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Microlocal Analysis and Calculations on Some Relatively Invariant Hyperfunctions Related to Zeta Functions Associated with the Vector Spaces of Quadratic Forms

By

Masakazu MURO*

Introduction

In this paper, we shall give explicit formulas of the Fourier transforms of some relatively invariant hyperfunctions on the vector spaces of quadratic forms and some similar Hermitian forms. These calculations are applicable to the calculations of the functional equations and residues of zeta functions associated with each vector spaces. (See Sato-Shintani [5] and Shintani [21]). Sato-Shintani [5] and Shintani [21] has calculated some of them in a classical way, but, in this paper, we shall give more results by making use of a different way.

Let $V_{\boldsymbol{R}}$ be the real vector space consisting of $n\times n$ symmetric matrices over $\boldsymbol{R}$. The real linear algebraic group $G_{\boldsymbol{R}}=GL(n,\boldsymbol{R})^{+}$ acts on this vector space $V_{\boldsymbol{R}}$ by

$$(g,x)\longmapsto g\cdot x\cdot{}^{t}g,\qquad (g\in G_{\boldsymbol{R}}\text{ and } x\in V_{\boldsymbol{R}}),$$

and it is a rational representation of $G_{\boldsymbol{R}}$ on $V_{\boldsymbol{R}}$. Then $V_{\boldsymbol{R}}$ decomposes into a finite number of orbits, and each orbit is parametrized by its signature. We denote by S_i^j the orbit consisting of the points whose signature is $(n-i-j,j)$. If $i=0$, then S_0^j is an open orbit in $V_{\boldsymbol{R}}$ and there exists a unique relatively invariant hyperfunction corresponding to the character $\chi(g)^s=\det(g)^{2s}(s\in\boldsymbol{C})$ which depends on s meromorphically and is supported on S_0^j. The calculations of the Fourier transforms of such relatively invariant hyperfunctions with a meromorphic

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*Department of Mathematics, Kochi University, Kochi 780, Japan.

parameter $s\in\boldsymbol{C}$ is equivalent to the computations of the functional equations of the zeta functions associated with quadratic forms which is denoted by $\xi_i^{(n)}(s, L)$ in Shintani [21]. If $i\geqq 1$, then $\boldsymbol{S}_i^j$ is a locally closed orbit and its dimension is less than the dimension of $V_{\boldsymbol{R}}$. We shall show that there exists a unique $\boldsymbol{SL}(n, \boldsymbol{R})$-invariant measure on each $\boldsymbol{S}_i^j$ and that it is extended as a tempered distribution on $V_{\boldsymbol{R}}$ supported on $\bar{\boldsymbol{S}}_i^j$ and compute the Fourier transform of it. By using the Poisson's summation formula, we can compute some contributions to the residues of $\xi_k^{(n)}(s, L)$ in terms of the explicit computations of the Fourier transforms. But, so far, we can not compute all the contributions by our results. For details, see Sato-Shintani [5] and Shintani [21]. Similarly, for the vector space $V_{\boldsymbol{R}}$ of complex Hermitian forms, we shall compute the same things. In such cases, we can compute the residues of the zeta functions associated with them completely from our calculations by making use of the method of Sato-Shintani [5].

In this paper, we start with the applications of the arguments in Sato-Kashiwara-Kimura-Oshima [6], which we abbreviate S-K-K-O, to the cases we shall deal with in this paper (§1). S-K-K-O [6] gave a method to examine a holonomic system of relatively invariant hyperfunctions. In §2, we restrict the holonomic systems to the real form $V_{\boldsymbol{R}}$ and investigate the real structures of them. The main result of this section is an application of Kashiwara's theorem to our cases (Proposition 2. 11). Kashiwara's theorem (Theorem 2. 8) was proved in Kashiwara and Miwa [8]. In Chapter II, we regard relatively invariant hyperfunctions as solutions of the holonomic systems which we shall investigate in Chapter I. We shall compute the Fourier transforms of $|\det x|^s$ (Theorem 3. 6) and construct the hyperfunctions $T_i^j(x)$ whose support is contained in a closure of $\boldsymbol{G}_{\boldsymbol{R}}$-orbit $\boldsymbol{S}_i^j$ and which is a relatively invariant measure on $\boldsymbol{S}_i^j$ (Theorem 4. 1). Moreover, we compute the Fourier transform of $T_i^j(x)$ (Theorem 4. 3). Among the above results, the Fourier transforms of $|\det x|^s$ have been computed by some authors. Above all, T. Suzuki [37] has computed them by utilizing microlocal analysis.

The main results of this paper are concentrated in §4. In almost all parts of the explanations on microlocal analysis are due to M. Kashiwara (Kashiwara-Miwa [8]) but the author added some complementaries for our computations.

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Notations

We denote by $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}$ and $\boldsymbol{C}$, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For any ring $\boldsymbol{F}$, we denote $\boldsymbol{M}(n, m, \boldsymbol{F})$ (resp. $\boldsymbol{M}(n, \boldsymbol{F})$) the set of $n\times m$ (resp. $n\times n$) matrices. We denote by $\boldsymbol{GL}(n, \boldsymbol{F})$ the group of invertible elements in $\boldsymbol{M}(n, \boldsymbol{F})$. For any finite dimensional real vector space V, $\mathscr{S}(V)$ is the space of rapidly decreasing functions on V.

Chapter I. Holonomic Systems

In this chapter, we shall examine the microlocal structures of relatively invariant hyperfunctions by analyzing the holonomic systems and holonomy diagrams.

§ 1. Holonomic Systems of Relative Invariants, (Reviews from the S-K-K-O [6] theory)

The contents of this section is essentially due to S-K-K-O [6] and T. Kimura [24]. For the details, see S-K-K-O [6] and T. Kimura[24].

Let $(\boldsymbol{G_C}, \rho, \boldsymbol{V_C})$ be a regular prehomogeneous vector space, which is one of the followings.

(1.1) 1) $\boldsymbol{G_C}=\boldsymbol{GL}(n, \boldsymbol{C})$,
$\boldsymbol{V_C}=\boldsymbol{Sym}(n, \boldsymbol{C}):=\{x\in \boldsymbol{M}(n, \boldsymbol{C})\,;\,{}^t x=x\}$,
$\rho(g)\,;\, x\longmapsto g\cdot x\cdot {}^t g,\ (g\in \boldsymbol{G_C},\ x\in \boldsymbol{V_C})$.
2) $\boldsymbol{G_C}=\boldsymbol{GL}(n, \boldsymbol{C})\times \boldsymbol{GL}(n, \boldsymbol{C})$,
$\boldsymbol{V_C}=\boldsymbol{M}(n, \boldsymbol{C})$,
$\rho(g)\,;\, x\longmapsto g_1\cdot x\cdot {}^t g_2,\ (g=(g_1, g_2)\in \boldsymbol{G_C},\ x\in \boldsymbol{V_C})$.
3) $\boldsymbol{G_C}=\boldsymbol{GL}(2n, \boldsymbol{C})$,
$\boldsymbol{V_C}=\boldsymbol{Alt}(2n, \boldsymbol{C})\,;=\{x\in \boldsymbol{M}(2n, \boldsymbol{C})\,;\ {}^t x=-x\}$,
$\rho(g)\,;\, x\longmapsto g\cdot x\cdot {}^t g,\ (g\in \boldsymbol{G_C}, x\in \boldsymbol{V_C})$.

Irreducible relative invariants $P(x)$ are by definition

(1.2) 1) $P(x)=(\det x)$,
2) $P(x)=(\det x)$,
3) $P(x)=(\mathrm{Pff}\, x)$,

respectively. Here $(\det x)$ means the determinant of a matrix x and $(\mathrm{Pff}\ x)$ means the Pffafian of an alternative matrix x. The corresponding characters χ, i.e., $P(\rho(g)x)=\chi(g)P(x)$, are

1) $\chi(g)=(\det g)^2$,
2) $\chi(g)=(\det g_1)(\det g_2)$,
3) $\chi(g)=\det g$,

respectively. The inner products $<\ ,\ >$ on $\boldsymbol{V_C}$ are defined by

$$<x,y>=\mathrm{tr}(x{}^t y), \tag{1.3}$$

and we identify $\boldsymbol{V_C}$ and its dual space $\boldsymbol{V_C^*}$ by (1.3). The contragredient representation ρ^* of ρ defines a prehomogeneous vector space having the same relative invariant $P(x)$, whose corresponding character is $\chi^{-1}(g)$.

Let $\mathscr{G}_C$ be the Lie algebra of $\boldsymbol{G_C}$, $d\rho$ the infinitesimal representation of ρ and $\delta\chi$ the infinitesimal character of χ. Then we have the equation,

$$(<d\rho(A)x,\ D_x>-s\delta\chi(A))P^s(x)=0\ (A\in \mathscr{G}_C).$$

Here $P^s(x)$ means a generator of $\mathscr{D}_{V_C}$-Module, where $\mathscr{D}_{V_C}$ is the sheaf of differential operators on $\boldsymbol{V_C}$, not a function. We consider the system of differential equations,

$$(1.4)\qquad \mathfrak{M}_s;\,(\langle d\rho(A)x, D_x\rangle - s\delta\chi(A))u=0 \qquad (A\in\mathscr{G}_C).$$

We shall examine the system of differential equation (1.4) following to the arguments in S-K-K-O [6].

The *characteristic variety* of $\mathfrak{M}_s$ is by definition,

$$(1.5)\qquad \{(x,y)\in T^*V_C;\langle d\rho(A)x,y\rangle=0,\ A\in\mathscr{G}_C\}$$

and we denote it by $\mathrm{ch}(\mathfrak{M}_s)$ or for simplicity, by $\mathfrak{C}$. We identify $T^*V_C^*$ and $V_C\times V_C^*$. Then the group G_C acts on T^*V_C naturally. The characteristic variety $\mathfrak{C}$ is an invariant subvariety under the actions of G_C because

$$\langle d\rho(A)\rho(g)x,\ \rho^*(g)y\rangle=\langle\rho(g)^{-1}d\rho(A)\rho(g)x,\ y\rangle,$$

and

$$d\rho(A)\longmapsto\rho(g)^{-1}d\rho(A)\rho(g)$$

defines an automorphism of $d\rho(\mathscr{G}_C)$. Therefore, the sets $\{(x,y)\in T^*V_C;$ $\langle d\rho(A)x,y\rangle=0,\ A\in\mathscr{G}_C\}$ and $\{(x,y)\in T^*V_C;\langle d\rho(A)\rho(g)x,\ \rho^*(g)y\rangle$ $=0,\ A\in\mathscr{G}_C\}$ coincide with each other. The characteristic variety $\mathfrak{C}$ decomposes into several G_C-orbits in T^*V_C. In fact, we can construct $\mathfrak{C}$ as a union of a finite number of G_C-orbits in T^*V_C. In order to write it down, we begin with the orbital decomposition of V_C.

The vector space $V_C=\boldsymbol{Sym}(n,C)$ decomposes into $(n+1)$ $GL(n,C)$-orbits $S_{iC}=\{x\in\boldsymbol{Sym}(n,C);\ \mathrm{rank}(x)=n-i\}$, $(i=0,1,\ldots,n)$. The orbit S_{iC} is generated by $x_i=\begin{bmatrix}I_{n-i} & \\ & 0_i\end{bmatrix}$. Similarly, the vector space $V_C=\boldsymbol{M}(n,C)$ (resp. $\boldsymbol{Alt}(2n,C)$) decomposes into $(n+1)\,GL(n,C)\times GL(n,C)$-(resp. $GL(2n,C)$-)orbits $S_{iC}=\{x\in\boldsymbol{M}(n,C);\ \mathrm{rank}(x)=n-i\}$ (resp. $S_{iC}=\{x\in\boldsymbol{Alt}(2n,C);\mathrm{rank}(x)=2(n-i)\}$), $(i=0,1,\ldots,n)$. The orbit S_{iC} is generated by $x_i=\begin{bmatrix}I_{n-i} & \\ & 0_i\end{bmatrix}$ (resp. $x_i=\begin{bmatrix} & I_{n-i} & \\ -I_{n-i} & & \\ & & 0_{2i}\end{bmatrix}$). Here $I_k=\begin{bmatrix} & & 1\\ & \cdot^{\cdot} & \\ 1 & & \end{bmatrix}$ (k columns). The dimension of V_C is $n(n+1)/2$ (resp. n^2, $n(2n-1)$) and we denote it by n', and the dimension of S_{iC} is $(n(n+1)-i(i+1))/2$ (resp. n^2-i^2, $n(2n-1)-i(2i-1)$) when $V_C=\boldsymbol{Sym}(n,C)$ (resp. $V_C=\boldsymbol{M}(n,C)$, $V_C=\boldsymbol{Alt}(2n,C)$). The orbit S_{0C} is an open dense orbit in V_C and coincides with the set $V_C-\{x\in V_C;P(x)=0\}$. The other orbits $S_{iC}(i\geqq 1)$ are contained in the set $\{x\in V_C;P(x)=0\}$. The Zariski closure $\bar{S}_{iC}$ of S_{iC} is $S_{iC}\cup S_{i+1C}\cup\ldots\cup S_{nC}$.

We identify $T^* V_C^*$ and $V_C \times V_C^*$ in the same way as the identification of $T^* V$ and $V_C \times V_C^*$. We denote by π(resp. π^*) the projection map

$$(1.6) \qquad \pi ; V_C \times V_C^* \longrightarrow V_C \ \text{(resp. } \pi^* ; V_C \times V_C^* \longrightarrow V_C^*).$$

The maps π and π^* are compatible with the action of G_C.

Let A be a non-singular subvariety in V_C. We define the *conormal bundle of* A by

$$(1.7) \qquad T_A^* V_C = \bigcup_{x \in A} \{(x, y) \in V_C \times V_C^* ; \langle a, y \rangle = 0 \text{ for all } a \in T_x A\},$$

which is clearly a non-singular subvariety of dimension n'. If A is irreducible, then $T_A^* V_C$ is also irreducible. So the conormal bundle $T_{S_i}^* V_C$ is an irreducible variety, and hence $\overline{T_{S_i}^* V_C}$ is irreducible. We denote $\Lambda_{iC} = \overline{T_{S_i}^* V_C}$. Moreover we have $\pi(\Lambda_{iC}) = \bar{S}_i$. In the same way, by defining $\Lambda_{jC}^* = \overline{T_{S_j}^* V_C^*}$, we have Λ_{jC}^* is an irreducible subvariety whose dimension is equal to n', and $\pi^*(\Lambda_{jC}^*) = \bar{S}_j$.

Let Σ_{ijC} be a G_C-orbit in $V_C \times V_C^*$ generated by

$$(1.8) \qquad \left(\begin{bmatrix} I_{n-i} & \\ & 0_i \end{bmatrix}, \begin{bmatrix} 0_j & \\ & I_{n-j} \end{bmatrix} \right) \in V_C \times V_C^*,$$

when $V_C = \boldsymbol{Sym}(n, \boldsymbol{C})$ or $V_C = \boldsymbol{M}(n, \boldsymbol{C})$, and

$$(1.9) \qquad \left(\begin{bmatrix} J_{n-i} & \\ & 0_{2i} \end{bmatrix}, \begin{bmatrix} 0_{2j} & \\ & J_{n-j} \end{bmatrix} \right) \in V_C \times V_C^*,$$

when $V_C = \boldsymbol{Alt}(2n, \boldsymbol{C})$. Here, $J_k = \begin{bmatrix} & I_k \\ -I_k & \end{bmatrix}$, and $n \geqq i \geqq 0, n \geqq j \geqq 0$ and $i + j \geqq n$. Then we have the orbital decompositions,

$$(1.10) \qquad T_{S_i}^* V_C = \bigcup_{j \geqq n-i} \Sigma_{ijC}, \qquad T_{S_{n-i}}^* V_C^* = \bigcup_{j \geqq i} \Sigma_{jn-iC},$$
$$\Lambda_{iC} = \bigcup_{\substack{k \geqq i \\ m \geqq n-i}} \Sigma_{kmC}, \qquad \Lambda_{n-iC}^* = \bigcup_{\substack{k \geqq i \\ m \geqq n-i}} \Sigma_{kmC},$$

and hence $\Lambda_{iC} = \Lambda_{n-iC}^*$. From the definition of the characteristic variety $\mathfrak{C}$, $\mathfrak{C}$ is a closed set and coincides with $\bigcup_{i \geqq 0} T_{S_i}^* V_C$, and hence

$$\mathfrak{C} = \overline{\bigcup_{i \geqq 0} T_{S_i}^* V_C} = \bigcup_{i \geqq 0} \overline{T_{S_i}^* V_C}.$$

Proposition 1.1. 1) *$\Lambda_{iC} = \Lambda_{n-iC}^*$, and $\mathfrak{C} = \bigcup_{i \geqq 0} \Lambda_{iC}$.*

2) *The varieties Λ_{iC} and Λ_{i+1C} have an intersection of dimension $(n'-1)$ and it contains a G_C-orbit of dimension $(n'-1)$. In a neighborhood of a point p of the $(n'-1)$-dimensional orbit, Λ_{iC} and Λ_{i+1C} are smooth varieties and*

$$(1.11)\qquad T_p\Lambda_{iC}\cap T_p\Lambda_{i+1C}=T_p(\Lambda_{iC}\cap\Lambda_{i+1C}).$$

Here, T_pX is the tangent space of X at p.

3) *Λ_{iC} and Λ_{jC} have no $(n'-1)$-dimensional intersection if $j-i\geqq 2$.*

The proof would be found in S-K-K-O [6].

We put W_C the Zariski closure of the set

$$(1.12)\qquad \{(x, s\cdot\mathrm{grad}_x\log P(x))\in V_C\times V_C^*;\ x\in V_C-S_C,\ s\in C\},$$

and a conormal bundle Λ_{iC} is called a *good Lagrangian subvariety* if it is contained in W_C and contains an open dense orbit in it. (See S-K-K-O [6], Definition 4.2, 4.5 and 4.14). Furthermore, for a good Lagrangian subvariety Λ_{iC}, the *order* of $\mathfrak{M}_s$ is defined and is given by

$$(1.13)\qquad \mathrm{ord}_{\Lambda_{iC}}(\mathfrak{M}_s)=s\delta\chi(A_0)-tr_{V^*_{x_i}}d\rho^*(A_0)+\frac{1}{2}\dim_C V^*_{x_i},$$

where A_0 is an element of $\mathscr{G}_C$ satisfying $d\rho(A_0)x_i=0$ and $d\rho^*(A_0)y_i=y_i$ for an element $(x_i,y_i)\in\Sigma_{i,n-iC}$. Let p be a point in $\mathfrak{C}$. We say that $\mathfrak{M}_s$ is a *simple holonomic system* at p if:

(1.14) 1) $\dim_C\mathfrak{C}=n$ and $\mathfrak{C}$ is non-singular.

2) the $\mathcal{O}_{T^*V_C}$-ideal generated by $\{\langle d\rho(A)x, y\rangle; A\in\mathscr{G}_C\}$ is a reduced ideal,

in a neighborhood of p. We say that two Lagrangian subvarieties Λ_{iC} and Λ_{jC} have a *good intersection* Σ if

(1.15) 1) Σ is a G_C-orbit in Λ_{iC} of dimension n'-1,

2) $\Lambda_{iC}, \Lambda_{jC}$ and W_C are non-singular in a neighborhood of any point $p\in\Sigma$.

3) $T_p\Lambda_{iC}\cap T_p\Lambda_{jC}=T_p\Sigma$ for any $p\in\Sigma$.

Proposition 1.2. 1) *The variety Λ_{iC} are all good Lagrangian subvarieties.*

2) *The order of $\mathfrak{M}_s$ on Λ_{iC} is*

$$(1.16)\qquad \mathrm{ord}_{\Lambda_{iC}}(\mathfrak{M}_s)=\begin{cases} -is-\dfrac{i(i+1)}{4} & (V_C=\mathbf{Sym}(n,C)),\\ -is-\dfrac{i}{2} & (V_C=\mathbf{M}(n,C)),\\ -is-\dfrac{i(2i-1)}{2} & (V_C=\mathbf{Alt}(n,C)),\end{cases}$$

3) *The Lagrangian subvarieties Λ_{iC} and Λ_{i+1C} have a good intersection.*

The proof would be found in S-K-K-O [6]. Now, by applying Theorems 6. 3, 6. 6 and 8. 3 in S-K-K-O, we have the following normal form of holonomic system near the point $z\in\Sigma_{i+1,n-iC}$.

Theorem 1. 3. *Let $z\in\Sigma_{i+1,n-iC}$. By a suitable quantized contact transformation, $\mathfrak{M}_s$ is transformed to the following holonomic system in a neighborhood of z.*

$$(1.17)\qquad \begin{aligned}&(x_1D_{x_1}-\lambda)u=0,\\ &(x_2D_{x_2}-\mu)u=0,\quad D_{x_3}u=..=D_{x_n}u=0,\end{aligned}$$

with

$$(1.18)\qquad \begin{aligned}&\Lambda_{i+1C}=\{(x,\xi)\in T^*V_C;x_1=x_2=\xi_3=...=\xi_{n'}=0\}\\ &\Lambda_{iC}=\{(x,\xi)\in T^*V_C;x_1=\xi_2=\xi_3=...=\xi_{n'}=0\}\\ &\Sigma_{i+1\ n-iC}=\Lambda_{i+1C}\cap\Lambda_{iC},\\ &z=(0,dx_2)\in T^*V_C\\ &\mu=\mathrm{ord}_{\Lambda_{iC}}(\mathfrak{M}_s)-\mathrm{ord}_{\Lambda_{i+1C}}(\mathfrak{M}_s)-\frac{1}{2}\\ &\quad=\begin{cases}s+\frac{i}{2} & (V_C=\boldsymbol{Sym}(n,\boldsymbol{C})),\\ s+i & (V_C=\boldsymbol{M}(n,\boldsymbol{C})),\\ s+2i & (V_C=\boldsymbol{Alt}(2n,\boldsymbol{C})).\end{cases}\end{aligned}$$

From those mentioned above, we have known the microlocal structures of holonomic systems $\mathfrak{M}_s$ on T^*V_C. It is simple on all the Lagrangian subvarieties and all the intersections of codimension one are good intersections.

Let $\mathfrak{M}_s$ be a holonomic system on an n'-dimensional complex manifold X_C. We write a circle ○ to represent an irreducible Lagrangian component of $\mathrm{ch}(\mathfrak{M}_s)$ in T^*X_C. We connect some of them by a line if and only if they have the same connected n'-1 dimensional variety. We call the diagram thus obtained the *complex holonomy diagram* of $\mathfrak{M}_s$. We write the codimension of the projection of a Lagrangian subvariety in the circle. We write its order beside the good Lagrangian subvariety. When two Lagrangian subvarieties have a good intersection, we write $(\mu+1)$, which is defined in (1. 18), beside the line connecting the two Lagrangian subvarieties.

Following to this definition, we have the complex holonomy diagrams in Figure 1.

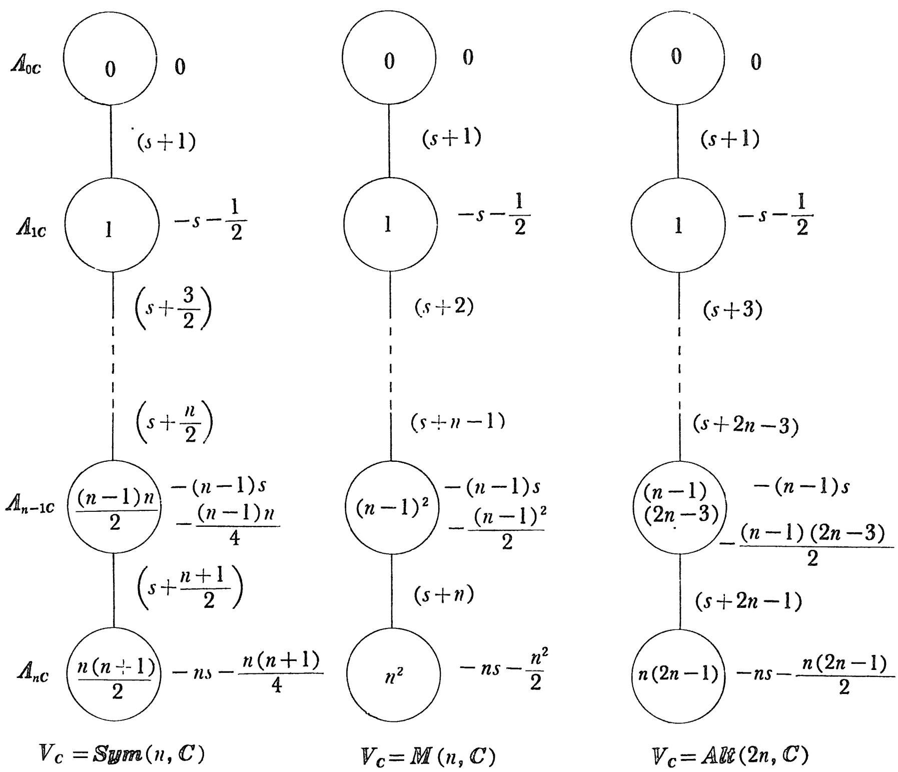


Figure 1.

The microlocal structure of the holonomic system is completely determined by the difference of orders when two Lagrangian subvarieties have an intersection of codimension one. From the method of the calculations of b-functions in S-K-K-O[6], we have the b-function of $P^s(x)$.

Proposition 1.4. 1) *There exists a polynomial $b(s)$, which we call a b-function, satisfying $P(D_x)\cdot P(x)^{s+1}=b(s)\cdot P(x)^s$.*

2) (*M. Kashiwara*) *The roots of $b(s)$ are negative rational numbers.*

The proofs of this Proposition is found in Sato-Shintani[5] and S-K-K-O[6] and Kashiwara[36].

Proposition 1.5. *The b-functions of $P^s(x)$ are*

1) $b(s)=\prod_{j=1}^{n}\left(s+\frac{1+j}{2}\right)$, $(V_C=\boldsymbol{Sym}(n,\boldsymbol{C}))$,

2) $b(s)=\prod_{j=1}^{n}(s+j)$, $(V_C=\boldsymbol{M}(n,\boldsymbol{C}))$,

3) $b(s)=\prod_{j=1}^{n}(s+2j-1)$, $(V_C=\boldsymbol{Alt}(2n,\boldsymbol{C}))$,

modulo constant multiples.

Though the proof of this Theorem is possible by the direct computation, we obtain these by drawing the complex holonomy diagrams Figure 1. (See S-K-K-O[6], T. Kimura[24]).

§ 2. Real Forms of Holonomic Systems and Principal Symbols (Applications of Kashiwara's Result)

The main purpose of this section is essentially an application of Kashiwara's theory in [8] to some concrete examples. The results from Definition 2.1 to Theorem 2.6 may have been done in his computations [37]. The explanations after Theorem 2.6 to Theorem 2.12 are quoted from Kashiwara-Miwa[8] in a slightly different form for our use.

In this section, we shall consider the real forms of (G_C, ρ, V_C) and examine the holonomic system on the real locus.

We say that (G_R, ρ, V_R) is a *real form* of (G_C, ρ, V_C) if G_R is a real form of the complex Lie group G_C, V_R is a real form of V_C and the restriction of ρ on G_R is a representation of G_R on V_R. We denote by G_R^+ the connected component of G_R containing the neutral element. We shall deal with (G_R^+, ρ, V_R) in place of (G_R, ρ, V_R) as a real form of (G_C, ρ, V_C) if G_R is not connected.

The followings are real forms of (G_C, ρ, V_C) in (1.1).

(2.1) 1) $G_C=\boldsymbol{GL}(n,\boldsymbol{C})$, $V_C=\boldsymbol{Sym}(n,\boldsymbol{C})$

(real symmetric bilinear forms)

$G_R^+=\boldsymbol{GL}(n,\boldsymbol{R})^+\ni g$

$V_R=\boldsymbol{Sym}(n,\boldsymbol{R})=\{x\in\boldsymbol{M}(n,\boldsymbol{R})\,;{}^t x=x\}$

$\rho(g)\,;x\longmapsto g\cdot x\cdot {}^t g$

2) $G_C = GL(n, C) \times GL(n, C)$, $V_C = M(n, C)$
(complex Hermitian forms)
$G_R = GL(n, C) \ni g$
$V_R = Her(n, C) = \{x \in M(n, C) ; {}^t\bar{x} = x\}$
$\rho(g) ; x \longmapsto g \cdot x \cdot {}^t\bar{g}$

3) $G_C = GL(2n, C)$, $V_C = Alt(2n, C)$
(quaternion Hermitian forms)
$G_R = GL(n, H) \ni g$
$V_R = Her(n, H) = \{x \in M(n, H) ; {}^t\bar{x} = x\}$
$\rho(g) ; x \longmapsto g \cdot x \cdot {}^t\bar{g}$

Here H stands for the quaternion field over R. We regard the real Lie group G_R (resp. the real vector space V_R) as a sugroup of G_C (resp. as a vector subspace of V_C) in the following way.

(2.2) 1)
$$\begin{array}{ccc} GL(n, R) & \longrightarrow & GL(n, C) \\ \cup\!\!| & & \cup\!\!| \\ g & \longmapsto & g \end{array}$$

$$\begin{array}{ccc} Sym(n, R) & \longrightarrow & Sym(n, C) \\ \cup\!\!| & & \cup\!\!| \\ x & \longmapsto & x \end{array}$$

2)
$$\begin{array}{ccc} GL(n, C) & \longrightarrow & GL(n, C) \times GL(n, C) \\ \cup\!\!| & & \cup\!\!| \\ g & \longmapsto & (g, \bar{g}) \end{array}$$

$$\begin{array}{ccc} Her(n, C) & \longrightarrow & M(n, C) \\ \cup\!\!| & & \cup\!\!| \\ x & \longmapsto & x \end{array}$$

3)
$$\begin{array}{ccc} GL(n, H) & \longrightarrow & GL(2n, C) \\ \cup\!\!| & & \cup\!\!| \\ g & \longmapsto & \iota(g) \end{array}$$

$$\begin{array}{ccc} Her(n, H) & \longrightarrow & Alt(2n, C) \\ \cup\!\!| & & \cup\!\!| \\ x & \longmapsto & \iota(x) \cdot J_n \end{array}$$

Here ι is the map $H \hookrightarrow M(2, C)$ defined by

$$(2.3) \qquad \iota ; x_0 \cdot 1 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2 \longmapsto \begin{bmatrix} x_0 + \sqrt{-1}\, x_1, & -x_2 - \sqrt{-1}\, x_3 \\ x_2 - \sqrt{-1}\, x_3, & x_0 - \sqrt{-1}\, x_1 \end{bmatrix}$$

where $e_1^2 = e_2^2 = -1$, $e_1 e_2 = -e_2 e_1$, and J_n is the $2n \times 2n$ alternative matrix defined by $\iota(-I_n e_2)$. We define the determinant of $x \in M(n, H)$ by

$\det x = \mathrm{Pff}(\iota(x)\cdot J_n)$.

The inner product $\langle\ ,\ \rangle$ on $V_{\boldsymbol{R}}$ are defined by

$$(2.4)\qquad \begin{aligned} &1)\ \langle x,y\rangle = \mathrm{tr}(x^t y) && (x,y\in \boldsymbol{Sym}(n,\boldsymbol{R}))\\ &2)\ \langle x,y\rangle = \mathrm{tr}(x^t \bar{y}) && (x,y\in \boldsymbol{Her}(n,\boldsymbol{C}))\\ &3)\ \langle x,y\rangle = \mathrm{tr}(x^t \bar{y}) && (x,y\in \boldsymbol{Her}(n,\boldsymbol{H}))\end{aligned}$$

Note that the restrictions of the inner products by (2.3) coincide with the inner products in (2.4) up to constant multiples. We identify $V_{\boldsymbol{R}}^*$ and $V_{\boldsymbol{R}}$ by the inner products. Then $V_{\boldsymbol{R}}\times V_{\boldsymbol{R}}^*$ is a real form of the complex vector space $V_{\boldsymbol{C}}\times V_{\boldsymbol{C}}^*$ on which $G_{\boldsymbol{R}}^+$ operates by $\rho\oplus\rho^*$ and the triple $(G_{\boldsymbol{R}}^+, \rho^*, V_{\boldsymbol{R}}^*)$ is a real form of $(G_{\boldsymbol{C}}, \rho^*, V_{\boldsymbol{C}})$.

Let x be a point in $V_{\boldsymbol{R}}$. The real locus $\rho(G_{\boldsymbol{C}})\cdot x\cap V_{\boldsymbol{R}}$ is a $G_{\boldsymbol{R}}^+$-invariant set and consists of a finite number of $G_{\boldsymbol{R}}^+$-orbits. Each orbit is characterized by the signature of the orbit which is defined by the following.

Definition 2.1. (Signatures of forms)
Let x be an element of $V_{\boldsymbol{R}}=\boldsymbol{Sym}(n,\boldsymbol{R})$ (resp. $\boldsymbol{Her}(n,\boldsymbol{C})$, $\boldsymbol{Her}(n,\boldsymbol{H})$). We say that a pair of non-negative integers (k,m) is the *signature* of x, and denote is by $\mathrm{sign}(x)$ if there exist a basis $y_1,\dots,y_n\in\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$, $\boldsymbol{H}^n$) of the real vector space $\boldsymbol{R}^n$ (resp. the complex vector space $\boldsymbol{C}^n$, the left quaternion vector space $\boldsymbol{H}^n$) satisfying ${}^t\bar{y}_i\cdot x\cdot y_j=\varepsilon_i\delta_{ij}$ with $\varepsilon_i=1$, -1 or 0 for $1\leqq i\leqq k$, $k+1\leqq i\leqq k+m$ or $i\geqq k+m+1$, respectively (δ_{ij}; Kroneker's δ). In particular, we say that k (resp. m) is the positive (resp. negative) signature of x.

The following Proposition 2.2 is well know and Proposition 2.3 is easily verified.

Proposition 2.2. (Orbital decompositions)
The real vector space $V_{\boldsymbol{R}}=\boldsymbol{Sym}(n,\boldsymbol{R})$ (resp. $\boldsymbol{Her}(n,\boldsymbol{C})$, $\boldsymbol{Her}(n,\boldsymbol{H})$) *decomposes into* $\boldsymbol{GL}(n,\boldsymbol{R})^+$*-orbits* (resp. $\boldsymbol{GL}(n,\boldsymbol{C})$-orbits, $\boldsymbol{GL}(n,\boldsymbol{H})$-*orbits*),

$$(2.5)\qquad S_i^j=\{x\in\boldsymbol{Sym}(n,\boldsymbol{R})\ \ (\text{resp. } x\in\boldsymbol{Her}(n,\boldsymbol{C}),\ x\in\boldsymbol{Her}(n,\boldsymbol{H}));\ \mathrm{sign}(x)=(j,n-i-j)\},\quad (i=0,1,\dots,n,\ j=0,\dots,n-i).$$

The orbit S_i^j *is generated by* $x_i^{(j)}=\begin{bmatrix} I_{n-1}^{(j)} & \\ & 0_i \end{bmatrix}\in V_R$, *where* $I_{n-i}^{(j)}=\begin{bmatrix} I_j & \\ & -I_{n-i-j}\end{bmatrix}$. *The real locus* $S_i\cap V_R=\bigcup_{0\leqq j\leqq n-i} S_i^j$.

In particular, among these G_R^+-orbits, $S_0^j(j=0,\ldots,n)$ are open orbits in V_R. We denote by $V_j^{(n)}$ instead of S_0^j. If $i>0$, then S_i^j is contained in the singular set $\{x\in V_R;P(x)=0\}$. We denote it by S_R. We have

$$(2.6)\qquad V_n^{(n)}\cup V_{n-1}^{(n)}\cup\ldots\cup V_0^{(n)}=V_R-S_R,$$
$$S_R=\bigcup_{n\geqq i\geqq 1} S_{iR},\ S_{iR}=\bigcup_{n-i\geqq j\geqq 0} S_i^j.$$

Since the dual space V_R^* is identified with V_R, and hence V_R^* has the same decomposition. We denote them by

$$(2.7)\qquad V_n^{*(n)}\cup V_{n-1}^{*(n)}\cup\ldots\cup V_0^{*(n)}=V_R^*-S_R^*,$$
$$S_R^*=\bigcup_{n\geqq i\geqq 1} S_{iR}^*,\ S_{iR}^*=\bigcup_{n-i\geqq j\geqq 0} S_i^{*j},$$

when we have to distinguish V_R from V_R^*. We say that $V_i^{(n)}$ are *open orbits*, and that S_i^j are *singular orbits.*

Proposition 2.3. *Let* S_i^j *be the* G_R^+*-orbit defined in* (2.6). *Then we have*

$$(2.8)\qquad \overline{S_i^j}=\bigcup_{\substack{n-i\geqq p\geqq 0\\ \mathrm{Min}\{j,p\}\geqq q\geqq \mathrm{Max}\{0,j-n+i+p\}}} S_{i+p}^{j-q}.$$

Now, consider the holonomic system $\mathfrak{M}_s$ in (1.4) and the hyperfunction solutions to $\mathfrak{M}_s$. Then the singular spectrum of the solution is contained in $\mathfrak{C}_R=\mathrm{ch}(\mathfrak{M}_s)\cap T^*V_R$. We may regard the solution as a microfunction whose support in contained in $\mathfrak{C}_R$. From Proposition 1.1, we have $\mathfrak{C}=\bigcup_{i=0}^{n}\Lambda_{iC}$ and

$$\Lambda_{iC}=\bigcup_{\substack{j\geqq n-i\\ k\geqq i}}\Sigma_{kjC}\ \text{(disjoint union)},$$

where Σ_{kjC} is the G_C-orbit in $V_C\times V_C^*$ generated by

$$\left(\begin{bmatrix} I_{n-k} & \\ & 0_k\end{bmatrix},\begin{bmatrix} 0_j & \\ & I_{n-j}\end{bmatrix}\right)\in V_R\times V_R^*.$$

Among the orbits in Λ_{iC}, the orbit $\Sigma_{i,n-iC}$ is an open dense subset in Λ_{iC} and any point in $\Sigma_{i,n-iC}$ does not contained in any other Lagrangian subvariety Λ_{jC} $(j\neq i)$. Especially, we deonote by Λ_{iC}° the G_C-orbit

$\Sigma_{i,n-iC}$. We note that

$$\hat{\Lambda}_{iC}=(\bigcup_{j\geqq n-i}\Sigma_{ijC})\cup(\bigcup_{j\geqq i}\Sigma_{j,n-iC}),$$

and that $\hat{\Lambda}_{iC}$ is an open dense non-singular subvariety of Λ_{iC}. (See Proposition 1.1). We have the following theorem on the real loci of the orbits Σ_{ijC}.

The following Proposition 2.4 is fundamental in order to draw the real holonomy diagrams of $\mathfrak{M}_s$.

Proposition 2.4. 1) *The real locus $\Sigma_{ijR}=\Sigma_{ijC}\cap V_R\times V_R^*$ decomposes into $(n-i+1)\times(n-j+1)$ G_R^+-orbits. The generators of G_R^+-orbits in Σ_{ijR} are*

$$\left(\begin{bmatrix} I_{n-i}^{(p)} & \\ & 0_i\end{bmatrix},\begin{bmatrix} 0_j & \\ & I_{n-j}^{(q)}\end{bmatrix}\right)\in V_R\times V_R^* \quad \begin{pmatrix}0\leqq p\leqq n-i\\ 0\leqq q\leqq n-j\end{pmatrix}. \tag{2.9}$$

We denote by Σ_{ijR}^{pq} the G_R^+-orbits generated by (2.9). Especially, for $\Lambda_{iR}^{\circ}=\Sigma_{in-iR}$, each connected component is a real Lagrangian submanifold. We denote Σ_{in-iR}^{pq} by Λ_{iR}^{pq} $(0\leqq p\leqq n-i,\ 0\leqq q\leqq i)$.

2) *The real locus $\hat{\Lambda}_{iR}=\hat{\Lambda}_{iC}\cap(V_R\times V_R^*)$ is a non-singular subvariety. The intersection $\hat{\Lambda}_{iR}\cap\hat{\Lambda}_{i+1R}$ consists of $(n'-1)$-dimensional G_R^+-orbits and it is $\Sigma_{i+1\,n-iR}$.*

3)

$$\bar{\Lambda}_{iR}^{pq}\cap(\hat{\Lambda}_{iR}\cap\hat{\Lambda}_{i+1R})=\Sigma_{i+1,n-iR}^{pq}\cup\Sigma_{i+1\,n-iR}^{p-1q} \tag{2.10}$$

$$\bar{\Lambda}_{iR}^{pq}\cap(\hat{\Lambda}_{i-1R}\cap\hat{\Lambda}_{iR})=\Sigma_{in-i+1R}^{pq-1}\cup\Sigma_{in-i+1R}^{pq}.$$

Here we set $\Sigma_{ij}^{pq}=\phi$ when $p>n-i$ or $q>n-j$.

Proof. 1) Let u be a point in Σ_{ijR}. Since $\pi(u)\in S_{iR}$, $\pi(u)$ is reduced to one of the points

$$x_i^{(p)}=\begin{bmatrix} I_{n-i}^{(p)} & \\ & 0_i\end{bmatrix},\quad (0\leqq p\leqq n-i), \tag{2.11}$$

by the actions of G_R^+. On the other hand, from the definition of Σ_{ijR}, we have $u\in T_{S_{iR}}^{*i}V_R$ and hence $\pi^*(u)$ is contained in the conormal vector space $V_{x_i}^*(p)$ of S_{iR} at $x_i^{(p)}$. Let $G_{x_i}(p)$ be the isotropy subgroup of G_R^+ at $x_i^{(p)}$. Then $G_{x_i}(p)$ acts on $V_{x_i}^*(p)$ by the contragredient representation, and the action $\rho_{x_i}^*(p)(g)$ is given in the following way.

$$(2.12)\quad G_{x_i}(p)=\left\{\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}\in GL(n,K)^+;\ \begin{array}{c} A\ I^{(p)}_{n-i}\ {}^tA=I^{(p)}_{n-i}\\ A\in GL(n-i,K),\\ B\in M(n-i,i,K)\\ C\in GL(i,K)\end{array}\right\}$$

$$V^*_{x_i}(p)=\left\{\begin{bmatrix}0 & 0\\ 0 & X\end{bmatrix}\in V^*_R; X\in M(i,K),\ {}^t\bar{X}=X\right\},$$

$$\rho^*_{x_i}(p)(g);\begin{bmatrix}0 & 0\\ 0 & X\end{bmatrix}\longmapsto\begin{bmatrix}0 & 0\\ 0 & {}^tC^{-1}XC^{-1}\end{bmatrix},$$

$$\text{with } g=\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}\in G_{x_i}(p).$$

Here, K is the field R (resp. C, H) when $V_R=Sym(n,R)$ (resp. $Her(n,C)$, $Her(n,H)$). From the assumption, the matrix $\begin{bmatrix}0 & 0\\ 0 & X\end{bmatrix}$ is an element of S^*_{jR} and hence X is of rank n-j. Therefore it is reduced to the matrix,

$$(2.13)\qquad \begin{bmatrix}0_{i-n+j} & \\ & I^{(q)}_{n-j}\end{bmatrix}.$$

Thus we have that any point in Σ_{ijR} is reduced to one of the points,

$$(2.14)\qquad \left(\begin{bmatrix}I^{(p)}_{n-i} & \\ & 0_i\end{bmatrix},\begin{bmatrix}0_j & \\ & I^{(q)}_{n-j}\end{bmatrix}\right)\in V_R\times V^*_R,\ \begin{pmatrix}0\leqq p\leqq n-i\\ 0\leqq q\leqq n-j\end{pmatrix}.$$

Next we shall show that the G^+_R-orbits generated by the points (2.14) differ from one another. Let u and v be the points in Σ_{ijR} given by

$$(2.15)\qquad u=\left(\begin{bmatrix}I^{(p)}_{n-i} & \\ & 0_i\end{bmatrix},\begin{bmatrix}0_j & \\ & I^{(q)}_{n-j}\end{bmatrix}\right),$$

$$v=\left(\begin{bmatrix}I^{(r)}_{n-i} & \\ & 0_i\end{bmatrix},\begin{bmatrix}0_j & \\ & I^{(s)}_{n-j}\end{bmatrix}\right).$$

We assume that $p\neq r$ or $q\neq s$. Then we have that $\pi(u)$ and $\pi(v)$ are not contained in the same G^+_R-orbit or that $\pi^*(u)$ and $\pi^*(v)$ are not contained in the same G^+_R-orbit. Therefore, $G^+_R\cdot u$ and $G^+_R\cdot v$ are not the same.

2) Since $\hat{\Lambda}_{iC}$ is a non-singular variety, it is evident that $\hat{\Lambda}_{iR}$ is non-singular. The second statement follows from that $\hat{\Lambda}_{iC}\cap\hat{\Lambda}_{i+1C}=\Sigma_{in-i+1C}$.

3) First we shall show that

$$(2.16)\qquad \overline{\Lambda}^{pq}_{iR}\supset\Sigma^{pq}_{i+1n-iR}\cup\Sigma^{p-1q}_{i+1n-iR}.$$

Let x be a point in $\boldsymbol{\Sigma}^{pq}_{i+1n-iR}$. Then x is reduced to the point $\left(\begin{bmatrix} I^{(p)}_{n-i-1} & \\ & 0_{i+1}\end{bmatrix}, \begin{bmatrix} 0_{n-i} & \\ & I^{(q)}_i\end{bmatrix}\right)$. The point $x_\varepsilon=\left(\begin{bmatrix} I^{(p)}_{n-i-1} & & \\ & -\varepsilon & \\ & & 0_{i+1}\end{bmatrix}, \begin{bmatrix} 0_{n-i} & \\ & I^{(q)}_i\end{bmatrix}\right)$ is an element in $\boldsymbol{A}^{pq}_{iR}$ if $\varepsilon>0$, and converges to a point in $\boldsymbol{\Sigma}^{pq}_{i+1n-iR}$ when $\varepsilon\to 0$. Thus we have,

$$\bar{\boldsymbol{A}}^{pq}_{iR}\supset \boldsymbol{\Sigma}^{pq}_{i+1n-iR}. \tag{2.17}$$

In the same way, $x'_\varepsilon=\left(\begin{bmatrix} I^{(p-1)}_{n-i-1} & & \\ & \varepsilon & \\ & & 0_i\end{bmatrix}, \begin{bmatrix} 0_{n-i} & \\ & I^{(q)}_i\end{bmatrix}\right)\in \boldsymbol{A}^{pq}_{iR}$ for $\varepsilon>0$, and x'_ε converges to a point in $\boldsymbol{\Sigma}^{p-1q}_{i+1,n-iR}$ when $\varepsilon\to 0$. Then we have

$$\bar{\boldsymbol{A}}^{pq}_{iR}\supset \boldsymbol{\Sigma}^{p-1q}_{i+1n-iR}. \tag{2.18}$$

Thus (2.16) is obtained.

Next we shall show that

$$\bar{\boldsymbol{A}}^{pq}_{iR}\cap \boldsymbol{\Sigma}^{fg}_{i+1n-iR}=\phi, \text{ if } f\neq p,\ p-1 \text{ or } g\neq q. \tag{2.19}$$

In fact, we have

$$\pi(\bar{\boldsymbol{A}}^{pq}_{iR})\subset \bar{\boldsymbol{S}}^{p}_{i},\ \pi(\boldsymbol{\Sigma}^{fg}_{i+1n-iR})=\boldsymbol{S}^{f}_{i+1},$$

and

$$\pi^*(\bar{\boldsymbol{A}}^{pq}_{iR})\subset \boldsymbol{S}^{*q}_{n-i},\ \pi^*(\boldsymbol{\Sigma}^{fg}_{i+1n-iR})=\boldsymbol{S}^{*g}_{n-i}$$

From Proposition 2.3, we have $\bar{\boldsymbol{S}}^{p}_{i}\cap \boldsymbol{S}^{f}_{i+1}=\phi$ if $f\neq p,\ p-1$ and $\boldsymbol{S}^{*g}_{n-i}\cap \bar{\boldsymbol{S}}^{*q}_{n-i}=\phi$ if $g\neq q$, and hence (2.19) is followed. Thus we have

$$\begin{aligned}\bar{\boldsymbol{A}}^{pq}_{iR}\cap(\hat{\boldsymbol{A}}_{iR}\cap\hat{\boldsymbol{A}}_{i+1R})&=\bar{\boldsymbol{A}}^{pq}_{iR}\cap\boldsymbol{\Sigma}_{i+1n-iR}\\&=\boldsymbol{\Sigma}^{pq}_{i+1n-iR}\cup\boldsymbol{\Sigma}^{p-1q}_{i+1n-iR},\end{aligned} \tag{2.20}$$

and hence the first line of (2.10) is obtained.

By taking

$$x_\varepsilon=\left(\begin{bmatrix} I^{(p)}_{n-i} & \\ & 0_i\end{bmatrix}, \begin{bmatrix} 0_{n-i} & & \\ & -\varepsilon & \\ & & I^{(q)}_{i-1}\end{bmatrix}\right)\in\bar{\boldsymbol{A}}^{pq}_{iR}\ (\varepsilon>0), \tag{2.21}$$

and

$$x'_\varepsilon=\left(\begin{bmatrix} I^{(p)}_{n-i} & \\ & 0_i\end{bmatrix}, \begin{bmatrix} 0_{n-i} & & \\ & \varepsilon & \\ & & I^{(q-1)}_{i-1}\end{bmatrix}\right)\in\bar{\boldsymbol{A}}^{pq}_{iR}\ (\varepsilon>0),$$

and bringing ε to zero, x_ε and x'_ε converges to a point in $\boldsymbol{\Sigma}^{pq-1}_{in-i-1R}$ and in $\boldsymbol{\Sigma}^{pq}_{in-i+1R}$, respectively. On the other hand, we have

$$\bar{\boldsymbol{A}}^{pq}_{iR}\cap\boldsymbol{\Sigma}^{fg}_{in-i+1R}=\phi \text{ if } g\neq q,\ q-1 \text{ or } f\neq p, \tag{2.22}$$

because

$$\pi(\bar{\Lambda}^{pq}_{iR}) \subset \bar{S}^{p}_{i}, \ \pi(\Sigma^{fg}_{in-i+1R}) = S^{f}_{i}$$

$$\pi^{*}(\bar{\Lambda}^{pq}_{iR}) \subset \bar{S}^{*q}_{n-i}, \ \pi^{*}(\Sigma^{fg}_{in-i+1R}) = S^{*g}_{n-i+1}$$

and

$$\begin{aligned} &\bar{S}^{p}_{i} \cap S^{f}_{i} = \phi, && \text{if } f \neq p, \\ &S^{*q}_{n-i} \cap S^{*g}_{n-i+1} = \phi, && \text{if } g \neq q, \ q-1. \end{aligned}$$

Thus we have

$$\bar{\Lambda}^{pq}_{iR} \supset \Sigma^{pq}_{in-i+1R} \cup \Sigma^{pq}_{in-i+1R}, \tag{2.23}$$

and hence we obtain the second line of (2.10). q.e.d.

From Proposition 1.2, Λ_{iC} and Λ_{i+1C} have a good intersection $\Sigma_{i+1n-iC}$. By restricting $\Sigma_{i+1n-iC}$ to the real domain, the real locus $\Sigma_{i+1n-iR}$ decomposes into $(n-i)\times(i+1)$-G^{+}_{R}-orbits $\Sigma^{pq}_{i+1n-iR}$ $(0 \leqq p \leqq n-i-1, \ 0 \leqq q \leqq i)$. From Proposition 2.4-3), The G^{+}_{R}-orbits in Λ_{iR} which contains $\Sigma^{pq}_{i+1n-iR}$ are Λ^{p+1q}_{iR} and Λ^{pq}_{iR}, and the G^{+}_{R}-orbits in Λ_{i+1R} which contains $\Sigma^{pq}_{i+1n-iR}$ are Λ^{pq+1}_{i+1R} and Λ^{pq}_{i+1R}. Thus we have the following proposition.

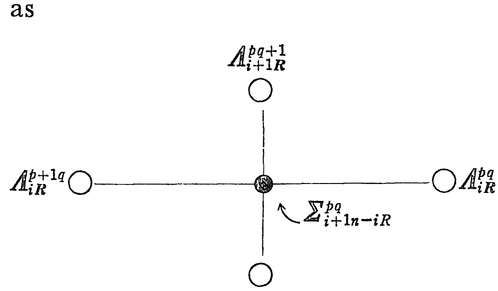
Proposition 2.5. *Let $z=(x,y)$ be a point of $\Sigma^{pq}_{i+1n-iR}$. Then there exists a neighborhood U_z of z in $T^{*}V_{R}$ such that*

$$\begin{aligned} U_z \cap \mathfrak{C}_{R} &= (\Lambda_{iR} \cup \Lambda_{i+1R}) \cap U_z \\ &= ((\Lambda^{p+1q}_{iR} \cup \Lambda^{pq}_{iR}) \cup \Sigma^{pq}_{i+1n-iR} \cup (\Lambda^{pq+1}_{i+1R} \cup \Lambda^{pq}_{i+1R})) \cap U_z, \end{aligned} \tag{2.24}$$

in U_z.

We write the Lagrangian subvarieties satisfying the conditions of Proposition 2.5 as

(2.25)



and call it a *real holonomy diagram*.

In the complex holonomy diagram, we write ○ to represent an irreducible Lagrangian component. But, in the real holonomy diagram, it means a connected component of the real locus of Λ_{iC}. In our cases, each ○ is a G_R^+-orbit. By the diagram (2.25), we express the intersection in (Figure 2).

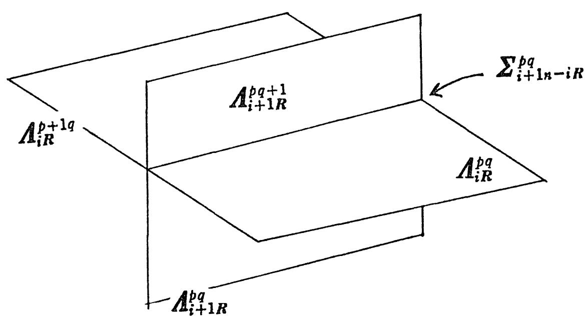


Figure 2.

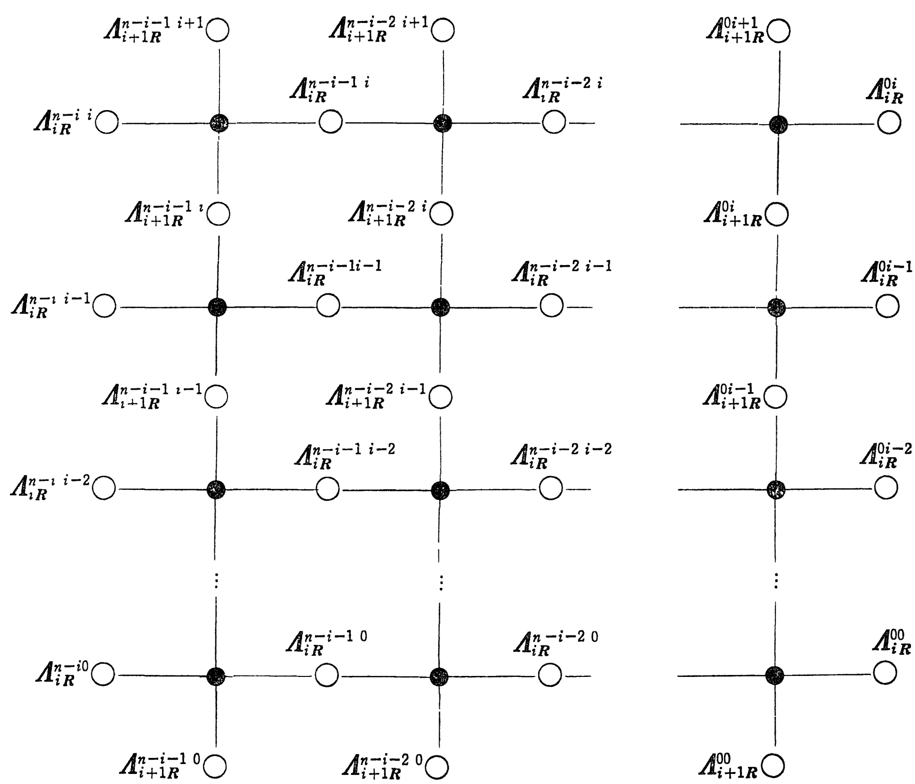


Figure 3. Real holonomy diagram of the intersection of Λ_{iR} and Λ_{i+1R}

The complete real holonomy diagram of intersections between Λ_{iR} and Λ_{i+1R} is (Figure 3).

In Theorem 1.3, we have shown that, in a neighborhood of a point of $\Sigma_{i+1n-iC}$, the holonomic system $\mathfrak{M}_s$ is transformed to a holonomic system (1.17) by a suitable contact transformation. This contact transformation gives a real contact transformation in T^*V_R and it transforms $\mathfrak{M}_s$ in the real locus.

Theorem 2.6. *Let $z \in \Sigma^{pq}_{i+1n-iR}$. By a suitable real quantized contact transformation, $\mathfrak{M}_s$ is transformed to the following form in a neighborhood of z.*

$$(2.26) \qquad \begin{cases}(x_1 D_{x_1} - \lambda) u = 0, \\ (x_2 D_{x_2} - \mu) u = 0, \ D_{x_3} u = D_{x_4} u = \cdots = D_{x_{n'}} u = 0, \end{cases}$$

With

$$(2.27) \qquad \begin{aligned} &\Lambda^{pq+1}_{i+1R} = \{(x, \xi) \in T^*V_R; x_1 = x_2 = \xi_3 = \cdots = \xi_{n'} = 0, \ \xi_2 > 0\} \\ &\Lambda^{pq}_{i+1R} = \{(x, \xi) \in T^*V_R; x_1 = x_2 = \xi_3 = \cdots = \xi_{n'} = 0, \ \xi_2 < 0\} \\ &\Lambda^{p+1q}_{iR} = \{(x, \xi) \in T^*V_R; x_1 = \xi_2 = \xi_3 = \cdots = \xi_{n'} = 0, \ x_2 > 0\} \\ &\Lambda^{pq}_{iR} = \{(x, \xi) \in T^*V_R, \ x_1 = \xi_2 = \xi_3 = \cdots = \xi_{n'} = 0, \ x_2 < 0\} \\ &\Sigma^{pq}_{i+1n-iR} = \{(x, \xi) \in T^*V_R; x_1 = x_2 = \xi_2 = \xi_3 = \cdots = \xi_{n'} = 0\} \end{aligned}$$

$$z = (0, dx_2) \in T^*V_R, \ \mu = \begin{cases} s + \dfrac{i}{2} & (V_R = \boldsymbol{Sym}(n, \boldsymbol{R})) \\ s + i & (V_R = \boldsymbol{Her}(n, \boldsymbol{C})) \\ s + 2i & (V_R = \boldsymbol{Her}(n, \boldsymbol{H})) \end{cases}$$

Proof. In the following proof, we shall always consider all things in a *sufficiently small neighborhood* of z. The proof of the existence of the contact transformation of the holonomic system $\mathfrak{M}_s$ to the normal form (2.26) can be carried out in the same way as the proof of Theorem 1.3. In fact, our holonomic system $\mathfrak{M}_s$ has an involutory basis of micro-differential operators *with real valued real analytic* coefficients on T^*V_R, and hence the contact transformation in T^*V_C defined in Theorem 1.3 preserves T^*V_R. Thus, there exists a contact coordinate transformation Ψ which transforms $\mathfrak{M}_s$ to (2.26). That is to say, we can write,

$$(2.28) \qquad \begin{aligned} &\Psi(\Lambda_{i+1R}) = \{(x, \xi) \in T^*V_R; \ x_1 = x_2 = \xi_3 = \cdots = \xi_{n'} = 0\}, \\ &\Psi(\Lambda_{iR}) = \{(x, \xi) \in T^*V_R; \ x_1 = \xi_2 = \xi_3 = \cdots = \xi_{n'} = 0\}, \end{aligned}$$

$$\Psi(z)=(0, dx_2)\in T^* V_R,$$

and the holonomic system $\mathfrak{M}_s$ is written as (2.26). Thus we have only to show that if $\Psi(\Lambda_{iR}^{p+1q})=\Psi(\Lambda_{iR})\cap\{x_2>0\}$, then $\phi(\Lambda_{i+1R}^{pq+1})=\phi(\Lambda_{i+1R})\cap\{\xi_2>0\}$.

Let f, g be real analytic, real valued functions such that

$$df(z)\neq 0,\ \{f=0\}\supset\Lambda_{i+1R} \tag{2.29}$$
$$dg(z)\neq 0,\ \{g=0\}\supset\Lambda_{iR}.$$

It is obvious that such functions exist. The Poisson bracket $\{f, g\}(z)$ does not vanish and we may assume that $\{f,g\}(z)>0$ by taking $-g$ instead of g if necessary. Then we have

(2.30) 1) $f|_{\Lambda_{iR}}$ has a zero of order one on $\Lambda_{iR}\cap\Lambda_{i+1R}$, and does not vanish on $\Lambda_{iR}-(\Lambda_{iR}\cap\Lambda_{i+1R})$.

2) $g|_{\Lambda_{i+1R}}$ has a zero of order one on $\Lambda_{iR}\cap\Lambda_{i+1R}$ and does not vanish on $\Lambda_{iR}-(\Lambda_{iR}\cap\Lambda_{i+1R})$.

Then we have

(2.31) (the signature of f on Λ_{iR}^{p+1q})
$\times$(the signature of g on Λ_{i+1R}^{pq+1})

does not depend on the choice of f,g. In fact, let f',g' be other real analytic, real valued functions satisfying (2.30) and $\{f', g'\}(z)>0$. Then there exist real valued, real analytic functions a, b which do not vanish at z and satisfy

$$f'|_{\Lambda_{iR}}=a\cdot f|_{\Lambda_{iR}},\quad g'|_{\Lambda_{i+1R}}=b\cdot g|_{\Lambda_{i+1R}}. \tag{2.32}$$

We have

$$\{f',g'\}(z)=\{af,bg\}(z)=a(z)b(z)\{f,g\}(z). \tag{2.33}$$

Therefore if $\{f',g'\}(z)>0$, then $a(z)b(z)>0$ and hence we have $ab>0$ in a neighborhood of z. Thus we obtain that the signature of (2.31) is the same as the signature of (2.31) calculated by using f',g' instead of f,g.

We may take

$$z=\left(\begin{bmatrix} I_{n-i-1}^{(p)} & \\ & 0_{i+1}\end{bmatrix},\begin{bmatrix} 0_{n-i} & \\ & I_i^{(q)}\end{bmatrix}\right)\in T^* V_R. \tag{2.34}$$

Setting

$$(2.35)\qquad \phi_1(t)=\left(\begin{bmatrix} I^{(p)}_{n-i-1} & & \\ & t & \\ & & 0_i\end{bmatrix},\begin{bmatrix} 0_{n-i} & \\ & I^{(q)}_i\end{bmatrix}\right)$$

$$\phi_2(t)=\left(\begin{bmatrix} I^{(p)}_{n-i-1} & \\ & 0_{i+1}\end{bmatrix},\begin{bmatrix} 0_{n-i-1} & & \\ & t & \\ & & I^{(q)}_i\end{bmatrix}\right)$$

we have

$$(2.36)\qquad \phi_1(t)\in \Lambda^{p+1q}_{iR},\quad \phi_2(t)\in\Lambda^{pq+1}_{i+1R},\ \text{if } t>0,$$

$$\phi_1(0)=\phi_2(0)=z.$$

We can take f, g satisfying

$$(2.37)\qquad f(\phi_1(t))=t,\quad g(\phi_2(t))=t,$$

$$df(z)=\left(\begin{bmatrix} 0_{n-i-1} & & \\ & 1 & \\ & & 0_i\end{bmatrix},\begin{bmatrix} 0 \end{bmatrix}\right)=(x_1,\xi_1)\in T^*_z(T^*V_R)$$

$$dg(z)=\left(\begin{bmatrix} 0 \end{bmatrix},\begin{bmatrix} 0_{n-i-1} & & \\ & 1 & \\ & & 0_i\end{bmatrix}\right)=(x_2,\xi_2)\in T^*_z(T^*V_R).$$

Then we have $\{f,g\}(z)=\{\langle x_1,\xi_2\rangle-\langle x_2,\xi_1\rangle\}=1>0$, and hence f, g satisfy $\{f,g\}(z)>0$. We have

$$(2.38)\qquad f|_{\Lambda^{p+1\,q}_{iR}}>0 \text{ and } g|_{\Lambda^{p\,q+1}_{i+1R}}>0.$$

On the other hand, by the contact transformation, we may take

$$(2.39)\qquad f=x_2 \text{ and } g=\xi_2.$$

In fact $\{f,g\}(\Psi(z))=1>0$. We have

$$(2.40)\qquad x_2|_{\phi(\Lambda_{iR})\cap\{x_2>0\}}>0,$$

$$\xi_2|_{\phi(\Lambda_{i+1R})\cap\{\xi_2>0\}}>0.$$

From (2.38) and (2.40), if

$$\Psi(\Lambda^{p+1q}_{iR})=\Psi(\Lambda_{iR})\cap\{x_2>0\},$$

then

$$\Psi(\Lambda^{pq}_{iR})=\Psi(\Lambda_{iR})\cap\{x_2<0\},$$
$$\Psi(\Lambda^{pq+1}_{i+1R})=\Psi(\Lambda_{i+1R})\cap\{\xi_2>0\},$$
$$\Psi(\Lambda^{pq}_{i+1R})=\Psi(\Lambda_{i+1R})\cap\{\xi_2<0\}.$$

Thus we have the result. q. e. d.

In order to consider the hyperfunction solutions on V_R to the holonomic system $\mathfrak{M}_s$ by "lifting up" $\mathfrak{M}_s$ on T^*V_R, we introduce the *real principal symbol* of the microfunction solution. First, remember the definition of the principal symbol of a simple holonomic system. Let X be a real analytic manifold of dimension n and let X_C be its complex neighborhood. Let $\mathfrak{M}$ be a holonomic system of differential equations on X_C. It defines naturally $\mathscr{E}_X\otimes_{\pi^{-1}(\mathscr{D}_X)}\mathfrak{M}$ of microdifferential equations on T^*X_C. We also denote it by $\mathfrak{M}$ if there is no fear of confusion.

We denote by $\mathrm{ch}(\mathfrak{M})$ the characteristic variety of $\mathfrak{M}$. Let Λ_C be an irreducible component of $\mathrm{ch}(\mathfrak{M})$, and let Λ_C° be the open subset of Λ_C consisting of non-singular points in $\mathrm{ch}(\mathfrak{M})$. We suppose that $\mathfrak{M}$ is simple characteristic at any point of Λ_C°.

Now, remember the definition of complex principal symbols of $\mathfrak{M}$ on Λ_C°. As in the definition 3.11. in S-K-K-O [6], we correspond a local holomorphic section of $\Omega_{\Lambda_C}^{\otimes 1/2}\otimes\Omega_{X_C}^{\otimes -1/2}$ on Λ_C° at each point to the holonomic system $\mathfrak{M}$. Here, Ω_{Λ_C} and Ω_{X_C} are the sheaves of holomorphic n-forms on Λ_C and on X_C, respectively. We call it the (complex) *principal symbol* of $\mathfrak{M}$ on Λ_C. It is defined as a solution of a system of differential equations and it is defined modulo constant multiples. In other words, for a simple holonomic system $\mathfrak{M}$ on Λ_C°, we have a locally constant sheaf of rank one which is a subsheaf of $\Omega_{\Lambda_C}^{\otimes 1/2}\otimes\Omega_{X_C}^{\otimes -1/2}$ and the principal symbol is a local section of it. There does not always exists a global non-trivial section on Λ_C°.

We consider the hyperfunction or microfunction solutions to a holonomic system $\mathfrak{M}$. Let $\mathscr{B}_X$ be the sheaf of hyperfunctions on X. We denote by $\mathrm{Supp}(f(x))$ the support of a section $f(x)\in\mathscr{B}_X$ on X. We denote by $\mathrm{S.S.}(f(x))$ the singular spectrum of $f(x)$ *on* $T^*X\text{-}X$. Namely; for a section $f(x)\in\mathscr{B}_X$, the section $\mathrm{sp}(f(x))$ of the sheaf of microfunctions on $T^*X\text{-}X$ is defined. We denote it by $f(x)$ for simplicity. Then the support of $f(x)$ on $T^*X\text{-}X$ is the singular spectrum on $T^*X\text{-}X$. Moreover, $f(x)$ is naturally considered as a section on T^*X by corresponding $f(x)$ on X and $\mathrm{sp}(f(x))$ on $T^*X\text{-}X$. This section $f(x)$ on T^*X is called a *microfunction on* T^*X. By this correspondence, any hyperfunction is naturally viewed as a microfunction on T^*X. We denote by $\mathrm{S.\check{S}.}(f(x))$ the support of the microfunction $f(x)$ *on* T^*X. If $f(x)$ is a solution of a holonomic system $\mathfrak{M}$, then

(2.41) $\overset{\vee}{\mathrm{S.S.}}\,(f(x))\subset \mathrm{ch}(\mathfrak{M})$.

For any hyperfunction $f(x)$, we have

(2.42) $\mathrm{Supp}(f(x))=\pi(\overset{\vee}{\mathrm{S.S.}}\,(f(x)))$,

where π; $T^*X\to X$ is the projection map.

Let Λ_C be an irreducible component of $\mathrm{ch}(\mathfrak{M})$. We suppose that $\Lambda=\Lambda_C\cap T^*X$ is a real Lagrangian subvariety in T^*X. Let $z_0=(x_0,y_0)$ be a point in $\Lambda^\circ=\Lambda_C^\circ\cap T^*X$ and let u be a microfunction solution to $\mathfrak{M}$ supported on Λ° defined near z_0. We suppose that, in a neighborhood of x_0 in X, $\pi(\Lambda^\circ)=\{\tilde{x}_1=\cdots=\tilde{x}_k=0\}$ by a local coordinate $(\tilde{x}_1,\ldots,\tilde{x}_k,\tilde{x}_{k+1},\ldots,\tilde{x}_n)$, and Λ is an open set in $T^*_{\pi(\Lambda)}V_R=\{(\tilde{x},\tilde{\xi})\in T^*V_R;\tilde{x}_1=\cdots=\tilde{x}_k=0,\ \tilde{\xi}_{k+1}=\cdots=\tilde{\xi}_n=0\}$. By using a microdifferential operator of fractional order (defined in S-K-K-O[6] §2 as $\mathscr{E}^{\infty}_{V}(\lambda)$) $P=\sum_{j=0}^{\infty}P_{\lambda-j}(\tilde{x}'',\tilde{D}_x')$ defined in a neighborhood of (x_0,y_0), we have an expression of u near (x_0,y_0),

$$u=\int_{\{|\tilde{\xi}'|=1\}}\sum_{j=0}^{\infty}P_{\lambda-j}(\tilde{x}'',\tilde{\xi}')\,\Phi_{\lambda-j+k}(\sqrt{-1}\,(\langle\tilde{x}',\tilde{\xi}'\rangle+i0))\,d\omega(\tilde{\xi}').$$

This is the plane wave expansion of microfunction u with respect to the coordinate (x',ξ'). Here, we set;

1) $\tilde{x}'=(\tilde{x}_1,\ldots,\tilde{x}_k)$, $\tilde{x}''=(\tilde{x}_{k+1},\ldots,\tilde{x}_n)$
   $\tilde{\xi}'=(\tilde{\xi}_1,\ldots,\tilde{\xi}_k)$, $\tilde{\xi}''=(\tilde{\xi}_{k+1},\ldots,\tilde{\xi}_n)$.
2) $\Phi_\lambda(z)=\Gamma(\lambda)(-z)^{-\lambda}$. This function is defined for $-\pi+\varepsilon<\arg(z)<\pi-\varepsilon$ and we take the branch satisfying $\Phi_\lambda(-1)=\Gamma(\lambda)$.
3) $d\omega(\tilde{\xi}')=\sum_{j=1}^{k}(-1)^j\tilde{\xi}_j\,d\tilde{\xi}_1\wedge\cdots\wedge d\tilde{\xi}_{j-1}\wedge d\tilde{\xi}_{j+1}\wedge\cdots\wedge d\tilde{\xi}_k$, is the measure on $(k-1)$ dimensional sphere $\{|\tilde{\xi}'|=1\}$.

Let $|\Omega_\Lambda|$ (resp. $|\Omega_X|$) be the line bundle of the volume element on Λ (resp. X). We can regard $|\Omega_X|$ as the line bundle on Λ, whose transition function is defined by pulling back the transition function on X by the projection map π.

Definition 2.7. (Kashiwara) Let u be a local section of a microfunction solution on Λ° defined near z_0 expressed as (2.41). We define a local section $\sigma_\Lambda(u)$ of $\sqrt{|\Omega_\Lambda|}\times\sqrt{|\Omega_X|}^{-1}$ by

(2.43) $$\sigma_\Lambda(u)=(2\pi)^{k/2}P_\lambda(\tilde{x}'',\tilde{\xi}')\sqrt{|d\tilde{x}''\wedge d\tilde{\xi}'|}/\sqrt{|dx|},$$

and call it the *real principal symbol* of u. This definition does not depend on the choice of the local coordinate $(\tilde{x},\tilde{\xi})$ on $\boldsymbol{T}^*X$.

The real principal symbol (2.43) is obtained as a restriction of a complex principal symbol. Namely, let z_0 be a point in Λ° and let A be a non-trivial local section of complex principal symbol of the simple holonomic system $\mathfrak{M}$ defined near z_0. Then, for any microfunction solution u defined near z_0, we have $\sigma_\Lambda(u)=c\cdot A|_\Lambda$ with a constant c. This is easily verified by proving that $\sigma_\Lambda(u)\sqrt{|dx|}$ satisfies the differential equation for the principal symbol.

Conversely, let A be a local section of a principal symbol on Λ° defined near z_0. Then there exists a unique microfunction solution u defined near z_0 such that $A=\sigma_\Lambda(u)$. In fact, let $\mathrm{Sol}(\mathfrak{M})_{z_0}$ be the vector space of microfunction solutions of $\mathfrak{M}$ near z_0 and let $\mathrm{Symbol}(\mathfrak{M})_{z_0}$ be the vector space of principal symbols of $\mathfrak{M}$ near z_0. Then$\mathrm{Sol}(\mathfrak{M})_{z_0}$ and $\mathrm{Symbol}(\mathfrak{M})_{z_0}$ are one dimensional vector spaces over $\boldsymbol{C}$ because $\mathfrak{M}$ is simple characteristic. Moreover,

$$\begin{array}{ccc}\sigma_\Lambda;u & \longmapsto & \sigma_\Lambda(u)\\ \Cap & & \Cap\\ \mathrm{Sol}(\mathfrak{M})_{z_0} & \longrightarrow & \mathrm{Symbol}(\mathfrak{M})_{z_0},\end{array}$$

is a linear isomorphism. Thus, for a point $z_0\in\Lambda^\circ$, if

(2.44) Λ is written as the conormal bundle of the non-singular subvariety $\pi(\Lambda)$ in a neighborhood of z_0,

then we have a one to one correspondence between $\mathrm{Sol}(\mathfrak{M})_{z_0}$ and $\mathrm{Symbol}(\mathfrak{M})_{z_0}$ through the map σ_Λ. Henceforth, we suppose that

(2.45) for any point $z\in\Lambda^\circ$, the condition (2.44) is satisfied.

Let $\Lambda^1\cup\ldots\cup\Lambda^k=\Lambda^\circ$ be the connected component decomposition. Let $\mathrm{Sol}(\mathfrak{M})(\Lambda^i)$ be the vector space of global sections of microfunction solutions on Λ^i and let $\mathrm{Symbol}(\mathfrak{M})(\Lambda^i)$ be the vector space of global sections of principal symbols on Λ^i. Since Λ^i is connected, $\mathrm{Sol}(\mathfrak{M})(\Lambda^i)$ and $\mathrm{Symbol}(\mathfrak{M})(\Lambda^i)$ are at most one-dimensional and there exists a one to one correspondence between them by the linear isomorphism σ_Λ for each i. Therefore any microfunction solution u on $\Lambda^\circ=$

$\varLambda^1 \cup \ldots \cup \varLambda^k$ is determined by the global section of the real principal symbol $\sigma_\varLambda(u)$ on $\varLambda^\circ$. Namely, we have the following proposition.

Proposition 2.8. *Let $\mathfrak{M}$ be a holonomic system of differential equations with the characteristic variety $\mathrm{ch}(\mathfrak{M}) = \bigcup_i \varLambda_{iC}$ with $\varLambda_{iC}$ an irreducible component. For each irreducible component $\varLambda_{iC}$, we denote by $\varLambda_{iC}^\circ$ the subset consisting of nonsingular points in $\mathrm{ch}(\mathfrak{M})$. We suppose that $\mathfrak{M}$ is simple on each $\varLambda_{iC}^\circ$ and the condition (2.45) is satisfied on $\varLambda_i^\circ = \varLambda_{iC}^\circ \cap T^*X$. We put $\varLambda_i^1 \cup \ldots \cup \varLambda_i^{m_i}$ be the connected component decomposition of $\varLambda_i^\circ$. Let $u(x)$ and $v(x)$ be two hyperfunction solutions to $\mathfrak{M}$. If their principal symbols $\sigma_{\varLambda_i}(u(x))$ and $\sigma_{\varLambda_i}(v(x))$ coincides with each other on every connected component $\varLambda_i^j$, then $u(x)$ coincides with $v(x)$ as a microfunction on the subset $\bigcup_i \varLambda_i^\circ \subset \mathrm{ch}(\mathfrak{M})_R$.*

Remark 2.9. 1) It is not yet proved that $u(x)$ coincides with $v(x)$ as a hyperfunction on X. Later, we will prove that $u(x)$ actually coincides with $v(x)$ in some special cases. See Theorem 2. 14.

2) Let $\mathfrak{M}$ be a holonomic system and let $\varLambda$ be an irreducible component of $\mathrm{ch}(\mathfrak{M})$. For a microfunction solution $u(x)$ on $\varLambda^\circ$, we do not yet have defined the real principal symbol $\sigma_\varLambda(u(x))$ at the point $z_0 \in \varLambda^\circ - \varLambda_{\mathrm{reg}}^\circ$. Here, $\varLambda_{\mathrm{reg}}^\circ$ is the subset of the points satisfying the condition (2.44). Then $\varLambda_{\mathrm{reg}}^\circ$ is an open dense subset of $\varLambda^\circ$. In fact, we can not extend the real principal symbol on $\varLambda_{\mathrm{reg}}^\circ$ as a real analytic section on $\varLambda^\circ$ in general. In order to correct the section on $\varLambda_{\mathrm{reg}}^\circ$ so as to be extendable real analytically to $\varLambda^\circ$, we have to multiply a locally constant function on $\varLambda_{\mathrm{reg}}^\circ$ which is written by Maslov index. See for detail Kashiwara-Miwa [8].

Now, we go back to the case of the holonomic system $\mathfrak{M}_s$:

$$(\langle d\rho(A)x, D_x\rangle - s\delta\chi(A))u = 0.$$

Each Lagrangian subvariety $\varLambda_{iC}$ in $\mathrm{ch}(\mathfrak{M}_s)$ is an irreducible component of $\mathrm{ch}(\mathfrak{M}_s)$ and $\varLambda_{iC}^\circ = \varSigma_{in-iC}$ is the subset of $\varLambda_{iC}$ consisting of the non-singular points in $\mathrm{ch}(\mathfrak{M}_s)$. From Propositions 4. 12 and 4. 14 of S-K-K-O [6], we have

Proposition 2.10. *The locally constant sheaf of complex principal symbol of $\mathfrak{M}_s$ on $\varLambda_{iC}^\circ$ is generated by the nonzero section,*

$$P^s_{\Lambda_{iC}}\sqrt{\omega_{\Lambda_{iC}}}/\sqrt{dx},$$

with

$$P_{\Lambda_{iC}}=P\circ\pi/\sigma^{m_{\Lambda_i}}|_{\Lambda_{iC}},$$

$$\omega_{\Lambda_{iC}}=\frac{\pi^{-1}(dx)\wedge d\sigma}{\sigma^{\mu_{\Lambda_i}}}\Big/ d\sigma|_{\Lambda_{iC}}.$$

Here,

$$\sigma=\langle x,y\rangle/n,$$

$$\begin{array}{ccccc} \pi; \boldsymbol{T}^*V_C=V_C\times V_C^*\supset W_C & \longrightarrow & V_C^* & & (\textit{the projection map}) \\ \Cup & & \Cup & & \\ (x,y) & \longrightarrow & x & & \end{array}$$

$dx=dx_1\wedge\ldots\wedge dx_{n'}$ *with a linear coordinate* $(x_1,\ldots,x_{n'})$,

and m_{Λ_i} *and* μ_{Λ_i} *are the constants such that* $-m_{\Lambda_i}s-\frac{\mu_{\Lambda_i}}{2}$ *is the order of* $\mathfrak{M}_s$ *on* Λ_{iC}.

Next, consider $\mathfrak{M}_s$ in the real domain. The real locus $\Lambda^\circ_{iR}=\Lambda^\circ_{iC}\cap\boldsymbol{T}^*V_R$ is a real Lagrangian subvariety satisfying the condition (2.45), and has the connected component decomposition,

$$\Lambda^\circ_{iR}=\bigsqcup\Lambda^{jk}_{iR},\quad \begin{pmatrix}0\leqq j\leqq n-i\\ 0\leqq k\leqq i\end{pmatrix},$$

(see Proposition 2.4, 1)). On each connected component Λ^{jk}_{iR}, $|P_{\Lambda_i}|^s$ and $\sqrt{|\omega_{\Lambda_i}|}$ are a real analytic function and a real analytic half-volume element, respectively. Here, P_{Λ_i} and ω_{Λ_i} are the restrictions of $P_{\Lambda_{iC}}$ and $\omega_{\Lambda_{iC}}$ to Λ_{iR}. Then the section $|P_{\Lambda_i}|^s\sqrt{|\omega_{\Lambda_i}|}$ gives a non-zero real analytic section of $\sqrt{|\Omega_{\Lambda_i}|}$ defined on Λ^{jk}_{iR}.

Proposision 2.11. *On each connected component* Λ^{jk}_{iR},

$$|P_{\Lambda_i}|^s\sqrt{|\omega_{\Lambda_i}|}/\sqrt{|dx|}, \tag{2.46}$$

is a basis of the vector space Symbol$(\mathfrak{M}_s)(\Lambda^{jk}_{iR})$ *of global sections of real principal symbols. Therefore, for any microfunction solution u of* $\mathfrak{M}_s$, *the real principal symbol,*

$$\sigma_{\Lambda^{jk}_{iR}}(u)=c\,|P_{\Lambda_i}|^s\sqrt{|\omega_{\Lambda_i}|},$$

with a constant c.

Definition 2.12. (the coefficient or the associated number)
For a microfunction solution of $\mathfrak{M}_s$, we put

$$\sigma_{A_{iR}^{pq}}(u)=c_{A_{iR}^{pq}}|P_{A_i}|^s\sqrt{|\omega_{A_i}|}/\sqrt{|dx|}, \tag{2.47}$$

and we call the constant $c_{A_{iR}^{pq}}$ the *coefficient* or the *associated number* of u on Λ_{iR}^{pq} with respect to the basis $|P_{A_i}|^s\sqrt{|\omega_{A_i}|}/\sqrt{|dx|}$. The constant term $c_{A_{iR}^{pq}}$ depends on the solution u, so we often denote it by $c_{A_{iR}^{pq}}(u)$.

Now, we have the following distinguished relations between the associated numbers. This theorem is obtained by an adaptation of the relation formula of real principal symbols in Kashiwara-Miwa [8] (p. 139, formula (3.5)).

Theorem 2.13. (Relations of the coefficients)

$$\begin{aligned}\begin{bmatrix} c_{A_{i+1R}^{j-1k+1}} \\ c_{A_{i+1R}^{j-1k}} \end{bmatrix} &= \frac{\Gamma(s-s_{i+1})}{\sqrt{2\pi}} \\ &\times\begin{bmatrix} \exp\left(-\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\right) \\ \exp\left(\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\right) \end{bmatrix} \\ &\times\begin{bmatrix} \exp\left(\frac{\pi}{4}\sqrt{-1}(i-2k)v\right) & \\ & \exp\left(-\frac{\pi}{4}\sqrt{-1}(i-2k)v\right) \end{bmatrix} \begin{bmatrix} c_{A_{iR}^{jk}} \\ c_{A_{iR}^{j-1k}} \end{bmatrix}\end{aligned} \tag{2.49}$$

where

$$s_i=\begin{cases} -(1+i)/2 & (V_R=\boldsymbol{Sym}(n,\mathbb{R})) \\ -i & (V_R=\boldsymbol{Her}(n,\mathbb{C})) \\ -2i+1 & (V_R=\boldsymbol{Her}(n,\mathbb{H})) \end{cases} \tag{2.50}$$

$$v=\begin{cases} 1 & (V_R=\boldsymbol{Sym}(n,\mathbb{R})) \\ 2 & (V_R=\boldsymbol{Her}(n,\mathbb{C})) \\ 4 & (V_R=\boldsymbol{Her}(n,\mathbb{H})). \end{cases}$$

Here, for $s=(s_{i+1}-k)$ $(k=0, 1, 2, \dots)$ the matrix in (2.49) is not well defined because the Γ-function has a pole. However, by computing the inverse matrix of it, we have,

(2.51)

$$\frac{\Gamma(1+s_{i+1}-s)}{\sqrt{2\pi}}\begin{bmatrix}\exp\left(\frac{\pi}{2}\sqrt{-1}(1+s_{i+1}-s)\right), & \exp\left(-\frac{\pi}{2}\sqrt{-1}(1+s_{i+1}-s)\right)\\ \exp\left(-\frac{\pi}{2}\sqrt{-1}(1+s_{i+1}-s)\right), & \exp\left(\frac{\pi}{2}\sqrt{-1}(1+s_{i+1}-s)\right)\end{bmatrix}$$

$$\times\begin{bmatrix}c_{A^{j-1k+1}_{i+1R}}\\ c_{A^{j-1k}_{i+1R}}\end{bmatrix}=\begin{bmatrix}\exp\left(\frac{\pi}{4}\sqrt{-1}(i-2k)v\right), & \\ & \exp\left(-\frac{\pi}{4}\sqrt{-1}(i-2k)v\right)\end{bmatrix}\begin{bmatrix}c_{A^{jk}_{iR}}\\ c_{A^{j-1k}_{iR}}\end{bmatrix},$$

For the number $s=(s_{i+1}-k)$ $(k=0,1,2,\dots)$, (2.51) is well defined and we interpret the relation matrix (2.49) as the relation matrix (2.51). Then the relations among the coefficients $\{c_{A^{jk}_{iR}}\}$ are well defined for all $s\in\boldsymbol{C}$.

Let $\{c^{jk}_i\}$ be a set of the coefficients on $\{\boldsymbol{A}^{jk}_{iR}\}$ which are compatible with the relation matrices (2.49), i.e., $\{c^{jk}_i\}$ satisfies the relations defined by (2.49). Then, by Proposition 2.8, there exists a unique microfunction solution $u(x)$ on $\bigcup_{i=0}^{n}\boldsymbol{A}^{\circ}_i\subset\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}$ whose coefficients $c_{A^{jk}_{iR}}(u(x))$ on $\boldsymbol{A}^{jk}_{iR}$ are c^{jk}_i. However, this statement does not give guarantee for the existence of the microfunction solution to $\mathfrak{M}_s$ on $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}=\bigcup_{i=0}^{n}\boldsymbol{A}_i$ whose coefficients on $\boldsymbol{A}^{jk}_{iR}$ are c^{jk}_i. If the existence of the microfunction solution globally defined on $\mathrm{ch}(\mathfrak{M}_s)_{\boldsymbol{R}}$ is proved, then it means the existence of the hyperfunction solution on $\boldsymbol{V}_{\boldsymbol{R}}$. In the next section, we shall show the *global existence* of the hyperfunction solution on $\boldsymbol{V}_{\boldsymbol{R}}$.

We conclude this section by showing the *uniqueness* of the hyperfunction solution, i.e., if there exists a hyperfunction solution $u(x)$ such that $c_{A^{jk}_{iR}}(u(x))=c^{jk}_i$, then such hyperfunction solution is uniquely determined. Namely, we have the following theorem.

Theorem 2.14. *Let $u_1(x)$ and $u_2(x)$ be two hyperfunction solutions to the holonomic system $\mathfrak{M}_s$. Then the following three conditions are equivalent.*

(2.52) *i)* $u_1(x)=u_2(x)$.

ii) The real principal symbol $\sigma_{A_i}(u_1(x))$ *and* $\sigma_{A_i}(u_2(x))$ *coincides* *with each other on every connected component* A^{jk}_{iR}.

iii) $c_{A^{jk}_{iR}}(u_1(x)) = c_{A^{jk}_{iR}}(u_2(x))$ *for every* A^{jk}_{iR}.

Proof. The conditions ii) and iii) are appearently equivalent. The condition i) clearly implies ii). We shall show the converse ii)$\Rightarrow$i). Note the following lemma.

Lemma 2.15. (Holmgren's type theorem) *Let X be a real analytic manifold and let $u(x)$ be a hyperfunction defined near a point $x_0 \in X$. Let $p(x)$ be a real valued real analytic function defined near x_0 such that $p(x_0)=0$ and $dp(x_0)\neq 0$. We suppose that*

(2.53) i) $\mathrm{Supp}(u(x)) \subseteqq \{p(x) \geqq 0\}$,

ii) $\mathrm{S.S.}\ (u(x)) \not\ni (x_0, dp(x_0))$ *or* $\mathrm{S.S.}\ (u(x)) \not\ni (x_0, -dp(x_0))$.

Here, S.S. $(u(x))$ *means the singular spectrum of* $u(x)$ *in* $T^*X - X$. *Then we have* $u(x)=0$ *near* x_0.

The proof of this lemma is given in S-K-K [7] p. 471 Proposition 2.1.3 and the next remark.

Corollary 2.16. *Let X be a real analytic manifold and let Y be a non-singular real analytic subvariety in X near $x_0 \in X$. Let $u(x)$ be a hyperfunction defined near x_0. We suppose that*

1) $\mathrm{Supp}(u(x)) \subseteqq Y$.
2) $\mathrm{S.S.}\ (u(x)) \not\supset (T^*_Y X - X) \cap \pi^{-1}(x_0)$.

Here π is the projection map $T^*X \to X$. Then $u(x)=0$ near x_0.

Proof. There exists a local coordinate $(p_1, \dots, p_n)$ difined near x_0 such that $Y = \{p_1 = \cdots = p_m = 0\}$, and $p_i(x_0)=0$. From the condition 2), there exists a point $(x_0, y_0) \in (T^*_Y X - X)$ such that $(x_0, y_0) \notin \mathrm{S.S.}(u(x))$. We can take $p(x) = \sum_{i=1}^{m} c_i p_i(x)$ with $c_i \in \mathbf{R}$ such that $dp(x_0) = y_0$. Then the condition (2.53) i), ii) are satisfied. Thus we have the result.

Now we go to the proof of the theorem. We put $v(x) = u_1(x) - u_2(x)$. Since $v(x)$ is a hyperfunction solution to $\mathfrak{M}_s$, we have

$$\mathrm{S.S.}\,(v(x)) \subset (\mathrm{ch}(\mathfrak{M}_s) \cap \boldsymbol{T}^*\boldsymbol{V}_R) - \boldsymbol{V}_R = \bigcup_{i=1}^{n} (\overline{\boldsymbol{T}^*_{S_{iR}}\boldsymbol{V}_R} - \boldsymbol{V}_R),$$

by (2.41). On the other hand, $v(x) = u_1(x) - u_2(x) = 0$ on $\boldsymbol{V}_R - \boldsymbol{S}_R$ because $\sigma_{A_0}(u_1(x)) = \sigma_{A_0}(u_2(x))$ on $\boldsymbol{V}_R - \boldsymbol{S}_R \times \{0\} \subset \boldsymbol{T}^*\boldsymbol{V}_R$. Therefore,

$$\mathrm{Supp}(v(x)) \subset \pi(\bigcup_{i\geq 1} \boldsymbol{\Lambda}_i) = \boldsymbol{S}_R = \boldsymbol{S}_{1R} \cup \boldsymbol{S}_{2R} \cup \ldots \cup \boldsymbol{S}_{nR},$$

by (2. 42).

We shall prove that $v(x) = 0$ by induction. Suppose that

$$\mathrm{Supp}(v(x)) \subset \bar{\boldsymbol{S}}_{iR} = \boldsymbol{S}_{iR} \cup \ldots \cup \boldsymbol{S}_{nR}. \tag{2.54}$$

Then, for any point $x_0 \in \boldsymbol{S}_{iR}$, $\boldsymbol{S}_{iR}$ is a non-singular subvariety near x_0. Since the real principal symbol $\sigma_{A_i}(v(x)) = 0$ on $\boldsymbol{\Lambda}_i^\circ$, we have

$$\mathrm{S.S.}\,(v(x)) \cap \boldsymbol{\Lambda}_i^\circ = \phi.$$

The variety $\boldsymbol{\Lambda}_i^\circ$ is an open dense subset of $\boldsymbol{\Lambda}_i = \overline{\boldsymbol{T}^*_{S_{iR}}\boldsymbol{V}_R}$ and hence

$$\mathrm{S.S.}(v(x)) \supset \boldsymbol{T}^*_{S_{iR}}\boldsymbol{V}_R \cap \pi^{-1}(x_0) \supset \boldsymbol{\Lambda}_i^\circ \cap \pi^{-1}(x_0).$$

Thus, by Corollary 2. 16, we have $v(x) = 0$ near x_0, and hence we have

$$\mathrm{Supp}(v(x)) \subset \bar{\boldsymbol{S}}_{i+1R} = \boldsymbol{S}_{i+1R} \cup \ldots \cup \boldsymbol{S}_{nR}. \tag{2.55}$$

Then, by induction on i, we have $\mathrm{Supp}(v(x)) = \phi$, i. e., $v(x) = 0$ on $\boldsymbol{V}_R$.

q. e. d.

Chapter II. Constructions of Relatively Invariant Hyperfunctions and the Fourier Transforms

The purpose of this chapter is to construct some hyperfunction solutions to $\mathfrak{M}_s$ and to calculate the Fourier transforms of them. The results of the Fourier transforms in §3 were first computed by M. Sato and T. Shintani [5] and T. Shintani [21] by another method when $\boldsymbol{V}_R = \boldsymbol{Her}(n, \boldsymbol{C})$ and $\boldsymbol{V}_R = \boldsymbol{Sym}(n, \boldsymbol{R})$, respectively. As far as the results in Theorem 3. 9, T. Suzuki has obtained them by using Kashiwara's method. But he did not state nothing about the results after Theorem 3. 10.

The coefficients of real principal symbols on a Lagrangian subvariety is always with respect to the basis in (2. 47). In this chapter, we shall always deal with the real forms, so we often omit $\boldsymbol{R}$ beside the notations.

For example, we denote simply V instead of V_R.

§ 3. The Hyperfunctions $|P|_i^s(x)$ and Their Fourier Transforms

We begin with the definition of tempered distributions with meromorphic parameter $s \in \boldsymbol{C}$.

Definition 3.1. Let Ω be a domain in $\boldsymbol{C}$ and let X be a finite dimensional real vector space.

1) We say that $u_s(x)$ is a tempered distribution with *a holomorphic parameter* $s \in \Omega$ if

i) For any $s \in \Omega$, $u_s(x)$ is a tempered distribution on X.

ii) For any $f \in \mathscr{S}(X)$, $T_s(f) = \int u_s(x) f(x) dx$ is holomorphic in $s \in \Omega$.

2) We say that $u_s(x)$ is a tempered distribution with a *meromorphic* parameter $s \in \Omega$ if $u_s(x)$ is written as $m(s) \times h_s(x)$ where $m(s)$ is a meromorphic function on Ω and $h_s(x)$ is a tempered distribution with a holomorphic parameter $s \in \Omega$. We say that $u_s(x)$ has a pole at $s = s_0$ if $m(s)$ has a pole at s_0 and $u_s(x)$ is not a tempered distribution with a holomorphic parameter at $s = s_0$.

Then we have the following propositions.

Proposition 3.2. *Let Ω be a domain in $\boldsymbol{C}$ and let X be a finite dimensional real vector space.*

1) *Let $u_s(x)$ and $v_s(x)$ be two tempered distributions on X with a meromorphic parameter $s \in \Omega$. If $u_s(x) = v_s(x)$ for any s in an open subset $\Omega' \subset \Omega$, then $u_s(x) = v_s(x)$ for any $s \in \Omega$.*

2) *Let $u_s(x)$ be a tempered distribution on X with a meromorphic parameter $s \in \Omega$. Then the Fourier transform $\hat{u}_s(x^*)$ with respect to the variable $x \in X$ is a tempered distribution on X^* (the dual vector space to X) with a meromorphic parameter $s \in \Omega$ whose poles are located at the same place as $u_s(x)$.*

3) *Let $u_s(x)$ be a tempered distribution on X with a meromorphic parameter $s \in \Omega$ and let $P(s, x, D_x)$ be a differential operator on X whose coefficients are polynomials with respect to x and holomorphic with respect to s. Then $P(s, x, D_x) u_s(x)$ is a tempered distribution on X with a meromorphic parameter $s \in \Omega$ and the set of the locations of the poles of $P(s, x, D_x)$ is contained in the set of the locations of the poles of $u_s(x)$.*

Proof. 1) It is trivial from the uniqueness of the analytic continuation.

2) We denote by $\hat{u}(y)$ the Fourier transform of a tempered distribution $u(x)$. First, we suppose that $u_s(x)$ is a tempered distribution with a holomorphic parameter $s\in\boldsymbol{\Omega}$. We have to show that

$$f \longmapsto \hat{T}_s(f) = \int \hat{u}_s(y) f(y)\, dy \qquad (f\in\mathcal{S}(X^*)),$$

defines a tempered distribution for any $s\in\Omega$, and that $T_s(f)$ is holomorphic with respect to $s\in\boldsymbol{\Omega}$ for any $f\in\mathcal{S}(X^*)$. In fact, $\hat{T}_s(f) = T_s(\hat{f})$ by definition and $f\longrightarrow\hat{f}$ is a linear continuous isomorphism from $\mathcal{S}(X^*)$ to $\mathcal{S}(X)$. Therefore,

$$f \longmapsto \hat{T}_s(f) = T_s(\hat{f}),$$

defines a tempered distribution on X^* and $T_s(\hat{f})$ is holomorphic with respect to $s\in\boldsymbol{\Omega}$ for any $f\in\mathcal{S}(X^*)$. Thus $\hat{u}_s(y)$ is a tempered distribution with a holomorphic parameter $s\in\boldsymbol{\Omega}$.

Next, consider the case of $u_s(x)$ with a meromorphic parameter $s\in\boldsymbol{\Omega}$. From the definition, $u_s(x) = m(s)\times v_s(x)$ with $m(s)$ a meromorphic function on $\boldsymbol{\Omega}$ and $v_s(x)$ a tempered distribution on X with a holomorphic parameter $s\in\boldsymbol{\Omega}$. The Fourier transform is $\hat{u}_s(y) = m(s)\times\hat{v}_s(y)$ and hence $\hat{u}_s(y)$ is a tempered distribution with a meromorphic parameter $s\in\Omega$ whose poles are located at the same place as $u_s(x)$. In fact, if $\hat{u}_s(y)$ does not have a pole at $s=s_0$, then $u_s(x)$ does not have a pole at $s=s_0$, i.e., holomorphic, because it is the inverse Fourier transform of $\hat{u}_s(y)$.

3) First, we suppose that $u_s(x)$ is a tempered distribution with a holomorphic parameter $s\in\boldsymbol{\Omega}$. Let

$$P(s, x, D_x) = \sum_{\alpha} a_\alpha(s, x) D_x^\alpha, \text{ (a finite sum)},$$

where $a_\alpha(s, x)$ is a polynomial whose coefficients are holomorphic functions in $s\in\boldsymbol{\Omega}$. Namely, we have

$$a_\alpha(s, x) = \sum_{\beta} a_{\alpha\beta}(s) x^\beta, \quad \text{(a finite sum)},$$

with $a_{\alpha\beta}(s)$ a holomorphic function in $s\in\boldsymbol{\Omega}$. Therefore we have to show that $v_{\alpha\beta s}(x) = a_{\alpha\beta}(s) x^\beta D_x^\alpha u_s(x)$ is a tempered distribution with a holomorphic parameter $s\in\boldsymbol{\Omega}$. Apparently, $v_{\alpha\beta s}(x)$ is a tempered distribution for any fixed $s\in\boldsymbol{\Omega}$. Consider the integral

$$T_{\alpha\beta s}(x)=\int v_{\alpha\beta s}(x)f(x)\,dx=\int a_{\alpha\beta}(s)\,(x^{\beta}D_x^{\alpha}u_s(x))f(x)\,dx$$
$$=(-1)^{|\alpha|}a_{\alpha\beta}(s)\int u_s(x)\,(D_x^{\alpha}(x^{\beta}f(x)))\,dx.$$

Here, $f(x)\in\mathscr{S}(X)$. Then $D_x^{\alpha}(x^{\beta}f(x))\in\mathscr{S}(X)$ and hence $T_{\alpha\beta s}(x)$ is holomorphic in $s\in\Omega$. Thus $v_{\alpha\beta s}(x)$ is a tempered distribution with a holomorphic parameter $s\in\Omega$.

Next, consider the case that $u_s(x)$ is a tempered distribution with a meromorphic parameter $s\in\Omega$. However, it is evident from the definition that $P(s, x, D_x)u_s(x)$ is a tempered distribution with a meromorphic parameter $s\in\Omega$, because

$$\begin{aligned}P(s, x, D_x)u_s(x) &= P(s, x, D_x)\,(m(s)\times v_s(x))\\ &= m(s)\times P(s, x, D_x)v_s(x),\end{aligned}$$

where $m(s)$ is a meromorphic function on Ω and $v_s(x)$ is a tempered distribution with a holomorphic parameter $s\in\Omega$. The locations of the poles of $P(s, x, D_x)u_s(x)$ is continued in those of $m(s)$ and we have the result. q. e. d.

Let us consider some examples of tempered distributions with a meromorphic parameter. Recall the connected component decomposition of V_R-S_R in the preceding section;

$$V_n^{(n)}\cup V_{n-1}^{(n)}\cup\dots\cup V_0^{(n)}=V_R-S_R \tag{3.1}$$

where $V_i^{(n)}$ is the connected component of V_R-S_R consisting of the elements of signature $(i, n-i)$. We define the hyperfunction,

$$|P|_i^s(x)=\begin{cases}|P(x)|^s & \text{if } x\in V_i^{(n)},\\ 0 & \text{if } x\notin V_i^{(n)}.\end{cases} \tag{3.2}$$

This hyperfunction $|P|_i^s(x)$ is a continuous function when s has a sufficiently large real part and is a tempered distribution on V_R with a holomorphic parameter $s\in\Omega_k=\{s\in C;\ \mathrm{Re}(s)>k\}$. Here we put k to be sufficiently large so as that $|P|_i^s(x)$ is a continuous function on V_R. In fact, if $s\in\Omega_k$, then

$$\phi\longmapsto\int|P|_i^s(x)\,\phi(x)\,dx \qquad (\phi\in\mathscr{S}(V_R))$$

is convergent and defines a tempered distribution on V_R. We shall continue $|P|_i^s(x)$ as a tempered distribution with a meromorphic parameter $s\in \boldsymbol{C}$ in the following way. When $s\in\boldsymbol{\Omega}_k$, the tempered distribution $|P|_i^s(x)$ is a continuous function on $\boldsymbol{V}$. By Proposition 1. 4, we have, modulo constant multiples,

$$P(D_x)\,|P(x)|_i^{s+1}=b(s)\,|P(x)|_i^s,$$

where $b(s)$ is the b-function of $P^s(x)$. When $\phi(x)\in\mathscr{S}(V)$, we have

$$(3.3)\qquad \int |P|_i^s(x)\phi(x)dx=\int b(s)^{-1}(P(D_x)\,|P|_i^{s+1}(x))\phi(x)dx$$
$$=b(s)^{-1}\int |P|_i^{s+1}(x)\,(P(-D_x)\phi(x))dx,$$

and hence $|P(x)|_i^s$ is defined by this formula for $s\in\Omega_{k-1}$. We can define $|P|_i^s(x)$ for $s\in\boldsymbol{\Omega}_{k-m}$ inductively by

$$(3.4)\qquad \int |P|_i^s(x)\phi(x)dx=\prod_{j=0}^{m-1} b(s+j)^{-1}\int |P|_i^{s+m}(x)\,(P(-D_x)^m\phi(x))dx.$$

Thus $|P|_i^s(x)$ is well defined as a tempered distribution with a meromorphic parameter $s\in\boldsymbol{C}$. The poles of $|P|_i^s(x)$ with respect to s are located in the set,

$$(3.5)\qquad \{s\in\boldsymbol{C};\ s \text{ is a root of } b(s+j)=0 \text{ with some } j=0,1,2,..\}.$$

Definition 3. 3 (critical points). We say that $s\in\boldsymbol{C}$ is a *critical point* for $P(x)^s$ if

$$(3.6)\qquad s\in\{s_i-j\in\boldsymbol{C};\ s_i \text{ is a root of } b(s)=0 \text{ and } j=1,2,3,\dots\},$$

and we denote by $\mathrm{Crit}(P(x)^s)$ the set of critical points for $P(x)^s$. The hyperfunction $|P|_i^s(x)$ is well defined for any $s\notin\mathrm{Crit}(P(x)^s)$ and has a possible poles at the points in $\mathrm{Crit}(P(x)^s)$.

Proposition 3. 4. 1) *Let s be a point in the complement of* $\mathrm{Crit}(P(x)^s)$. *Then the hyperfunction $|P|_i^s(x)$ is a solution to the holonomic system $\mathfrak{M}_s$, i. e., $(\langle d\rho(A)x, D_x\rangle-s\delta\chi(A))\,|P|_i^s(x)=0$ for any $A\in\mathscr{G}_{\boldsymbol{R}}$.*

2) *Let $\Gamma_q(s)=\prod_{i=1}^{q}\Gamma(s-s_i)$. Here s_i is the roots of the b-function, which is defined in* (2. 50) *explicitly. Then $\Gamma_q(s)^{-1}|p|_j^s(x)$ $(j=0,1,\dots,n)$ is a tempered distribution with a holomorphic parameter s in the domain* $\{s\in\boldsymbol{C};\mathrm{Re}(s)>s_{q+1}\}$. *In particular, when $q=n$, $\Gamma_n(s)^{-1}|P|_j^s(x)$ is a tempered distribution with a holomorphic parameter s in $\boldsymbol{C}$. Moreover, if* $\mathrm{Re}(s)>s_{q+1}$ *(resp. $s\in\boldsymbol{C}$), then $\Gamma_q(s)^{-1}|P|_j^s(x)$ (resp. $\Gamma_n(s)^{-1}|P|_j^s(x)$) is a solution to the holonomic system $\mathfrak{M}_s$.*

Proof. 1) Let $A\in\mathscr{G}_R$. Then $\exp(tA)\in G_R^+$ for sufficiently small $t\in\boldsymbol{R}$. We have

$$(3.7)\qquad |P|_i^s(\rho(\exp(tA)x)=\chi(\exp(tA))\,|P|_i^s(x),$$

for any $s\in\boldsymbol{C}$ with a sufficiently large real part since $|P|_i^s(x)$ is a continuous function on V_R. By differentiating (3.7) with t and by putting $t=0$, we have

$$(3.8)\qquad \langle d\rho(A)x, D_x\rangle |P|_i^s(x)=s\delta\chi(A)\,|P|_i^s(x),$$

for $s\in\boldsymbol{C}$ with a sufficiently large real part so as that $|P|_i^s(x)$ is C^1-class. Thus, $|P|_i^s(x)$ is a solution of $\mathfrak{M}_s$ for any $s\in\boldsymbol{C}$ with a sufficiently large real part. Since $|P|_i^s(x)$ is continued to the complex plane as a tempered distribution with a meromorphic parameter $s\in\boldsymbol{C}$, both hands of (3.8) are tempered distributions with a meromorphic parameter $s\in\boldsymbol{C}$ by Proposition 3.2. The equation (3.8) holds for any $s\in\boldsymbol{C}$-$\mathrm{Crit}(P(x)^s)$ by the analytic continuation.

2) Note that,

$$(3.9)\qquad b(s)b(s+1)\dots b(s+m)\cdot|P|_i^s(x),$$

is a tempered distribution with a holomorphic parameter s in $\boldsymbol{\Omega}_{-m-1}=\{s\in\boldsymbol{C};\mathrm{Re}(s)>-m-1\}$. In fact, if $|P|_i^s(x)$ is holomorphic with respect to s in $\boldsymbol{\Omega}_k$, then (3.9) is holomorphic with respect to s in $\boldsymbol{\Omega}_{k-m}$. We take m a sufficiently large integer. All the zeros of $b(s), b(s+1),\dots,b(s+m)$ are contained in $\mathrm{Re}(s)\leqq-1$, hence $|P|_i^s(x)$ actually does not have a pole in $\mathrm{Re}(s)>-1$, i.e., holomorphic. Similarly, all the zeros of $b(s+1), b(s+2),\dots,b(s+m)$ are contained in $R(s)\leqq-2$, and hence $b(s)\,|P|_i^s(x)$ does not have a pole in $\mathrm{Re}(s)\geqq-2$. Moreover, since all the zeros of $b(s+k)$, $b(s+k+2)$, $\dots,b(s+m)$ are contained in $\mathrm{Re}(s)\leqq-(k+1)$,

$$(3.10)\qquad b(s)b(s+1)\dots b(s+k-1)\,|P|_i^s(x),$$

is holomorphic in $\mathrm{Re}(s)>-(k+1)$.

Next we consider $\Gamma_q(s)^{-1}|P|_i^s(x)$. $\Gamma_1(s)^{-1}=\Gamma(s-s_1)^{-1}$ has a simple pole at $s=s_1-k$ $(k=0, 1, 2,\dots)$. Namely, among the poles of

$$(3.11)\qquad b(s+k)^{-1}=(s-s_1+k)^{-1}(s-s_2+k)^{-1}\dots(s-s_n+k)^{-1},$$

the first pole $(s-s_1+k)^{-1}$ is canceled by $\Gamma(s-s_1)^{-1}$. Therefore, for any integer $m\geqq0$, the poles of

$$\Gamma_1(s)^{-1}b(s)^{-1}b(s+1)^{-1}\dots b(s+m)^{-1},$$

are located in $\mathrm{Re}(s)\leqq s_2$ since $s_1>s_2>s_3\ldots>s_n$. Similarly, among the poles of (3.11), the poles $(s-s_1+k)^{-1}$, $(s-s_2+k)^{-1},\ldots,$ $(s-s_q+k)^{-1}$ are canceled by $\Gamma_q(s)^{-1}=\Gamma(s-s_1)^{-1}\Gamma(s-s_2)^{-1}\ldots\Gamma(s-s_q)^{-1}$. Then, for any integer $m\geqq 0$, the poles of

$$\Gamma_q(s)^{-1}b(s)^{-1}b(s+1)^{-1}\ldots b(s+m)^{-1} \tag{3.12}$$

is located in $\mathrm{Re}(s)\leqq s_{q+1}$. In particular, if $q=n$, then all the poles of (3.11) are canceled by $\Gamma_n(s)^{-1}$, and hence $\Gamma_n(s)^{-1}b(s)^{-1}b(s+1)^{-1}\ldots b(s+m)^{-1}$ is an entire function for any integer $m\geqq 0$. Therefore,

$$\Gamma_q(s)^{-1}|P|_i^s(x)=\Gamma_q(s)^{-1}\prod_{j=0}^{m}b(s+j)^{-1}\Big(\prod_{j=0}^{m}b(s+j)\Big)|P|_i^s(x), \tag{3.13}$$

is a tempered distribution with a holomorphic parameter s in $\{s\in\boldsymbol{C};\ \mathrm{Re}(s)>\max\{-m-1,s_{q+1}\}\}$, for any integer $m\geqq 0$, and hence it is holomorphic in $\mathrm{Re}(s)>s_{q+1}$. In particular, if $q=n$, then (3.13) is holomorphic in $\boldsymbol{\Omega}_{-m-1}$ for any integer $m\geqq 0$, and hence it is entire with respect to s.

Lastly, we shall show that $\Gamma_q(s)^{-1}|P|_i^s(x)$ is a solution to $\mathfrak{M}_s$ for any s in $\mathrm{Re}(s)>s_{q+1}$. In fact, since

$$(\langle d\rho(A)x, D_x\rangle-s\delta\chi(A))\Gamma_q(s)^{-1}|P|_i^s(x)=0, \tag{3.14}$$

for any $s\in\boldsymbol{C}$ with a sufficiently large real part, we have the result by an analytic continuation to $\{s\in\boldsymbol{C};\mathrm{Re}(s)>s_{q+1}\}$ (Proposition 3.2, 1)). In particular, if $q=n$, then (3.14) is valid for any $s\in\boldsymbol{C}$.

q. e. d.

We introduce the Euclidean measure dx on $\boldsymbol{V_R}$ by

(3.15) 1) $dx=|(\bigwedge_{i=1}^{n}dx_{ii})\wedge(\bigwedge_{i<j}dx_{ij})|$, when $\boldsymbol{V_R}=\boldsymbol{Sym}(n,\boldsymbol{R})$.

2) $dx=|(\bigwedge_{i=1}^{n}dx_{ii})(\bigwedge_{i<j}(d\mathrm{Re}(x_{ij})\wedge d\mathrm{Im}(x_{ij}))|$, when $\boldsymbol{V_R}=\boldsymbol{Her}(n,\boldsymbol{C})$.

3) $dx=|(\bigwedge_{i=1}^{n}dx_{ii})(\bigwedge_{i<j}(dx_{ij}^1\wedge dx_{ij}^2\wedge dx_{ij}^3\wedge dx_{ij}^4))|$, when $\boldsymbol{V_R}=\boldsymbol{Her}(n,\boldsymbol{H})$. Here, we write $x_{ij}=x_{ij}^1+x_{ij}^2e_1+x_{ij}^3e_2+x_{ij}^4e_1e_2$ with $x_{ij}^k\in\boldsymbol{R}$ and $e_1^2=e_2^2=-1$, $e_1e_2=-e_2e_1$.

We define the Fourier transform of $u(x)\in\mathscr{S}(\boldsymbol{V_R})$ by

$$\hat{u}(y)=\int u(x)\ \exp(2\pi\sqrt{-1}\langle x,y\rangle)\,dx, \tag{3.16}$$

and the inverse Fourier transform for $u(y)\in\mathscr{S}(V_R^*)$, by

$$\check{u}(x)=\int u(y)\ \exp(-2\pi\sqrt{-1}\langle x,y\rangle)\,dy. \tag{3.17}$$

Thus, as a well known result, we have

$$\overset{\diamond}{u}(x)=2^{-n(n-1)v/2}u(x)\ \text{ and }\ \overset{\times}{u}(y)=2^{-n(n-1)v/2}u(y), \tag{3.18}$$

where $v=1$ $(V_R=\boldsymbol{Sym}(n,\boldsymbol{R}))$, 2 $(V_R=\boldsymbol{Her}(n,\boldsymbol{C}))$ and 4 $(V_R=\boldsymbol{Her}(n,\boldsymbol{H}))$.

Recall the holonomic system

$$\mathfrak{M}_s;\ (\langle d\rho(A)x,\ D_x\rangle-s\delta\chi(A))u(x)=0.$$

Here, $d\rho$ and $\delta\chi$ are the infinitesimal representation of ρ and the infinitesimal character of χ; $\langle\ ,\ \rangle$ is the bilinear form on $V_R\times V_R^*$ defined in (2.4); s is a complex number. We define the "dual" holonomic system on V_R^* of $\mathfrak{M}_s$ by

$$\mathfrak{M}_s^*;\ (\langle d\rho^*(A)y,\ D_y\rangle-s\delta\chi(A))v(y)=0. \tag{3.19}$$

Here, $d\rho^*$ is the infinitesimal representation of the contragredient representation of ρ. Then we have the following propositions.

Proposition 3.5. 1) *If we identify V_R and V_R^* by the inner product $\langle x,y\rangle$ defined in* (2.4), *then the holonomic systems $\mathfrak{M}_s$ and $\mathfrak{M}_{-s}^*$ are the same.*

2) *We denote by* $\mathrm{Sol}(\mathfrak{M}_s)_{tem}$ *the tempered distribution solution space to the holonomic system $\mathfrak{M}_s$. Then we have*

(3.20) (1) $u(x)\in\mathrm{Sol}(\mathfrak{M}_s)_{tem}\ \Leftrightarrow\ \hat{u}(y)\in\mathrm{Sol}(\mathfrak{M}^*_{s+(n'/n)})_{tem}$.

(2) $v(y)\in\mathrm{Sol}(\mathfrak{M}_s^*)_{tem}\ \Leftrightarrow\ \check{v}(x)\in\mathrm{Sol}(\mathfrak{M}_{s-(n'/n)})_{tem}$.

(3.21) (1) $u(x)\in\mathrm{Sol}(\mathfrak{M}_s)_{tem}\ \Leftrightarrow\ u(x)$ *is a tempered distribution on V_R and $u(\rho(g)x)=\chi(g)^s u(x)$ for all $g\in G_R^+$.*

(2) $v(y)\in\mathrm{Sol}(\mathfrak{M}_s^*)_{tem}\ \Leftrightarrow\ v(y)$ *is a tempered distribution on V_R^* and $v(\rho^*(g)y)=\chi(g)^s v(y)$ for all $g\in G_R^+$.*

3) *Let* $u(x)\in\mathrm{Sol}(\mathfrak{M}_s)_{tem}$ *(resp.* $v(y)\in\mathrm{Sol}(\mathfrak{M}^*_{s+(n'/n)})_{tem}$*). Then $u(x)$ (resp. $v(y)$) is real analytic on V_R-S_R (resp. $V_R^*-S_R^*$) and*

(3.22) (1) $u(x)\,|_{V_R-S_R}=\sum_i a_i\,|P|_i^s(x)\,|_{V_R-S_R}$,

(2) $v(y)\,|_{V_R^*-S_R^*}=\sum_i b_i\,|P|_i^{-s-(n'/n)}(y)\,|_{V_R^*-S_R^*}$,

with some constants $a_i \in \boldsymbol{C}$ *(resp.* $b_i \in \boldsymbol{C}$*).*

Proof. 1) Note that we identify V_R and V_R^* by the inner product $\langle x, y\rangle = \mathrm{tr}(x^t\bar{y})$. Then the Lie algebra $\mathscr{G}_R$ is naturally identified with a Lie subalgebra in $\mathfrak{gl}(V_R) = \mathfrak{gl}(V_R^*)$ by $d\rho$ and $d\rho^*$. However, the images of $\mathscr{G}_R$ by $d\rho$ and $d\rho^*$ coincides with each other by the automorphism $\phi; A \to -{}^t\bar{A}$. In fact, we have

$$(3.23)\qquad \begin{aligned} &d\rho(A)x = Ax + x^tA,\ d\rho^*(A)x = -{}^t\bar{A}x - x\bar{A},\\ &\delta\chi(A) = -\delta\chi(-{}^t\bar{A}) = \delta\chi({}^t\bar{A}). \end{aligned}$$

Therefore, we have

$$(3.24)\qquad \begin{aligned} &\{\langle d\rho(A)x, D_x\rangle - s\delta\chi(A)\ ;\ A \in \mathscr{G}_R\}\\ &= \{\langle d\rho^*(A)x, D_x\rangle + s\delta\chi({}^t\bar{A})\ ;\ A \in \mathscr{G}_R\}. \end{aligned}$$

Thus we have the result.

2) First we shall show that (3.21) (1). Then (3.21) (2) is evident by 1). Let $u(x) \in \mathrm{Sol}(\mathfrak{M}_s)_{tem}$. Then, for any $f(x) \in \mathscr{S}(V_R)$, we have

$$(3.25)\qquad \int u(x)\langle d\rho(A)x, D_x\rangle^* f(x)\,dx = s\delta\chi(A)\int u(x)f(x)\,dx.$$

Here, $\langle d\rho(A)x, D_x\rangle^*$ stands for the formal adjoint operator of $\langle d\rho(A)x, D_x\rangle$. Since

$$\begin{aligned} \langle d\rho(A)x, D_x\rangle^* &= -\langle D_x, d\rho(A)x\rangle\\ &= -\langle d\rho(A)x, D_x\rangle - tr(d\rho(A))\\ &= -\langle d\rho(A)x, D_x\rangle - (n'/n)\delta\chi(A), \end{aligned}$$

we have

$$(3.26)\qquad \int u(x)(-\langle d\rho(A)x, D_x\rangle)f(x)\,dx = (s + (n'/n))\delta\chi(A)\int u(x)f(x)\,dx.$$

On the other hand, for a sufficiently small $t \in \boldsymbol{R}$, the element $g = \exp(tA) \in G_R^+$ is defined and

$$(3.27)\qquad \begin{aligned} &\sum_{m=0}^{\infty}(t^m/m!)(-\langle d\rho(A)x, D_x\rangle)^m\ f(x) = f(\rho(g)^{-1}x),\\ &\sum_{m=0}^{\infty}(t^m/m!)(s + (n'/n))^m\delta\chi(A)^m\ f(x) = \chi(g)^{s+(n'/n)}f(x), \end{aligned}$$

are convergent in $\mathscr{S}(V_R)$. Thus we have

$$(3.28)\qquad \int u(x)f(\rho(g)^{-1}x)\,dx$$

$$=\int u(x)\sum_{m=0}^{\infty}(t^m/m!)(-\langle d\rho(A)x, D_x\rangle)^m f(x)\,dx$$

$$=\int u(x)\sum_{m=0}^{\infty}(t^m/m!)(s+(n'/n))^m\delta\chi(A)^m f(x)\,dx$$

$$=\int u(x)\chi(g)^{s+(n'/n)}f(x)\,dx.$$

Since

$$\int u(x)f(\rho(g)^{-1}x)\,dx$$
$$=\int u(\rho(g)x)f(x)\,d(\rho(g)x)=\chi(g)^{(n'/n)}\int u(\rho(g)x)f(x)\,dx,$$

we have

$$\int u(\rho(g)x)f(x)\,dx=\chi(g)^s\int u(x)f(x)\,dx, \tag{3.29}$$

for all $f(x)\in\mathscr{S}(V_{\boldsymbol{R}})$. Thus we have

$$u(\rho(g)x)=\chi(g)^s u(x), \tag{3.30}$$

with $g=\exp(tA)$ for sufficiently small $t\in\boldsymbol{R}$. Any element $g\in G_{\boldsymbol{R}}^{+}$ is written as

$$g=\exp(t_1A_1)\ldots\exp(t_kA_k), \tag{3.31}$$

with $t_i\in\boldsymbol{R}$ and $A_i\in\mathscr{G}_{\boldsymbol{R}}$. Therefore (3.30) is valid for all $g\in G_{\boldsymbol{R}}^{+}$.

Conversely, if (3.29) is valid, then (3.28) is also true. By differentiating (3.28) by t and by putting $t=0$, we have (3.26). Thus, we have $u(x)\in\mathrm{Sol}(\mathfrak{M}_s)_{tem}$.

Next we shall show (3.20), (1). We have to show that

$$u(\rho(g)x)=\chi(g)^s u(x)\Leftrightarrow\hat{u}(\rho^*(g)y)=\chi(g)^{s+(n'/n)}\hat{u}(y). \tag{3.32}$$

We suppose that $u(\rho(g)x)=\chi(g)^s u(x)$. Then we have

$$\int\hat{u}(\rho^*(g)y)f(y)\,dy=\int\hat{u}(y)f(\rho^*(g)^{-1}y)\,d(\rho^*(g)^{-1}y) \tag{3.33}$$
$$=\chi(g)^{(n'/n)}\int u(x)\widehat{f(\rho^*(g)^{-1}y)}\,dx$$
$$=\int u(x)\hat{f}(\rho(g)^{-1}x)\,dx$$
$$=\int u(\rho(g)x)\hat{f}(x)\,d(\rho(g)x)=\chi(g)^{s+(n'/n)}\int u(x)\hat{f}(x)\,dx,$$

because,

$$\widehat{f(\rho^*(g)^{-1}y)}=\int f(\rho^*(g)^{-1}y)\exp(2\pi\sqrt{-1}\langle x,y\rangle)\,dy$$
$$=\int f(y)\exp(2\pi\sqrt{-1}\langle x,\rho^*(g)y\rangle)\,d(\rho^*(g)y)$$
$$=\chi(g)^{-(n'/n)}\int f(y)\exp(2\pi\sqrt{-1}\langle\rho(g)^{-1}x,y\rangle)\,dy$$
$$=\chi(g)^{-(n'/n)}\hat{f}(\rho(g)^{-1}x).$$

Thus we have

$$\hat{u}(\rho^*(g)y)=\chi(g)^{s+(n'/n)}\hat{u}(y).$$

The converse is shown in the same way. For (3.20), (2), the proof is the same.

3) In fact, $\mathfrak{M}_s$ is an elliptic system on V_R-S_R, and hence $u(x)$ is real analytic on V_R-S_R. Moreover, since $\mathfrak{M}_s$ is a simple holonomic system on V_R-S_R and $|P|_i^s(x)$ is a basis of the solution on $V_i^{(n)}$, we have (3.22), (1). For (3.22), (2), we can show it in the same way.

q. e. d.

Proposition 3.6. *Let* $s\notin\mathrm{Crit}(P(x)^s)$. *Then we have*

$$(3.34)\qquad \sigma_{V_j^{(n)}\times\{0\}}(|P|_i^s(x))=\begin{cases}|P(x)|^s & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

$$(3.35)\qquad \sigma_{\{0\}\times V_j^{*(n)}}(|P|_i^s(x))=(2\pi)^{ns+(n'/2)}2^{n(n-1)/2}$$
$$\times|\check{P}|_i^s(y)\sqrt{|dy|}/\sqrt{|dx|}\;\Big|_{V_j^*}$$

Here, $\{0\}\times V_j^{*(n)}$ *(resp.* $V_i^{(n)}\times\{0\}$*) is the Lagrangian subvariety* Λ_n^{0i} *(resp.* Λ_0^{i0}*).*

Proof. (3.34) is evident by definition, because $|P|_i^s(x)$ is a real analytic function on $V_j^{(n)}$ (Definition 2.7).

Now we consider the tempered distribution

$$(3.36)\qquad |P|_i^s(x)=\int|\check{P}|_i^s(y)\exp(2\pi\sqrt{-1}\langle x,y\rangle)\,dy\,(2^{n(n-1)v/2})$$

The distribution $|\check{P}|_i^s(y)$ is a homogeneous distribution on V_R^* of degree $-sn-n'$. In fact, it is proved by that $|P|_i^s(x)$ is a homogeneous distribution on V_R of degree sn. Therefore, we can view $|\check{P}|_i^s(y)\,|_{V_R^*-\{0\}}$ as a distribution $|\check{P}|_i^s(r\omega)=r^{-ns-n}|\check{P}|_i^s(\omega)$ on $(r,\omega)\in R_+\times S^{n'-1}$ where $S^{n'-1}$ is the unit sphere in V_R^* by the map

$$(3.37)\qquad \begin{array}{ccc} (r,\omega) & \longmapsto & r\omega \\ \cap\!\!\!\cap & & \cap\!\!\!\cap \\ \mathbb{R}_+\times S^{n'-1} & \to & V_R^*-\{0\} \end{array}$$

Moreover, we can extend this as a distribution on $R\times S^{n'-1}$ by

$$(3.38)\qquad |\check{P}|_i^s(r\omega)=|\check{P}|_i^s(\omega)r_+^{-ns-n'},$$

where $r_+^{-ns-n'}$ is a tempered distribution that is equal to $r^{-ns-n'}$ on $r>0$ and is equal to 0 on $r<0$. (Of course such distribution is not uniquely determined.) The distribution (3.38) is naturally a tempered distribution on V_R^* by

$$(3.39)\qquad f(x)\longmapsto\int_{-\infty}^{+\infty}\int_{S^{n'-1}}|\check{P}|_i^s(\omega)r_+^{-ns-n}f(r\omega)r^{n'-1}dr d\omega,$$

with $f(x)\in\mathscr{S}(V_R^*)$. Here $d\omega$ is the rotation invariant measure on $S^{n'-1}$. The tempered distribution (3.39) coincides with $|\check{P}|_i^s(y)$ except for the origin. Therefore, we have

$$|P|_i^s(x)=\int_{S^{n'-1}}|\check{P}|_i^s(\omega)\left(\int_{-\infty}^{+\infty}r_+^{-ns-n'+n'-1}\exp(2\pi\sqrt{-1}r\langle x,\omega\rangle)dr\right)d\omega \\ +O(x),$$

where $O(x)$ is a real analytic function on V_R. In fact, the Fourier transform of any compactly supported distribution is an entire function. Note that the Fourier transform of r_+^λ $(\lambda\in\mathbb{C})$ is calculated to be

$$(3.40)\qquad \int_{-\infty}^{+\infty}r_+^\lambda\exp(\sqrt{-1}r\sigma)dr=\exp\left(\frac{\pi}{2}\sqrt{-1}(\lambda+1)\right)\Gamma(\lambda+1)(\sigma+i0)^{-\lambda-1} \\ +O(\sigma)$$

$$(3.41)\qquad =\Phi_{\lambda+1}(\sqrt{-1}(\sigma+i0))+O(\sigma).$$

Here $O(\sigma)$ is a real analytic function in σ and $\Phi_{\lambda+1}(\sqrt{-1}(\sigma+i0))$ is well defined for any $\lambda\in\mathbb{C}$ as a microfunction whose singular support is $(0, d\sigma)$ and the meaning of the expression (3.41) is a hyperfunction which coincides with $\Phi_{\lambda+1}(\sqrt{-1}(\sigma+i0))$ modulo real analytic functions. (For the definition of Φ_λ, see §2, and for the proof, for example, see Gelfand-Shilov [3].) Thus we have

$$(3.42)\qquad |P|_i^s(x)=\int\Phi_{-ns}(\sqrt{-1}(2\pi\langle x,\omega\rangle+i0))|\check{P}|_i^s(\omega)d\omega\circ(2^{n(n-1)v/2}) \\ +O(x)$$

$$= (2\pi)^{ns}\cdot 2^{n(n-1)v/2}\int |\check{P}|_i^s(\omega)\,\Phi_{-ns}(\sqrt{-1}(\langle x,\omega\rangle+i0))\,d\omega$$
$$+O(x).$$

Therefore, the microfunction $|P|_i^s(x)$ is expressed on $\{0\}\times V_j^*$ by the integral (3.42) since $|P|_i^s(\omega)$ is real analytic on V_j^*. Thus, from the definition of the real principal symbol, we have

$$(3.43)\qquad \sigma_{\{0\}\times V_j^*}(|P|_i^s(x))$$
$$= (2\pi)^{(n'/2)+ns}\, 2^{n(n-1)v/2}\,|\check{P}|_i^s(\omega)\,r^{-ns-n}\sqrt{|dy|}/\sqrt{|dx|}\,|_{V_j^*}$$
$$= (2\pi)^{(n'/2)+ns}\, 2^{n(n-1)v/2}\,|\check{P}|_i^s(y)\sqrt{|dy|}/\sqrt{|dx|}\,|_{V_j^*}.$$

q. e. d.

Proposition 3.7.

$$(3.44)\qquad |P_{\Lambda_{nR}^{0i}}|^s\sqrt{|\omega_{\Lambda_{nR}^{0i}}|} = |K_0|^s\sqrt{|K_1|}\,|P|_i^{-s-\frac{n'}{n}}(y)\sqrt{|dy|},$$

where $K_0=1$ *and* $K_1=2^{\frac{n(n-1)v}{2}}$.

Proof. Note that $\Lambda_n=\{0\}\times V^*$ and that

W = Zariski closure of
$\{(x, s\cdot\mathrm{grad}\cdot\log P(x))\in T^*V\,;\ s\in \boldsymbol{R},\ \det(x)\neq 0\}$,
= Zariski closure of
$\{(s\cdot\mathrm{grad}\cdot\log P(y),\ y)\in V\times V^*;\ s\in\boldsymbol{R},\ \det(y)\neq 0\}$,
= Zariski closure of
$\{(s\cdot y^{-1}, y)\in V\times V^*;\ s\in R,\ \det(y)\neq 0\}$.

Hence from (2.44), we have

$$\sigma=\frac{\langle x,y\rangle}{n}=\frac{s\langle y^{-1},y\rangle}{n}=\frac{s\,\mathrm{tr}(y^{-1}y)}{n}=s.$$

We have

$$P_{\Lambda_n}=\frac{P\circ\pi}{\sigma^n}\,|_{\Lambda_n}=\frac{s^n(\det y^{-1})}{s^n}=\det(y)^{-1},$$

and

$$\omega_{\Lambda_n}=\frac{\pi^*(dx)\wedge d\sigma}{\sigma^{n'}}\Big/ d\sigma\,|_{\Lambda_n}=2^{\frac{n(n-1)v}{2}}\cdot\frac{d(s\cdot y^{-1})}{s^{n'}}$$
$$=2^{\frac{n(n-1)v}{2}}\cdot(\det(y))^{-\frac{2n'}{n}}dy.$$

Thus we have the result. q. e. d.

Let $c_{ij}^{(n)}(s)$ be the coefficient of $|P|_i^s(x)$ on $\Lambda_n^{0j}=\{0\}\times V_j^{*(n)}$. We have the following formula of the Fourier transforms by (3.35) and (3.44).

$$(3.45)\qquad (2\pi)^{ns+\frac{n'}{2}}\cdot 2^{\frac{n(n-1)v}{2}}\,|\check{P}|_i^s(y)\,|_{V_R^*-S_R^*}$$
$$=\sum_{j=0}^{n} 2^{\frac{n(n-1)v}{4}} c_{ij}^{(n)}(s)\,|P|_j^{-s-\frac{n'}{n}}(y).$$

Thus we have

Theorem 3.8 (Formulas of Fourier transforms). *Let* $s\notin \mathrm{Crit}(P(x)^s)$. *Then we have,*

$$(3.46)\qquad \int |P|_i^s(x)\exp(-2\pi\sqrt{-1}\langle x,y\rangle)\,dx\,|_{V_R^*-S_R^*}$$
$$=(2\pi)^{-ns-\frac{n'}{2}}\cdot 2^{-\frac{n(n-1)v}{4}}\sum_{j=0}^{n} c_{ij}^{(n)}(s)\,|\check{P}|_j^{-s-\frac{n'}{n}}(y),$$

where $n'=\dim V$ *and* $v=\begin{cases}1 & (V=\boldsymbol{Sym}(n,\boldsymbol{R}))\\ 2 & (V=\boldsymbol{Her}(n,\boldsymbol{C})).\\ 4 & (V=\boldsymbol{Her}(n,\boldsymbol{H}))\end{cases}$

The calculation of the Fourier transform is reduced to the calculation of the coefficient $c_{ij}(s)$. By the formula (2.49) (Chapter I Proposition 2.13), we have the relations of the coefficients on Λ_i and those on Λ_{i+1}. Thus we can compute the relations between the coefficients on Λ_n and those on Λ_0 inductively.

Now, we shall calculate $c_{ij}^{(n)}(s)$. Let t be a variable, and we set $c_{\Lambda_0^{i0}}=t^i$. Then we have

$$(3.47)\qquad c_{\Lambda_n^{0j}}=\sum_{i=1}^{n} c_{ij}^{(n)}(s)\cdot c_{\Lambda_0^{i0}}=\sum_{i=1}^{n} c_{ij}^{(n)}(s)\cdot t^i.$$

Conversely, from the relation $c_{\Lambda_n^{0j}}=\sum_{i=1}^{n} c_{ij}^{(n)}(s)t^i$, we have $c_{\Lambda_n^{0j}}=\sum_{i=1}^{n} c_{ij}^{(n)}(s)c_{\Lambda_0^{i0}}$ for arbitrary coefficients. On the other hand, when we give the associated numbers $c_{\Lambda_0^{i0}}=t^i$ we have $c_{\Lambda_i^{jk}}:c_{\Lambda_i^{j-1k}}=t:1$. This is proved by induction on i using the relation matrix (2.49), which does not depend on j. Therefore we have from (2.49).

$$(3.48)\qquad c_{\Lambda_{i+1}^{j-1k+1}}=F_{ik}^{+}(s)\,c_{\Lambda_i^{j-1k}}$$
$$c_{\Lambda_{i+1}^{j-1k}}=F_{ik}^{-}(s)\,c_{\Lambda_i^{j-1k}}$$

with

(3.49)

$$F_{ik}^{+}(s)=\frac{\Gamma(s-s_{i+1})}{\sqrt{2\pi}}\Big(\exp\Big(-\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}(2k-i)v\Big)t$$
$$+\exp\Big(\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}(i-2k)v\Big)\Big),$$
$$F_{ik}^{-}(s)=\frac{\Gamma(s-s_{i+1})}{\sqrt{2\pi}}\Big(\exp\Big(\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}(2k-i)v\Big)t$$
$$+\exp\Big(-\frac{\pi}{2}\sqrt{-1}(s-s_{i+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}(i-2k)v\Big)\Big).$$

Then we have

$$c_{A_n^{0j}}=F_{n-1j-1}^{+}(s)F_{n-2j-2}^{+}(s)\dots F_{n-j0}^{+}(s)\times F_{n-j-10}^{-}(s)F_{n-j-20}^{-}(s)\dots F_{00}^{-}(s)c_{A_0^{00}}.\tag{3.50}$$

$$F_{n-1\ j-1}^{+}(s)F_{n-2\ j-2}^{+}(s)\dots F_{n-j\ 0}^{+}(s)\times F_{n-j-1\ 0}^{-}(s)F_{n-j-2\ 0}^{-}(s)\dots F_{00}^{-}(s)\tag{3.51}$$

is equal to;

$$(\sqrt{2\pi})^{-n}\Gamma(s+1)\Gamma\Big(s+\frac{3}{2}\Big)\dots\Gamma\Big(s+\frac{n+1}{2}\Big)\times\prod_{p=1}^{j}\exp\Big(-\frac{\pi}{2}\sqrt{-1}p\Big)\Big(t\cdot\exp\Big(-\frac{\pi}{2}\sqrt{-1}s\Big)+(-1)^{p}\exp\Big(\frac{\pi}{2}\sqrt{-1}s\Big)\Big)\times\prod_{q=1}^{n-j}\exp\Big(\frac{\pi}{2}\sqrt{-1}q\Big)\Big(t\cdot\exp\Big(\frac{\pi}{2}\sqrt{-1}s\Big)+(-1)^{q}\exp\Big(-\frac{\pi}{2}\sqrt{-1}s\Big)\Big)\tag{3.52}$$

when $V=\boldsymbol{Sym}(n,\boldsymbol{R})$;

$$(\sqrt{2\pi})^{-n}\Gamma(s+1)\Gamma(s+2)\dots\Gamma(s+n)(\sqrt{-1})^{-n}(-1)^{\frac{n(n+1)}{2}-nj}\times\Big(t\cdot\exp\Big(-\frac{\pi}{2}\sqrt{-1}s\Big)-\exp\Big(\frac{\pi}{2}\sqrt{-1}s\Big)\Big)^{j}\times\Big(t\cdot\exp\Big(\frac{\pi}{2}\sqrt{-1}s\Big)-\exp\Big(-\frac{\pi}{2}\sqrt{-1}s\Big)\Big)^{n-j},\tag{3.53}$$

when $V=\boldsymbol{Her}(n,\boldsymbol{C})$;

$$(\sqrt{2\pi})^{-n}\Gamma(s+1)\Gamma(s+3)\dots\Gamma(s+2n-1)(\sqrt{-1})^{n}\times\Big(t\cdot\exp\Big(-\frac{\pi}{2}\sqrt{-1}s\Big)-\exp\Big(\frac{\pi}{2}\sqrt{-1}s\Big)\Big)^{j}\tag{3.54}$$

$$\times\left(t\cdot\exp\left(\frac{\pi}{2}\sqrt{-1}s\right)-\exp\left(-\frac{\pi}{2}\sqrt{-1}s\right)\right)^{n-j}$$

when $V=\boldsymbol{Her}(n, \boldsymbol{H})$.

Theorem 3.9 (Explicit computations of $c_{ij}(s)$).

1. *In the case of* $V=\boldsymbol{Sym}(n, \boldsymbol{R})$.

Putting

$$(3.55)\quad c_{ij}^{(n)}(s)=(2\pi)^{-\frac{n}{2}}\prod_{p=1}^{n}\Gamma\left(s+\frac{p+1}{2}\right)\exp\left(\frac{\pi}{2}\sqrt{-1}\left(\frac{n}{2}-i\right)(n+1)\right)$$
$$\times\exp\left(\frac{\pi}{2}\sqrt{-1}\,(2j+2i-n)\,s\right)a_{ij}^{(n)}(s)$$

we have

(3.56) *If* $n\equiv 1$ (mod. 2), *then*:

1) $$a_{ij}^{(n)}(s)=(-1)^{(n+j-2)/2}\sum_{l=max(0,\,(j+i-n-1)/2)}^{min((j-1)/2,\,(i-1)/2)}$$
$$\times\begin{pmatrix}\frac{i-1}{2}\\ l\end{pmatrix}\begin{pmatrix}\frac{n-i}{2}\\ \frac{j-1}{2}-l\end{pmatrix}\exp(-4\pi\sqrt{-1}ls)\exp(-2\pi\sqrt{-1}s)$$

if $i\equiv 1$ *and* $j\equiv 1$ (mod. 2).

2) $$a_{ij}^{(n)}(s)=(-1)^{(n+j+1)/2}\sum_{l=max(0,\,(j+i-n)/2)}^{min(j/2,\,(i-1)/2)}$$
$$\times\begin{pmatrix}\frac{i-1}{2}\\ l\end{pmatrix}\begin{pmatrix}\frac{n-i}{2}\\ \frac{j}{2}-l\end{pmatrix}\exp(-4\pi\sqrt{-1}ls)$$

if $i\equiv 1$ *and* $j\equiv 0$ (mod. 2).

3) $$a_{ij}^{(n)}(s)=(-1)^{(n+j-2)/2}\sum_{l=max(0,\,(j+i-n)/2)}^{min((j-1)/2,\,i/2)}$$
$$\times\begin{pmatrix}\frac{i}{2}\\ l\end{pmatrix}\begin{pmatrix}\frac{n-i-1}{2}\\ \frac{j-1}{2}-l\end{pmatrix}\exp(-4\pi\sqrt{-1}\,ls)$$

if $i\equiv 0$ *and* $j\equiv 1$ (mod. 2).

4) $$a_{ij}^{(n)}(s)=(-1)^{(n+j+1)/2}\sum_{l=max(0,\,(j+i-n-1)/2)}^{min(j/2,\,i/2)}$$

$$\times\binom{\frac{i}{2}}{l}\binom{\frac{n-i-1}{2}}{\frac{j}{2}-l}\exp(-4\pi\sqrt{-1}ls)$$

if $i\equiv 0$ *and* $j\equiv 0 \pmod{2}$.

(3.57) *If* $n\equiv 0 \pmod{2}$, *then*:

1) $a_{ij}^{(n)}(s)=(-1)^{(n+j)/2}\sum_{l=max(0,(j-n+i)/2)}^{min(j/2,i/2)}$

$$\times\binom{\frac{i}{2}}{l}\binom{\frac{n-i}{2}}{\frac{j}{2}-l}\exp(-4\pi\sqrt{-1}ls)$$

if $i\equiv 0$ *and* $j\equiv 0 \pmod{2}$

2) $a_{ij}^{(n)}(s)=0$

if $i\equiv 0$ *and* $j\equiv 1 \pmod{2}$

3) $a_{ij}^{(n)}(s)=(-1)^{(n+j)/2}\sum_{l=max(0,(j+i-3)/2)}^{min(j/2,(i-1)/2)}$

$$\times\binom{\frac{i-1}{2}}{l}\left(\binom{\frac{n-i-1}{2}}{\frac{j}{2}-l-1}\exp(-2\pi\sqrt{-1}s)-\binom{\frac{n-i-1}{2}}{\frac{j}{2}-l}\right)$$

$$\times\exp(-4\pi\sqrt{-1}ls)$$

if $i\equiv 1$ *and* $j\equiv 0 \pmod{2}$

4) $a_{ij}^{(n)}(s)=(-1)^{(n+j-1)/2}\sum_{l=max(0,(j-n+i)/2)}^{min((j-1)/2,(i-1)/2)}$

$$\times\binom{\frac{i-1}{2}}{l}\binom{\frac{n-i-1}{2}}{\frac{j-1}{2}-l}(\exp(\pi\sqrt{-1}s)$$

$$+\exp(-\pi\sqrt{-1}s))\exp(-4\pi\sqrt{-1}\,ls)$$

if $i\equiv 1$ *and* $j\equiv 1 \pmod{2}$

2. *In the case of* $V=\boldsymbol{Her}(n,\boldsymbol{C})$.

(3.58) $c_{ij}^{(n)}(s)=(2\pi)^{-n/2}\prod_{p=1}^{n}\Gamma(s+p)\exp\left(\frac{\pi}{2}\sqrt{-1}(-n^2+2nj)\right)$

$$\times \exp\left(\frac{\pi}{2}\sqrt{-1}\,(2j+2i-n)\,s\right)\sum_{l=max(0,\,j+i-n)}^{min(j,i)}$$
$$\times (-1)^{j}\binom{j}{l}\binom{n-j}{i-l}\exp(-2\pi l\sqrt{-1}s).$$

3. *In the case of* $V=\boldsymbol{Her}(n, \boldsymbol{H})$.

$$c_{ij}^{(n)}(s)=(2\pi)^{-n/2}\prod_{p=1}^{n}\Gamma(s+(2p-1))\exp\left(\frac{\pi}{2}\sqrt{-1}\,(n-2j)\right) \tag{3.59}$$
$$\times \exp\left(\frac{\pi}{2}\sqrt{-1}\,(2j+2i-n)\,s\right)\sum_{l=max(0,\,i+j-n)}^{min(i,j)}$$
$$\times (-1)^{n+j}\binom{j}{l}\binom{n-j}{i-l}\exp(-2\pi\sqrt{-1}ls)$$

Theorem 3.10. *Let* $s\notin \mathrm{Crit}(P(x)^{s})$ *(resp.* $-s-(n'/n)\notin \mathrm{Crit}(P(x)^{s})$*).*

1) *Let* $u(x)$ *(resp.* $v(y)$*) be a hyperfunction solution to* $\mathfrak{M}_s$ *on* $V_{\boldsymbol{R}}-\boldsymbol{S}_{\boldsymbol{R}}$ *(resp. to* $\mathfrak{M}^{*}_{s+(n'/n)}$ *on* $V^{*}_{\boldsymbol{R}}-\boldsymbol{S}^{*}_{\boldsymbol{R}}$*). Then,* $u(x)$ *(resp.* $v(y)$*) is uniquely extended to* $V_{\boldsymbol{R}}$ *(resp.* $V^{*}_{\boldsymbol{R}}$*) as a hyperfunction solution.*

2) *The hyperfunction* $|P|_i^{s}(x)$ *(resp.* $|P|_i^{-s-(n'/n)}(y)$*)* $(i=1,\dots,n)$ *forms a basis of the hyperfunction solution to* $\mathfrak{M}_s$ *on* $V_{\boldsymbol{R}}$ *(resp. to* $\mathfrak{M}^{*}_{s+(n'/n)}$ *on* $V^{*}_{\boldsymbol{R}}$*).*

Proof. 1) Let $u(x)$ be a hyperfunction solution on $V_{\boldsymbol{R}}-\boldsymbol{S}_{\boldsymbol{R}}$ to $\mathfrak{M}_s$ with $s\notin \mathrm{Crit}(P(x)^{s})$. From Proposition 3.5, 3), we have

$$u(x)=\sum_{i=1}^{n}a_i\cdot |P|_i^{s}(x)\,|_{V_{\boldsymbol{R}}-S_{\boldsymbol{R}}}.$$

Since $s\notin \mathrm{Crit}(P(x)^{s})$, $\sum_i a_i|P|_i^{s}(x)$ is well defined as a tempered distribution on $V_{\boldsymbol{R}}$ by Proposition 3.4, 1), and hence $u(x)$ is extendable to $V_{\boldsymbol{R}}$ as a solution of $\mathfrak{M}_s$.

Next, we shall show the uniqueness of the extension. Since $u(x)$ is a solution of the linear differential equation $\mathfrak{M}_s$, we have to show that if $u(x)\,|_{V_{\boldsymbol{R}}-S_{\boldsymbol{R}}}=0$, then $u(x)=0$ on $V_{\boldsymbol{R}}$.

If $u(x)\,|_{V_{\boldsymbol{R}}-S_{\boldsymbol{R}}}=0$, then the coefficients on $V_j^{(n)}\times\{0\}$, $c_{V_j^{(n)}\times\{0\}}(u(x))$ are zero for any j. Note that the matrices in (2.49) is well defined for any $s\notin \mathrm{Crit}(P(x)^{s})$, and for any $0\leqq i\leqq n-1$. Then we have

$$c_{A_1^{jk}}(u(x))=0,$$

for any $0\leqq j\leqq n-1$ and $0\leqq k\leqq 1$ by the matrix (2.49) with $i=0$.

Moreover, since the coefficients of $u(x)$ on Λ_l° are determined by those on Λ_{l-1}° by the matrices (4.29) with $i=l$, we have that the coefficients of $u(x)$ on Λ_l° are all zero if the coefficients of $u(x)$ on Λ_{l-1}° are all zero. Thus, by induction on l, we have that all the coefficients $c_{\Lambda_i^{jk}}(u(x))$ are zero. Therefore, by Theorem 2.4, we have $u(x)=0$.

By Proposition 3.5, 1) this theorem is true for the solution $v(y)$ to $\mathfrak{M}^*_{s+(n'/n)}$ on V_R^*.

2) Any solution $u(x)$ to $\mathfrak{M}_s$ on V_R is written as

$$u(x)\,|_{V_R-S_R}=\sum_i a_i\,|P|_i^s(x)\,|_{V_R-S_R},$$

by Proposition 3.5, 3) and hence, by 1), we have

$$u(x)=\sum_i a_i\,|P|_i^s(x),$$

on V_R. Thus we have the result. Similarly, we can prove the theorem for a solution $v(y)$ to $\mathfrak{M}^*_{s+(n'/n)}$.

q. e. d.

Theorem 3.11. *The formulas of Fourier transforms in Theorem 3.8 are valid not only on $V_R^*-S_R^*$ but also on V_R^* and not only for $s\notin \mathrm{Crit}(P(x)^s)$ but also for all $s\in C$. That is to say, we have*

$$(3.60)\qquad \int |P|_i^s(x)\exp(-2\pi\sqrt{-1}\langle x,y\rangle)\,dx$$
$$=(2\pi)^{-ns-(n'/2)}2^{-n(n-1)v/4}\sum_{j=0}^{n}c_{ij}^{(n)}(s)\;|P|_j^{-s-(n'/n)}(y),$$

for any $s\in C$ by considering $|P|_i^s(x)$ to be a tempered distribution with a meromorphic parameter $s\in C$. Here, we *use the same notations as in Theorem 3.8.*

Proof. We suppose that $s\notin\mathrm{Crit}(P(x)^s)$ and $-s-(n'/n)\notin\mathrm{Crit}(P(x)^s)$. Then both $|P|_i^s(x)$ and $|P|_j^{-s-(n'/n)}(y)$ are well defined, and the formula (3.60) is valid on $V_R^*-S_R^*$. However, the left hand side of (3.60) is a solution to the holonomic system $\mathfrak{M}^*_{s+(n'/n)}$ on V_R^* by Proposition 3.5, 2), and the right hand side of (3.60) is a solution to $\mathfrak{M}^*_{s+(n'/n)}$ on $V_R^*-S_R^*$. Therefore, since $-s-(n'/n)\notin\mathrm{Crit}(\mathrm{P}(x)^s)$, the right hand side is extended uniquely as a solution to $\mathfrak{M}^*_{s+(n'/n)}$ on V_R^* by Theorem 3.10. Thus the formula (3.60) is valid for any s in

$$(3.61)\qquad A=\{s\in C\,;\,s\notin\mathrm{Crit}(P(x)^s)\ \text{and}\ -s-(n'/n)\notin\mathrm{Crit}(P(x)^s)\}.$$

Since A is the complement of a discrete set in $\boldsymbol{C}$, (3.60) is valid for any $s\in\boldsymbol{C}$ by the analytic continuation of Theorem 3.2, 1).

q. e. d.

§4. Invariant Measures on Singular Orbits and Their Fourier Transforms

In this section, we use the term *associated numbers* instead of *coefficients* in order to avoid confusions.

Theorem 4.1. *Let* $S_i^j=\{x\in V; \operatorname{sign}(x)=(j, n-i-j)\}$. *There exist hyperfunctions* $T_{is}^j(x)$ $(i=0, 1,.., n,\ j=0, 1,.., n-i)$ *with meromorphic parameter* $s\in\boldsymbol{C}$ *satisfying the following properties.*

(4.1) 1) $T_{is}^j(\rho(g)x)=\chi(g)^s T_{is}^j(x)$.

2) *For* $i\geqq 1$, *the support of* $T_{is_i}^j(x)$ *is contained in* $\bar{S}_i^j$ *and* $T_{is_i}^j(x)$ dx *gives a non-zero measure on* S_i^j.

3) *The associated number of* $T_{is_i}^j(x)$ *on* A_i^{jk} *is* 1.

Proof. We define $T_{is}^j(x)$ by induction on i. First, we set

$$T_{0s}^j(x)=|P|_j^s(x) \qquad (j=0, 1, \ldots, n). \tag{4.2}$$

Then $T_{0s}^{n-p}(x)$ is a hyperfunction whose support is $V_{n-p}^{(n)}$. Suppose that we have defined $T_{is}^j(x)$ for $i=0, 1,.., q$ and $j=0, 1,.., n-i$. We define

$$\begin{aligned} T_{q+1s}^j(x) = a_q(s)\,(b_q(s)T_{qs}^{j+1}(x) - b_q(s)^3 T_{qs}^{j+2}(x)+\ldots \\ \ldots+(-1)^{n-q-1-j} b_q(s)^{2(n-q-j)-1} T_{qs}^{n-q}(x)), \end{aligned} \tag{4.3}$$

for $j=0, 1,.., n-q-1$. Here, we set

$$\begin{aligned} a_q(s) &= \sqrt{2\pi}\ \Gamma(s-s_{q+1})^{-1}, \\ b_q(s) &= \exp\Big(\frac{\pi}{2}\sqrt{-1}(s-s_{q+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}q\cdot v\Big) \end{aligned} \tag{4.4}$$

where

$$v=\begin{cases} 1 & (V=\boldsymbol{Sym}(n, \boldsymbol{R})) \\ 2 & (V=\boldsymbol{Her}(n, \boldsymbol{C})) \\ 4 & (V=\boldsymbol{Her}(n, \boldsymbol{H})). \end{cases}$$

From the definition, $T^j_{is}(x)$ is a linear combination of $|P|^s_k(x)$ with entire function coefficients on $s\in\boldsymbol{C}$. Therefore $T^j_{is}(x)$ is a hyperfunction solution of $\mathfrak{M}_s$, and hence satisfies the condition 1).

Lemma 4.1.1. *Let* $c_{\Lambda^{jk}_i}(T^q_{ps}(x))$ *be the associated number of* $T^q_{ps}(x)$ *on* Λ^{jk}_i. *Then, for* $p\geqq 1$, *we have*

(4.5) 1) $c_{\Lambda^{jk}_i}(T^q_{ps}(x))\,|_{s=s_p}=0$ *if* $i<p$.

2) $c_{\Lambda^{jk}_p}(T^q_{ps}(x))\,|_{s=s_p}=\begin{cases}1, & j=q,\\ 0, & j\neq q.\end{cases}$

3) $c_{\Lambda^{jk}_i}(T^q_{ps}(x))\,|_{s=s_p}=0$ *if* $i>p$ *and* $j<q-(i-p)$ *or if* $i>p$ *and* $j>q$.

Proof. First, in order to prove (4.5), we shall show that the associated numbers of $T^q_{ps}(x)$ on Λ^{jk}_i are entire functions in s and that they vanish at $s=s_p$ if $i<p$. We shall show these by induction on p. Consider the hyperfunction $a_0(s)\,|P|^s_q(x)\;(q=0,1,\ldots,n)$. Their associated numbers on $\Lambda^{j0}_0\,(j=0,1,\ldots,n)$ are

$$\begin{cases}a_0(s) & (q=j),\\ 0 & (q\neq j).\end{cases} \tag{4.6}$$

They all vanish at $s=s_1$ because $a_0(s)$ is an entire function with the zero of order 1 at $s=s_1$. Therefore the associated numbers of $T^k_{1s}(x)$ also vanish on all Λ^{j0}_0 because $b_0(s)$ is an entire function. The associated numbers on Λ^{jk}_1 $(j=0,1,\ldots,n-1,k=0,1)$ are all entire functions on $s\in\boldsymbol{C}$, since they can be written as $a_0(s)\cdot\Gamma(s-s_1)\times$ (*an entire function*) from the relations of the associated numbers (2.49). Thus we obtain that our assertion is valid for $p=1$.

Suppose that

(4.7) 1) The associated numbers of $T^q_{ps}(x)$ on Λ^{jk}_i are entire function on $s\in\boldsymbol{C}$ when $i\leqq p$,

2) They vanish at $s=s_p$ if $i<p$.

Consider the hyperfunction $a_p(s)T^q_{ps}(x)$. Since $a_p(s)$ is an entire function with a zero of order 1 at $s=s_{p+1}$, the associated numbers on all $\Lambda^{jk}_i\,(i\leqq p)$ are entire functions in $s\in\boldsymbol{C}$ and vanish at $s=s_{p+1}$. Moreover, the associated numbers on Λ^{jk}_{i+1} are entire functions in s

because they can be written as $a_p(s)\cdot\Gamma(s-s_{p+1})\times$ (*an entire function*). Since $T^q_{p+1s}(x)$ are written as a linear combination of $a_p(s)T^q_{ps}(x)$ $(q=0,..,n-p)$ with coefficients of entire functions, we obtain that

(4.8) 1) The associated numbers of $T^q_{p+1s}(x)$ on Λ^{jk}_i $(i\leqq p+1)$ are entire functions on $s\in\boldsymbol{C}$.

2) They vanish at $s=s_{q+1}$ if $i\leqq p$.

Thus the assertion (4.5) 1) has been proved by induction.

Next we shall show that

(4.9) The associated numbers $c_{\Lambda^{jk}_p}(T^q_{ps}(x))|_{s=s_p}$ $(0\leqq k\leqq p)$ coincide with one another.

Note that $\Gamma(s-s_p)c_{\Lambda^{j+1k}_{p-1}}(T^q_{ps}(x))$ and $\Gamma(s-s_p)c_{\Lambda^{jk}_{p-1}}(T^q_{ps}(x))$ are holomorphic at $s=s_p$. We denote by a_1 and a_2 the values of them at $s=s_p$, respectively. From the relation matrix of associated numbers (2, 49), we have

$$(4.10)\qquad c_{\Lambda^{jk+1}_p}(T^q_{ps}(x))|_{s=s_p}=c_{\Lambda^{jk}_p}(T^q_{ps}(x))|_{s=s_p}$$
$$=(\sqrt{2\pi})^{-1}(a_1+a_2).$$

Thus we have that $c_{\Lambda^{jk}_p}(T^q_{ps}(x))|_{s=s_p}$ $(0\leqq k\leqq p)$ coincide with one another.

We shall show that

$$(4.11)\qquad c_{\Lambda^{jp}_p}(T^q_{ps}(x))=\begin{cases}1, & j=q,\\ 0, & j\neq q.\end{cases}$$

by induction on p. When $p=0$, it is evident from the definition. We assume that (4.11) is valid for an integer p. Then the associatep numbers $c_{\Lambda^{jp}_p}(T^k_{ps})$ are all identically zero if $k\geqq j+1$, and $c_{\Lambda^{j+1p}_p}(T^k_{ps})$ are all identically zero if $k\geqq j+2$. Therefore we have

$$\begin{aligned}(4.12)\qquad & c_{\Lambda^{jp+1}_{p+1}}(T^j_{p+1s})\\ &=\frac{\Gamma(s-s_{p+1})}{\sqrt{2\pi}}\exp\Big(-\frac{\pi}{2}\sqrt{-1}(s-s_{p+1})\Big)\exp\Big(\frac{\pi}{4}\sqrt{-1}pv\Big)\\ &\quad\times a_p(s)b_p(s)c_{\Lambda^{j+1p}_p}(T^{j+1}_{ps})\\ &=c_{\Lambda^{j+1p}_p}(T^{j+1}_{ps})=1,\end{aligned}$$

and

$$(4.13)\qquad c_{\Lambda^{kp+1}_{p+1}}(T^{j+1}_{p+1s})=0\quad\text{for } k<j,$$

from the induction hypothesis. For $k>j$, the associated number of T^j_{p+1s} on Λ^{kp+1}_{p+1} are determined by the associated numbers on Λ^{k+1p}_p and Λ^{kp}_p. We have

$$
\begin{aligned}
(4.14)\qquad & c_{\Lambda^{kp+1}_{p+1}}(T^j_{p+1s}) \\
&=\frac{\Gamma(s-s_{p+1})}{\sqrt{2\pi}}\Big(\exp\Big(-\frac{\pi}{2}\sqrt{-1}(s-s_{p+1})\Big)\exp\Big(\frac{\pi}{4}\sqrt{-1}pv\Big)(-1)^{k-j} \\
&\times a_p(s)b_p(s)^{2k-2j+1}c_{\Lambda^{k+1p}_p}(T^{k+1}_{ps}) \\
&\quad+\exp\Big(\frac{\pi}{2}\sqrt{-1}(s-s_{p+1})\Big)\exp\Big(-\frac{\pi}{4}\sqrt{-1}pv\Big)(-1)^{k-j-1} \\
&\quad\times a_p(s)b_p(s)^{2k-2j-1}c_{\Lambda^{kp}_p}(T^k_{ps})\Big) \\
&=(-1)^{k-j}b_p(s)^{2k-2j}(c_{\Lambda^{k+1p}_p}(T^{k+1}_{ps})-c_{\Lambda^{kp}_p}(T^k_{ps})) \\
&=0,
\end{aligned}
$$

for $k>j$. Thus we have

$$
(4.15)\qquad c_{\Lambda^{kp+1}_{p+1}}(T^j_{p+1s})=\begin{cases}1 & k=j,\\ 0 & k\neq j.\end{cases}
$$

Thus we complete the proof by induction. By (4.9) and (4.11), we have the assertion (4.5) 2).

Now we shall show (4.5) 3) by induction on i. From (4.5) 2) and the relation matrix of associated numbers, we have

$$
(4.16)\qquad c_{\Lambda^{jk}_{p+1}}(T^q_{ps})_{s=s_p}=0,
$$

if $j<q-1$ or $j>q$. Suppose that for a fixed i, $c_{\Lambda^{jk}_i}(T^q_{ps}(x))\,|_{s=s_p}=0$ if $j<q-i+p$ or $j>q$. Then, the associated number $c_{\Lambda^{jk}_{i+1}}(T^q_{ps}(x))\,|_{s=s_p}$ ($j<q-(i+1-p)$ or $j>q$) are written as linear combinations of the associated numbers on Λ^{jk}_i ($j<q-(i-p)$ or $j>q$). Then we have

$$
(4.17)\qquad c_{\Lambda^{jk}_{i+1}}(T^q_{ps}(x))\,|_{s=s_p}=0,
$$

if $j<q-(i+1-p)$ or $j>q$. Thus by induction we have (4.5) 3).

Lemma 4. 1. 1 q. e. d.

From Lemma 4. 1. 1, the singular spectrum of $T^q_{ps_i}(x)$ is contained in

$$
\overline{\bigcup_{i=0}^{n-p}\Big(\bigcup_{i\geq j\geq 0}\Big(\bigcup_{k=0}^{p+i}\Lambda^{q-jk}_{p+i}\Big)\Big)}.
$$

Therefore the support of $T^q_{ps_i}(x)$ is contained in

$$(4.18) \qquad \overline{\pi\Big(\bigcup_{i=0}^{n-p}\Big(\bigcup_{i\geqq j\geqq 0}\Big(\bigcup_{k=0}^{p+i}\Lambda^{q-j,k}_{p+i}\Big)\Big)\Big)} = S^q_p \cup (S^q_{p+1}\cup S^{q-1}_{p+1}) \cup \dots \cup (S^0_n) = \bar{S}^q_p$$

by Proposition 2.3. Thus we have $\mathrm{Supp}(T^q_{ps_p}(x)) \subset \bar{S}^q_p$ by (2.42).

Lemma 4.1.2. *Let p be a point in V and let K be a non-singular variety in V of codimension m defined in a neighborhood of p. Suppose that a hyperfunction $u(x)$ defined in a neighborhood of p satisfies* $\mathrm{Supp}(u(x)) = K$.

Let Λ be a Lagrangian variety $T^*_K V$, and we denote by Λ_C its complexification. We suppose that the hyperfunction $u(x)$ satisfies a holonomic system whose characteristic variety contains Λ_C, and the holonomic system is simple on Λ_C and the order of $u(x)$ is $m/2$. Then there exists a local coordinate system $(x_1, \dots, x_{n'})$ near p satisfying

(4.19) i) $K = \{x_1 = \dots = x_m = 0\}$.

ii) $u(x) = P(x_{m+1}, \dots, x_{n'})\delta(x_1)\dots\delta(x_m)$, *where $P(x_{m+1}, \dots, x_{n'})$ is a non zero real analytic function.*

Proof. Since the hyperfunction $u(x)$ satisfies simple holonomic system on Λ_C, we can write as a microfunction $u(x) = P(x, D_x)\delta(x_1)\dots\delta(x_m)$ on Λ by using a local coordinate system $(x_1, \dots, x_{n'})$ satisfying $K = \{x_1 = \dots = x_m = 0\}$, and a microdifferential operator of fractional order $P(x, D_x)$. We may assume that $P(x, D_x)$ is written as $P(x_{m+1}, \dots, x_{n'}, D_{x_1}, \dots, D_{x_m})$ and it is uniquely determined by this expression (see the definition of real principal symbol (2.43)). Since the support of $u(x)$ is contained in K, $P(x, D_x)$ is a proper differential operator, i.e., without any terms of order less than -1 and with terms of integer order. The order of $u(y)$ on Λ is $\mathrm{ord}(\sigma(P)) + \frac{m}{2}$, and hence P is of order 0, i.e., an analytic function. Therefore we obtain the expression

$$u(x) = P(x_{m+1}, \dots, x_n)\delta(x_1)\dots\delta(x_m),$$

with some analytic function in a neighborhood of p.

Lemma 4.1.2 q.e.d.

We apply this Lemma to $T^q_{ps_p}(x)$. Let $z \in \boldsymbol{S}^q_p$. Then $\boldsymbol{S}^q_p$ is a non-singular subvariety in a neighborhood of z, and the codimension is $p(p+1)/2$ (resp. p^2, $p(2p-1)$) when $\boldsymbol{V}=\boldsymbol{Sym}(n,\boldsymbol{R})$ (resp. $\boldsymbol{V}=\boldsymbol{Her}(n,\boldsymbol{C})$, $\boldsymbol{V}=\boldsymbol{Her}(n,\boldsymbol{H})$). The order of $T^q_{ps_p}(x)$ on $T^*_{S^q_p}\boldsymbol{V}$ is $-ps_p-\frac{p(p+1)}{4}$ $\left(\text{resp. } -ps_p-\frac{p^2}{2},\ -ps_p-p(2p-1)\right)$ and hence coincides with a half of the codimension of $\boldsymbol{S}^q_p$. Therefore Lemma 4.1.2 can be applied, and hence we have the result (4.1) 2).

q. e. d.

We define the hyperfunction

$$T^j_i(x) := T^j_{is_i}(x). \tag{4.20}$$

This hyperfunction is relatively invariant and its support coincides with $\bar{\boldsymbol{S}}^j_i$ and it defines a $\boldsymbol{G}^1$-invariant measure on $\boldsymbol{S}^j_i$. That is to say, we have the following theorem.

Theorem 4.2. *There exists a $\boldsymbol{G}^1$-invariant measure $d\nu^j_i$ on $\boldsymbol{S}^j_i$ and satisfies*

$$\int_V f(x)\,T^j_i(x)\,dx = \int_{S^j_i} f|_{S^j_i}(x)\,d\nu^j_i(x), \tag{4.21}$$

for any $f(x) \in C^\infty_0(V)$ such that $f|_{S^j_i} \in C^\infty_0(\boldsymbol{S}^j_i)$.

Conversely, for any $\boldsymbol{G}^1$-invariant measure $d\nu^j_i$ on $\boldsymbol{S}^j_i$,

$$\int_{S^j_i} f|_{S^j_i}(x)\,d\nu^j_i(x) \tag{4.22}$$

is absolutely convergent and coincides with

$$(const.)\int_V f(x)\,T^j_i(x)\,dx, \tag{4.23}$$

for any $f(x) \in \mathscr{S}(V)$.

Proof. In this proof, we denote $\boldsymbol{G}^{(n)1}$ and $\boldsymbol{V}^{(n)}$ instead of $\boldsymbol{G}^1$ and $\boldsymbol{V}$, respectively for the later convenience.

The formula (4.21) is the direct consequence of Theorem 4.1. Namely, $T^j_i(x)$ is a hyperfunction on $\boldsymbol{V}$, whose support is contained in $\bar{\boldsymbol{S}}^j_i$. Therefore,

$$f \longmapsto \int f(x)\,T^j_i(x)\,dx, \quad (f \in C^\infty_0(\boldsymbol{V}),\ f|_{S^j_i} \in C^\infty_0(\boldsymbol{S}^j_i)),$$

gives a measure on S_i^j. It follows from that $T_i^j(\rho(g)x)d(\rho(g)x) = T_i^j(x)dx$ ($g \in G^{(n)1}$) that this measure is a $G^{(n)1}$-invariant measure.

Next we shall show the converse.

We consider the relatively invariant measure $dx^{(n)}$ on $V_i^{(n)}$. By Sato-Shintani [5] p. 138, there exists a $G^{(n)1}$-invariant $n'-1$ form $\omega_n(x)$ on $V_i^{(n)}$ such that

(4.24) $$|d(P(x)) \wedge \omega_n(x)| = dx^{(n)}.$$

Here, n' is the dimension of $V^{(n)}$ and $P(x)$ is the restriction of the relative invariant defined in (1.2). Then, for any $f(x) \in \mathscr{S}(V^{(n)})$, we have

(4.25) $$\begin{aligned}&\int_{V_i^{(n)}} f(x)\,|P(x)|^s dx^{(n)}\\ &= \int_0^\infty t^s dt \int_{\{x\in V_i^{(n)};\, |\det(x)|=t\}} f(x)\,\omega_n(x),\end{aligned}$$

and they are absolutely convergent when $\mathrm{Re}(s) > -1$.

Let $f(x) \in \mathscr{S}(V^{(n)})$. We put

(4.26) $$f^{co}(x) = \int_{U^{(n)}\ni g} f(\rho(g)x)\,dg \times \frac{1}{\mathrm{Volume}(U^{(n)})},$$

where $U^{(n)}$ is the compact group $O(n, \boldsymbol{R}) \cap G^{(n)1}$ (resp. $U(n, \boldsymbol{C})$, $U(n, \boldsymbol{H})$) in the case of $V^{(n)} = \boldsymbol{Sym}(n, \boldsymbol{R})$ (resp. $\boldsymbol{Her}(n, \boldsymbol{C})$, $\boldsymbol{Her}(n, \boldsymbol{H})$), and dg stands for a $U^{(n)}$-invariant measure. Then $f^{co}(x)$ also belongs to $\mathscr{S}(V^{(n)})$, and $\int f^{co}(x)\,dx^{(n)} = \int f(x)\,dx^{(n)}$.

We put

(4.27) $$M_t^{(i,j)} = \left\{ \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix};\ \begin{array}{l} L \in \boldsymbol{Sym}(n-i, \boldsymbol{R}) \text{ (resp. } \boldsymbol{Her}(n, \boldsymbol{C}),\\ \boldsymbol{Her}(n, \boldsymbol{H})),\ |p(L)| = t, \mathrm{sign}(L)\\ = (j, n-i-j). \end{array} \right\}.$$

Then there exists a $G^{(n)1}$-invariant measure ω_{n-i} on $M_t^{(i,j)}$ given by (4.21). We put

(4.28) $$\tilde{M}_t^{(i,j)} = \{g \cdot M_t^{(i,j)} \cdot {}^t g;\ g \in U^{(n)}\}.$$

We can regard $\tilde{M}_t^{(i,j)}$ as

(4.29) $$U^{(n)}/(U^{(n-i)} \times U^{(i)}) \cap U^{(n)} \times M_t^{(i,j)}.$$

Here $U^{(n-i)} \times U^{(i)}$ is a subgroup of $U^{(n)}$ by

(4.30) $$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\ A \in U^{(n-i)},\ B \in U^{(i)} \right\}.$$

There exists a $U^{(n)}$-invariant measure ω on $U^{(n)}/(U^{(n-i)}\times U^{(i)})\cap U^{(n)}$ and we introduce a measure $\tilde{\omega}_{n-i}$ on $\tilde{M}_t^{(i,j)}$ by

(4.31) $$\tilde{\omega}_{n-i}=|\omega\wedge\omega_{n-i}|.$$

Note that

(4.32) $$S_i^j=\bigcup_{t>0}\tilde{M}_t^{(i,j)}.$$

Lemma 4.2.1. *The measure*

(4.33) $$|\tilde{\omega}_{n-i}\wedge t^{-1/2}dt| \quad (resp.\ |\tilde{\omega}_{n-i}\wedge dt|,\ |\tilde{\omega}_{n-i}\wedge t\,dt|)$$

is a $G^{(n)1}$-invariant measure on S_i^j.

Proof. We put $(A,B)=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\in G^{(n)1}$. Then, by the action of (A,B), the volume form $|\tilde{\omega}_{n-i}\wedge dt|$ is multiplied by the factor $|\det A|^{-1}$ (resp. 1, $|\det A|^2$). Therefore, $|\tilde{\omega}_{n-i}\wedge \mathrm{d}t|$ is invariant by the action of (A,B). From the definition, it is invariant under the actions of $U^{(n)}$. Since any point in S_i^j is reduced to the point $\begin{pmatrix}I_{n-i}^j & \\ & 0\end{pmatrix}$ by the actions of an (A,B) and an element in $U^{(n)}$, and since there exists a $G^{(n)1}$-invariant measure on S_i^j, the measure $|\tilde{\omega}_{n-i}\wedge t^a dt|$ with $a=-1/2$ (*resp.* $a=0, a=1$) is a $G^{(n)1}$-invariant measure on S_i^j.

q.e.d. of Lemma 4.2.1

Now let $f\in\mathcal{S}(V^{(n)})$ and define $f^{co}(x)$ by (4.26). Then $f^{co}(x)$ is invariant by the actions of $U^{(n)}$ and

(4.34) $$\begin{aligned}\int_{\tilde{M}_t^{(i,j)}} f(x)\tilde{\omega}_{n-i}(x) &=\int_{\tilde{M}_t^{(i,j)}} f^{co}(x)\tilde{\omega}_{n-i}(x)\\ &=\mathrm{Vol}(U^{(n)}/(U^{(n-i)}\times U^{(i)})\cap U^{(n)})\\ &\quad\times\int_{M_t^{(i,j)}} f^{co}(x)\,|_{V^{(n-i)}}\omega_{n-i}(x).\end{aligned}$$

Here, by the inclusion map

(4,35) $$\begin{array}{ccc} x & \longmapsto & \begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix},\\ \cap & & \cap\\ V^{(n-i)} & & V^{(n)}\end{array}$$

$V^{(n-i)}$ can be viewed as a subspace of $V^{(n)}$. The restriction of $f^{co}(x)$ on $V^{(n-i)}$ is an element of $\mathcal{S}(V^{(n-i)})$ and it is $U^{(n-i)}$-invariant.

Lemma 4.2.2. *Let $f \in \mathscr{S}(V^{(n)})$. Then,*

$$(4.36)\qquad \tilde{f}_i(t) = \int_{\{x \in V_i^{(n)}; |P(x)| = t\}} f(x)\,\omega_n(x)$$

is a C^∞-function on $t > 0$ and rapidly decreasing at infinity. Moreover,

$$(4.37)\qquad \tilde{f}_i(0) = \lim_{t \to 0} \tilde{f}_i(t)$$

exists and is finite.

Proof. In fact, $\tilde{f}_i(t)$ is absolutely convergent for $t > 0$, and

$$(4.38)\qquad \frac{\partial}{\partial t}\tilde{f}_i(t) = \left(\widetilde{\frac{\partial}{\partial P(x)} f}\right)_i(t),$$

where

$$(4.39)\qquad \frac{\partial}{\partial P(x)} = |\mathrm{grad}\, P(x)|^{-2} \left\langle \mathrm{grad}\, P(x), \frac{\partial}{\partial x}\right\rangle.$$

Since $\frac{\partial}{\partial P(x)} f(x)$ is a C^∞-function except at the origin and since $\frac{\partial}{\partial P(x)} f(x)$ is rapidly decreasing at infinity, $\left(\widetilde{\frac{\partial}{\partial P(x)} f}\right)_i(t)$ is absolutely convergent for $t > 0$, and hence it is continuous in t and rapidly decreasing at infinity. In the same way, we can show that $\left(\frac{\partial}{\partial t}\right)^n \tilde{f}_i(t)$ is continuous on $t > 0$ and rapidly decreasing at infinity.

Now we shall show (4.37) by induction on n. Suppose that (4.37) is proved for $n \leqq k$. We put $n = k+1$. From the definition, $\lim_{t \to 0} \tilde{f}_i(t)$ coincides with

$$(4.40)\qquad \int_{\bar{V}_i^{(k+1)} - V_i^{(k+1)}} f(x)\, d\nu_1(x),$$

where $d\nu_1$ is a $G^{(k+1)1}$-invariant measure on the orbit of codimension one $S_1 = \{x \in V^{(k+1)}; P(x) = 0, \mathrm{rank}(x) = k\}$ which is naturally induced by $\tilde{\omega}_k$, the $(k+1)' - 1$ form defined by (4.24). From Lemma 4.2.1 and (4.34), the integrability of $\int_{S_1^j} f(x)\, d\nu_1^j$ is reduced to the integrability of

$$(4.41)\qquad \int_0^\infty t^a dt \int_{M_t^{(1,j)}} f^{co}(x)\, |_{V^{(k)}} \omega_k(x),$$

with $a = -1/2$ (resp. $a = 0$, $a = 1$). From the induction hypothesis,

$$\int_{M_t^{(1,j)}} f^{co}(x) \mid_{V^{(k)}} \omega_k(x) = (\widetilde{f^{co}(x)} \mid_{V^{(k)}})_j(t)$$

is a bounded C^∞-function which is rapidly decreasing at infinity. Hence (4.40) is integrable.

q. e. d. of Lemma 4.2.2

From Lemma 4.2.1 and (4.34), the integrability of $\int_{S_i^j} f(x) d\nu_i^j$ is reduced to the integrability of

$$\int_0^\infty t^a dt \int_{M_t^{(i,j)}} f^{co}(x) \mid_{V^{(n-i)}} \omega_{n-i}(x), \tag{4.42}$$

where $a=-1/2$ (resp. $a=0$, $a=1$). From Lemma 4.2.2,

$$(f^{co}(x) \mid_{V^{(n-i)}})_j = \int_{M_t^{(i,j)}} f^{co}(x) \mid_{V^{(n-i)}} \omega_{n-i}(x)$$

is a bounded C^∞-function on $t>0$ and rapidly decreasing at infinity, hence (4.42) is absolutely integrable. Thus,

$$f \longmapsto \int_{S_i^j} f(x) \mid_{S_i^j} d\nu_i^j(x) \qquad (f \in \mathscr{S}(V^{(n)})) \tag{4.43}$$

defines a hyperfunction which is also a tempered distribution with support $\bar{S}_i^j$. We put it $T_i^{j\prime}(x)$. From (4.1) 1), 2), the hyperfunction $T_i^j(x)\ dx$ defines a $G^{(n)1}$-invariant measure on S_i^j, and hence

$$T(x) = T_i^j(x) - T_i^{j\prime}(x)$$

is a hyperfunction whose support is contained in $\bar{S}_i^j - S_i^j$. Moreover $T(x)$ is a solution of $\mathfrak{M}_{s_i}$ because we have $T_i^j(\rho(g)x) = \chi(g)^{s_i} T_i^j(x)$. If $T(x)$ is not zero, the singular spectrum of $T(x)$ is contained in $\bigcup_{k>i} \overline{T_{S_k}^* V_R}$.

Lemma 4.2.3. *Let* $u(x)$ *be a hyperfunction solution to* $\mathfrak{M}_{s_i}$. *Suppose that* $\mathrm{Supp}(u(x)) \subset \bar{S}_{i+1}$. *Then* $u(x)=0$.

Proof. Since $u(x)$ is a hyperfunction solution to $\mathfrak{M}_{s_i}$, it is determined by the associated numbers on Λ_i^{jk}. (Theorem 2.14). From the assumption, the support of $u(x)$ is contained in $\bar{S}_{i+1}$, and hence $u(x)$ is zero on the real Lagrangian subvariety Λ_j° for $0 \leqq j \leqq i$. Therefore, the associated numbers $c_{\Lambda_j^{kl}}(u(x))=0$ for $0 \leqq j \leqq i$ and for any k, l. On the other hand, the relation matrix (2.49) is well defined if $s>s_{i+1}$. Since $s_i>s_{i+1}$, (2.49) is well defined for $s=s_i$, and hence we have

$$(4.44)\qquad c_{A_{i+1}^{kl}}(u(x))=0,$$

for any k, l. Next, consider the matrix (2.49) substituting $i+1$ to i. Then, it is well defined for $s=s_i$ because $s_i>s_{i+2}$. Hence we have $c_{A_{i+2}^{kl}}(u(x))=0$ for any k, l by (4.44). Similarly, by induction, we have $c_{A_j^{kl}}(u(x))=0$ for any j and for any k, l. After all, we have $u(x)=0$ by Theorem 2.14.

q. e. d.

Note that $T(x)$ satisfies the condition of Lemma 4.2.3. In fact,

$$\bar{S}_i^j-S_i^j\subset\bar{S}_{i+1},$$

by Proposition 2.3, and hence the support of $T(x)$ is contained in $\bar{S}_i$. Then, by applying Lemma 4.2.3, we have $T(x)=0$. Therefore we have

$$T_i^j(x)=T_i^{j'}(x).$$

Thus we have the result.

q. e. d.

The hyperfunction $T_i^j(x)$ is a solution of a holonomic system $\mathfrak{M}_{s_i}$. When we give associated numbers on Λ_i of a solution, then we can calculate the associated numbers on Λ_{i+1}, and inductively we know the associated numbers on $\Lambda_{i+2},\ldots,\Lambda_n$. In order to calculate these associated numbers, we set $c_i^{jk}=t^j$, where c_i^{jk} is the associated number on Λ_i^{jk}. Then we have

$$(4.45)\qquad c_{i'}^{jk}:c_{i'}^{j-1k}=t:1\qquad(i'>i),$$

inductively. We define $F_{i'k}^{\pm}(s)$ as defined in (3.29). We have

$$(4.46)\qquad c_{i'+1}^{j-1k+1}=F_{i'k}^{+}(s_i)\,c_{i'}^{j-1k}\qquad(i'\geqq i),$$

$$c_{i'+1}^{j-1k}=F_{i'k}^{-}(s_i)\cdot c_{i'}^{j-1k}\qquad(i'\geqq i)$$

from the relation matrix of associated numbers (2.49).

Therefore we have

$$(4.47)\qquad c_n^{0k}=F_{n-1k-1}^{+}(s_i)\cdots F_{n-k0}^{+}(s_i)\,F_{n-k-10}^{-}(s_i)\cdots F_{i0}^{-}(s_i)\cdot c_i^{00}$$

$$=\sum_{j=0}^{n-i}a_{ik}^{j}t^{j}c_i^{00}=\sum_{j=0}^{n-i}a_{ik}^{j}c_i^{j0},\quad(k<n-i).$$

$$c_n^{0k}=F^+_{n-1k-1}(s_i)\cdots F^-_{ik-n+i}(s_i)\,c_i^{0k-n+i}$$
$$=\sum_{j=0}^{n-i}a^j_{ik}t^jc_i^{0k-n+i}=\sum_{j=0}^{n-i}a^j_{ik}c_i^{jk-n+i},\qquad (k\geqq n-i).$$

Here a^j_{ik} is the coefficient of t^j of the polynomial

$$F^+_{n-1k-1}(s_i)\cdots F^+_{n-k0}(s_i)\,F^-_{n-k-10}(s_i)\cdots F^-_{i0}(s_i)$$

if $k<n-i$, or

$$F^+_{n-1k-1}(s_i)\cdots F^+_{ik-n+i}(s_i)$$

if $k\geqq n-i$.

The support of $T^j_i(x)$ is contained in $\bar{S}^j_i$, and the associated numbers on Λ^{jk}_i $(k=0, 1,.., n)$ are all 1. The associated numbers on $\Lambda^{jk}_i(j'\neq j)$ are all zero, i. e., there are no singular spectrums. Therefore, the principal symbol on Λ^{0k}_n is given by

$$\sigma_{\Lambda^{0k}_n}(T^j_i(x))=a^j_{ik}\,|P_{\Lambda^{0k}_n}|^s\sqrt{|\omega_{\Lambda^{0k}_n}|}, \tag{4.48}$$

with $k=0, 1,\dots, n$.

Thus, as in the same way of Theorem 3. 8, we have the formula of the Fourier transforms,

$$\int T^j_i(x)\exp(-2\pi\sqrt{-1}\langle x,y\rangle)\,dx|_{V^*_R-S^*_R}$$
$$=(2\pi)^{-n'/2-ns_i}2^{-n(n-1)v/4}\sum_{k=0}^{n}a^j_{ik}|\det y|_k^{-s_i-\frac{n+1}{2}}, \tag{4.49}$$

where n', v are defined in (3. 46) and s_i is defined in (2. 50).

By computing a^j_{ik} explicitely and by putting $b^{(n)j}_{ik}=(2\pi)^{-\frac{n'}{2}-ns_i}2^{-\frac{n(n-1)v}{4}}\cdot a^j_{ik}$, we have the following formula of the Fourier transforms of $T^j_i(x)$.

Theorem 4. 3. 1. *In the case of* $V=\boldsymbol{Sym}(n, \boldsymbol{R})$.

$$\int T^j_i(x)\exp(-2\pi\sqrt{-1}\langle x, y\rangle)\,dx|_{V^*_R-S^*_R}$$
$$=\sum_{k=0}^{n}b^{(n)j}_{ik}|\det y|_k^{-(n-i)/2} \tag{4.50}$$

where

(4. 51) 1) $n-i\equiv 0, j\equiv 0 \pmod{2}$

$$b^{(n)j}_{ik}=(2\pi)^{(n+1)(2i-n)/4}2^{-n(n-1)/4}\prod_{p=1}^{n-i}\Gamma(p/2)$$
$$\times\exp\Big(\frac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2ij)\Big)\binom{(n-i)/2}{j/2}$$

2) $n-i\equiv 0,\ j\equiv 1 \pmod{2}$

$$b_{ik}^{(n)j}=0$$

3) $n-i\equiv 1,\ j\equiv 1 \pmod{2}$

$$b_{ik}^{(n)j}=(2\pi)^{(n+1)(2i-n)/4}2^{-n(n-1)/4}\prod_{p=1}^{n-i}\Gamma(p/2)$$
$$\times\exp\left(\frac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2i(j+1))\right)\binom{(n-i-2)/2}{j(-1)/2}$$

4) $n-i\equiv 1,\ j\equiv 0 \pmod{2}$

$$b_{ik}^{(n)j}=(2\pi)^{(n+1)(2i-n)/4}2^{-n(n-1)/4}\prod_{p=1}^{n-i}\Gamma(p/2)$$
$$\times\exp\left(\frac{\pi}{4}\sqrt{-1}((n-i)(n-2k+2i)+2i(j+1)+4k-2)\right)\binom{(n-i-2)/2}{j/2}$$

2. *In the case of* $V=\boldsymbol{Her}(n,\boldsymbol{C})$.

$$\int T_i^j(x)\exp(-2\pi\sqrt{-1}\langle x,y\rangle)dx|_{V^*-S^*}=\sum_{k=0}^{n}b_{ik}^{(n)j}|\det y|_k^{i-n} \tag{4.52}$$

where

$$b_{ik}^{(n)j}=(2\pi)^{-(n(n-2i)+(n-i))/2}2^{-n(n-1)/2}\prod_{p=1}^{n-i}\Gamma(p)$$
$$\times\exp\left(\frac{\pi}{2}\sqrt{-1}((n-i)(2k-n)+2(i+1)j)\right)\binom{n-i}{j} \tag{4.53}$$

3. *In the case of* $V=\boldsymbol{Her}(n,\boldsymbol{H})$.

$$\int T_i^j(x)\exp\left(-2\pi\sqrt{-1}\langle x,y\rangle\right)dx|_{V^*-S^*}=\sum_{k=0}^{n}b_{ik}^{(n)j}|\det y|_k^{2(i-n)} \tag{4.54}$$

where

$$b_{ik}^{(n)j}=(2\pi)^{n(-n+2i-1)+\frac{i}{2}}2^{-n(n-1)}\sum_{p=1}^{n-i}\Gamma(2p)\,(-1)^{(i+1)(n-i)}\binom{n-i}{j} \tag{4.55}$$

Lastly, we shall give a relation formula between the $G^{(n)1}$-invariant measure $d\nu_i^j$ on S_i^j and the invariant measure $dg^{(n)1}$ on $G^{(n)1}$. From now on, we denote by $G^{(n)}$ or $G^{(n)1}$ instead of G_R or G_R^1, respectively and V_R is denoted by $V^{(n)}$.

The $G^{(n)1}$-orbit S_i^j is a homogeneous space. The integration on $G^{(n)1}$ by a $G^{(n)1}$-invariant measure $dg^{(n)1}$ is divided into the integration on S_i^j by the $G^{(n)1}$ invariant measure and the integration on the fiber $G_x^{(n)1}$ $(x\in S_i^j)$ by the left invariant measure. We shall determine the measure on $G_x^{(n)1}$ naturally defined by determining the measure on S_i^j.

Let $x_0^{(j)}=I_n^{(j)}$. Then x_0 is a generic point of the $G^{(n)}$-orbit $V_j^{(n)}$. We can identify $V_j^{(n)}$ and $G^{(n)}/G^{(n)}_{x_0^{(j)}}$ where $G^{(n)}_{x_0^{(j)}}$ is the isotropy subgroup of $G^{(n)}$ at $x_0^{(j)}$. There exists the $G^{(n)}$-invariant measure $|P(x)|^{-n'/n}dx^{(n)}$ on $V-S$. Here $n'=\dim V^{(n)}$ and $dx^{(n)}$ is the Euclidean volume element on $V^{(n)}$ which is defined in (3.6). Then we have

$$(4.56)\quad |P(x)|^{-n'/n}dx^{(n)}|_{x=x_0^{(j)}}=|\bigwedge_{i\leq j} dX_{ij}|,$$
$$\text{(resp. } =|(\bigwedge_{i=1}^{n} dX_{ii})\wedge(\bigwedge_{i<j}(dX_{ij}^1\wedge dX_{ij}^2))|,$$
$$=|(\bigwedge_{i=1}^{n} dX_{ii})\wedge(\bigwedge_{i<j}(dX_{ij}^1\wedge dX_{ij}^2\wedge dX_{ij}^3\wedge dX_{ij}^4))|),$$

with

$$x=(X_{ij})\in \boldsymbol{Sym}(n,\boldsymbol{R})=\boldsymbol{T}_{x_0^{(j)}}V^{(n)}$$
$$\text{(resp. } x=(X_{ij})\in \boldsymbol{Her}(n,\boldsymbol{C})=\boldsymbol{T}_{x_0^{(j)}}V^{(n)}$$
$$\text{and } X_{ij}=X_{ij}+\sqrt{-1}X_{ij}\ \ (i<j)),$$
$$x=(X_{ij})\in \boldsymbol{Her}(n,\boldsymbol{H})=\boldsymbol{T}_{x_0^{(j)}}V^{(n)}$$
$$\text{and } X_{ij}=X_{ij}^1+e_1X_{ij}^2+e_2X_{ij}^3+e_1e_2X_{ij}^4\ \ (i<j)).$$

Thus, there exists a left invariant measure $dg^{(n)}_{x_0^{(j)}}$ on the isotropy subgroup $G^{(n)}_{x_0^{(j)}}$ satisfying

$$(4.57)\quad \int f(g)\,dg^{(n)}=\int\Big(\int f(x\cdot h)\,dg^{(n)}_{x_0^{(j)}}(h)\Big)\,|P(x)|^{-n'/n}dx^{(n)},$$
$$(f\in C_0^\infty(\boldsymbol{G})),$$

where $dg^{(n)}$ is a $G^{(n)}$-invariant measure on $G^{(n)}$ and $x\in G^{(n)}/G^{(n)}_{x_0^{(j)}}\cong V_j^{(n)}$. Let $\mathscr{G}^{(n)}$ be the Lie algebra of $G^{(n)}$, which is identified with the tangent space of $G^{(n)}$ at the neutral element $e\in G^{(n)}$. We normalize the $G^{(n)}$-invariant measure $dg^{(n)}$ on $G^{(n)}$ by

$$(4.58)\quad dg^{(n)}(e)=|\bigwedge_{1\leq i,j\leq n} dA_{ij}|\ \text{ with } (A_{ij})\in\mathscr{G}^{(n)}=\boldsymbol{M}(n,\boldsymbol{R})$$
$$\text{(resp. } dg^{(n)}(e)=|\bigwedge_{1\leq i,j\leq n}(dA_{ij}^1\wedge dA_{ij}^2)|$$
$$\text{with } (A_{ij}^1+\sqrt{-1}A_{ij}^2)\in\mathscr{G}^{(n)}=\boldsymbol{M}(n,\boldsymbol{C}),$$
$$dg^{(n)}(e)=|\bigwedge_{1\leq i,j\leq n}(dA_{ij}^1\wedge dA_{ij}^2\wedge dA_{ij}^3\wedge dA_{ij}^4)|$$
$$\text{with } (A_{ij}^1+e_1A_{ij}^2+e_2A_{ij}^3+e_1e_2A_{ij}^4)\in\mathscr{G}^{(n)}=\boldsymbol{M}(n,\boldsymbol{H})).$$

The vector space $\mathscr{G}^{(n)}/\mathscr{G}_{x_0}^{(n)}$ is identified with $T_{x_0}V^{(n)}$ for $x_0 \in V-S$. Then $dx^{(n)}$ is regarded as the volume form on $\mathscr{G}^{(n)}/\mathscr{G}_{x_0}^{(n)}$, and hence we have,

$$(4.59) \qquad dg^{(n)}(e) = |dx^{(n)}(x_0) \wedge dg_{x_0}^{(n)}(e)|,$$

by normalizing $dg_{x_0}^{(n)}$ suitably. Especially, for $x_0 = I_n^{(j)}$, the isotropy subgroup is $O(j, n-j, R)$ (resp. $U(j, n-j, C)$, $U(j, n-j, H)$). The Lie algebra $\mathscr{G}_{x_0}^{(n)}$ is written as

$$(4.60) \qquad \mathscr{G}_{x_0}^{(n)} = \{A \in M(n, R);\ Ax_0 + x_0{}^tA = 0\}.$$
$$(\text{resp. } \mathscr{G}_{x_0}^{(n)} = \{A \in M(n, C);\ Ax_0 + x_0{}^t\bar{A} = 0\},$$
$$\mathscr{G}_{x_0}^{(n)} = \{A \in M(n, H);\ Ax_0 + x_0{}^t\bar{A} = 0\}.)$$

We define the volume form $dg_{x_0}^{(n)}(e)$ on $\mathscr{G}_{x_0}^{(n)}$ by the relation (4.59) and denote it by $dA_{x_0}^{(n)}$, i.e., $dg^{(n)}(e) = |dx^{(n)} \wedge dA_{x_0}^{(n)}|$ with $A \in \mathscr{G}_{x_0}^{(n)}$.

We put $x_i^{(j)} = \begin{bmatrix} I_{n-i}^{(j)} & \\ & 0_i \end{bmatrix}$. The tangent space of S_i^j at $x_i^{(j)}$ is written as

$$(4.61) \qquad T_{x_i^{(j)}}S_i^j = \left\{\begin{bmatrix} X_1 & X_2 \\ {}^tX_2 & 0_i \end{bmatrix};\ \begin{matrix} X_1 \in Sym(n-i, R) \\ X_2 \in M(n-i, i, R) \end{matrix}\right\}$$
$$(\text{resp. } T_{x_i^{(j)}}S_i^j = \left\{\begin{bmatrix} X_1 & X_2 \\ {}^t\bar{X}_2 & 0_i \end{bmatrix};\ \begin{matrix} X_1 \in Her(n-i, C) \\ X_2 \in M(n-i, i, C) \end{matrix}\right\}.$$
$$T_{x_i^{(j)}}S_i^j = \left\{\begin{bmatrix} X_1 & X_2 \\ {}^t\bar{X}_2 & 0_i \end{bmatrix};\ \begin{matrix} X_1 \in Her(n-i, H) \\ X_2 \in M(n-i, i, H) \end{matrix}\right\}).$$

We define the volume form $|dX_1 \wedge dX_2|$ on $T_{x_i^{(j)}}S_i^j$ by

$$(4.62) \qquad dX_1 = \bigwedge_{j \leq k} dX_{jk} \text{ with } X_1 = (X_{jk}) \in Sym(n-i, R).$$
$$(\text{resp. } dX_1 = (\bigwedge_{j=1}^{n-i} dX_{jj}) \wedge (\bigwedge_{j<k} (dX_{jk}^1 \wedge dX_{jk}^2)),$$
$$\text{with } (X_{jk}) \in Her(n-i, C) \text{ and } X_{jk} = X_{jk}^1 + \sqrt{-1}X_{jk}^2,$$
$$dX_1 = (\bigwedge_{j=1}^{n-i} dX_{jj}) \wedge (\bigwedge_{j<k} (dX_{jk}^1 \wedge dX_{jk}^2 \wedge dX_{jk}^3 \wedge dX_{jk}^4))$$
$$\text{with } (X_{jk}) \in Her(n-i, H) \text{ and } X_{jk} = X_{jk}^1 + e_1X_{jk}^2 + e_2X_{jk}^3 + e_1e_2X_{jk}^4)$$

and

$$(4.63)\quad dX_2=\wedge dX_{jk} \text{ with } X_2=(X_{jk})\in \boldsymbol{M}(n-i,\ i,\ \boldsymbol{R}).$$
$$\text{(resp. } dX_2=\wedge(dX^1_{jk}\wedge dX^2_{jk})$$
$$\text{with } (X_{jk}+\sqrt{-1}X_{jk})\in \boldsymbol{M}(n-i,i,\boldsymbol{C}),$$
$$dX_2=\wedge(dX^1_{jk}\wedge dX^2_{jk}\wedge dX^3_{jk}\wedge dX^4_{jk})$$
$$\text{with } (X^1_{jk}+e_1X^2_{jk}+e_2X^3_{jk}+e_1e_2X^4_{jk})\in \boldsymbol{M}(n-i,i,\boldsymbol{H}).).$$

Then, for the $G^{(n)1}$-invariant measure $d\nu^j_i$ on S^j_i defined in Theorem 4. 2, we have

Lemma 4. 4.

$$(4.64)\qquad d\nu^j_i(x^{(j)}_i)=(2\pi)^{\frac{i(i+1)}{4}}|dX_1\wedge dX_2|.$$
$$\text{(resp. } d\nu^j_i(x^{(j)}_i)=(2\pi)^{\frac{i^2}{2}}|dX_1\wedge dX_2|,$$
$$d\nu^j_i(x^{(j)}_i)=(2\pi)^{\frac{i(2i-1)}{2}}|dX_1\wedge dX_2|)$$

Proof. We shall prove this lemma only in the case of $V=\boldsymbol{Sym}(n,\boldsymbol{R})$. Similar proofs are possible in the cases of $V=\boldsymbol{Her}(n,\ \boldsymbol{C})$ and $V=\boldsymbol{Her}(n,\boldsymbol{H})$.

We denote $\begin{array}{cc} \overbrace{}^{n-i} & \overbrace{}^{i} \end{array}$ $\left[\begin{array}{c|c} x_1 & x_2 \\ \hline {}^t x_2 & x_3 \end{array}\right]\in V^{(n)}$.

We put $S^{j\perp}_i=\left\{\begin{bmatrix} I^{(j)}_{n-i} & \\ & x_3\end{bmatrix};\ x_3\in \boldsymbol{Sym}(i,\ \boldsymbol{R})\right\}$.

Then, $S^{j\perp}_i$ intersects with S^j_i at $x^{(j)}_i$ transversally, and

$$\boldsymbol{T}_{x^{(j)}_i}S^{j\perp}_i=\left\{\begin{bmatrix} 0_{n-i} & \\ & X_3\end{bmatrix};\ X_3\in \boldsymbol{Sym}(i,\boldsymbol{R})\right\}.$$

We take a coordinate system $u=(u_{km})_{1\leqq k\leqq m\leqq i}$ which satisfies

$$(4.65)\qquad u_{km}|_{S^{j\perp}_i}=x_{n-i+kn-i+m}|_{S^{j\perp}_i},$$

and

$$S^j_i=\{u_{km}=0 \text{ for all } 1\leqq k\leqq m\leqq i\},$$

in a neighborhood of $x^{(j)}_i$. Then

$$du_{km}(x^{(j)}_i)=dX^{km}_3,$$

where $X_3=(X^{km}_3)$.

Let $d\mu$ be the $G^{(n)1}$-invariant measure on S_i^j normalizing by $d\mu(x_i^{(j)}) = |dX_1 \wedge dX_2|$. When we regard $d\mu$ as a measure on $V^{(n)}$, we can write it as

$$(4.66) \qquad d\mu = f(x) \prod_{k \leq m} \delta(u_{km}) \, |du \wedge dx_1 \wedge dx_2|,$$

by using a local coordinate system (u, x_1, x_2) in a neighborhood of $x_i^{(j)}$, where $f(x)$ is an analytic function with $f(x_i^{(j)}) = 1$, and $du = \wedge du_{km}$, $dx_1 = \bigwedge_{1 \leq k \leq m \leq i} dx_{km}$ and $dx_2 = \bigwedge_{\substack{1 \leq k \leq n-i \\ n-i+1 \leq m \leq n}} dx_{km}$. Moreover, by regarding (4.66) as a hyperfunction on $V^{(n)}$, the principal symbol is by definition

$$(4.67) \qquad (2\pi)^{-\frac{i(i+1)}{4}} f(x) \sqrt{|du^*|} / \sqrt{|du|},$$

where (u_{km}^*) is the dual coordinate of (u_{km}) and $du^* = \bigwedge_{k \leq m} du_{km}^*$. Therefore, the principal symbol at x is

$$(4,68) \qquad (2\pi)^{-\frac{i(i+1)}{4}} \sqrt{|du^*|} / \sqrt{|du|} \, \Big|_{x = x_i^{(j)}}$$

$$= (2\pi)^{-\frac{i(i+1)}{4}} \sqrt{|dX_3^*|} / \sqrt{|dX_3|},$$

where $|dX_3^*|$ is the volume form $|\bigwedge_{k \leq m} d(X_3^*)_{km}|$ on the cotangent vector space $T^*_{x_i^{(j)}} S_i^j$ by the dual coordinate $X_3^* = ((X_3^*)_{km})$ of X_3. On the other hand, the measure $d\nu_i^j$ on S_i^j is also a $G^{(n)1}$-invariant measure. The principal symbol of the measure $d\nu_i^j$, regarding as a hyperfunction on $V^{(n)}$ defined near $x_i^{(j)}$, is

$$(4.69) \qquad |P_\Lambda| S^i \sqrt{|\omega_\Lambda|} / \sqrt{|dx^{(n)}|}$$

from (4.1). Here Λ is the conormal bundle of S_i^j. In order to write (4.69) by using the coordinate system (u_{km}), we restrict (4.69) to $S_i^{j\perp}$. We have $P(x)|_{S_i^{j\perp}} = \det(u_{km})$. Therefore we have

$$(4.70) \qquad W \cap T^* S_i^j = \text{The Zariski closure of}$$
$$\{(u, \sigma \operatorname{grad}_u \log(\det(u)) \in T^* S_i^{j\perp};\ \det u \neq 0,\ \sigma \in \mathbb{R}\}.$$

Let π be the projection map from $W \cap T^* S_i^{j\perp}$ to $S_i^{j\perp}$. Then,

$$(4.71) \qquad P_\Lambda = P \circ \pi / \sigma^i |_\Lambda,$$

$$\omega_\Lambda = \frac{\pi^*(du) \wedge dx_1 \wedge dx_2 \wedge d\sigma}{\sigma^{i'}} \Big/ d\sigma |_\Lambda$$

in $T^* S_i^{j\perp}$ where $i' = \dim V^{(i)} = \dfrac{i(i+1)}{2}$.

By a similar calculations in (3.24), we have

$$|P_A|^{s_i}\sqrt{|\omega_A|}/\sqrt{|dx|}\;|_{T^*S_i^{j\perp}}=c_0^{s_i}c_1^{1/2}\sqrt{|du^*|}/\sqrt{|du|},$$

with

$$c_0=|(\det u)\cdot\det(\mathrm{grad}_u\log(\det u))|=1,$$

and

$$c_1=|(\det u)^{2i'/i}.\ (\mathrm{Hess}_u\log(\det u)|=1,$$

and hence

$$|P_A|^{s_i}\sqrt{|\omega_A|}/\sqrt{|dx|}\;|_{x=x_i^{(j)}}=\sqrt{|dX_3^*|}/\sqrt{|dX_3|}. \tag{4.72}$$

Thus we have

$$\begin{aligned} d\nu_i^j(x_i^{(j)}) &= (2\pi)^{\frac{i(i+1)}{4}}d\mu(x_i^{(j)}) \\ &= (2\pi)^{\frac{i(i+1)}{4}}|dX_1\wedge dX_2|. \end{aligned} \tag{4.73}$$

q. e. d.

We can identify S_i^j and $G^{(n)1}/G^{(n)1}_{x_i^{(j)}}$. Let $\mathscr{G}^{(n)1}$ be the Lie algebra of $G^{(n)1}$. We can write

$$\mathscr{G}^{(n)1}=\{A\in\mathscr{G}^{(n)};\ \mathrm{Re}(\mathrm{Tr}(A))=0\}, \tag{4.74}$$

and $\mathscr{G}^{(n)1}$ is regarded as the tangent space of $G^{(n)1}$ at the neutral element e. We denote the invariant measure on $G^{(n)1}$ by $dg^{(n)1}$ and suppose that it is normalized by

$$\begin{aligned} dg^{(n)1}(e) &= |\bigwedge_{(i,j)\neq(n,n)} dA_{ij}|, \\ (\text{resp. } dg^{(n)1}(e) &= |\bigwedge_{(i,j)\neq(n,n)} (dA_{ij}^1\wedge dA_{ij}^2\wedge dA_{nn}^2|, \\ dg^{(n)1}(e) &= |\bigwedge_{(i,j)\neq(n,n)} (dA_{ij}^1\wedge dA_{ij}^2\wedge dA_{ij}^3\wedge dA_{ij}^4) \\ &\quad \wedge(dA_{nn}^2\wedge dA_{nn}^3\wedge dA_{nn}^4)|). \end{aligned} \tag{4.75}$$

Let $\mathscr{G}^{(n)1}_{x_i^{(j)}}$ be the Lie algebra of $G^{(n)1}_{x_i^{(j)}}$. Then $\mathscr{G}^{(n)1}/\mathscr{G}^{(n)1}_{x_i^{(j)}}$ is identified with the tangent space of S_i^j at $x_i^{(j)}$. The volume form $|dX_1\wedge dX_2|$ on $T_{x_i^{(j)}}S_i^j$ is regarded as that on $\mathscr{G}^{(n)1}/\mathscr{G}^{(n)1}_{x_i^{(j)}}$. Note that $\mathscr{G}^{(n)1}=\mathscr{G}^{(n)1}_{x_i^{(j)}}\oplus\mathscr{G}^{(n)1}/\mathscr{G}^{(n)1}_{x_i^{(j)}}$. There exists a left invariant measure $dg^{(n)1}_{x_i^{(j)}}$ on $G^{(n)1}_{x_i^{(j)}}$ normalized by

(4.76) $$dg^{(n)1}(e) = |(dX_1 \wedge dX_2) \wedge dg^{(n)1}_{x_i^{(j)}}(e)|.$$

We can write the Lie algebra $\mathscr{G}^{(n)1}_{x_i^{(j)}}$ as

(4.77) $$\mathscr{G}^{(n)1}_{x_i^{(j)}} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} ; \begin{array}{l} A \in \mathscr{G}^{(n-i)}_{x_0^{(j)}} B \in M(n-i, i, \mathbf{R}) \\ C \in M(i, \mathbf{R}),\ \mathrm{Tr}(A) + \mathrm{Tr}(C) = 0 \end{array} \right\}$$

$$\text{(resp. } \mathscr{G}^{(n)1}_{x_i^{(j)}} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} ; \begin{array}{l} A \in \mathscr{G}^{(n-i)}_{x_0^{(j)}} B \in M(n-i, i, \mathbf{C}) \\ C \in M(i, \mathbf{C}), \mathrm{Re}(\mathrm{Tr}(A) + \mathrm{Tr}(C)) = 0 \end{array} \right\},$$

$$\mathscr{G}^{(n)1}_{x_i^{(j)}} = \left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} ; \begin{array}{l} A \in \mathscr{G}^{(n-i)}_{x_0^{(j)}} B \in M(n-i, i, \mathbf{H}) \\ C \in M(i, \mathbf{H}), \mathrm{Re}(\mathrm{Tr}(A) + \mathrm{Tr}(C)) = 0 \end{array} \right\})$$

with $x_0^{(j)} = I_{n-i}^{(j)}$. Then we have

Lemma 4.5.

(4.78) $$dg^{(n)1}_{x_i^{(j)}}(e) = |dA^{(n-i)}_{x_0^{(j)}} \wedge dB \wedge dC|.$$

Here, $dA^{(n-i)}_{x_0^{(j)}}$ is the volume form $dx^{(n-i)}(x_0^{(j)})$ defined by (4.59), dB is the volume form defined by $\wedge dB_{ij}$ and dC is the volume form $dg^{(i)1}$ in (4.75).

Proof. Let $h = \begin{bmatrix} A & B \\ D & C \end{bmatrix} \in \mathscr{G}^{(n)1}$ and put $dD = \wedge dD_{ij}$. Then $dg^{(n)1}(e) = |dA \wedge dB \wedge dC \wedge dD|$. The action of h to $x_i^{(j)}$ is

(4.79) $$d\rho(h) x_i^{(j)} = \begin{bmatrix} AI_{n-i}^{(j)} + I_{n-i}^{(j)}\,{}^t\bar{A} & I_{n-i}^{(j)}\,{}^t\bar{D} \\ DI_{n-i}^{(j)} & 0_i \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ {}^t\bar{X}_2 & 0_i \end{bmatrix}.$$

So we have

(4.80) $$dA = dX_1 \wedge dA^{(n-i)}_{x_0^{(j)}} \text{ and } dD = dX_2.$$

Therefore we have

(4.81) $$dg^{(n)1}(e) = |(dX_1 \wedge dX_2) \wedge (dA^{(n-i)}_{x_0^{(j)}} \wedge dB \wedge dC)|,$$

and hence we have the result. q. e. d.

Let $d\nu_{x_i^{(j)}}$ be the left invariant measure on $G^{(n)1}_{x_0}$ normalized by

(4.82) $$dg^{(n)1}(e) = |d\nu_i^j(x_i^{(j)}) \wedge d\nu_{x_i^{(j)}}(e)|.$$

Then we have from (4.64),

(4.83) $$d\nu_{x_i^{(j)}} = (2\pi)^{\frac{-(i+1)}{4}} dg^{(n)1}_{x_i^{(j)}}.$$

$$(\text{resp. } d\nu_{x_i^{(j)}} = (2\pi)^{-\frac{i^2}{2}} dg^{(n)1}_{x_i^{(j)}},$$

$$d\nu_{x_i^{(j)}} = (2\pi)^{-\frac{i(2i-1)}{2}} dg^{(n)1}_{x_i^{(j)}}.)\,.$$

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