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# Derivations in Covariant Representations of C*-algebras

By

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### Abstract

Let $\alpha$ be an action of a locally compact abelian or compact group $G$ on a C*-algebra $\mathscr{A}$ and $\pi$ a representation of $\mathscr{A}$ which induces an action $\tilde{\alpha}$ of $\overline{\pi(\mathscr{A})}$ from $\alpha$. If $\delta$ is a locally bounded (in a sense) *-derivation in $\mathscr{A}$ defined on $\mathscr{A}_F$, then there exists a locally $\sigma$-weakly continuous *-derivation $\tilde{\delta}$ in $\overline{\pi(\mathscr{A})}$ defined on $\overline{\pi(\mathscr{A})}_F$ such that $\tilde{\delta}\circ\pi\supset\pi\circ\delta$.

Let $\alpha$ be an action of a locally compact abelian group $G$ on a C*-algebra. Let $\delta$ be a *-derivation in $\mathscr{A}$ which is defined on $\mathscr{A}_F$ and bounded on each spectral subspace $\mathscr{A}^{\alpha}(K)$ of $\alpha$ corresponding to a compact subset $K$ of the dual group $\hat{G}$ of $G$, where $\mathscr{A}_F$ denotes the union of all spectral subspaces of $\alpha$ corresponding to compact subsets of $\hat{G}$.

Then the second adjoint $\alpha^{**}$ of $\alpha$ does not necessarily continuously act on the second dual $\mathscr{A}^{**}$ of $\mathscr{A}$, and $\delta$ is not necessarily $\sigma$-weakly closable in $\mathscr{A}^{**}$. Nevertheless, denoting by $\mathscr{B}$ the norm closure of the union of all $\sigma$-weak closures $\overline{\mathscr{A}^{\alpha}(K)}$ of spectral subspaces $\mathscr{A}^{\alpha}(K)$ with $K$ compact, $\alpha^{**}$ strongly continuously acts on $\mathscr{B}$ and $\delta$ can be extended to a *-derivation $\delta_{\mathscr{B}}$ in $\mathscr{A}^{**}$ which is defined on $\mathscr{B}_F$ and $\sigma$-weakly continuous on $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)$ for any compact subset $K$ of $\hat{G}$. If, in addition, a representation $\pi$ induces an action $\tilde{\alpha}$ on the weak closure $\mathscr{M}$ of $\pi(\mathscr{A})$ from $\alpha$, then, from $\delta_{\mathscr{B}}$ the canonical extension $\tilde{\pi}$ of $\pi$ onto $\mathscr{A}^{**}$ induces a *-derivation $\tilde{\delta}$ in $\mathscr{M}$ defined on $\mathscr{M}_F$, namely $\bigcup_{K:compact}\mathscr{M}^{\tilde{\alpha}}(K)$, such that $\tilde{\delta}\circ\tilde{\pi}=\tilde{\pi}\circ\delta_{\mathscr{B}}\supset\pi\circ\delta$ and $\tilde{\delta}$ is $\sigma$-weakly continuous on $\mathscr{M}^{\tilde{\alpha}}(K)$ for any compact subset $K$ of $\hat{G}$. This remains valid even if $G$ is compact.

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In [3] Kishimoto showed the above when $G$ is locally compact abelian and $\pi$ is irreducible and $\alpha$-covariant. Moreover, he proved there that $\delta$ is a pregenerator if there exists a faithful family of $\alpha$-covariant irreducible representations of $\mathscr{A}$. We generalize this and [1, Theorem 3. 1] also.

If, in particular, $G=\boldsymbol{R}$ and $\delta$ is defined on the domain of the generator $\delta_0$ of $\alpha$, then $\delta$ is relatively bounded with respect to $\delta_0$ [4], and hence is bounded on $\mathscr{A}^{\alpha}(K)$ for any compact subset $K$ of $\hat{G}$. The relative bound of $\tilde{\tilde{\delta}}$ with respect to the generator of $\tilde{\alpha}$ does not exceed that of $\delta$ with respect to $\delta_0$, in virtue of the Kaplansky's density theorem and the functional calculus for $*$-derivations [2]. However we obtain a more precise estimate concerning the relative boundedness of $\tilde{\tilde{\delta}}$.

Throughout the whole, let a group $G$ be either locally compact abelian or compact, $\alpha$ an action of $G$ on a C*-algebra $\mathscr{A}$, and $\pi$ a representation of $\mathscr{A}$ which induces the action $\tilde{\alpha}$ of $G$ on the weak closure $\mathscr{M}$ of $\pi(\mathscr{A})$ such that $\tilde{\alpha}_t\circ\pi=\pi\circ\alpha_t$. Let $\delta$ be a *-derivation in $\mathscr{A}$ which is defined on $\mathscr{A}_F$ and bounded on $\mathscr{A}^{\alpha}(K)$ (resp., $\mathscr{A}^{\alpha}(\gamma)$) for any compact subset $K\subset\hat{G}(\gamma\in\hat{G})$, when $G$ is abelian (compact). When $G$ is compact, let $\mathscr{A}^{\alpha}(\gamma)$ and $\mathscr{M}^{\tilde{\alpha}}(\gamma)$ denote the spectral subspaces of $\alpha$ and $\tilde{\alpha}$ corresponding to $\gamma\in\hat{G}$, and $\mathscr{A}_F$ and $\mathscr{M}_F$ the unions of these, respectively.

**Theorem 1.** *If $G$ is abelian, then there exists a unique *-derivation $\tilde{\tilde{\delta}}$ in $\mathscr{M}$ such that $\tilde{\tilde{\delta}}$ is defined on $\mathscr{M}_F$, $\tilde{\tilde{\delta}}\circ\pi\supset\pi\circ\delta$ and $\tilde{\tilde{\delta}}$ is $\sigma$-weakly continuous on $\mathscr{M}^{\tilde{\alpha}}(K)$ for any compact subset $K$ of $\hat{G}$. Furthermore we have*

$$\|\tilde{\tilde{\delta}}\,|\,\mathscr{M}^{\tilde{\alpha}}(K)\|\leqq\inf_{V}\|\delta\,|\,\mathscr{A}^{\alpha}(K+V)\|,$$

*where $V$ runs over all compact neighbourhoods of 0 in $\hat{G}$.*

*Even if $G$ is compact, the above consequences hold, provided that $\mathscr{A}^{\alpha}(K)$ and $\mathscr{M}^{\tilde{\alpha}}(K)$ should be replaced with $\mathscr{A}^{\alpha}(\gamma)$ and $\mathscr{M}^{\tilde{\alpha}}(\gamma)$ corresponding $\gamma\in\hat{G}$ respectively, and the inequality becomes as follows:*

$$\|\tilde{\tilde{\delta}}\,|\,\mathscr{M}^{\tilde{\alpha}}(\gamma)\|\leqq\|\delta\,|\,\mathscr{A}^{\alpha}(\gamma)\|.$$

*Proof.* First assume that $G$ is abelian. Let $\mathscr{B}$ denote the norm closure in $\mathscr{A}^{**}$ of the union of all $\sigma(\mathscr{A}^{**},\mathscr{A}^{*})$-closures $\overline{\mathscr{A}^{\alpha}(K)}$ of

$\mathscr{A}^\alpha(K)$ with $K$ compact, so that $\mathscr{B}$ is an $\alpha^{**}$-invariant C*-subalgebra of $\mathscr{A}^{**}$. The mapping $G\ni t\mapsto\alpha_t|\mathscr{A}^\alpha(K)$ is uniformly continuous, and so does $t\mapsto(\alpha_t|\mathscr{A}^\alpha(K))^{**}$. Identifying $\mathscr{A}^\alpha(K)^{**}$ with $\overline{\mathscr{A}^\alpha(K)}$, $t\mapsto\alpha_t{}^{**}|\overline{\mathscr{A}^\alpha(K)}$ is also uniformly continuous, and hence the restriction $\alpha^{**}|\mathscr{B}$ is a strongly continuous action of $G$ on $\mathscr{B}$.

For any compact subset $K$ of $\hat{G}$, we shall show that

$$\overline{\mathscr{A}^\alpha(K)}\subset\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)=\bigcap_V\overline{\mathscr{A}^\alpha(K+V)},$$

where $V$ runs over all compact neighbourhoods of 0 in $\hat{G}$; then note that $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)$ is $\sigma(\mathscr{A}^{**},\mathscr{A}^*)$-closed. For any $f\in L^1(G)$ we have

$$\begin{aligned}(\alpha^{**}|\mathscr{B})(f)|\overline{\mathscr{A}^\alpha(K)}&=(\alpha^{**}|\overline{\mathscr{A}^\alpha(K)})(f)=(\alpha|\mathscr{A}^\alpha(K))^{**}(f)\\&=(\alpha(f)|\mathscr{A}^\alpha(K))^{**}=\alpha(f)^{**}|\overline{\mathscr{A}^\alpha(K)},\end{aligned}$$

and hence $(\alpha^{**}|\mathscr{B})(f)=\alpha(f)^{**}|\mathscr{B}$, which implies the $\sigma(\mathscr{A}^{**},\mathscr{A}^*)$-continuity of $(\alpha^{**}|\mathscr{B})(f)$. Therefore we have $\overline{\mathscr{A}^\alpha(K)}\subset\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)$. Moreover, if $f\in L^1(G)$, Supp $\hat{f}\subset K+V$ for a compact neighbourhood $V$ of 0 and $\hat{f}(\gamma)=1$ on some neighbourhood of $K$, then it follows from the above that

$$\begin{aligned}\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)&\subset(\alpha^{**}|\mathscr{B})(f)(\mathscr{B})=\alpha(f)^{**}(\mathscr{B})\subset\overline{\alpha(f)(\mathscr{A})}\\&\subset\overline{\mathscr{A}^\alpha(K+V)}\subset\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K+V),\end{aligned}$$

so that $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)=\bigcap_V\overline{\mathscr{A}^\alpha(K+V)}$.

Similarly we have $\mathscr{M}^{\bar{\alpha}}(K)=\bigcap_V\overline{\pi(\mathscr{A}^\alpha(K+V))}$.

Let $\tilde{\pi}$ be the canonical representation of $\mathscr{A}^{**}$ onto extending $\pi$, so that $\bar{\alpha}\circ\tilde{\pi}=\tilde{\pi}\circ\alpha^{**}$. Let $(e_\iota)$ be an approximate identity of ker $\tilde{\pi}\cap\mathscr{B}$; we may assume that $e_\iota\in\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K')$ for some compact subset $K'$, because, if $f\geqq0,\int f\,\mathrm{dt}=1$ and Supp $\hat{f}\subset K'$ then $((\alpha^{**}|\mathscr{B})(f)(e_\iota))$ is an approximate identity of ker $\tilde{\pi}\cap\mathscr{B}$. Since $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K')$ is $\sigma(\mathscr{A}^{**},\mathscr{A}^*)$-closed, $(e_\iota)$ $\sigma$-weakly converges to the identity $e$ of ker $\tilde{\pi}\cap\mathscr{B}$, and hence $e$ belongs to the fixed point algebra $\mathscr{B}^{\alpha^{**}|\mathscr{B}}$ and is a central projection of $\mathscr{A}^{**}$. Therefore, for any compact subset $K$ of $\hat{G}$, $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)(1-e)$ and $\tilde{\pi}(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K))$ are isometrically isomorphic and $\sigma$-weakly homeomorphic under $\tilde{\pi}$, because these are $\sigma$-weakly closed. Since $e\in\mathscr{B}^{\alpha^{**}|\mathscr{B}}$,

$$\begin{aligned}\tilde{\pi}(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K))&=\tilde{\pi}(\bigcap_V\overline{\mathscr{A}^\alpha(K+V)})\subset\bigcap_V\overline{\pi(\mathscr{A}^\alpha(K+V))}\\&\subset\bigcap_V\tilde{\pi}(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K+V))=\bigcap_V\tilde{\pi}(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K+V)(1-e))\\&=\tilde{\pi}(\bigcap_V(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K+V)(1-e)))\end{aligned}$$

$$\subset\tilde{\pi}(\bigcap_{V}\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K+V))=\tilde{\pi}(\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)),$$

where $V$ runs over all compact neighbourhoods of 0. Thus $\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)(1-e)$ is isometrically isomorphic and $\sigma$-weakly homeomorphic to $\mathscr{M}^{\alpha}(K)$ under $\tilde{\pi}$.

Now, since $\delta|\mathscr{A}^{\alpha}(K)$ is bounded for a compact subset $K$ of $\hat{G}$, its second adjoint is a bounded linear mapping of $\overline{\mathscr{A}^{\alpha}(K)}$ into $\mathscr{A}^{**}$ with the same norm. Hence there exists a $*$-derivation $\delta_{\mathscr{B}}$, namely $\bigcup_{K:compact}(\delta|\mathscr{A}^{\alpha}(K))^{**}$, in $\mathscr{A}^{**}$ extending $\delta$ and defined on $\mathscr{B}_F$. Put $\tilde{\delta}=\tilde{\pi}\circ\delta_{\mathscr{B}}\circ(\tilde{\pi}|\mathscr{B}_F(1-e))^{-1}$. Then $\tilde{\delta}$ is a $*$-derivation in $\mathscr{M}$ which is defined on $\mathscr{M}_F$ and $\sigma$-weakly continuous on $\mathscr{M}^{\alpha}(K)$ for any compact subset $K$ of $\hat{G}$. Moreover we have

$$\begin{aligned}||\tilde{\delta}|\mathscr{M}^{\alpha}(K)||&=||\delta_{\mathscr{B}}|\mathscr{B}^{\alpha^{**}|\mathscr{B}}(K)(1-e)||\leqq\inf_{V}||\delta_{\mathscr{B}}|\overline{\mathscr{A}^{\alpha}(K+V)}||\\&=\inf_{V}||\delta|\mathscr{A}^{\alpha}(K+V)||,\end{aligned}$$

where $V$ runs over all compact neighbourhoods of 0 in $\hat{G}$. Since $e$ is a central projection, $\delta_{\mathscr{B}}(e)=0$ and $\delta_{\mathscr{B}}(x(1-e))=\delta_{\mathscr{B}}(x)(1-e)$ for any $x\in\mathscr{B}_F$. Therefore we have, for any $x\in\mathscr{B}_F$,

$$\begin{aligned}\tilde{\delta}\circ\tilde{\pi}(x)&=\tilde{\pi}\circ\delta_{\mathscr{B}}\circ(\tilde{\pi}|\mathscr{B}_F(1-e))^{-1}\circ\tilde{\pi}(x)=\tilde{\pi}(\delta_{\mathscr{B}}(x(1-e)))\\&=\tilde{\pi}(\delta_{\mathscr{B}}(x)(1-e))=\tilde{\pi}\circ\delta_{\mathscr{B}}(x).\end{aligned}$$

When $G$ is compact, put $P_{\gamma}=\int\dim\gamma\ \mathrm{Tr}\ \gamma(t)^{-1}\,\alpha_t dt$ for $\gamma\in\hat{G}$. Then $P_{\gamma}$ is a projection onto $A^{\alpha}(\gamma)$. By using $P_{\gamma}$ instead of $\alpha(f)$, we obtain the consequences similarly. Thus we complete the proof of the theorem.

*Remark 2.* Let $E$ denote the set of $\phi\in\mathscr{A}^*$ such that $t\mapsto\alpha_t^*\phi$ is continuous in norm. Then the polar $E^{\circ}$ of $E$ in $\mathscr{A}^{**}$ is a $\sigma$-weakly closed $\alpha^{**}$-invariant ideal of $\mathscr{A}^{**}$, and hence there is a $\sigma$-weakly continuous action on the von Neumann algebra $\mathscr{A}^{**}/E^{\circ}$ induced from $\alpha^{**}$, and $\mathscr{A}$ may be imbedded in $\mathscr{A}^{**}/E^{\circ}$. By Theorem 1 we obtain a $*$-derivation in $\mathscr{A}^{**}/E^{\circ}$ extending $\delta$, as in Theorem 1. However, directly it can be obtained; clearly existence of such a $*$-derivation in $\mathscr{A}^{**}/E^{\circ}$ implies Theorem 1. Indeed, for any compact subset $K$ of $\hat{G}$, $\delta|\mathscr{A}^{\alpha}(K)$ is $\sigma(\mathscr{A},E)$-continuous and $E/\mathscr{A}^{\alpha}(K)^{\circ}\cap E$ is isometrically isomorphic to $\mathscr{A}^*/\mathscr{A}^{\alpha}(K)^{\circ}$, because for any $\varepsilon>0$ there is an element $f\in L^1(G)$ such that $\hat{f}(\gamma)=1$ on some neighbourhood of

$K$, Supp $\hat{f}$ is compact and $\|f\|_1 \leqq 1+\varepsilon$. Consequently there exists a $\sigma(\mathscr{A}^{**}/E^\circ, E)$-continuous extension of $\delta|\mathscr{A}^\alpha(K)$ onto the $\sigma$-weak closure of $\mathscr{A}^\alpha(K)$ in $\mathscr{A}^{**}/E^\circ$ with the same norm as it.

The following corollary is an immediate consequence of a series of lemmas in [3] and Theorem 1 because $u_t^\iota$ and $u_s^\iota$ as below commute, and generalizes [3, Theorem] and [1, Theorem 3. 1].

**Corollary 3.** *Suppose that $G$ is abelian. Suppose that there exist a faithful family $(\pi_\iota)$ of representations of $\mathscr{A}$ and a family $(\alpha^\iota)$ such that $\alpha^\iota$ is an action of $G$ on $\overline{\pi_\iota(\mathscr{A})}$, $\alpha_t^\iota \circ \pi_\iota = \pi_\iota \circ \alpha_t$ and each $\alpha_t^\iota$ is implemented by a unitary $u_t^\iota$ fixed by $\alpha^\iota$.*

*Then $\delta$ is closable and its closure is a generator. Furthermore, for any finite measure $\mu$ on $G$ with $\hat{\mu}(0)=0$, the *-derivation $\delta_\mu$ on $\mathscr{A}_F$, defined by*

$$\delta_\mu = \int \alpha_t \circ \delta \circ \alpha_{-t} \mathrm{d}\mu(t),$$

*is bounded and $\|\delta_\mu\| \leqq \inf_V \|\delta|\mathscr{A}^\alpha(K+V)\| \|\mu\|$, where $V$ runs over all compact neighbourhoods of 0 in $\hat{G}$.*

**Proposition 4.** *Suppose $G=\mathbf{R}$ and let $\delta_0$ and $\tilde{\delta}_0$ be the generators of $\alpha$ and $\tilde{\alpha}$ respectively.*

*Suppose that $\|\delta(x)\| \leqq a\|x\| + b\|\delta_0(x)\|$ on $\mathscr{A}_F$ for real numbers $a$, $b \geqq 0$.*

*Then there exists a unique *-derivation $\tilde{\delta}$ in $\mathscr{M}$ defined on $\mathrm{D}(\tilde{\delta}_0)$ such that the mapping $(x, \tilde{\delta}_0(x)) \mapsto \tilde{\delta}(x)$ $(x \in \mathrm{D}(\tilde{\delta}_0))$ is $\sigma$-weakly continuous and $\|\tilde{\delta}(x)\| \leqq a\|x\| + b\|\tilde{\delta}_0(x)\|$ on $\mathrm{D}(\tilde{\delta}_0)$.*

*Proof.* Let $E$ be as in Remark 2 and $I$ the polar of $E$ in $\mathscr{A}^{**}$; then the canonical extension $\tilde{\pi}$ of $\pi$ to $\mathscr{A}^{**}$ is contained in the canonical homomorphism of $\mathscr{A}^{**}$ onto $\mathscr{A}^{**}/I$, and hence we may assume without loss of generality that $\tilde{\pi}$ is the canonical homomorphism of $\mathscr{A}^{**}$ onto $\mathscr{A}^{**}/I = \mathscr{M}$. Since $\tilde{\pi}|\mathscr{B}$ is isometric, where $\mathscr{B}$ is the C*-subalgebra of $\mathscr{A}^{**}$ in the proof of Theorem 1, we may regard as $\mathscr{B} \subset \mathscr{M}$. Let $p$ denote the central projection such that $I = \mathscr{A}^{**}p$.

Identify $\delta_0$ with the subalgebra $\left\{\begin{pmatrix} x & \delta_0(x) \\ 0 & x \end{pmatrix} \middle| x \in \mathrm{D}(\delta_0)\right\}$ of $\mathscr{A} \otimes M_2$, where $\mathrm{M}_2$ denotes the $2\times 2$ matrix algebra, and equip it with the norm defined by $\|x\| = a\|x\| + b\|\delta_0(x)\|$. Then the second dual $\delta_0^{**}$ of

$\tilde{\delta}_0$ can be identified with the $\sigma$-weak closure of $\delta_0$ in $\mathscr{A}^{**}\otimes M_2$, and so is an algebra.

First we shall show that

$$\left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix} \mid x\in\mathscr{A}^{**}\right\}\cap\delta_0^{**}=\left\{\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix} \mid x\in I\right\}$$

and

$$\left\{\begin{pmatrix}x & y\\ 0 & x\end{pmatrix} \mid x\in I,\ y\in I\right\}\cap\delta_0^{**}=\left\{\begin{pmatrix}0 & y\\ 0 & 0\end{pmatrix} \mid y\in I\right\}.$$

If $\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\in\delta_0^{**}$, then $(1-\delta_0)^{-1**}x=0$, and hence $x\in I$, because $\tilde{\pi}\circ(1-\delta_0)^{-1**}=(1-\tilde{\delta}_0)^{-1}\circ\tilde{\pi}$. Since $(1-\delta_0)^{-1*}\phi=\int_0^\infty e^{-t}\alpha_t^*\phi\ dt\in E$ for all $\phi\in A^*$, we have for any $x\in I$ and $\phi\in\mathscr{A}^*$

$$\langle(1-\delta_0)^{-1**}x,\ \phi\rangle=\langle x,\ (1-\delta_0)^{-1*}\phi\rangle=0,$$

and hence $(1-\delta_0)^{-1**}x=0$ and $\begin{pmatrix}0 & x\\ 0 & 0\end{pmatrix}\in\delta_0^{**}$. We have thus $\begin{pmatrix}0 & \mathscr{A}^{**}\\ 0 & 0\end{pmatrix}\cap\delta_0^{**}=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$. If $x,\ y\in I$ and $\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}\in\delta_0^{**}$, then $x=(1-\delta_0)^{-1**}(x-y)$. Since $x-y\in I$, we have $x=0$. Thus we have $\begin{pmatrix}I & I\\ 0 & I\end{pmatrix}\cap\delta_0^{**}=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$.

Second we shall show that the $\sigma$-weakly closed subalgebra $\left\{\begin{pmatrix}x & y\\ 0 & x\end{pmatrix}\in\delta_0^{**} \mid yp=0\right\}$ of $\mathscr{A}^{**}\otimes M_2$, denoted by $\Delta$, is $\sigma$-weakly homeomorphic to $\tilde{\delta}_0$ as a $\sigma$-weakly closed subalgebra of $\mathscr{M}\otimes M_2$ under $\tilde{\pi}\otimes id$. By $\begin{pmatrix}I & I\\ 0 & I\end{pmatrix}\cap\delta_0^{**}=\begin{pmatrix}0 & I\\ 0 & 0\end{pmatrix}$, $\tilde{\pi}\otimes id\,|\,\Delta$ is injective. For $x\in D(\tilde{\delta}_0)$ there is a bounded filter $\mathscr{F}$ on $\delta_0$ $\sigma$-weakly converging to $\begin{pmatrix}x & \tilde{\delta}_0(x)\\ 0 & x\end{pmatrix}$, because of the boundedness of $(1-\tilde{\delta}_0)^{-1}$ and the Kaplansky's density theorem. Since $\begin{pmatrix}y & \delta_0(y)(1-p)\\ 0 & y\end{pmatrix}\in\Delta$ for $y\in D(\delta_0)$, $(\tilde{\pi}\otimes id\,|\,\Delta)^{-1}\mathscr{F}$ is a bounded filter base on $\Delta$, and hence it has a cluster point $z\in\Delta$. Since $\tilde{\pi}\otimes id$ is $\sigma$-weakly continuous, $(\tilde{\pi}\otimes id)z$ is a cluster point of $\mathscr{F}$ and so $\begin{pmatrix}x & \tilde{\delta}_0(x)\\ 0 & x\end{pmatrix}=(\tilde{\pi}\otimes id)z$. Thus $(\tilde{\pi}\otimes id)\Delta=\tilde{\delta}_0$. Since $\Delta$ and $\tilde{\delta}_0$ have preduals, $\tilde{\pi}\otimes id\,|\,\Delta$ is a homeomorphism.

Now, consider $\delta$ as a linear mapping from $\delta_0$ into $\mathscr{A}$ with the norm smaller than 1; then its second adjoint $\delta^{**}$ is a $\sigma$-weakly continuous linear mapping from $\delta_0^{**}$ into $A^{**}$ with the norm smaller than 1 such that

$$\delta^{**}\left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\begin{pmatrix} z & w \\ 0 & z \end{pmatrix}\right\} = \delta^{**}\left\{\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}\right\} z + x\delta^{**}\left\{\begin{pmatrix} z & w \\ 0 & z \end{pmatrix}\right\}$$

if $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$, $\begin{pmatrix} z & w \\ 0 & z \end{pmatrix} \in \delta_0^{**}$. Therefore we have for $x \in \mathrm{D}(\delta_0)$ and $y \in I$

$$\delta^{**}\left\{\begin{pmatrix} 0 & xy \\ 0 & 0 \end{pmatrix}\right\} = \delta^{**}\left\{\begin{pmatrix} x & \delta_0(x) \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right\} = x\delta^{**}\left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right\}.$$

Tending $x$ to $p$, $\delta^{**}\left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right\} = p\delta^{**}\left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right\} \in I$ for all $y \in I$.

Define $\tilde{\delta}$ by

$$\tilde{\delta}(x) = \tilde{\pi} \circ \delta^{**} \circ (\tilde{\pi} \otimes id \,|\, \Delta)^{-1}\left\{\begin{pmatrix} x & \tilde{\delta}_0(x) \\ 0 & x \end{pmatrix}\right\}, \quad x \in \mathrm{D}(\tilde{\delta}_0);$$

then $\tilde{\delta}$ is a $*$-derivation in $\mathscr{M}$ and the mapping $(x, \tilde{\delta}_0(x)) \mapsto \tilde{\delta}(x)$ is $\sigma$-weakly continuous. Moreover

$$\begin{aligned} \tilde{\delta}(x) &= \tilde{\pi}\left\{\delta^{**}\left\{\begin{pmatrix} x & \delta_0(x)\,(1-p) \\ 0 & x \end{pmatrix}\right\}\right\} \\ &= \tilde{\pi}\left\{\delta^{**}\left\{\begin{pmatrix} x & \delta_0(x) \\ 0 & x \end{pmatrix}\right\}\right\} + \tilde{\pi}\left\{\delta^{**}\left\{\begin{pmatrix} 0 & \delta_0(x)p \\ 0 & 0 \end{pmatrix}\right\}\right\} \\ &= \tilde{\pi}(\delta_0(x)) = \delta_0(x), \quad x \in \mathrm{D}(\delta_0); \end{aligned}$$

and

$$\begin{aligned} \|\tilde{\delta}(x)\| &\leqq \|\pi\|\|\delta^{**}\|\|(\tilde{\pi} \otimes id \,|\, \Delta)^{-1}\left\{\begin{pmatrix} x & \tilde{\delta}_0(x) \\ 0 & x \end{pmatrix}\right\}\| \\ &\leqq \left\|\begin{pmatrix} x & \tilde{\delta}_0(x)\,(1-p) \\ 0 & x \end{pmatrix}\right\| \leqq a\|x\| + b\|\tilde{\delta}_0(x)\|, \quad x \in \mathscr{B}_F = \mathscr{M}_F. \end{aligned}$$

Let $(f_\iota)$ be an approximate identity of $\mathrm{L}^1(\mathbb{R})$ with Supp $f_\iota$ compact. We have then for any $x \in \mathrm{D}(\tilde{\delta}_0)$

$$\begin{aligned} \|\tilde{\delta}(\tilde{\alpha}(f_\iota)(x))\| &\leqq a\|\tilde{\alpha}(f_\iota)(x)\| + b\|\tilde{\delta}_0(\tilde{\alpha}(f_\iota)(x))\| \\ &= a\|\tilde{\alpha}(f_\iota)(x)\| + b\|\tilde{\alpha}(f_\iota)(\tilde{\delta}_0(x))\| \\ &\leqq a\|x\| + b\|\tilde{\delta}_0(x)\|. \end{aligned}$$

Since $x = \lim_\iota \tilde{\alpha}(f_\iota)(x)$ and $\tilde{\delta}_0(x) = \lim_\iota \tilde{\delta}_0(\tilde{\alpha}(f_\iota)(x))$, $\tilde{\delta}(x) = \lim_\iota \tilde{\delta}(\tilde{\alpha}(f_\iota)(x))$ and hence $\|\tilde{\delta}(x)\| \leqq a\|x\| + b\|\tilde{\delta}_0(x)\|$. Thus $\tilde{\delta}$ is as desired.

We do not know whether $\delta$ and $\tilde{\delta}$ in Theorem 1 are norm-closable and $\sigma$-weakly closable, respectively. However we obtain the following:

**Lemma 5.** *Suppose that G is abelian. Let $\tilde{\delta}$ be as in Theorem 1.*

*For a finite measure* $\mu$ *on* $G$ *with* Supp $\hat{\mu}$ *compact, we put* $\delta_\mu=\int\alpha_t\circ\delta\circ\alpha_{-t}\mathrm{d}\mu(t)$ *and* $\tilde{\tilde{\delta}}_\mu=\int\alpha_t\circ\tilde{\tilde{\delta}}\circ\alpha_{-t}\mathrm{d}\mu(t)$.

*Then* $\delta_\mu$ *is norm-closable and* $\tilde{\tilde{\delta}}_\mu$ *is* $\sigma$*-weakly closable.*

*Proof.* It suffices to show that $\delta_\mu$ is $\sigma$-weakly closable in $\mathscr{A}^{**}/E^\circ$, or equivalently that the domain of the adjoint of $\delta_\mu$ in $E$ is dense in norm, where $E$ is the norm-closed subspace of $\mathscr{A}^*$ in Remark 2.

For any $f\in \mathrm{L}^1(G)$ with Supp $\hat{f}$ compact, we have

$$\alpha(f)\circ\delta_\mu=\int\mathrm{d}s\alpha_s\circ\delta\circ\int\mathrm{d}\mu(t)f(s-t)\alpha_{-t},$$
$$\|\alpha(f)\circ\delta_\mu\|\leqq\|\mu\|\|f\|_1\|\delta|\mathscr{A}^\alpha(\mathrm{Supp}\ \hat{f}-\mathrm{Supp}\ \hat{\mu})\|,$$

and

$$\alpha(f)\circ\delta_\mu\circ\alpha_t=\alpha(f(\cdot-t))\circ\delta_{\mu(\cdot+t)}.$$

Consequently the domain of the adjoint of $\delta_\mu$ in $E$ contains $\phi\circ\alpha(f)$ for any $\phi\in E$ and $f\in\mathrm{L}^1(G)$ with Supp $\hat{f}$ compact, and so is dense in $E$.

**Proposition 6.** *Suppose* $G=\boldsymbol{R}$, *and let* $\delta_0$ *and* $\tilde{\tilde{\delta}}_0$ *be the generators of* $\alpha$ *and* $\tilde{\alpha}$ *respectively. Let* $\phi$ *be an* $\alpha$*-invariant state of* $A$ *and* $(\pi, H, \xi)$ *the* $\alpha$*-covariant representation associated with* $\phi$.

*Suppose that there exists a directed family* $(u^\iota)_{\iota\in I}$ *of unitary representations of* $G$ *satisfying the following four conditions*:

(i) $u^\iota_t\in\mathrm{D}(\tilde{\tilde{\delta}}_0)$ *for any* $\iota\in I$ *and* $t\in G$;

(ii) $\lim_\iota\|\mathrm{Ad}\ u^\iota_t(x)-\tilde{\alpha}_t(x)\|=0$ *for any* $x\in\pi(\mathscr{A})$ *and* $t\in G$;

(iii) $\sup_{\iota,t}\|\tilde{\tilde{\delta}}_0(u^\iota_t)\|<+\infty$; *and*

(iv) $\lim_\iota\|\omega_\xi\circ\mathrm{Ad}\ u^\iota_t-\omega_\xi\|=0$ *for any* $t\in G$,

*where* $\mathrm{Ad}\ u^\iota_t(x)=u^\iota_txu^{\iota*}_t$ *and* $\omega_\xi(x)=(x\xi|\xi)$ *for* $x\in\mathscr{M}$.

*Furthermore, suppose that* $\delta$ *is relatively bounded with respect to* $\delta_0$.

*Then there exists a self-adjoint element* $h\in\mathscr{M}$ *such that* $\tilde{\tilde{\delta}}-\delta_{ih}$ *commutes with* $\tilde{\alpha}$, *where* $\delta_{ih}(x)=i[h, x]$ *for* $x\in\mathscr{M}$. *Moreover* $\tilde{\tilde{\delta}}|\pi(\mathscr{A})$ *is norm-closable and* $\sigma$*-weakly closable, and its closures are generators in* $\pi(\mathscr{A})$ *and* $\mathscr{M}$ *respectively.*

*Proof.* Let $f$ be an element of $\mathrm{L}^1(G)$ with Supp $\hat{f}$ compact. Then, by Lemma 5 and Proposition 4, $\tilde{\tilde{\delta}}_f$ is $\sigma$-weakly closable and its closure

$\bar{\bar{\delta}}_f$ is relatively bounded with respect to $\tilde{\delta}_0$, and hence $\sup_{\iota,t} ||\bar{\bar{\delta}}_f(u_t^\iota)|| \leqq c||f||_1$ for some positive number $c$. Therefore, by [1, Lemma 3.5], there exists a family $(h_f^\iota)_\iota$ of self-adjoint elements in $\overline{co}\{\bar{\bar{\delta}}_f(u_t^\iota)u_t^{\iota *}|t\in G\}$ such that $\bar{\bar{\delta}}_f(u_t^\iota)=i[h_f^\iota, u_t^\iota]$ for any $\iota\in I$ and $t\in G$, and $\sup_\iota ||h_f^\iota||\leqq c||f||_1$. Hence $\bar{\bar{\delta}}_f-\delta_{ih_f^\iota}$ commutes with Ad $u_t^\iota$, that is, for any $x\in\mathscr{M}_F$

(*) $\bar{\bar{\delta}}_f(\mathrm{Ad}\ u_t^\iota(x))=\mathrm{Ad}\ u_t^\iota((\bar{\bar{\delta}}_f-\delta_{ih_f^\iota})(x))+\delta_{ih_f^\iota}(\mathrm{Ad}\ u_t^\iota(x))$.

On the other hand, it follows that $\lim_\iota ||\phi\circ(\mathrm{Ad}\ u_t^\iota-\tilde{\alpha}_t)||=0$ for $\phi\in\mathscr{M}_*$ and $t\in G$, and hence $(\mathrm{Ad}\ u_t^\iota(x))_\iota$ $\sigma$-strongly converges to $\tilde{\alpha}_t(x)$ for any $x\in\mathscr{M}$. Indeed, for any $x\in\mathscr{M}$ and any $y$, $z\in\pi(\mathscr{A})$,

$$\begin{aligned}
&|\omega_{y\xi,z\xi}(\mathrm{Ad}\ u_t^\iota(x)-\tilde{\alpha}_t(x))|\\
&\leqq|\omega_\xi(\mathrm{Ad}\ u_t^\iota(\mathrm{Ad}\ u_{-t}^\iota(z^*)x\mathrm{Ad}\ u_{-t}^\iota(y)-\tilde{\alpha}_{-t}(z^*)x\tilde{\alpha}_{-t}(y)))|\\
&\quad+|\omega_\xi((\mathrm{Ad}\ u_t^\iota-\tilde{\alpha}_t)(\tilde{\alpha}_{-t}(z^*)x\tilde{\alpha}_{-t}(y)))|\\
&\leqq(||\mathrm{Ad}\ u_{-t}^\iota(z^*)-\tilde{\alpha}_{-t}(z^*)||||y||+||z^*||||\mathrm{Ad}\ u_{-t}^\iota(y)-\tilde{\alpha}_{-t}(y)||\\
&\quad+||\omega_\xi\circ\mathrm{Ad}\ u_t^\iota-\omega_\xi||||z^*||||y||)||x||,
\end{aligned}$$

and hence by (ii) and (iv) we have $\lim_\iota ||\omega_{y\xi,z\xi}\circ(\mathrm{Ad}\ u_t^\iota-\tilde{\alpha}_t)||=0$. Since $\{\omega_{y\xi,z\xi}|y, z\in\pi(\mathscr{A})\}$ is total in $\mathscr{M}_*$, $\lim_\iota ||\phi\circ(\mathrm{Ad}\ u_t^\iota-\tilde{\alpha}_t)||=0$ for any $\phi\in\mathscr{M}_*$.

Thus, taking a cluster point $h_f$ of $(h_f^\iota)_\iota$, $\tilde{\alpha}_t((\bar{\bar{\delta}}_f-\delta_{ih_f})(x))+\delta_{ih_f}(\tilde{\alpha}_t(x))$ is a cluster point of the right hand side of the equality (*). Therefore it follows from the $\sigma$-weak closability of $\tilde{\delta}_f$ that $\bar{\bar{\delta}}_f-\delta_{ih_f}$ commutes with $\tilde{\alpha}$.

Put $f_\varepsilon(t)=\varepsilon^{-1}f(\varepsilon^{-1}t)$ for $\varepsilon>0$. If $\int fdt=1$, then $(\tilde{\delta}_{f_\varepsilon}(x))$ $\sigma$-weakly converges to $\tilde{\delta}(x)$ as $\varepsilon\to0$ for any $x\in\mathscr{M}_F$, because the function $t\mapsto\tilde{\alpha}_t\circ\bar{\delta}\circ\tilde{\alpha}_{-t}(x)$ is $\sigma$-weakly continuous and bounded in virtue of Theorem 1. Taking again a cluster point $h$ of $(h_{f_\varepsilon})$, we conclude that $\bar{\delta}-\delta_{ih}$ commutes with $\tilde{\alpha}$. Then the remaining consequences follow from a series of lemmas in [3].

*Remarks 7.* (1) In Quantum statistical mechanics, condition (ii) is fulfilled for models with bounded surface energy. If $u_t^\iota=e^{ith_\iota}$, $\tilde{\delta}_0(u_t^\iota)=\delta_{ik_\iota}(u_t^\iota)$ and $\sup_\iota||k_\iota-h_\iota||<+\infty$, then

$$\sup_\iota||\tilde{\delta}_0(u_t^\iota)||=\sup_\iota ||\delta_{i(k_\iota-h_\iota)}(u_t^\iota)||\leqq2\sup_\iota||k_\iota-h_\iota||<+\infty.$$

(2) If $\phi$ is an $\alpha$-*KMS* state at $\beta\in\mathbb{R}\setminus\{0\}$ and $u_t^\iota\in\mathscr{M}^{\tilde{\alpha}}$, then $\omega_\xi$ is invariant under Ad $u_t^\iota$. In this case $\delta$ need not be relatively bounded

with respect to $\delta_0$ to get the conclusion.

(3) For a general locally compact abelian group $G$, if (i) and (iii) are replaced by $u_t^\iota \in \mathcal{M}_F$ and $\sup_{\iota,s,t} \|\tilde{\delta}(\tilde{\alpha}_s(u_t^\iota))\| < +\infty$, in particular, by $u_t^\iota \in \mathcal{M}^{\tilde{\alpha}}$, then the consequences in Proposition 6 remain valid.

In the same way as the proof of the above proposition, for any $f \in L^1(G)$ with Supp $\hat{f}$ compact there exists a self-adjoint element $h_f$ of $\mathcal{M}$ such that $\tilde{\delta}_f - \delta_{ih_f}$ commutes with $\tilde{\alpha}$. Since the set of such $f$ is dense in $L^1(G)$, this remains valid for any $f \in L^1(G)$. Since there is a directed family $(f_\kappa)$ such that $\|f_\kappa\|_1 \leqq 1$ and $g(0) = \lim_\kappa \int f_\kappa g dt$ for any bounded continuous function $g$ on $G$, it follows that $\tilde{\delta} - \delta_{ih}$ commutes with $\tilde{\alpha}$ for some self-adjoint element $h$ of $\mathcal{M}$.

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