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Passive Quasi-free States of the CAR Algebra with Discrete Hamiltonians

By

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Abstract

The spectrally passive, gauge-invariant, quasi-free stases ω on the C*-algebra of anti-commutation relations with respect to a one-parameter quasi-free action τ are described. If the one-particle Hamiltonian H is discrete, the precise condition on the one-particle density Q of ω is combinatorial, but if the Connes spectrum of τ is non-zero, it implies that $Q=(I+e^{\beta(H+T)})^{-1}$ for some $\beta\geq 0$ and some operator T of bounded trace norm, apart from some degenerate possibilities. If H has both discrete and continuous parts, these results can be combined with those of de Cannière for the purely continuous case.

§ 1. Introduction

The notion of spectral passivity of an invariant state ω of a C*-algebra $\mathfrak{A}$ with respect to a one-parameter group of automorphisms of $\mathfrak{A}$ was introduced by de Cannière [3] as a part of the KMS condition. Thus ω is spectrally passive if

$$\omega(x^*x)\leq\omega(xx^*) \tag{1.1}$$

for all x in the spectral subspace $R(-\infty, 0)$ of $\mathfrak{A}$. All KMS states at inverse temperature β, where $0\leq\beta\leq\infty$, are spectrally passive. Conversely, the condition

$$x_i\in R(-\infty, \lambda_i),\ \sum_{i=1}^{n}\lambda_i\leq 0\Rightarrow\prod_{i=1}^{n}\omega(x_i^*x_i)\leq\prod_{i=1}^{n}\omega(x_ix_i^*)$$

implies that ω is a KMS state at some $0\leq\beta\leq\infty$ (see [1] for a short proof). The condition (1.1) is closely related to the condition of passivity, as defined by Pusz and Woronowicz [7] based on the

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Second Law of Thermodynamics. Passive states are spectrally passive [3], and there is no known example of a spectrally passive state which is not passive (see [1] for some partial results).

Now de Cannière [4] has investigated the condition (1.1) in the specific context of a (gauge-invariant) quasi-free state and quasi-free automorphisms on the CAR algebra $\mathfrak{A}(\mathscr{H})$ associated with a one-particle Hilbert space $\mathscr{H}$ (or the gauge-invariant part $\mathfrak{A}(\mathscr{H})^{(0)}$). Thus $\mathfrak{A}(\mathscr{H})$ is generated by annihilation and creation operators $a(f)$, $a^*(g)$ $(f, g\in\mathscr{H})$ satisfying the canonical anticommutation relations

$$a(f)a(g)+a(g)a(f)=0$$
$$a(f)a^*(g)+a^*(g)a(f)=(f, g)I.$$

These conditions, together with linearity of a^* and the relation $a^*(f)=(a(f))^*$, determine $\mathfrak{A}(\mathscr{H})$ uniquely up to isomorphism. The quasi-free one-parameter group of automorphisms is given by

$$\tau_t(a(f))=a(e^{itH}f) \tag{1.2}$$

where e^{itH} is a unitary group on $\mathscr{H}$ with one-particle Hamiltonian H. The gauge action is the quasi-free one-parameter action with $H=I$. The gauge-invariant quasi-free states ω are in one-to-one correspondence with the positive contractions Q on $\mathscr{H}$, the correspondence being given by

$$\omega(a^*(g_1)\ldots\ldots a^*(g_n)a(f_n)\ldots\ldots a(f_1))=\det[(f_i, Qg_j)]. \tag{1.3}$$

The operator Q is known as the *one-particle density* of ω. Invariance of ω under τ corresponds to Q commuting with H. All these, and other, properties of $\mathfrak{A}(\mathscr{H})$ may be found in [2, 5.2].

It was shown in [4] that if H has no eigenvectors then ω is spectrally passive on $\mathfrak{A}(\mathscr{H})^{(0)}$ if and only if either $Q=P(-\infty, \mu)$ for some $-\infty\leq\mu\leq\infty$, where P is the spectral measure of H, or $Q=(I+e^{\beta H-\mu I})^{-1}$ for some $\beta\geq 0$ and some $-\infty<\mu<\infty$. Furthermore, if ω is spectrally passive on $\mathfrak{A}(\mathscr{H})$, then $Q=P(-\infty, 0)$ or $Q=(I+e^{\beta H})^{-1}$. Some partial results were given in [4] if H has some eigenvectors.

This paper continues this programme by considering all possible combinations of eigenvalues of H. Thus a complete description is given in Theorem 4.3 (see Corollary 5.2 for the converse result) of all the one-particle densities Q whose associated states are spectrally passive on $\mathfrak{A}(\mathscr{H})$. If H is diagonalisable (so that H has a complete orthonormal set of eigenvectors), then Q is also diagonalisable, and the eigenvalues of H, Q are paired in a certain combinatorial way (the "passive" pairs of § 3). If the eigenvalues of H are evenly spaced or of large multiplicity, this requires that $Q=(I+e^{\beta(H+T)})^{-1}$ for some $\beta\geq 0$ and some operator T of trace class, apart from some degenerate possibilities. The assumptions on the eigenvalues are essentially that the Connes spectrum $\Gamma(\tau)$ of τ should not be (0), and the conclusion is that $T=0$ if $\Gamma(\tau)=\boldsymbol{R}$ and $\operatorname{tr}|T|\leq\gamma$ if $\Gamma(\tau)=\gamma\boldsymbol{Z}$ for some $\gamma>0$. If H has some eigenvectors but is not diagonalisable, then either $Q=(I+e^{\beta H})^{-1}$ or $Q(I-P(0))=P(-\infty, 0)$ or H has a direct sum decomposition $H=H_1\oplus H_2$ where H_1 is of trace class and $(-\operatorname{tr}|H_1|, \operatorname{tr}|H_1|)$ is contained in the resolvent set of H_2, and $Q=Q_1\oplus P_2(-\infty, 0)$, where (H_1, Q_1) is a passive pair and P_2 is the spectral measure of H_2. In the final section, corresponding results for $\mathfrak{A}(\mathscr{H})^{(0)}$ are stated.

The techniques of proof in this paper are mostly derived from [4], with certain extra combinatorial complications.

§ 2. Exponentials of Trace Class Operators

In the sequel, it will be seen that spectral passivity of ω implies inequalities of the form

$$\prod_{i=1}^{m}(f_i,Qf_i)\prod_{j=m+1}^{n}(1-(f_j,Qf_j))\leq c\prod_{i=1}^{m}(1-(f_i,Qf_i))\prod_{j=m+1}^{n}(f_j,Qf_j) \tag{2.1}$$

for certain constants c, integers $0\leq m\leq n$ and orthonormal sets $\{f_i: 1\leq i\leq n\}$ with $(f_i,Qf_j)=0$ for $1\leq i<j\leq n$. The results in this section describe some of the consequences of (2.1). For an operator T on $\mathscr{H}$, let $\|T\|_1=\operatorname{tr}|T|$ if T is of trace class, $\|T\|_1=\infty$ otherwise.

Lemma 2.1. *Let Q be a positive contraction on $\mathscr{H}$ such that Q and $I-Q$ are invertible, and let b be a positive real number. The following are equivalent:*

(*i*) $\log(b(Q^{-1}-I))$ *is of trace class,*

(*ii*) $Q-b(b+1)^{-1}I$ *is of trace class,*

(*iii*) *Q has a complete orthonormal set of eigenvectors $\{f_\alpha: \alpha\in A\}$ with eigenvalues ρ_α, and there are constants $0<c_1\leq 1\leq c_2<\infty$ such that*

$$c_1\leq\prod_{\alpha\in B}b(\rho_\alpha^{-1}-1)\leq c_2 \tag{2.2}$$

for all finite subsets B of A.

In this case,

$$\exp(\|\log(b(Q^{-1}-I))\|_1) = \sup\{\prod_{\alpha\in A_1} b(\rho_\alpha^{-1}-1)/\prod_{\alpha\in A_2} b(\rho_\alpha^{-1}-1)\} \quad (2.3)$$

$$= \prod_{\rho_\alpha<b(b+1)^{-1}} b(\rho_\alpha^{-1}-1)/\prod_{\rho_\alpha>b(b+1)^{-1}} b(\rho_\alpha^{-1}-1) \quad (2.4)$$

where the supremum in (2.3) *is taken over all finite (disjoint) subsets A_1, A_2 of A.*

Proof. The equivalence of (i) and (ii) follows from the inequality

$$4|t-b(b+1)^{-1}| \le |\log(b(t^{-1}-1))| \le \varepsilon^{-1}(1-\varepsilon)^{-1}|t-b(b+1)^{-1}|$$

which is valid for $\varepsilon\le t\le 1-\varepsilon$, $0<\varepsilon\le\min(1, b)/(b+1)$.

The equivalence of (i) and (iii) follows from the fact that (2.2) is equivalent to the condition

$$\sum_{\alpha\in A} |\log(b(\rho_\alpha^{-1}-1)| \le \log(c_2/c_1).$$

This also establishes (2.3) and (2.4). □

Lemma 2.2. *Let Q be a positive contraction on $\mathscr{H}$, $Q\neq 0, I$.*

1. *Suppose that there is a constant $c\ge 1$ such that* (2.1) *is valid for all integers $0\le m\le n$ and orthonormal sets $\{f_i\colon 1\le i\le n\}$ with $(f_i, Qf_j)=0$ for $1\le i<j\le n$. Then Q and $I-Q$ are invertible, and $\|\log(Q^{-1}-I)\|_1\le$ log c.*

2. *Suppose that there is a constant $c\ge 1$ such that* (2.1) *is valid for $n=2m\ge 0$, and for all orthonormal sets $\{f_i\colon 1\le i\le 2m\}$ with (f_i, Qf_j) $=0$ for $1\le i<j\le 2m$. Then Q and $I-Q$ are invertible, and there exists b >0 such that $\|\log(b(Q^{-1}-I))\|_1\le\log c$.*

The conclusions of (1) *and* (2) *remain valid if $\mathscr{H}$ has a complete orthonormal set of eigenvectors for Q and* (2.1) *is assumed only to hold when each f_i belongs to this set.*

Proof. 1. Taking $m=0$, $n=1$ in (2.1) shows that Q is invertible (and $Q\ge(c+1)^{-1}I$). Taking $m=1$, $n=1$ shows that $I-Q$ is invertible (and $Q\le c(c+1)^{-1}I$).

Suppose that $Q-\frac{1}{2}I$ is not compact. By spectral theory, there exist $\varepsilon>0$ and an infinite orthonormal sequence $\{f_i\}$ such that $(f_i, Qf_j)=$

0 if $i\neq j$, and either $(f_i, Qf_i)\leq\frac{1}{2}-\varepsilon$ for all i, or $(f_i, Qf_i)\geq\frac{1}{2}+\varepsilon$ for all i. Putting $m=0$ or $m=n$ in (2.1), it follows that

$$\left(\frac{1}{2}+\varepsilon\right)^n\leq c\left(\frac{1}{2}-\varepsilon\right)^n$$

for all n. This contradiction shows that $Q-\frac{1}{2}I$ is compact, and in particular Q has a complete set of eigenvectors $\{g_\alpha: \alpha\in A\}$ with corresponding eigenvalues ρ_α. If A_1 and A_2 are finite disjoint subsets of A, then (2.1) gives

$$\prod_{\alpha\in A_1}(\rho_\alpha^{-1}-1)/\prod_{\alpha\in A_2}(\rho_\alpha^{-1}-1)\leq c,$$

so $\|\log(Q^{-1}-I)\|_1\leq\log c$, by Lemma 2.1.

2. If Q were not injective, there would be orthonormal vectors f_1, f_2 with $Qf_1\neq 0$, $Qf_2=0$. Applying (2.1) with $m=1$, $n=2$ would lead to a contradiction. Similarly, $I-Q$ is injective.

Suppose that $Q-b'I$ is not compact for any b'. By spectral theory, there exist $0<b_1<b_2<1$ and infinite orthonormal sequences $\{f_i\}$ and $\{g_j\}$ such that

$$\begin{aligned}&(f_i, Qf_j)=(g_i, Qg_j)=0 && \text{if } i\neq j\\ &(f_i, g_j)=(f_i, Qg_j)=0 && \text{for all } i, j\\ &(f_i, Qf_i)>b_2,\ (g_j, Q g_j)<b_1 && \text{for all } i, j.\end{aligned}$$

Applying (2.1) to $\{f_1, \dots f_m, g_1, \dots g_m\}$,

$$b_2^m(1-b_1)^m\leq c(1-b_2)^m b_1^m.$$

Since $b_2(1-b_1)>(1-b_2)b_1$, this is a contradiction, showing that $Q-b'I$ is compact for some b', and Q has a complete orthonormal set of eigenvectors $\{h_\alpha: \alpha\in A\}$ with corresponding eigenvalues ρ_α.

If $\mathscr{H}$ is infinite-dimensional, b' is unique, and $0<b'<1$. Let $b=b'(1-b')^{-1}$. Let A_1, A_2 be finite disjoint subsets of A with cardinalities m, n, respectively, and A_3 be a subset of $A\backslash(A_1\cup A_2)$ of cardinality $|m-n|$ such that ρ_α is arbitrarily close to b' for all α in A_3. Applying (2.1) to $\{h_\alpha: \alpha\in A_2\cup A_3\cup A_1\}$ and taking a limit,

$$(b')^{m-n}\prod_{\alpha\in A_2}\rho_\alpha\prod_{\alpha\in A_1}(1-\rho_\alpha)\leq c(1-b')^{m-n}\prod_{\alpha\in A_2}(1-\rho_\alpha)\prod_{\alpha\in A_1}\rho_\alpha$$

or

$$\prod_{\alpha\in A_1}b(\rho_\alpha^{-1}-1)/\prod_{\alpha\in A_2}b(\rho_\alpha^{-1}-1)\leq c,$$

so $\|\log(b(Q^{-1}-1))\|_1 \leq \log c$, by Lemma 2.1.

If $\mathscr{H}$ is finite-dimensional, choose b' so that Q has equal numbers of eigenvalues greater than (respectively, less than) b', and let $b = b'(1-b')^{-1}$. In the argument above, consider all the eigenvectors (except possibly one) simultaneously, and deduce that $\|\log(b(Q^{-1}-I))\|_1 \leq \log c$.

The proof of the final statement is included in the proof above. □

Remark. There are converse results to Lemma 2.2, obtained by applying the theory of trace class operators to log $(b(Q^{-1}-I))$. Thus (2.1) holds with $c = \exp \|\log(b(Q^{-1}-I))\|_1$ for any orthonormal set $\{f_i\}$, provided that $m=n$ if $b \neq 1$.

§ 3. Passive Pairs of Operators

Let H be a (possibly unbounded) self-adjoint operator on $\mathscr{H}$ with spectral measure P, and Q be a positive contraction on $\mathscr{H}$. The pair (H, Q) will be said to be *passive* if $\mathscr{H}$ has a complete orthonormal set $\{f_\alpha: \alpha \in A\}$ of common eigenvectors for H, Q with eigenvalues λ_α, ρ_α respectively, and

$$\prod_{\alpha\in A_1} \rho_\alpha \prod_{\alpha\in A_2} (1-\rho_\alpha) \leq \prod_{\alpha\in A_1} (1-\rho_\alpha) \prod_{\alpha\in A_2} \rho_\alpha \tag{3.1}$$

for any finite (disjoint) subsets A_1, A_2 of A such that $\lambda(A_1) > \lambda(A_2)$, where $\lambda(B) = \sum_{\alpha\in B} \lambda_\alpha$ for a finite subset B of A.

If $\rho_\alpha = 0$ whenever $\lambda_\alpha > 0$ and $\rho_\alpha = 1$ whenever $\lambda_\alpha < 0$, then (H, Q) is a passive pair. Such a passive pair will be said to be *trivial*.

If H has a complete orthonormal set of eigenvectors, then $(H, (I+e^{\beta H})^{-1})$ is a passive pair for any $\beta \geq 0$.

If (H, Q) is a passive pair, then (3.1) gives

$$\lambda_\alpha > 0 \Rightarrow 0 \leq \rho_\alpha \leq \frac{1}{2} \tag{3.2}$$

$$\lambda_\alpha < 0 \Rightarrow \frac{1}{2} \leq \rho_\alpha \leq 1 \tag{3.3}$$

$$\lambda_{\alpha_1} > \lambda_{\alpha_2} \Rightarrow \rho_{\alpha_1} \leq \rho_{\alpha_2} \tag{3.4}$$

$$\lambda_{\alpha_1} + \lambda_{\alpha_2} > 0 \Rightarrow \rho_{\alpha_1} + \rho_{\alpha_2} \leq 1 \tag{3.5}$$

$$\lambda_{\alpha_1} + \lambda_{\alpha_2} < 0 \Rightarrow \rho_{\alpha_1} + \rho_{\alpha_2} \geq 1. \tag{3.6}$$

Lemma 3.1. *Suppose that* (H, Q) *is a non-trivial passive pair, and let* $\mathscr{H}_0=P(0)\mathscr{H}$, $Q_0=Q|\mathscr{H}_0$. *Then* Q_0 *and* $I-Q_0$ *are invertible and* $\log(Q_0^{-1}-I_{\mathscr{H}_0})$ *is of trace class on* $\mathscr{H}_0$. *Furthermore*

$$\prod_{\alpha\in A_1}\rho_\alpha\prod_{\alpha\in A_2}(1-\rho_\alpha)\leq\exp(-\|\log(Q_0^{-1}-I_{\mathscr{H}_0})\|_1)\prod_{\alpha\in A_1}(1-\rho_\alpha)\prod_{\alpha\in A_2}\rho_\alpha \tag{3.7}$$

for any finite subsets A_1, A_2 *of* $\{\alpha\in A: \lambda_\alpha\neq 0\}$ *such that* $\lambda(A_1)>\lambda(A_2)$.

Proof. Since (H, Q) is non-trivial, there is some κ in A such that either $\lambda_\kappa>0$ and $\rho_\kappa>0$ or $\lambda_\kappa<0$ and $\rho_\kappa<1$. Let $c=\rho_\kappa^{-1}-1$ or $c=(1-\rho_\kappa)^{-1}-1$, respectively.

Let B_1 and B_2 be finite subsets of $\{\alpha\in A: \lambda_\alpha=0\}$. Applying (3.1) to $A_1\cup B_1$, $A_2\cup B_2$,

$$\prod_{A_1}\rho_\alpha\prod_{A_2}(1-\rho_\alpha)\prod_{B_1}\rho_\alpha\prod_{B_2}(1-\rho_\alpha)\leq\prod_{A_1}(1-\rho_\alpha)\prod_{A_2}\rho_\alpha\prod_{B_1}(1-\rho_\alpha)\prod_{B_2}\rho_\alpha. \tag{3.8}$$

The special case when $A_1=\{\kappa\}$, $A_2=\phi$ or $A_1=\phi$, $A_2=\{\kappa\}$ shows that

$$\prod_{B_1}\rho_\alpha\prod_{B_2}(1-\rho_\alpha)\leq c\prod_{B_1}(1-\rho_\alpha)\prod_{B_2}\rho_\alpha.$$

By Lemma 2.2(1) applied to $\mathscr{H}_0$ and Q_0, Q_0 and $I_{\mathscr{H}_0}-Q_0$ are invertible, and $\log(Q_0^{-1}-I_{\mathscr{H}_0})$ is of trace class. On rearranging (3.8) and taking an infimum over all choices of B_1, B_2, (3.7) follows. □

The inequalities (3.2), (3.3) may now be strengthened to give

$$\lambda_\alpha>0\Rightarrow 0\leq\rho_\alpha\leq(1+\delta)^{-1} \tag{3.9}$$

$$\lambda_\alpha<0\Rightarrow\delta(1+\delta)^{-1}\leq\rho_\alpha\leq 1 \tag{3.10}$$

where $\delta=\exp(\|\log(Q_0^{-1}-1_{\mathscr{H}_0})\|_1)$.

Proposition 3.2. *Let* (H, Q) *be a non-trivial passive pair, and suppose that either* Q *or* $I-Q$ *is not injective. There is a decomposition* $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, $H=H_1\oplus H_2$, $Q=Q_1\oplus P_2(-\infty, 0)$, *where* P_2 *is the spectral measure of* H_2, H_1 *is of trace class,* $(-\|H_1\|_1, \|H_1\|_1)$ *is contained in the resolvent set of* H_2, Q_1 *and* $I_{\mathscr{H}_1}-Q_1$ *are invertible and* (H_1, Q_1) *is a passive pair.*

Proof. Let $A_1=\{\alpha\in A: 0<\rho_\alpha<1\}$, $A_2=A\backslash A_1$, $\gamma=\inf\{|\lambda_\alpha|: \alpha\in A_2\}$.

By assumption, A_2 is non-empty, so γ is finite. It follows from (3.2)-(3.6) that

$$|\lambda_\alpha|>\gamma\Rightarrow\alpha\in A_2.$$

Since (H, Q) is non-trivial, it follows that $\gamma>0$. If $\mathscr{H}_i$ is the closed linear span of $\{f_\alpha:\ \alpha\in A_i\}$ then $H=H_1\oplus H_2$, $Q=Q_1\oplus P_2(-\infty, 0)$ (using (3.2), (3.3)), $\|H_1\|\leq\gamma$, $(-\gamma, \gamma)$ is contained in the resolvent set of H_2, Q_1 and $I_{\mathscr{H}_1}-Q_1$ are injective, and (H_1, Q_1) is a passive pair.

Let B_1, B_2 be finite subsets of A_1, and suppose that $\lambda(B_1)-\lambda(B_2)>\gamma$. By definition of γ, there exists an index κ in A_2 such that $|\lambda_\kappa|<\lambda(B_1)-\lambda(B_2)$. Applying (3.1) to B_1, $B_2\cup\{\kappa\}$ or to $B_1\cup\{\kappa\}$, B_2 (depending on whether $\lambda_\kappa>0$ or $\lambda_\kappa<0$,

$$\prod_{B_1}\rho_\alpha\prod_{B_2}(1-\rho_\alpha)\leq 0$$

since $\rho_\kappa=0$ or $\rho_\kappa=1$. This contradicts the definition of A_1. Hence $\|H_1\|_1\leq\gamma$.

Let $\gamma'=\max\{|\lambda_\alpha|:\ \alpha\in A_1\}$. It follows from (3.4)-(3.6) that

$$\|Q_1^{-1}\|, \|(I_{\mathscr{H}_1}-Q_1)^{-1}\|\leq\max\{\rho_\alpha^{-1}\vee(1-\rho_\alpha)^{-1}:\ \alpha\in A_1,\ |\lambda_\alpha|=\gamma'\},$$

the right-hand side being finite since there are only finitely many indices α in A_1 with $|\lambda_\alpha|=\gamma'$. □

Proposition 3.2 reduces the problem of describing passive pairs to the case when Q and $I-Q$ are injective. The aim is to show then that

$$Q=(I+e^{\beta(H+T)})^{-1}$$

for some $\beta\geq 0$ and some trace class operator T with $\|T\|_1$ bounded. Thus the problem is to show that

$$T_\beta:=\log(Q^{-1}-I)-\beta H \tag{3.11}$$

is trace class (and to obtain a bound on its trace norm). Writing

$$\sigma_\alpha=\log(\rho_\alpha^{-1}-1)$$
$$\sigma(B)=\sum_B\sigma_\alpha$$

for a finite subset B of A, (3.1) becomes

$$\lambda(A_1)>\lambda(A_2)\Rightarrow\sigma(A_1)\geq\sigma(A_2), \tag{3.12}$$

and the objective is to show that

$$\sup_{A_1, A_2} \{\sigma(A_1) - \sigma(A_2) - \beta(\lambda(A_1) - \lambda(A_2))\} < \infty. \tag{3.13}$$

The obstruction to (3.13) relates to the possibility of the quasi-free action τ on $\mathfrak{A}(\mathscr{H})$ associated with H having zero Connes spectrum. Let $\Gamma(H)$ be the set of all real numbers γ such that there are sequences of mutually disjoint finite subsets A_n, B_n of A such that $\lambda(A_n) - \lambda(B_n) \to \gamma$. Elementary combinatorial considerations show that $\Gamma(H)$ is a closed subgroup of $\boldsymbol{R}$. In this case where H is diagonalisable, $\mathfrak{A}(\mathscr{H})$ may be identified with $\otimes_{\alpha \in A} M_2$ and τ with

$$\otimes\ Ad \begin{bmatrix} 1 & 0 \\ 0 & e^{it\lambda_\alpha} \end{bmatrix}$$

(see [2, 5.2.5] and Section 5). It was observed in [6] that $\Gamma(H)$ is the Connes spectrum of this action.

If H is of trace class, then $\Gamma(H) = (0)$; if H is compact, but not of trace class, then $\Gamma(H) = \boldsymbol{R}$; if H is bounded but not of trace class, then $\Gamma(H) \neq (0)$; in general, $\Gamma(H)$ contains all eigenvalues of infinite multiplicity.

The first case when it is possible to establish (3.11) is when H itself is of trace class. This result may be applied to the pair (H_1, Q_1) of Proposition 3.2.

Proposition 3.3. *Let (H, Q) be a non-trivial passive pair, where H is of trace class, and Q and I-Q are injective. Then* $\log(Q^{-1} - I)$ *is of trace class, and, for any $\beta \geq 0$, $Q = (I + e^{\beta(H+T)})^{-1}$ for some operator T of trace class.*

Proof. It is possible to consider separately the restrictions to $P(0)\mathscr{H}$, $P(-\infty, 0)\mathscr{H}$ and $P(0, \infty)\mathscr{H}$. The first of these is covered by Lemma 3.1, and the other two are similar to each other, so we consider only the latter. Assume therefore that $\lambda_\alpha > 0$ for all α, so that $\sigma_\alpha \geq 0$ by (3.2). Let A_1 be a finite subset of A such that $\lambda(A_1) > \frac{1}{2}\|H\|_1$, and A_2 be any finite subset of $A \backslash A_1$. Then $\lambda(A_1) > \lambda(A_2)$, so by (3.12) $\sigma(A_1) \geq \sigma(A_2)$. Thus $\sum_A \sigma_\alpha \leq 2\sigma(A_1)$, as required. □

The next case is when $\Gamma(H) \neq (0)$.

Proposition 3.4. *Let (H, Q) be a passive pair, and suppose that Q*

and $I-Q$ *are injective.*

1. *If* $\Gamma(H)=\gamma\mathbf{Z}$ *where* $\gamma>0$, *then* $Q=(I+e^{\beta(H+T)})^{-1}$ *for some* $\beta\geq 0$ *and some operator* T *with* $\|T\|_1\leq\gamma$.
2. *If* $\Gamma(H)=\mathbf{R}$, *then* $Q=(I+e^{\beta H})^{-1}$ *for some* $\beta\geq 0$.

Proof. Let $\gamma>0$ be any member of $\Gamma(H)$, and A_n, $B_n(n\geq 1)$ be disjoint finite subsets of A such that $\lambda(A_n)-\lambda(B_n)\to\gamma$. By (3.12), $\sigma(A_n)-\sigma(B_n)\geq 0$ for large n.

Let α be any index with $\lambda_\alpha>0$. Choose m so that $\alpha\notin A_m\cup B_m$, $\lambda(A_m)-\lambda(B_m)>\gamma-\lambda_\alpha$. If $\alpha\notin A_n\cup B_n$, $n\neq m$, then

$$\begin{aligned}\lambda(A_m\cup B_n\cup\{\alpha\})&=\lambda(A_m)+\lambda(B_n)+\lambda_\alpha\\&>\lambda(B_m)+\lambda(A_n)\\&=\lambda(B_m\cup A_n)\end{aligned}$$

if n is large. By (3.12)

$$\sigma(A_m)+\sigma(B_n)+\sigma_\alpha\geq\sigma(B_m)+\sigma(A_n).$$

Thus

$$0\leq\sigma(A_n)-\sigma(B_n)\leq\sigma(A_m)-\sigma(B_m)+\sigma_\alpha.$$

If $\lambda_\alpha<0$, a similar argument shows that

$$0\leq\sigma(A_n)-\sigma(B_n)\leq\sigma(A_m)-\sigma(B_m)-\sigma_\alpha.$$

Since $H\neq 0$, choosing α so that $\lambda_\alpha\neq 0$, it follows in either case that $\sigma(A_n)-\sigma(B_n)$ is bounded. Passing to a subsequence, assume that $\sigma(A_n)-\sigma(B_n)$ converges to a limit $\beta\gamma$, where $\beta\geq 0$.

Now let A_0, B_0 be disjoint finite subsets of A, and put $\lambda(A_0)-\lambda(B_0)=\gamma_0$. If $\gamma_0<0$, let $k\geq 0$ be the largest integer such that $k\gamma+\gamma_0<0$. For large n, the sets A_0, B_0, A_n, B_n, A_{n+1}, B_{n+1}, ... are disjoint. Let $C_n=A_0\cup A_n\cup\ldots\cup A_{n+k-1}$, $D_n=B_0\cup B_n\cup\ldots\cup B_{n+k-1}$. Then

$$\lambda(C_n)-\lambda(D_n)=\gamma_0+\sum_{i=n}^{n+k-1}(\lambda(A_i)-\lambda(B_i))<0$$

for large n. By (3.12)

$$\begin{aligned}0\geq\sigma(C_n)-\sigma(D_n)&=\sigma(A_0)-\sigma(B_0)+\sum_{i=n}^{n+k-1}(\sigma(A_i)-\sigma(B_i))\\&\to\sigma(A_0)-\sigma(B_0)+k\beta\gamma\end{aligned}$$

as $n\to\infty$. Hence

$$\sigma(A_0)-\sigma(B_0)-\beta(\lambda(A_0)-\lambda(B_0))\leq -k\beta\gamma-\beta\gamma_0\leq\beta\gamma. \tag{3.14}$$

If $\gamma_0\geq 0$, let $k'>0$ be the smallest integer such that $k'\gamma>\gamma_0$, let $C'_n=$

$B_0 \cup A_n \cup \ldots \cup A_{n+k'-1}$, $D'_n = A_0 \cup B_n \cup \ldots \cup B_{n+k'-1}$. For large n, $\lambda(C'_n) > \lambda(D'_n)$, so $\sigma(C'_n) \geq \sigma(D'_n)$. It follows that

$$\sigma(A_0) - \sigma(B_0) - \beta(\lambda(A_0) - \lambda(B_0)) \leq k'\beta\gamma - \beta\gamma_0 \leq \beta\gamma. \tag{3.15}$$

Together, (3.14) and (3.15) give (3.13) and $\|T_\beta\|_1 \leq \beta\gamma$. If $\Gamma(H) = \gamma\boldsymbol{Z}$, the proof is complete. If $\Gamma(H) = \boldsymbol{R}$, then H is not of trace class, so the value of β is independent of γ. Letting $\gamma \to 0$, it follows that $T_\beta = 0$, so $Q = (I + e^{\beta H})^{-1}$. □

Example 3.5. Suppose that H has a complete orthonormal set of eigenvectors $\{f_n: n = 0, 1, 2\ldots\}$ with eigenvalues $\lambda_n = n$. For example, the harmonic oscillator Hamiltonian $H = \frac{1}{2}\left(-\frac{d^2}{dt^2} + t^2 - 1\right)$ on $L^2(\boldsymbol{R})$ has eigenvectors

$$f_n(t) = (n!)^{-1/2}\ 2^{n/2} i^{-n} \pi^{-3/4} e^{2/2} \int u^n e^{-u^2 + 2itu} du$$

[5, Lemma 7.12]. Now $\Gamma(H) = \boldsymbol{Z}$. If (H, Q) is a passive pair, then f_n is an eigenvector of Q, and the eigenvalues ρ_n satisfy either (a) $\rho_n = 0$ for all $n \geq 3$, or (b) $0 < \rho_n < 1$ for all n (Proposition 3.2). Indeed (H, Q) is a passive pair if and only if either

(a) $Qf_0 = \rho_0 f_0$, $Qf_1 = \rho_1 f_1$, $Qf_2 = \rho_2 f_2$, $Qf_n = 0$ $(n \geq 3)$,
where $1 \geq \rho_0 \geq \rho_1 \geq \rho_2 \geq 0$, $\rho_0 + \rho_1 \leq 1$,
$(1-\rho_0)(1-\rho_1)\rho_2 \leq \rho_0\rho_1(1-\rho_2)$, $\rho_0(1-\rho_1)\rho_2 \leq (1-\rho_0)\rho_1(1-\rho_2)$,

or

(b) $Q = (I + e^{\beta(H+T)})^{-1}$ for some $\beta \geq 0$ and some T with $\|T\|_1 \leq 1$.

The necessity of (b) if $0 < \rho_n < 1$ follows from Proposition 3.4 and the sufficiency is established by verifying (3.12) directly. □

If H is unbounded and $\Gamma(H) = (0)$, there may be passive pairs (H, Q) where Q and $I - Q$ are injective but T_β is not of trace class for any $\beta \geq 0$.

Example 3.6. Suppose that the index set $A = \{0, 1, 2\ldots\}$, and $\lambda_n = 2^n$. Then $\Gamma(H) = (0)$, and (3.12) simplifies to

$$\sigma_0 \geq 0, \ \sigma_{n+1} \geq \sum_{i=0}^{n} \sigma_i.$$

If $\sigma_n = 3^n$, for example, this condition is satisfied, but $\sum |\sigma_n - \beta\lambda_n| = \infty$ for all β. Thus if $Qf_n = (1 + e^{\sigma_n})^{-1} f_n$, then (H, Q) is passive, but T_β

is not of trace class.

§4. Spectrally Passive Quasi-free States

In this section, let H be a Hamiltonian on the one-particle Hilbert space $\mathscr{H}$, with associated quasi-free one-parameter action τ on $\mathfrak{A}(\mathscr{H})$ given by (1.2). Let ω be a gauge-invariant quasi-free state of $\mathfrak{A}(\mathscr{H})$ with associated one-particle density Q, so that (1.3) is satisfied. Throughout this section, suppose that ω is spectrally passive with respect to τ. In particular, ω is τ-invariant, so Q commutes with H and its spectral measure P.

Lemma 4.1. *Let $0\leq m\leq n$ be integers, and $\{\lambda_i: 1\leq i\leq n\}$ be real numbers such that $\sum_{i=1}^{m}\lambda_i\geq\sum_{j=m+1}^{n}\lambda_j$. Let $\{f_i: 1\leq i\leq n\}$ be an orthonormal set, and suppose that $f_i\in P(\lambda_i,\infty)\mathscr{H}$ $(1\leq i\leq m)$, $f_j\in P(-\infty,\lambda_j)\mathscr{H}$ $(m<j\leq n)$ and $(f_i, Qf_j)=0$ $(1\leq i<j\leq n)$. Then*

$$\prod_{i=1}^{m}(f_i, Qf_i)\prod_{j=m+1}^{n}(1-(f_j, Qf_j))\leq\prod_{i=1}^{m}(1-(f_i, Qf_i))\prod_{j=m+1}^{n}(f_j, Qf_j). \tag{4.1}$$

If $\sum_{i=1}^{m}\lambda_i>\sum_{j=m+1}^{n}\lambda_j$, and $f_i\in P[\lambda_i,\infty)\mathscr{H}$ $(1\leq i\leq m)$, $f_j\in P(-\infty,\lambda_j]\mathscr{H}$ $(m<j\leq n)$, then (4.1) remains valid.

Proof. The proof is quite similar to [4, Lemma 3.8]. Consider

$$x=a(f_1)a(f_2)\dots a(f_m)a^*(f_{m+1})a^*(f_{m+2})\dots a^*(f_n)\in R(-\infty, 0).$$

Then (1.3), the canonical anticommutation relations, the orthonormality and the condition $(f_i, Qf_j)=0$ give

$$\omega(x^*x)=\prod_{i=1}^{m}(f_i, Qf_i)\prod_{j=m+1}^{n}(1-(f_j, Qf_j))$$
$$\omega(xx^*)=\prod_{i=1}^{m}(1-(f_i, Qf_i))\prod_{j=m+1}^{n}(f_j, Qf_j).$$

The condition (1.1) defining spectral passivity, gives (4.1).

The final statement follows on perturbing λ_i slightly. □

Now write $\mathscr{H}=\mathscr{H}_p\oplus\tilde{\mathscr{H}}$, $H=H_p\oplus\tilde{H}$, $Q=Q_p\oplus\tilde{Q}$, where $\mathscr{H}_p$ is the closed linear span of the eigenspaces of H. Identify $\mathfrak{A}(\tilde{\mathscr{H}})$ with the C^*-subalgebra of $\mathfrak{A}(\mathscr{H})$ generated by $\{a(f): f\in\tilde{\mathscr{H}}\}$. The restriction of ω to $\mathfrak{A}(\tilde{\mathscr{H}})$ is the gauge-invariant quasi-free state with one-particle density $\tilde{Q}$, and it is spectrally passive for the restriction of τ to $\mathfrak{A}(\tilde{\mathscr{H}})$. This restricted action is the quasi-free action with one-particle Hamiltonian $\tilde{H}$, which has no eigenvectors. The results of de Cannière [4] may therefore be applied to $\tilde{H}$, $\tilde{Q}$. He proved (in a slightly different formulation) that either $\tilde{Q}=\tilde{P}(-\infty, \mu)$ for some $-\infty\leq\mu\leq\infty$, or $\tilde{Q}=(\tilde{I}+e^{\beta\tilde{H}-\mu\tilde{I}})^{-1}$ for some $\beta\geq 0$ and some $-\infty<\mu<\infty$, where $\tilde{P}$ is the spectral measure of $\tilde{H}$, and $\tilde{I}$ is the identity operator on $\tilde{\mathscr{H}}$. He also stated without proof [4, Remark 3. 14] that in fact $\tilde{Q}=\tilde{P}(-\infty, 0)$ or $\tilde{Q}=(\tilde{I}+e^{\beta\tilde{H}})^{-1}$. This last fact is contained in the following lemma, which also removes some superfluous conditions from [4, Lemma 4. 3]. The lemma could be proved in a similar fashion to [4], using the known characterization of the KMS states of quasi-free actions and the inequality

$$\omega(x^*x)\leq e^{\beta\lambda}\omega(xx^*) \qquad (x\in R(-\infty, \lambda)),$$

valid for KMS states ω at inverse temperature β [3]. However, it is no great hardship to give a proof from first principles.

Lemma 4. 2. *Suppose that* $\tilde{\mathscr{H}}\neq(0)$.

1. *If* $\tilde{Q}=\tilde{P}(-\infty, \mu)$ *for some* $-\infty\leq\mu\leq\infty$, *then* $\tilde{Q}=\tilde{P}(-\infty, 0)$.
2. *If* $\tilde{Q}=(\tilde{I}+e^{\beta\tilde{H}-\mu\tilde{I}})^{-1}$ *for some* $\beta\geq 0$ *and* $-\infty<\mu<\infty$, *then* $\mu=0$ *and* $Q=(\tilde{I}+e^{\beta H})^{-1}$.

Proof. 1. For unit vectors f in $\tilde{P}(0, \infty)\tilde{\mathscr{H}}$, g in $\tilde{P}(-\infty, 0)\tilde{\mathscr{H}}$, Lemma 4. 1 with $n=1$ gives

$$\begin{aligned}(f, Qf)&\leq 1-(f, Qf)\\ 1-(g, Qg)&\leq(g, Qg).\end{aligned}$$

The result follows immediately.

2. Let S be the set of all real numbers λ such that $(\lambda-\varepsilon, \lambda)$ and $(\lambda, \lambda+\varepsilon)$ intersect the spectrum of H for all $\varepsilon>0$. Then S is uncountable and dense in itself, and generates a dense subgroup of $\mathbb{R}$.

Let f_0 be a normalised eigenvector of H with eigenvalue λ_0. For any $\varepsilon>0$, there exist integers $0\leq m\leq n$ and distinct λ_i in S $(1\leq i\leq n)$, not equal to λ_0, such that

$$0<\lambda_0+\sum_{i=1}^{m}\lambda_i-\sum_{j=m+1}^{n}\lambda_j<\varepsilon.$$

For $0<\eta<\min\left\{\frac{1}{2}|\lambda_i-\lambda_j| : 0\leq i<j\leq n\right\}$, let f_i be a unit vector in $\tilde{P}(\lambda_i, \lambda_i+\eta)\mathscr{H}$ $(1\leq i\leq m)$ or in $\tilde{P}(\lambda_i-\eta, \lambda_i)\mathscr{H}$ $(m<i\leq n)$. Applying Lemma 4.1,

$$(f_0, Qf_0)\prod_{i=1}^{m}\frac{(f_i, Qf_i)}{1-(f_i, Qf_i)}\leq(1-(f_0, Qf_0))\prod_{j=m+1}^{n}\frac{(f_j, Qf_j)}{1-(f_j, Qf_j)}.$$

But

$$\frac{(f_i, Qf_i)}{1-(f_i, Qf_i)}\geq\exp(-\beta(\lambda_i+\eta)+\mu)\quad(1\leq i\leq m)$$
$$\frac{(f_j, Qf_j)}{1-(f_j, Qf_j)}\leq\exp(-\beta(\lambda_j-\eta)+\mu)\quad(m<j\leq n),$$

so

$$(f_0, Qf_0)\exp\{-\beta\sum_{i=1}^{m}\lambda_i+m(\mu-\beta\eta)\}$$
$$\leq(1-(f_0, Qf_0))\exp\{-\beta\sum_{j=m+1}^{n}\lambda_j+(n-m)(\beta\eta+\mu)\}.$$

Letting $\eta\to 0$,

$$(f_0, Qf_0)\leq(1-(f_0, Qf_0))\exp\{\beta(\sum_{i=1}^{m}\lambda_i-\sum_{j=m+1}^{n}\lambda_j)+(n-2m)\mu\}$$
$$\leq(1-(f_0, Qf_0))\exp\{\beta(\varepsilon-\lambda_0)+(n-2m)\mu\}. \tag{4.2}$$

A similar argument, excluding f_0 and λ_0, shows that

$$\lambda_i\in S\ (1\leq i\leq n),\ 0<\sum_{i=1}^{m}\lambda_i-\sum_{j=m+1}^{n}\lambda_j<\varepsilon\Rightarrow\exp(\beta\varepsilon+(n-2m)\mu)\geq 1.$$

Since S is uncountable, one may find such λ_i with $2m-n$ arbitrary (see [4, Lemma 3.16]), and it follows that $\mu=0$.

Now in (4.2), letting $\varepsilon\to 0$,

$$(f_0, Qf_0)\leq(1+e^{\beta\lambda_0})^{-1}.$$

Similarly choosing $0\leq k\leq l$, distinct μ_i $(1\leq i\leq l)$ in S, not equal to $\mu_0=\lambda_0$, such that

$$0<-\lambda_0+\sum_{i=1}^{k}\mu_i-\sum_{j=k+1}^{l}\mu_j<\varepsilon$$

and unit vectors g_i in $\tilde{P}(\mu_i, \mu_i+\eta)\mathscr{H}(1\leq i\leq k)$, g_j in $\tilde{P}(\mu_j-\eta, \mu_j)\mathscr{H}$ $(k<j\leq l)$ where $0<\eta<\min\left\{\frac{1}{2}|\mu_i-\mu_j| : 0\leq i<j\leq l\right\}$, and applying Lemma 4.1 to $g_1,\dots g_l$, f_0 gives

$$1-(f_0, Qf_0)\leq(f_0, Qf_0)\exp\{\beta(\sum_{i=1}^{m}\mu_i-\sum_{j=k+1}^{l}\mu_j)\}$$

$$\leq (f_0, Qf_0) e^{\beta(\lambda_0+\varepsilon)}$$

and hence $(f_0, Qf_0) \geq (1+e^{\beta\lambda_0})^{-1}$. Thus $Q_p = (I_{\mathscr{H}_p} + e^{\beta H_p})^{-1}$, so $Q = (I + e^{\beta H})^{-1}$.

Remark. If $\tilde{\mathscr{H}} \neq (0)$, the Connes spectrum of τ is $\mathbb{R}$. (I am grateful to A. Kishimoto for confirmation of this.) Thus Lemma 4.2(2) is analogous to Proposition 3.4(2).

Theorem 4.3. *Let H be the one-particle Hamiltonian of a one-parameter group τ of quasi-free automorphisms of $\mathfrak{A}(\mathscr{H})$, P be the spectral measure of H, and ω be a gauge-invariant quasi-free state of $\mathfrak{A}(\mathscr{H})$ which is spectrally passive for τ. The one-particle density Q of ω satisfies one of the following conditions:*

(*i*) $Q = (I + e^{\beta H})^{-1}$ *for some* $\beta \geq 0$,

(*ii*) $Q(I - P(0)) = P(-\infty, 0)$,

(*iii*) *Q and $I - Q$ are injective, and (H, Q) is a passive pair (in particular, H and Q are diagonalisable),*

(*iv*) *There is a decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, $H = H_1 \oplus H_2$, $Q = Q_1 \oplus P_2(-\infty, 0)$ where H_1 is of trace class, $(-\|H_1\|_1, \|H_1\|_1)$ is contained in the resolvent set of H_2, Q_1 and $I_{\mathscr{H}_1} - Q_1$ are invertible and (H_1, Q_1) is a passive pair. (Here, P_2 is the spectral measure of H_2.)*

Proof. By Lemma 4.2 and the preceding remarks, either $Q = (I + e^{\beta H})^{-1}$ in which case (i) is satisfied, or $\tilde{Q} = \tilde{P}(-\infty, 0)$. In the remainder of the proof, assume that $\tilde{Q} = \tilde{P}(-\infty, 0)$. Assume also that (ii) is not satisfied, so that there is a unit vector f_0 in $P(0, \infty)\mathscr{H}$ with $(f_0, Qf_0) > 0$ or a unit vector g_0 in $P(-\infty, 0)$ with $(g_0, Qg_0) < 1$. Let $c = (f_0, Qf_0)^{-1} - 1$ or $c = (1 - (g_0, Qg_0))^{-1} - 1$, respectively.

For real λ, let $\mathscr{H}_\lambda = P(\lambda)\mathscr{H}$, $Q_\lambda = Q|\mathscr{H}_\lambda$. Let $\{f_i: 1 \leq i \leq 2m\}$ be an orthonormal set in $\mathscr{H}_\lambda$ such that $(f_i, Qf_j) = 0$ $(1 \leq i < j \leq 2m)$. Applying Lemma 4.1 to $f_0, f_1, \ldots f_{2m}$ or $f_1, \ldots f_{2m}, g_0$, gives (2.1) where $n = 2m$. By Lemma 2.2(2), $\mathscr{H}_\lambda$ has a complete orthonormal set of eigenvectors for Q_λ. It follows that $\mathscr{H}_p$ has a complete orthonormal set of common eigenvectors for H_p, Q_p, and Lemma 4.1 shows that (H_p, Q_p) is a passive pair. Since (ii) is not satisfied, the pair is non-trivial. By Proposition 3.2, there is a decomposition $\mathscr{H}_p=\mathscr{H}_1\oplus\mathscr{H}_3$, $H_p=H_1\oplus H_3$, $Q_p=Q_1\oplus P_3(-\infty, 0)$, where $(-||H_1||_1, ||H_1||_1)$ is contained in the resolvent set of H_3, Q_1 and $I_{\mathscr{H}_1}-Q_1$ are injective, and (H_1, Q_1) is a passive pair. ($\mathscr{H}_3$ may be (0), in which case $||H_1||_1$ may be infinite.) If $\tilde{\mathscr{H}}=(0)$, the proof is now complete, since either (iii) applies (if $\mathscr{H}_3=(0)$) or (iv) applies with $\mathscr{H}_2=\mathscr{H}_3$.

If $\tilde{\mathscr{H}}\neq(0)$ let $\gamma=\inf\{|\lambda| : \lambda\tilde{I}-\tilde{H}$ is not invertible$\}$. It suffices to show that $||H_1||_1\leq\gamma$, for then $\mathscr{H}_2$ may be taken to be $\mathscr{H}_3\oplus\tilde{\mathscr{H}}$. Suppose, on the contrary, that there is an orthonormal set $\{f_i: 1\leq i\leq n\}$ in $\mathscr{H}_1$, an integer $0\leq m\leq n$, and scalars λ_i, ρ_i such that $0<\rho_i<1$, $H_if_i=\lambda_if_i$, $Qf_i=\rho_if_i$ $(1\leq i\leq n)$, $\sum_{i=1}^{m}\lambda_i-\sum_{j=m+1}^{n}\lambda_j=\gamma'>\gamma$. There is a unit vector $\tilde{f}_0$ in $\tilde{P}(-\gamma', -\gamma)\tilde{\mathscr{H}}$ or a unit vector $\tilde{g}_0$ in $\tilde{P}(\gamma, \gamma')\tilde{\mathscr{H}}$. By assumption $(\tilde{f}_0, Q\tilde{f}_0)=1$ or $(\tilde{g}_0, Q\tilde{g}_0)=0$. Applying Lemma 4.1 to $\tilde{f}_0, f_1, f_2, \ldots f_n$ or to $f_1, f_2, \ldots f_n, \tilde{g}_0$, gives

$$\prod_{i=1}^{m}\rho_i\prod_{j=m+1}^{n}(1-\rho_j)\leq 0.$$

This contradicts the fact that $0<\rho_i<1$. □

§ 5. Quasi-free States of Passive Pairs

This section is devoted to establishing that if Q satisfies any of the conditions (i) – (iv) of Theorem 4.3, then the associated quasi-free state ω is spectrally passive.

If condition (i) is satisfied, then ω is the unique KMS state at inverse temperature β [2, 5.2.23], and is therefore spectrally passive [3]. If condition (ii) is satisfied, then ω is a ground state [2, 5.3.20], so that $\omega(x^*x)=0\leq\omega(xx^*)$ for x in $R(-\infty, 0)$. Thus ω is spectrally passive.

The remaining conditions (iii) and (iv) are both covered by the following result, where $\mathscr{H}_2$ may be (0), in which case the resolvent set of H_2 is $\mathbf{R}$, and $||H_1||_1$ may be infinite (condition (iii) of Theorem 4.3).

Theorem 5.1. *Suppose that* $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, $H=H_1\oplus H_2$, $Q=Q_1\oplus P_2(-\infty, 0)$, *where* P_2 *is the spectral measure of* H_2, (H_1, Q_1) *is a passive pair and* $(-||H_1||_1, ||H_1||_1)$ *is contained in the resolvent set of* H_2. *Then the gauge-invariant quasi-free state* ω *with one-particle density* Q *is spec-*

trally passive with respect to the one-parameter group of quasi-free automorphisms of $\mathfrak{A}(\mathscr{H})$ *with one-particle Hamiltonian* H.

Proof. First, suppose that $\mathscr{H}_1$ is finite-dimensional with an orthonormal basis $\{f_i: 1 \leq i \leq n\}$ of common eigenvectors for H_1, Q_1 with eigenvalues λ_i, ρ_i respectively. Let $\mathscr{G}$ be an orthonormal basis of $\mathscr{H}_2$. There is a standard construction [2, 5.2.5], by which $\mathfrak{A}(\mathscr{H}_2)$ is identified with $\bigotimes_{\mathscr{G}} M_2$ and $\mathfrak{A}(\mathscr{H})$ with $\bigotimes_{\{1,\dots n\}\cup\mathscr{G}} M_2$ and hence with the C^*-algebra of $2^n \times 2^n$ matrices with entries in $\mathfrak{A}(\mathscr{H}_2)$. Any integer $1 \leq j \leq 2^n$ has a binary representation $j = 1 + \sum_{r \in S_j} 2^{r-1}$ for some subset S_j of $\{1, \dots n\}$. The identification is made in such a way that

$$\tau_t(x) = [\exp(it(\lambda(S_j) - \lambda(S_k)))\tau_t^{(2)}(x_{jk})]_{1 \leq j, k \leq 2^n} \tag{5.1}$$

for any $x = [x_{jk}]$ in $\mathfrak{A}(\mathscr{H})$, where $\tau^{(2)}$ is the one-parameter quasi-free action on $\mathfrak{A}(\mathscr{H}_2)$ with one-particle Hamiltonian H_2. If $\mathfrak{A}(\mathscr{H}_2)$ is identified with the C^*-subalgebra of $\mathfrak{A}(\mathscr{H})$ generated by $\{a(f): f \in \mathscr{H}_2\}$, then $\tau^{(2)}$ is the restriction of τ. Furthermore

$$\omega(x) = \sum_{j=1}^{2^n} \theta_j \omega_2(x_{jj})$$

where

$$\theta_j = \prod_{i \in S_j} \rho_i \prod_{i \in S_j'} (1 - \rho_i)$$

$$S_j' = \{1, \dots n\} \backslash S_j$$

and ω_2 is the ground state for τ_2 (the gauge-invariant quasi-free state with one-particle density $P_2(-\infty, 0)$). These facts may be verified by induction on the dimension n of $\mathscr{H}_1$, using the results of [4, p. 142] for $n = 1$.

It follows from (5.1) that

$$x \in R(-\infty, 0) \Leftrightarrow x_{jk} \in R(-\infty, \lambda(S_k) - \lambda(S_j)) \text{ for all } j, k. \tag{5.2}$$

(The spectral spaces for $\tau^{(2)}$ are the parts of the spectral spaces for τ which are contained in $\mathfrak{A}(\mathscr{H}_2)$.) It is therefore sufficient to show that the condition (5.2) implies that

$$\sum_{j,k} \theta_k \omega_2(x_{jk}^* x_{jk}) \leq \sum_{j,k} \theta_j \omega_2(x_{jk} x_{jk}^*).$$

Considering one coordinate x_{jk} at a time, it suffices to show that

$$\theta_k\omega_2(x^*x)\leq\theta_j\omega_2(xx^*) \text{ for all } x \text{ in } R(-\infty, \lambda(S_k)-\lambda(S_j))\cap\mathfrak{A}(\mathscr{H}_2). \tag{5.3}$$

If $\lambda(S_k)\leq\lambda(S_j)$, (5.4) follows immediately from the fact that ω_2 is a ground state. If $\lambda(S_k)>\lambda(S_j)$ then $\theta_k\leq\theta_j$ by (3.1) applied to $S_k\backslash S_j$, $S_j\backslash S_k$. Since $\lambda(S_k)-\lambda(S_j)\leq||H_1||_1$, it suffices to show that

$$\omega_2(x^*x)\leq\omega_2(xx^*) \text{ for all } x \text{ in } R(-\infty, ||H_1||_1)\cap\mathfrak{A}(\mathscr{H}_2). \tag{5.4}$$

Since $(-||H_1||_1, ||H_1||_1)$ is contained in the resolvent set of H_2 and ω_2 is a ground state, it can be shown as in [4, p. 146] that

$$\omega_2(x^*x)=|\omega_2(x)|^2\leq\omega_2(xx^*)$$

as required.

Next, suppose that $\mathscr{H}_1$ is infinite-dimensional with a complete orthonormal set $\mathscr{F}$ of common eigenvectors for H_1, Q_1. For each finite subset F of $\mathscr{F}$, let $\mathscr{H}_F$ be the linear span of $F\cup\mathscr{H}_2$. Then $\mathfrak{A}(\mathscr{H}_F)$ is a τ-invariant C^*-subalgebra of $\mathfrak{A}(\mathscr{H})$, the restriction of τ to $\mathfrak{A}(\mathscr{H}_F)$ is the qausi-free action with one-particle Hamiltonian $H|\mathscr{H}_F$, and the restriction of ω is the gauge-invariant quasi-free state with one-particle density $Q|\mathscr{H}_F$. By the finite-dimensional case above, $\omega(x^*x)\leq\omega(xx^*)$ for all x in $R(-\infty, 0)\cap\mathfrak{A}(\mathscr{H}_F)$. Since $R(-\infty, 0)=[\bigcup_F(R(-\infty, 0)\cap\mathfrak{A}(\mathscr{H}_F))]^-$, it follows that ω is spectrally passive. □

Corollary 5.2. *Let H be the one-particle Hamiltonian of a one-parameter quasi-free action τ on $\mathfrak{A}(\mathscr{H})$, and Q be the one-particle density of a gauge-invariant quasi-free state ω on $\mathfrak{A}(\mathscr{H})$, and suppose that any of the conditions (i), (ii), (iii) and (iv) of Theorem* 4.3 *is satisfied. Then ω is spectrally passive for τ.*

Proof. Cases (i) and (ii) were discussed at the beginning of this section. Case (iii) follows from Theorem 5.1 with $\mathscr{H}_2=(0)$. Case (iv) is also covered by Theorem 5.1. □

§6. The Gauge-invariant CAR Algebra

The C*-algebra of primary concern in [4] was $\mathfrak{A}(\mathscr{H})^{(0)}$, and the corresponding results for $\mathfrak{A}(\mathscr{H})$ were merely stated. Here, detailed arguments have been given for $\mathfrak{A}(\mathscr{H})$, but it is possible to modify them for the gauge-invariant part. The main technical difference is that if ω is merely assumed to be spectrally passive on $\mathfrak{A}(\mathscr{H})^{(0)}$, then (4.1) can only be deduced if $n=2$m. A correspondingly weaker notion of passive pairs is needed.

The pair (H, Q) is *weakly passive* if $\mathscr{H}$ has a complete orthonormal set of common eigenvectors with eigenvalues λ_α, ρ_α, and (3.1) holds for any finite subsets A_1, A_2 of A such that $\lambda(A_1)>\lambda(A_2)$ and $|A_1|=|A_2|$, where $|B|$ denotes the cardinality of B.

The methods of Sections 4 and 5 now give the following modification of Theorem 4.3 and Corollary 5.2.

Theorem 6.1. *Let H be the one-particle Hamiltonian of a one-parameter quasi-free action τ, P be the spectral measure of H, and Q be the one-particle density of a gauge-invariant quasi-free state ω. The restriction of ω to $\mathfrak{A}(\mathscr{H})^{(0)}$ is spectrally passive with respect to the restriction of τ if and only if at least one of following conditions is satisfied*:

(i) $Q=(I+e^{\beta H-\mu I})^{-1}$ *for some* $\beta\geq 0$, $-\infty<\mu<\infty$,

(ii) $Q(I-P(\mu))=P(-\infty, \mu)$ *for some* $-\infty<\mu<\infty$,

(iii) Q *and* $I-Q$ *are injective, and* (H, Q) *is a weakly passive pair*,

(iv) *There exist* $-\infty\leq\mu_1<\mu_2\leq\infty$ *and a decomposition* $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$, $H=H_1\oplus H_2$, $Q=Q_1\oplus P_2(-\infty, \mu_2)$ *such that* Q_1 *and* $I_{\mathscr{H}_1}-Q_1$ *are injective*, (H_1, Q_1) *is a weakly passive pair*, (μ_1, μ_2) *is contained in the resolvent set of* H_2, *and the complete family* $\{\lambda_\alpha: \alpha\in A\}$ *of eigenvalues of* H_1 *satisfies*

$$\mu_1\leq\lambda(A_1)-\lambda(A_2)\leq\mu_2 \tag{6.1}$$

for any finite subsets A_1, A_2 *of* A *with* $|A_1|=|A_2|+1$.

(v) $Q=cI$ *for some* $0\leq c\leq 1$.

Remark. In case (iv), it can be assumed that μ_1 or μ_2 is finite (otherwise, case (iii) applies). Then, if $\mathscr{H}_1$ does not have finite even dimension, (6.1) is equivalent to the condition that

$$||H_1-\mu I_{\mathscr{H}_1}||_1\leq\min\ (\mu_2-\mu,\ \mu-\mu_1)$$

for some $\mu_2\leq\mu\leq\mu_1$. If $\mathscr{H}_1$ has dimension $2m$, and the eigenvalues of H_1 are $\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_{2m}$, then (6.1) is equivalent to the conditions that

$$||H_1-\lambda_m I_{\mathscr{H}_1}||_1\leq\mu_2-\lambda_m \text{ and } ||H_1-\lambda_{m+1}I_{\mathscr{H}_1}||_1\leq\lambda_{m+1}-\mu_1.$$

Finally, with the help of Lemma 2.2(2), the results of Section 3 can be modified to describe weakly passive pairs. Let $\Gamma(H)^{(0)}$ be the closed subgroup of $\boldsymbol{R}$ containing those γ such that there are sequences of mutually disjoint finite subsets A_n, B_n of A such that $|A_n| = |B_n|$ and $\lambda(A_n) - \lambda(B_n) \to \gamma$.

Proposition 6.2. *Let (H, Q) be a weakly passive pair, and suppose that Q and $I-Q$ are injective.*

1. *If $H-\lambda I$ is of trace class for some real λ, but $H \neq \lambda I$, then $\log(b(Q^{-1}-I))$ is of trace class for some $b>0$, and for any $\beta>0$, $Q=(I+e^{\beta(H+T)-\mu I})^{-1}$ for some real μ and some operator T of trace class.*
2. *If $\Gamma(H)^{(0)} = \gamma\boldsymbol{Z}$ for some $\gamma>0$, then $Q=(I+e^{\beta(H+T)-\mu I})^{-1}$ for some $\beta \geq 0$, some real μ, and some operator T with $\|T\|_1 \leq \gamma$.*
3. *If $\Gamma(H)^{(0)} = \boldsymbol{R}$, then $Q=(I+e^{\beta H-\mu I})^{-1}$ for some $\beta \geq 0$ and some real μ.*

Remark. If H is diagonalisable, $\Gamma(H)^{(0)}$ is the Connes spectrum of $\tau|\mathfrak{A}(\mathscr{H})^{(0)}$. If H is not diagonalisable, the Connes spectrum is $\boldsymbol{R}$. (I am grateful to A. Kishimoto for confirmation of this.)

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