Partial *-Algebras of Closed Linear Operators in Hilbert Space

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In a recent manuscript [1] K-D. Kürsten has produced counterexamples to two statements contained in our paper. In the light of those results, we will discuss here at some length the appropriate modifications to the paper, deferring to a further publication [2] a detailed analysis (and generalization) of the counterexamples. In the meantime a corrected version of our statements has been included in another work by one of us [3].

The present addition to our paper results from extensive discusions, both orally and by correspondance, with Dr. Kürsten, Prof. G. Lassner and Dr. F. Mathot. We express our gratitude to all of them.

1. Non-distributivity of Multiplication in $\mathfrak{C}(\mathscr{D})$

The statement (p. 213, 1-2) that $A\mathbin{\hat{+}}B=\overline{A+B}\supset A+B$ was based on the seemingly obvious fact that a subset $\mathscr{D}$ dense both in $D(A)$ and $D(B)$ is necessarily dense in $D(A+B)=D(A)\cap D(B)$ with respect to the projective topology t_+ given e. g. by the norm $\|f\|_+=\|f\|+\|Af\|$ $+\|Bf\|$. But this is incorrect and Kürsten [1] exhibits a (tricky) counterexample. The correct situation is the following.

On $D(A+B)$ we may consider both the projective topology t_+ and the graph topology t_{A+B} of $A+B$, which is coarser since the norm $\|\cdot\|_+$ majorizes the graph norm of $A+B$. Furthermore, $D(A+B)$ is complete for t_+, but need not be complete for t_{A+B}: its completion

is $D(\overline{A+B})$. Thus the identity on $D(A+B)$ extends to a continuous embedding $\iota: D(A+B) \to D(\overline{A+B})$. Taking restrictions to $\mathscr{D}$, denote by $\widetilde{\mathscr{D}}_+$, resp. $D(A\mathbin{\widehat{+}}B)$, the completion of $\mathscr{D}$ with respect to t_+, resp. t_{A+B}. Thus $\widetilde{\mathscr{D}}_+$ is a closed subspace of $D(A+B)[t_+]$, and $D(A\mathbin{\widehat{+}}B)$ is a closed subspace of $D(\overline{A+B})[t_{A+B}]$.

Finally the identity on $\mathscr{D}$ extends to a continuous embedding between the respective completions, $\tilde{\iota}: \widetilde{\mathscr{D}}_+ \to D(A\mathbin{\widehat{+}}B)$; the map $\tilde{\iota}$ is obviously continuous, and it is an embedding (i. e. it is injective) because it is simply the restriction to $\widetilde{\mathscr{D}}_+$ of the continuous injective map $\hat{\iota}$. As a consequence we get the following picture, where $\to$ denotes a continuous embedding:

$$\mathscr{D} \longrightarrow \widetilde{\mathscr{D}}_+ \underset{\tilde{\iota}}{\overset{}{\nearrow}} \begin{matrix} D(A+B) \overset{\hat{\iota}}{\searrow} \\ \\ D(A\mathbin{\widehat{+}}B) \nearrow \end{matrix} D(\overline{A+B})$$

So, in general, $A\mathbin{\widehat{+}}B$ need not be an extension of $A+B$, quite on the contrary. For instance, if $A+B$ is closed, $A\mathbin{\widehat{+}}B \subseteq A+B$. Also if B is relatively bounded with respect to A, $D(A+B) = D(A)$ with equivalent graph norms, so that $A\mathbin{\widehat{+}}B = A+B = \overline{A+B}$.

From this it follows that some sets of multipliers are not vector spaces. It might happen indeed [1] that $C \in R(A) \cap R(B)$ and $C \notin R(A\mathbin{\widehat{+}}B)$ and then $L(C)$ is not a vector space, i. e. the $\bullet$ multiplication is not distributive with respect to the $\widehat{+}$ addition. The consequences for our paper are threefold.

(1)   Proposition 3.1. must be replaced by the following:

Proposition 3.1′.   *Given a dense domain $\mathscr{D}$, let $\mathfrak{C} \equiv \mathfrak{C}(\mathscr{D})$ be the set of all $\mathscr{D}$-minimal operators. Equip $\mathfrak{C}$ with the $\widehat{+}$ addition, the involution $A \leftrightarrow A^+$ and the $\bullet$ multiplication restricted to those pairs (A, B) which verify conditions (M1), (M2). Then:*

   (i)   *$\mathfrak{C}$ is a vector space for $\widehat{+}$*

   (ii)   *$\neq$ is an involution for the $\bullet$ product: $A \in L(B)$ iff $B^+ \in L(A^+)$ and then $(A \bullet B)^+ = B^+ \bullet A$*

   (iii)   *the identity operator I is a unit*

   (iv)   *if $A \in L(C), B \in L(C)$ and $A\mathbin{\widehat{+}}B \in L(C)$, then distributivity holds: $(A\mathbin{\widehat{+}}B) \bullet C = (A \bullet C) \widehat{+} (B \bullet C)$*

(2) The definition of a *partial *-algebra of $\mathscr{D}$-minimal operators*, Definition 3.4, should include a requirement of distributivity:

Definition 3.4'. *To conditions (i), (ii), (iii) of Definition 3.4 add the following*:

(iv) if $A, B, C \in \mathfrak{M}$, then $A \in \mathrm{L}(C)$ and $B \in \mathrm{L}(C)$ imply $A \dotplus B \in \mathrm{L}(C)$.

Notice that, except $\mathfrak{C}$ itself, all examples given in the paper, including Example 3.5, verify this additional condition.

(3) In Proposition 5.2, the two cases $\mathfrak{N} \subset \mathfrak{C}$ and $\mathfrak{N} \subset \mathfrak{C}^*$ must be treated separately.

Proposition 5.2' (a) *Let* $\mathfrak{N} \subset \mathfrak{C}^*$. *Then the statement of Proposition 5.2 is valid.*

(b) *Let* $\mathfrak{N} \subset \mathfrak{C}$. *Then* $\mathscr{D}(\tilde{\mathfrak{N}}) \subset \mathscr{D}(\mathfrak{N})$, *but the topologies defined on* $\mathscr{D}(\tilde{\mathfrak{N}})$ *by* $\mathfrak{N}$ *and* $\tilde{\mathfrak{N}}$ *are equivalent. If, in addition,* $D(A \dotplus B) \supset D(A) \cap D(B)$ *for all* $A, B \in \mathfrak{N}$, *then* $\mathscr{D}(\tilde{\mathfrak{N}}) = \mathscr{D}(\mathfrak{N})$.

Furthermore, the statement before Proposition 5.3 (p. 226, lines -3 and -4) is valid for $\mathfrak{C}^*$ only. However, Proposition 5.3 itself is correct both in $\mathfrak{C}$ and in $\mathfrak{C}^*$.

2. Non-associativity of $\mathfrak{C}^*(\mathscr{D})$

Proposition 4.3 is incorrect: the *multiplication on $\mathfrak{C}^*(\mathscr{D})$ is *not* associative. The gap in the proof occurs on p. 223, line 2. The vector $A^\dagger \phi$ belongs to $D(B^\dagger)$ by assumption, but not to $\mathscr{D}$ in general. So the relevant adjoint is $(B^\dagger)^* = B^{\dagger *}$, instead of $B^{\dagger\dagger}$. However $C\phi$ need not belong to $D(B^{\dagger *})$, so that the last equality on line 2 does not hold. In [1] Kürsten produces an example where $A*B = I$ and $B*C = 0$, and indeed $C\phi \notin D(B^{\dagger *})$. The reader will notice the analogy with the famous example of L. Schwartz [4] for multiplication of distributions, and the similar one indicated by Grossmann for operators on nested Hilbert spaces [5].

This discussion as well as Proposition 3.2 shows that associativity is too strong a requirement. However, there are several indications

[2] [3] that the following less stringent property is sufficient. Let $\mathfrak{A}$ be a partial $*$-algebra or, more generally, a set equipped with a partial multiplication (the addition does not play any rôle here). We say that $\mathfrak{A}$ is *semi-associative* if $y \in R(x)$ implies $y \circ z \in R(x)$ for all $z \in R\mathfrak{A}$ and then $(x \circ y) \circ z = x \circ (y \circ z)$. Then we get the following result which replaces Proposition 4.3 (and improves Proposition 3.2)

Proposition 4.3′. (a) *Let* $\mathscr{D} = \mathscr{D}(\mathfrak{C})$. *Then* $\mathfrak{C}$ *is semi-associative for the · multiplication.*

(b) *Let* $\mathscr{D} = \mathscr{D}(\mathfrak{C}) = \mathscr{D}(\mathfrak{C}^*)$. *Then* $\mathfrak{C}^*$ *is semi-associative for the* $*$ *multiplication.*

Proof. (a) Since $\mathscr{D} = \mathscr{D}(\mathfrak{C})$, $R\mathfrak{C}$ consists of all bounded operators mapping $\mathscr{D}$ into itself. Let $B \in R(A)$ and $C \in R\mathfrak{C}$. Then $B.C \in R(A)$, for we have, $\forall \phi \in \mathscr{D}$:

(i) $(B.C)\phi = BC\phi \in D(A)$

(ii) $A\phi^+ \in D(B^+) \subset D((B.C)^+)$.

The last inclusion follows from the boundedness of C:

$$(B.C)^+ = C^+.B^+ = \overline{C^*(B^* \upharpoonright \mathscr{D})} = [C^*(B^* \upharpoonright \mathscr{D})]^{**}$$
$$= [(B^* \upharpoonright \mathscr{D})^*C]^* \supset C^*B^+,$$

hence

$$D((B.C)^+) \supset D(C^*B^+) = D(B^+).$$

Finally the relation $(A.B).C = A.(B.C)$ is obtained as in the proof of Proposition 3.2.

(b) The assumption $\mathscr{D} = \mathscr{D}(\mathfrak{C}^*)$ implies that $C \in R^*\mathfrak{C}^*$ is bounded and maps $\mathscr{D}$ into itself. Given $A, B \in \mathfrak{C}^*$ such that $B \in R^*(A)$ and $C \in R^*\mathfrak{C}^*$, we obtain $B*C \in R^*(A)$ by an argument similar to that in (a). Here $D((B*C)^\dagger) \supset D(B^\dagger)$ results from the following relations, if we note that $C\mathscr{D} \subset \mathscr{D} \subset D(B^{++})$:

$$(B*C)^\dagger = C^\dagger*B^\dagger = [B(C \upharpoonright \mathscr{D})]^* = [B^{++}(C \upharpoonright \mathscr{D})]^*$$
$$\supset C^*B^{+++} = C^*B^\dagger.$$

Finally the relation

$$(A*B)*C = A*(B*C)$$

follows from the argument of Proposition 4.3, which is now valid since $C\phi \in \mathscr{D}$. $\square$

Obviously the non-associativity of $\mathfrak{C}$ and $\mathfrak{C}^*$ invalidates some statements about commutants: in general they are not partial *-algebras. However the completeness results are not affected, at least for $\mathfrak{C}^*$. More precisely, Propositions 6.1 and 6.3 should be replaced by the following ones.

Proposition 6.1′.   *If* $\mathfrak{N} \subset \mathfrak{C}^*$, *then* $\mathfrak{N}'_*$ *is a vector subspace of* $\mathfrak{C}^*$. *If* $\mathfrak{N} = \mathfrak{N}^\dagger$, *then* $(\mathfrak{N}'_*)^\dagger = \mathfrak{N}'_*$.

Proposition 6.3′.   *Let* $\mathfrak{N} = \mathfrak{N}^\dagger \subset \mathfrak{C}^*$. *Then the bicommutant* $\mathfrak{N}''_{**}$ *is a vector subspace of* $\mathfrak{C}^*$, *stable under the involution* $\dagger$ *and complete in the topology* $\tau_*(\mathfrak{N}'_*)$.

Finally Propositions 6.4 and 6.5 should be deleted altogether, since commutants in $\mathfrak{C}$ need not be vector spaces, as we have seen above and, on the other hand, the proof of Proposition 6.5 shows only that $\mathfrak{N}'$ is closed in $M\mathfrak{N}$, *not* its completeness (see [2]).

References

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[4] Schwartz, L., *Théorie des distributions*, t. I, Hermann, Paris 1957.

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