rho_{N,M}\circ\psi$ where $\rho_{N,M}\colon \mathcal{L}(E', F)\to\mathcal{L}((E/N)', (F/M))$ is defined by

$$\rho_{N,M}(w):=\pi_M\circ w\circ{}^t\pi_N$$

Thus, since α satisfy (3.3)

$$\hat{\pi}_{N\otimes M}(u):=(\rho_{N,M}\circ\hat{\psi})\,(u)$$

defines a mapping from $E\hat{\otimes}_\alpha F$ into $(E/N)\otimes(F/M)$ which prolongues $\pi_{N\otimes M}$.

Hence, if α satisfies (3.3) and μ is a probability on $E\hat{\otimes}_\alpha F$, then μ defines a cylindrical measure on $E\otimes F$ by

$$\mu_{N\otimes M}:=\hat{\pi}_{N\otimes M}(\mu)$$

We denote this cylindrical measure by $\check{\mu}=(\mu_{N\otimes M})$.

The inverse connection is more interesting for us.

Definition. *Cylindrical measure λ on $E\otimes F$ is a Radon probability on $E\hat{\otimes}_\alpha F$* if there exists a unique Radon probability μ on $E\hat{\otimes}_\alpha F$ such that $\check{\mu}=\lambda$.

The condition on uniqueness is essential. Namely, if it exists a probability μ on $E\hat{\otimes}_\alpha F$ such that $\lambda=\check{\mu}$, it cannot be assumed a priori that μ is unique (as in the classical case). A sufficient condition for this is due by Prohorov (cf. [11], Theorem 22, p. 81): it is sufficient that the mappings $\{\hat{\pi}_{N\otimes M}, N\in FC(E), M\in FC(F)\}$ separate points of $E\hat{\otimes}_\alpha F$.

Proposition 3.3. *The following conditions are equivalent*:

(i) $\{\hat{\pi}_{N\otimes M}, N\in FC(E), M\in FC(F)\}$ *separate points of* $E\hat{\otimes}_\alpha F$.

(ii) $\{u\mapsto\langle u, x'\otimes y'\rangle, x'\in E', y'\in F'\}$ *separate points of* $E\hat{\otimes}_\alpha F$.

(iii) $\{u\mapsto[u, x'], x'\in E'\}$ *separate points of* $E\hat{\otimes}_\alpha F$.

(iv) $\hat{\psi}\colon E\hat{\otimes}_\alpha F\to\mathcal{L}(E', F)$ *is one-to-one*

Proof. Let us prove only (i)$\Rightarrow$(ii). Take $u\in E\hat{\otimes}_\alpha F$, $u\neq0$, and $N\in FC(E)$, $M\in FC(F)$ for which holds $\hat{\pi}_{N\otimes M}(u)\neq0$, i.e. $\pi_M\circ\hat{u}\circ{}^t\pi_N\neq0$. There exist $x'_N\in(E/N)'$, $y'_M\in(F/M)'$ such that

$$\langle(\pi_M \circ \hat{u} \circ {}^t\pi_N)(x'_N), y'_M\rangle \neq 0$$

Define $x' := {}^t\pi_N x'_N$, $y' := {}^t\pi_M y'_M$. Then it holds $\langle u, x'\otimes y'\rangle \neq 0$.

In the sequel we will assume that α satisfies (3.3) *and*

$$\hat{\psi}: E\hat{\otimes}_\alpha F \to \mathcal{L}(E', F) \text{ is one-to-one} \tag{3.4}$$

Thus, from the well known conditions for the fulfilment of (3.3) and (3.4) we obtain

Proposition 3.4. *Let α be a reasonable norm and λ a cylindrical measure on $E\otimes F$. If E or F have the metric approximation property, and if it exists a probability μ on $E\hat{\otimes}_\alpha F$ for which holds $\check{\mu}=\lambda$, then μ is unique.*

We will denote such probability by $\hat{\lambda}$. Hence, in this case we have $(\hat{\lambda})^\vee = \lambda$, $(\check{\mu})^\wedge = \mu$.

§4. Fourier Transform and Image of Cylindrical Measure

Let λ be a cylindrical measure on $E\otimes F$, and $w_1: E\to G$, $w_2: F\to H$ continuous linear operators. Denote $W := w_1\otimes w_2$. First we define the image $W(\lambda)$.

Take $X\in FC(G)$. Then $N := w_1^{-1}(X)\in FC(E)$ and all the operators in the following commutative diagram are continuous:

$$\begin{array}{ccc} E & \xrightarrow{\quad w_1 \quad} & G \\ \Big\downarrow{\scriptstyle \pi_N} & & \Big\downarrow{\scriptstyle \pi_X} \\ E/w_1^{-1}(X) & \xrightarrow[\quad (w_1)_X \quad]{} & G/X \end{array} \tag{4.1}$$

If we take now $Y\in FC(H)$, then $M := w_2^{-1}(Y)\in FC(F)$ and the Radon probability $\lambda_{N\otimes M}$ on the space $(E/w_1^{-1}(X))\otimes(F/w_2^{-1}(Y))$ is well defined. Denote $W_{X\otimes Y} := (w_1)_X\otimes(w_2)_Y$.

Definition. The *image* $(w_1\otimes w_2)(\lambda)$ of a cylindrical measure λ is a cylindrical measure on $G\otimes H$ defined by

$$(w_1\otimes w_2)(\lambda)_{X\otimes Y} := W_{X\otimes Y}(\lambda_{N\otimes M}) \tag{4.2}$$

The *Fourier transform* of a cylindrical measure is defined similarly as in the classical case: for $u'\in E'\otimes F'$ and $s\in \mathbb{R}$ we define

$$\mathcal{F}(\lambda)\,(s\,u') := \mathcal{F}(u'(\lambda))\,(s) \tag{4.3}$$

where $u'(\lambda)$ is a probability on $\boldsymbol{R}$, defined in Proposition 3.2.

Proposition 4.1. *For* $w_1 \in \mathcal{L}(E, G)$, $w_2 \in \mathcal{L}(F, H)$ *it holds*

$$\mathcal{F}((w_1 \otimes w_2)\,(\lambda))\,(u') = \mathcal{F}(\lambda)\,(({}^t w_1 \otimes {}^t w_2)\,(u'))$$

Proof. Let $u' = \sum x'_j \otimes y'_j$ be a representation of u'. We have

$$\begin{aligned} u' \circ (w_1 \otimes w_2) &= (\sum x'_j \otimes y'_j) \circ (w_1 \otimes w_2) \\ &= \sum (x'_j \circ w_1) \otimes (y'_j \circ w_2) \\ &= \sum {}^t w_1 x'_j \otimes {}^t w_2 y'_j = ({}^t w_1 \otimes {}^t w_2)\,(u') \end{aligned}$$

and the Proposition follows.

The Fourier transformation establishes a one-to-one correspondence between cylindrical measures on Banach (more generally, locally convex) space E and functions on E' of positive type, whose restrictions to the finite dimensional subspaces are continuous. This is an easy consequence of Bochner theorem (see [1], p. 19). The same proof gives:

Proposition 4.2. *A function* ϕ: $E' \otimes F' \to \boldsymbol{C}$ *is the Fourier transform of a cylindrical measure* λ *on* $E \otimes F$ *if and only if* ϕ *satisfies*

(i) $\phi(0) = 1$

(ii) ϕ *is of positive type, i.e., for all* $n \in \boldsymbol{N}$, $u'_1, \cdots, u'_n \in E' \otimes F'$ *and* $\zeta_1, \cdots, \zeta_n \in \boldsymbol{C}$ *it holds*

$$\sum_{i,j=1}^{n} \zeta_i \bar{\zeta}_j \phi(u'_j - u'_i) \geqslant 0$$

(iii) *The restriction of* ϕ *to finite dimensional subspaces of* $E' \otimes F'$ *is continuous.*

Lemma 4.3. *For a cylindrical measure* λ *on* $E \otimes F$, *the following is equivalent*:

(i) $(x', y') \mapsto \mathcal{F}(\lambda)\,(x' \otimes y')$ *is continuous on* $E' \times F'$.

(ii) *If* $(x'_j)_\gamma \to x'_j$ *and* $(y'_j)_\gamma \to y'_j$ $(j = 1, \cdots, n)$, *then*

$$\mathcal{F}(\lambda)\,(\sum_{j=1}^{n} (x'_j)_\gamma \otimes (y'_j)_\gamma) \to \mathcal{F}(\lambda)\,(\sum_{j=1}^{n} x'_j \otimes y'_j)$$

Proof. Denote $\phi := \mathcal{F}(\lambda)$. From the classical inequality

$$|\phi(u'_1) - \phi(u'_2)|^2 \leqslant 2\,|1 - \phi(u'_1 - u'_2)|$$

we obtain

$$|\phi(\sum_{j=1}^{n} u'_j)-1| \leqslant \sum_{k=1}^{n} |\phi(\sum_{j=1}^{k} u'_j)-\phi(\sum_{j=1}^{k-1} u'_j)| \leqslant \sqrt{2}\sum_{k=1}^{n} |1-\phi(u'_k)|^{1/2}$$

Thus, it holds

$$\begin{aligned}|\phi(\sum_{j=1}^{n} (x'_j)_\gamma\otimes(y'_j)_\gamma)-\phi(\sum_{j=1}^{n} x'_j\otimes y'_j)|^2 &\leqslant 2|1-\phi\{\sum_{j=1}^{n} ((x'_j)_\gamma-x'_j)\otimes(y'_j)_\gamma \\ &+\sum_{j=1}^{n} x'_j\otimes((y'_j)_\gamma-y'_j)\}| \leqslant 2\sqrt{2}\sum_{k=1}^{n} \{|1-\phi[((x'_k)_\gamma-x'_k)\otimes(y'_k)_\gamma]|^{1/2} \\ &+|1-\phi[x'_k\otimes((y'_k)_\gamma-y'_k)]|^{1/2}\}\end{aligned}$$

which converges to 0 if (i) is satisfied. The converse is obvious.

§5. Type and Approximability

A probability μ on a Banach space G is of *order* p $(0<p<\infty)$ if

$$||\mu||_p := \{\int_G ||z||^p d\mu(z)\}^{1/p}<\infty$$

Let λ be a cylindrical measure on $E\otimes F$. For $x'\in E', y'\in F'$, the image $(x'\otimes y')(\lambda)$ is a probability on $\mathbb{R}$. We say that λ is of *type* p $(0<p<\infty)$ if

$$||\lambda||_p^* := \sup\{||(x'\otimes y')(\lambda)||_p, ||x'||\leqslant 1, ||y'||\leqslant 1\}<\infty$$

λ is of type 0 if for every $\eta>0$ it exists $R>0$ such that

$$\sup\{[(x'\otimes y')(\lambda)](]R, \infty[), ||x'||\leqslant 1, ||y'||\leqslant 1\}<\eta$$

The set of all cylindrical measures on $E\otimes F$ of type p is denoted by $\mathcal{M}_p^c(E\otimes F)$.

It is evident that a cylindrical measure of type p_1 is also of type p_2, for $p_1>p_2$.

The following proposition is obvious:

Proposition 5.1. *Let λ be a cylindrical measure of type $p>0$ on $E\otimes F$, $w_1\in\mathcal{L}(E,G)$, $w_2\in\mathcal{L}(F,H)$ and $W=w_1\otimes w_2$. Then $W(\lambda)$ is a cylindrical measure of type p on $G\otimes H$, and*

$$||W(\lambda)||_p^*\leqslant||w_1|| \, ||w_2|| \, ||\lambda||_p^* \tag{5.1}$$

We say that a linear operator W is *p-Radonifying* if it maps every cylindrical measure on $E\otimes F$ of type p into a Radon probability of order p.

The following lemma establishes a connection between the type of a cylindrical measure and the continuity of its Fourier transform. The proof is identical to the classical one (cf. [1], p. 26) so we omit it:

Lemma 5.2. *A cylindrical measure λ on $E\otimes F$ is of type 0 if and only if* *the mapping* $(x', y') \mapsto \mathcal{F}(\lambda)(x' \otimes y')$ *is continuous on* $E' \times F'$.

Definition. A net $\{\lambda_\gamma, \gamma \in \Gamma\}$ of cylindrical measures converges $\otimes$-*cylindrically* to $\lambda \in \mathcal{M}^c(E \otimes F)$ if $(\lambda_\gamma)_{N \otimes M}$ converges to $\lambda_{N \otimes M}$ weakly, for every $N \in FC(E)$, $M \in FC(F)$.

The notion of $\otimes$-cylindrical convergence has also the sense for Radon probabilities on $E \hat{\otimes}_\alpha F$:

Proposition 5.3. *On* $\mathcal{M}(E \hat{\otimes}_\alpha F)$ *we have*

(i) *If* $\mu_\gamma \to \mu$ *weakly, then* $\mu_\gamma \to \mu$ $\otimes$-*cylindrically*

(ii) *If* $\mu_\gamma \to \mu$ $\otimes$-*cylindrically, then* μ *is unique.*

Proof. (i) is immediate. If $\mu_\gamma \to \mu$ $\otimes$-cylindrically, then $\check{\mu}_\gamma \to \check{\mu}$ $\otimes$-cylindrically, and thus $(\mu_\gamma)_{N \otimes M} \to \mu_{N \otimes M}$ weakly. Let us suppose $\mu_\gamma \to \nu$ $\otimes$-cylindrically. Then $\nu_{N \otimes M} = \mu_{N \otimes M}$ for all $N \in FC(E)$, $M \in FC(F)$ so that $\check{\nu} = \check{\mu}$ and hence $\nu = \mu$, since α satisfies (3.4).

Proposition 5.4. *If* $w_1 \in \mathcal{L}(E, G)$, $w_2 \in \mathcal{L}(F, H)$ *and* $\lambda_\gamma \to \lambda$ $\otimes$-*cylindrically, then* $(w_1 \otimes w_2)(\lambda_\gamma) \to (w_1 \otimes w_2)(\lambda)$ $\otimes$-*cylindrically.*

Proof. Denote $W = w_1 \otimes w_2$. Let $W_{X \otimes Y}$ be the continuous linear operator defined in (4.1). By definition, $(\lambda_\gamma)_{N \otimes M} \to \lambda_{N \otimes M}$ weakly, where $N = w_1^{-1}(X)$, $M = w_2^{-1}(Y)$. Then,

$$(W(\lambda_\gamma))_{X \otimes Y} = W_{X \otimes Y}((\lambda_\gamma)_{N \otimes M}) \to W_{X \otimes Y}(\lambda_{N \otimes M}) = (W(\lambda))_{X \otimes Y}$$

The following lemma represents an essential step in the Radonification problem:

Lemma 5.5. *Suppose* E' *and* F' *have the metric approximation property. If* λ *is a cylindrical measure on* $E \otimes F$ *of type* $p > 0$, *then there is a net* $\{\lambda_\gamma\}$ *of Radon probabilities on* $E \otimes F$ (*each of them is concentrated on some finite-dimensional subspace*) *such that* $\{\lambda_\gamma\}$ *converges* $\otimes$-*cylindrically to* λ, *and*

$$\|\lambda_\gamma\|_p^* \leqslant \|\lambda\|_p^* \tag{5.2}$$

Proof. There exist finite dimensional operators $p_\gamma\colon E' \to E'$ and $q_\gamma\colon F' \to F'$ which converge pointwise to the identities, and it holds $\|p_\gamma\| \leqslant 1$, $\|q_\gamma\| \leqslant 1$. We can further suppose that p_γ and q_γ are weakly*-continuous [9]. Thus, there exist finite-dimensional operators ${}^t p_\gamma\colon E \to E$, ${}^t q_\gamma\colon F \to F$.

Define $\lambda_\gamma := ({}^t p_\gamma \otimes {}^t q_\gamma)(\lambda)$. λ_γ is a Radon probability, concentrated on some finite-dimensional space. Moreover, Proposition 5.1 gives

$$||\lambda_\gamma||_p^* \leqslant ||^t p_\gamma|| \cdot ||^t q_\gamma|| \cdot ||\lambda||_p^* \leqslant ||\lambda||_p^*$$

It remains to prove that $\lambda_\gamma \to \lambda$ $\otimes$-cylindrically.

It is sufficient to obtain that $\mathscr{F}((\lambda_\gamma)_{N\otimes M})$ converges to $\mathscr{F}(\lambda_{N\otimes M})$ uniformly on compact sets. Let $u' \in ((E/N)\otimes(F/M))'$. Then

$$\begin{aligned}\mathscr{F}((\lambda_\gamma)_{N\otimes M})\,(u') &= \mathscr{F}(\pi_{N\otimes M}({}^t p_\gamma \otimes {}^t q_\gamma)\,(\lambda))\,(u') \\ &= \mathscr{F}(\lambda)\,((p_\gamma \otimes q_\gamma)\,({}^t\pi_{N\otimes M}\,(u'))\end{aligned}$$

Take a representation ${}^t\pi_{N\otimes M}(u') = \sum_{j=1}^{n} t_j x'_j \otimes y'_j$, where $\{x'_j \otimes y'_j\}$ is a basis of the space ${}^t\pi_{N\otimes M}[((E/N)\otimes(F/M))'] \subset E'\otimes F'$. x'_j and y'_j can be taken such that $||x'_j|| = ||y'_j|| = 1$ holds.

Denote $(x'_j)_\gamma := p_\gamma x'_j$, $(y'_j)_\gamma := q_\gamma y'_j$. Then $(x'_j)_\gamma \to x'_j$ and $(y'_j)_\gamma \to y'_j$.

By assumption, λ is of type $p>0$, hence also of type 0. By Lemma 5.2, $\mathscr{F}(\lambda)$ is continuous on $E'\times F'$. Hence, by Lemma 4.2

$$\mathscr{F}(\lambda)\,(\sum t_j (x'_j)_\gamma \otimes (y'_j)_\gamma) \to \mathscr{F}(\lambda)\,(\sum t_j x'_j \otimes y'_j)$$

uniformly on bounded $\{t_j\}$. The lemma is proved.

§6. Cylindrical Measures and $\tilde{p}$-summing Operators

Let (Ω, Σ, P) be a probability space, and $f\colon \Omega \to E\hat{\otimes}_\varepsilon F$ such that $\omega \mapsto \langle f(\omega), x'\otimes y'\rangle$ is measurable function, for all $x'\in E'$, $y'\in F'$. Define

$$||f||_p^* := \sup_{\substack{||x'||\leqslant 1\\ ||y'||\leqslant 1}} \{\int_\Omega |\langle f(\omega), x'\otimes y'\rangle|^p dP(\omega)\}^{1/p}$$

and denote by $L_p^*(\Omega, \Sigma, P;\ E\hat{\otimes}_\varepsilon F)$ the space of all such functions for which it holds $||f||_p^* < \infty$.

Proposition 6.1. *Let $f \in L_p^*(\Omega, \Sigma, P;\ E\hat{\otimes}_\varepsilon F)$. If $W\colon E\otimes F \to G$ is $\tilde{p}$-summing, then*

$$\{\int_\Omega ||W(f(\omega))||^p dP(\omega)\}^{1/p} \leqslant \tilde{\pi}_p(W)||f||_p^* \tag{6.1}$$

Proof. Denote by K_1 the unit ball of the space E', with the weak topology $\sigma(E', E)$, similarly for $K_2 \subset F'$. By Pietsch Majorization theorem for $\tilde{p}$-summing operators (cf. [2], Theorem 1), there exists a Radon probability μ on the compact space $K := K_1 \times K_2$, such that for every $u \in E\otimes F$ it holds

$$||W(u)|| \leqslant \tilde{\pi}_p(W)\{\int_K |\langle u, x'\otimes y'\rangle|^p d\mu(x', y')\}^{1/p} \tag{6.2}$$

Take $u \in E\hat{\otimes}_\varepsilon F$, and $\{u_k\} \subset E\otimes_\varepsilon F$, $u_k \to u$. Then $W(u)$ is well defined, since a $\tilde{p}$-summing operator can be extended by continuity on $E\hat{\otimes}_\varepsilon F$. Moreover, since the ε-norm topology on $E\otimes F$ is stronger than the weak topology $\sigma(E\otimes F, E'\otimes F')$, it holds $\langle u_k, x'\otimes y'\rangle \to \langle u, x'\otimes y'\rangle$ and, by Dominated Convergence theorem

$$\int_K |\langle u_k, x'\otimes y'\rangle|^p d\mu(x', y') \to \int_K |\langle u, x'\otimes y'\rangle|^p d\mu(x', y')$$

Thus, (6.2) holds for all $u\in E\hat{\otimes}_\varepsilon F$.

By Foubini's theorem, we have

$$\begin{aligned}
&\{\int_\Omega \|W(f(\omega))\|^p dP(\omega)\}^{1/p} \\
&\leqslant \tilde{\pi}_p(W)\{\int_\Omega\int_K |\langle f(\omega), x'\otimes y'\rangle|^p d\mu(x', y')dP(\omega)\}^{1/p} \\
&= \tilde{\pi}_p(W)\{\int_K\int_\Omega |\langle f(\omega), x'\otimes y'\rangle|^p dP(\omega)d\mu(x', y')\}^{1/p} \\
&\leqslant \tilde{\pi}_p(W) \sup_{\substack{\|x'\|\leqslant 1 \\ \|y'\|\leqslant 1}} \{\int_\Omega |\langle f(\omega), x'\otimes y'\rangle|^p dP(\omega)\}^{1/p} \\
&= \tilde{\pi}_p(W)\|f\|_p^*
\end{aligned}$$

which proves (6.1).

Take now $\Omega = E\hat{\otimes}_\varepsilon F$, $P=\mu$, $f\colon \Omega \to E\hat{\otimes}_\varepsilon F$ identity. Then

$$\|f\|_p^* = \sup_{\substack{\|x'\|\leqslant 1 \\ \|y'\|\leqslant 1}} \{\int_\Omega |\langle u, x'\otimes y'\rangle|^p d\mu(x', y')\}^{1/p} = \|\mu\|_p^*$$

Thus, Proposition 6.1 gives:

Corollary 6.2. *If $W\colon E\otimes F\to G$ is $\tilde{p}$-summing, μ a Radon probability on $E\hat{\otimes}_\varepsilon F$ of type p, then*

$$\|W(\mu)\|_p \leqslant \tilde{\pi}_p(W)\|\mu\|_p^* \tag{6.3}$$

Remark. The main difficulty in the proof above is crossing to the completion $E\hat{\otimes}_\varepsilon F$. This is necessary since the notion "Radon measure on $E\otimes F$" has no sense ($E\otimes F$ has no topology). But, if μ is concentrated on some finite dimensional space, and has the type p (thus, $\mu\in\mathcal{M}_p^c(E\otimes F)$), then Corollary 6.2 remains true. We will apply this corollary only for such measures.

We are ready to prove the main result:

Theorem 6.3. *Let E, F, G, H be Banach spaces, $1<p<\infty$ and α a norm on $G\otimes H$ which satisfies* (3.3) *and* (3.4). *$W := w_1\otimes w_2\colon E\otimes F\to G\hat{\otimes}_\alpha H$ is p-Radonifying if and only if it is $\tilde{p}$-summing, and for $\lambda\in\mathcal{M}_p^c(E\otimes F)$ it holds*

$$||W(\lambda)||_p \leqslant \tilde{\pi}_p(W)||\lambda||_p^* \tag{6.4}$$

Remark. If $w_1 \otimes w_2$ is $\tilde{p}$-summing, it is known that w_1 and w_2 must also be p-summing, and it holds $\tilde{\pi}_p(W) = \pi_p(w_1)\pi_p(w_2)$; moreover, if w_1 and w_2 are p-summing, then $w_1 \otimes w_2$ is $\tilde{p}$-summing whenever the $\otimes$-norm α satisfies $\alpha \leqslant /d_p$ or $\alpha \leqslant g_p\backslash$ (cf. [2], Theorem 3). See [7] for description of the norm $/d_p$ and $g_p\backslash$.

Another example of $\tilde{p}$-summing operators of the form $w_1 \otimes w_2$ gives (cf. [2], Corollary 1):

Corollary 6.4. *Let w_1: $E \to G$ be p-left-nuclear, and w_2: $F \to H$ p-summing, $1 < p < \infty$. Then $w_1 \otimes w_2$: $E \otimes F \to G \hat{\otimes}_{d_p} H$ and $w_2 \otimes w_1$: $F \otimes E \to H \hat{\otimes}_{g_p} G$ are p-Radonifying, and for $\lambda \in \mathcal{M}_p^c(E \otimes F)$ it holds*

$$||W(\lambda)||_p \leqslant g_p(w_1)\pi_p(w_2)||\lambda||_p^* \tag{6.5}$$

Proof of Theorem 6.3: The only if part follows as in the classical case, cf. [12], Théorème 3.4, p. 196. It is sufficient to take a sequence $\{c_n\}$ of positive number of the sum 1 and $\{u_n\} \subset E \otimes F$ such that $Q_p(u_n) < \infty$. Denote by δ_n the Dirac measure in the point $c_n^{-1/p} u_n$ and define $\lambda := \sum c_n \delta_n$. λ is obviously a cylindrical measure on $E \otimes F$ of type p:

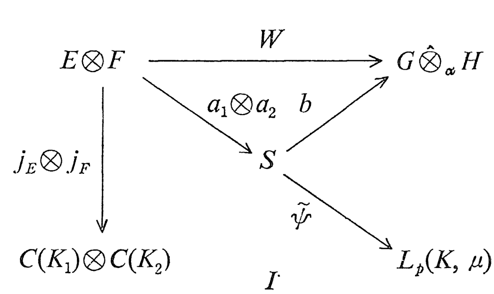
$$||\lambda||_p^* = \sup_{\substack{||x'|| \leqslant 1 \\ ||y'|| \leqslant 1}} \{\sum c_n |\langle x' \otimes y', c_n^{-1/p} u_n \rangle|^p\}^{1/p} = Q_p(u_n)$$

Since W is p-Radonifying, $W(\lambda)$ is a Radon probability on $G \hat{\otimes}_\alpha H$ of order p.

$$\begin{aligned} ||W(\lambda)||_p &= \{\int ||W(u)||^p d\lambda(u)\}^{1/p} = \{\sum c_n ||c_n^{-1/p} W(u_n)||^p\}^{1/p} \\ &= N_p(W(u_n)) \end{aligned}$$

Thus, $Q_p(u_n) < \infty$ implies $N_p(W(u_n)) < \infty$ and W is $\tilde{p}$-summing.

Let us prove the sufficiency. The operator W, being $\tilde{p}$-summing, has the factorization of the form (cf. [2], Theorem 2 and Theorem 3):



(6.6)

K_1, K_2, K are defined in the proof of Proposition 6.1, μ_1 and μ_2 are the Pietsch measures, and $\mu:=\mu_1\otimes\mu_2$. The space $L_p(K, \mu)$ is obtained as the completion $L_p(K_1, \mu_1)\hat{\otimes}_{s_p}L_p(K_2, \mu_2)$, where s_p is one of the norm $g_p\backslash$ or $/d_p$ which coincides on the space $L_p\otimes L_p$ (cf. [8], Corollaire 4). S is a closed subspace of $L_p(K, \mu)$, obtained as the closure of the space $S_1\otimes S_2$ in the norm s_p, where S_1 and S_2 are closed subspaces of $L_p(K_1, \mu_1)$ and $L_p(K_2, \mu_2)$, respectively. $\hat{\psi}=\psi_1\hat{\otimes}\psi_2$ is the canonical embedding, b a continuous linear operator, with $||b||\leqslant\tilde{\pi}_p(W)$ and $a_1\otimes a_2$ a $\tilde{p}$-summing operator. j_E (and similarly j_F) is defined by $j_Ex:=(x'\mapsto\langle x, x'\rangle)$. I is defined by

$$I(\sum f_k\otimes g_k):=((x', y')\mapsto \sum f_k(x')g_k(y'))$$

Let λ be a cylindrical measure on $E\otimes F$ of type p. By Proposition 5.1 $\tilde{\lambda}:=(j_E\otimes j_F)(\lambda)$ is a cylindrical measure on $C(K_1)\otimes C(K_2)$ of type p. Since $C(K_1)'$ and $C(K_2)'$ have the metric approximation property, by Lemma 5.5 there exists a net $\{\lambda_\gamma\}$ of Radon probabilities on $C(K_1)\otimes C(K_2)$ (each of them is concentrated on some finite-dimensional space) which converges $\otimes$-cylindrically to λ and

$$||\lambda_\gamma||_p^*\leqslant||\tilde{\lambda}||_p^* = ||(j_E\otimes j_F)(\lambda)||_p^*\leqslant||j_E||\,||j_F||\,||\lambda||_p^* = ||\lambda||_p^*$$

The mapping I is $\tilde{p}$-summing, with $\tilde{\pi}_p(I)\leqslant 1$ (cf. [2], Lemma 1). Thus, Corollary 6.2 gives for the Radon probabilities $I(\lambda_\gamma)$ on $L_p(K, \mu)$:

$$||I(\lambda_\gamma)||_p\leqslant\tilde{\pi}_p(I)||\lambda_\gamma||_p^*\leqslant||\lambda||_p^*$$

Let $L_p(K, \mu)_\sigma$ be the space $L_p(K, \mu)$ with the weak topology. We can observe $I(\lambda_\gamma)$ as a Radon measure on $L_p(K, \mu)_\sigma$. By a version of Prohorov's theorem, see e.g. [4] Proposition 4, $\{I(\lambda_\gamma)\}$ is relatively compact in the topology of the weak convergence of probability measures. Hence, it exists a Radon probability ν on $L_p(K, \mu)_\sigma$ which lies in the closure of $\{I(\lambda_\gamma)\}$. We may suppose $I(\lambda_\gamma)\to\nu$ weakly, and hence also a $\otimes$-cylindrically. By Phillips theorem ([11], Theorem 3, p. 162) (weak and strong topology on a Banach space are Radon-equivalent), ν is a Radon probability on $L_p(K, \mu)$.

On the other hand, $I(\tilde{\lambda})$ defines a cylindrical measure on $L_p(K_1, \mu_1)\otimes L_p(K_2, \mu_2)$. Since L_p-spaces have the metric approximation property, by Proposition 3.4 there exists at most one Radon probability $I(\tilde{\lambda})^\wedge$ on $L_p(K, \mu)\simeq L_p(K_1, \mu_1)\hat{\otimes}_{d_p}L_p(K_2, \mu_2)$. Since it holds $I(\tilde{\lambda})=\lim I(\lambda_\gamma)$, Proposition 5.3 ensures $\nu=I(\tilde{\lambda})^\wedge$, i.e. $I(\tilde{\lambda})=\check{\nu}$. Hence, $I(\tilde{\lambda})$ is a Radon probability on $L_p(K, \mu)$. We must show that ν is concentrated on S.

Let $\mathcal{V}$ be a topology on $L_p(K,\mu)$ with the following basis of neighborhoods of zero:

$$V=\{h\in L_p(K,\mu):\ |\langle h,f'_i\cdot g'_j\rangle|<\varepsilon,\quad \varepsilon>0,$$
$$f'_1,\cdots,f'_n\in L_{p'}(K_1,\mu_1),\ g'_1,\cdots,g'_m\in L_{p'}(K_2,\mu_2)\}$$

Since $f'_i\cdot g'_j\in L_{p'}(K_1\times K_2,\mu_1\otimes\mu_2)$, the topology $\mathcal{V}$ is weaker than the weak topology on $L_p(K,\mu)$. But, the converse is also true: take $h'\in L_{p'}(K,\mu)$. Since $L_{p'}(K,\mu)$ is isometrically isomorphic to the space $L_{p'}(K_1,\mu_1)\hat{\otimes}_{s_{p'}}L_{p'}(K_2,\mu_2)$, h' can be approximated in $L_{p'}$-norm by functions of the form $\sum_{i=1}^{n}f'_i\otimes g'_i$, $f'_i\in L_{p'}(K_1,\mu_1)$, $g'_i\in L_{p'}(K_2,\mu_2)$, i.e., by the functions $(x,y)\mapsto\sum_{i=1}^{n}f'_i(x)g'_i(y)$, and our statement follows easily.

Consider the space $S:=S_1\hat{\otimes}_{s_p}S_2$. Let $N\in FC(L_p(K_1,\mu_1)),M\in FC(L_p(K_2,\mu_2))$ be arbitrary. The norm s_p satisfy (3.3), thus $\pi_{N\otimes M}$ can be extended to a continuous linear operator $\hat{\pi}_{N\otimes M}$: $L_p(K,\mu)\to(L_p(K_1,\mu_1)/N)\otimes(L_p(K_2,\mu_2)/M)$. Also, the Radon probability ν on $L_p(K,\mu)$ defines a unique cylindrical measure $\check{\nu}$ on $L_p(K_1,\mu_1)\otimes L_p(K_2,\mu_2)$. It holds $\hat{\pi}_{N\otimes M}(\nu)=\check{\nu}_{N\otimes M}=\pi_{N\otimes M}(\check{\nu})$, and commutative diagram (6.6) shows that $\hat{\pi}_{N\otimes M}(\nu)$ is concentrated on $\pi_{N\otimes M}(S_1\otimes S_2)$.

Suppose that ν is not concentrated on S. Then it exists $h^0\notin S$ which lies in the support of ν. Since S is closed in the weak topology of the space $L_p(K,\mu)$, we can choose $\eta>0,f'_1,\cdots,f'_n\in L_{p'}(K_1,\mu_1)$, $g'_1,\cdots,g'_m\in L_{p'}(K_2,\mu_2)$ such that it holds

$$|\langle h-h^0,f'_i\otimes g'_j\rangle|\leqslant\eta \tag{6.7}$$

for all i,j and $h\in S_1\otimes S_2$.

Define $N\in FC(L_p(K_1,\mu_1))$ by

$$N:=\{f:\langle f,f'_i\rangle=0,\ i=1,\cdots,n\}$$

and similarly $M\in FC(L_p(K_2,\mu_2))$. Suppose $\hat{\pi}_{N\otimes M}(h^0)\in\pi_{N\otimes M}(S_1\otimes S_2)$. Then $\hat{\pi}_{N\otimes M}(h-h^0)=0$ for some $h\in S_1\otimes S_2$, hence $h-h^0\in N\otimes L_p(K_2,\mu_2)+L_p(K_1,\mu_1)\otimes M$ which contradicts (6.7). Since $\hat{\pi}_{N\otimes M}(h^0)$ belongs to the support of $\hat{\pi}_{N\otimes M}(\nu)\subset\pi_{N\otimes M}(S_1\otimes S_2)$, we get a contradiction again. Thus, ν is a Radon probability on S.

Finally, $W(\lambda)=b(\nu)$ is a Radon probability on $G\hat{\otimes}_\alpha H$, for which it holds

$$||W(\lambda)||_p\leqslant||b||\ ||\nu||_p\leqslant\tilde{\pi}_p(W)||\lambda||_p^*$$

The proof is complete.

Corollary 6.5. *Let* $1<p<\infty$, $\lambda\in\mathcal{M}_p^c(l_1\otimes l_1)$ *and* i: $l_1\to l_2$ *be the canon-*

ical injection. Then $(i \otimes i)(\lambda)$ *is a Radon probability on the space* $HS(l_2, l_2)$ *of Hilbert-Schmidt operators.*

Proof. By Grothendieck's result, $i: l_1 \to l_2$ is p-summing, for all $p > 1$. By Theorem 6.3, $i \otimes i: l_1 \otimes l_1 \to l_2 \hat{\otimes}_{g_p \backslash} l_2$ is p-Radonifying. This space coincides with the space of all p-summing operators: $l_2 \to l_2$ (cf. [7], p. 91). The corollary follows since p-summing operator between Hilbert spaces is Hilbert-Schmidt operator—this is a known result of Pietsch-Pelczyński.

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