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On Lie Algebras of Vector Fields on Smooth Orbifolds

By

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§ 0. Introduction

In the previous paper [1], we studied Pursell-Shanks type theorem for orbit spaces of G-manifolds. The purpose of this paper is to prove that this still holds for smooth orbifolds and for fibrations over smooth orbifolds with compact generic fibre.

Let B and B' be connected smooth orbifolds. Let $\mathfrak{D}(B)$ (resp. $\mathfrak{X}(B)$) be the Lie algebra of all smooth vector fields (resp. strata preserving smooth vector fields) on B with compact support.

Theorem 0.1. *The following statements are equivalent.*
(1) *There exists a Lie algebra isomorphism* $\Phi:\mathfrak{D}(B)\to\mathfrak{D}(B')$.
(2) *There exists a Lie algebra isomorphism* $\Phi:\mathfrak{X}(B)\to\mathfrak{X}(B')$.
(3) *There exists a diffeomorphism* $\sigma:B\to B'$.

E admits a natural orbifold structure. Let $p:E\to B$ (resp. $p':E'\to B'$) be a fibration over B (resp. B') with generic fibre F (resp. F'), a connected closed smooth manifold. Let $\mathfrak{D}(E;p)$ (*resp.* $\mathfrak{X}(E:p)$) be the subalgebra of $\mathfrak{D}(E)$ (resp. $\mathfrak{X}(E)$) consisting of fibration preserving vector fields (see § 1). Using Theorem 0.1, we prove the following.

Theorem 0.2. *The following statements are equivalent.*
(1) *There exists a Lie algebra isomorphism* $\Phi:\mathfrak{D}(E;p)\to\mathfrak{D}(E';p')$.
(2) *There exists a Lie algebra isomorphism* $\Phi:\mathfrak{X}(E;p)\to\mathfrak{X}(E';p')$.
(3) *There exists a fibration preserving diffeomorphism* $\sigma:E\to E'$.

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Theorem 0.1 is a special case of Theorem 0.2, but we prove Theorem 0.2 by using Theorem 0.1. Theorem 0.2 was proved by Omori [6] for the case that E and E' are smooth fibre bundles.

The paper is organized as follows. In § 1 we define smooth vector fields on smooth orbifolds and on fibrations over smooth orbifolds. § 2 is devoted to preliminaries. In § 3 and § 4 we determine the maximal ideals of $\mathfrak{X}(B)$, $\mathfrak{X}(E;p)$ and some subalgebras of $\mathfrak{X}(E;p)$. In § 5 we prove that Theorem 0.2 (2) implies (3). In § 6 and § 7 we determine the maximal ideals of $\mathfrak{D}(B)$ and $\mathfrak{D}(E;p)$, and prove Theorem 0.1 and Theorem 0.2.

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§ 1. Definitions

In this section we shall define smooth vector fields on smooth orbifolds and fibrations over smooth orbifolds.

Definition 1.1 (See Satake [7] and Thurston [10])

A paracompact Hausdorff space M is called a smooth orbifold if there exists an open covering $\{U_i\}$ of M, closed under finite intersections, satisfying the following conditions.

(1) For each U_i, there are a finite group Γ_i, a smooth action of Γ_i on an open set $\tilde{U}_i$ of R^n and a homeomorphism $\phi_i: U_i \to \tilde{U}_i/\Gamma_i$.

(2) Whenever $U_i \subset U_j$, there is a smooth embedding $\phi_{ij}: \tilde{U}_i \to \tilde{U}_j$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{U}_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j \\ \downarrow & & \downarrow \\ \tilde{U}_i/\Gamma_i & & \tilde{U}_j/\Gamma_j \\ \downarrow{\scriptstyle \phi_i^{-1}} & & \downarrow{\scriptstyle \phi_j^{-1}} \\ U_i & \hookrightarrow & U_j \end{array}$$

Remark. We can choose the finite group Γ_i such that the action $\Gamma_i \times \tilde{U}_i \to \tilde{U}_i$ is effective, and then there is a unique injective group homomorphism $f_{ij}: \Gamma_i \to \Gamma_j$ such that ϕ_{ij} is equivariant with respect to f_{ij}.

Two coverings give rise to the same orbifold structure if they can be combined consistently to give a larger cover still satisfying the conditions.

Let F be a smooth manifold. A paracompact Hausdorff space E with a continuous map $p:E\to M$ is called a smooth fibrations over a smooth orbifold M with generic fibre F if the following conditions are satisfied.

(1) For each U_i, there exists a smooth Γ_i-action on F such that $p^{-1}(U_i)=U_i\times F/\Gamma_i$ and the following diagram commutes:

$$\begin{array}{ccc} \tilde{U}_i\times F & \longrightarrow & p^{-1}(U_i) \\ \downarrow & & \downarrow \\ \tilde{U}_i & \longrightarrow & U_i \end{array}$$

(2) Whenever $U_i\subset U_j$, the reexists a smooth embedding $\phi_{ij}:\tilde{U}_i\times F\to\tilde{U}_j\times F$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{U}_i\times F & \xrightarrow{\phi_{ij}} & \tilde{U}_j\times F \\ \downarrow & & \downarrow \\ \tilde{U}_i\times F/\Gamma_i & & \tilde{U}_j\times F/\Gamma_j \\ \| & & \| \\ p^{-1}(U_i) & \hookrightarrow & p^{-1}(U_j) \end{array}$$

Definition 1.2 (See [1], § 1) Let M be a smooth orbifold given in Definition 1.1, A function $f:M\to R$ is said to be smooth if the composition $\tilde{U}_i\to\tilde{U}_i/\Gamma_i=U_i\xrightarrow{f}R$ is smooth for any U_i. Let $C^\infty(M)$ denote the algebra of all smooth functions on M. As in [1], § 1 for any $b\in M$ we can define a tangent space $\tau_b(M)$. We define a smooth vector field on M to be a real linear derivation on $C^\infty(M)$. Let $\mathfrak{D}(M)$ be the Lie algebra of all smooth vector fields on M with compact support. A smooth vector field X on M is called strata preserving if X preserves the ideals in $C^\infty(M)$ of smooth functions which vanish on the strata of $U_i=\tilde{U}_i/\Gamma_i$ for any U_i(cf. [10], § 2). Let $\mathfrak{X}(M)$ denote the Lie algebra of strata preserving vector fields on M with compact support.

Let $p:E\to M$ be a smooth fibration over a smooth orbifold M with generic fibre F. Then E has a natural smooth orbifold structure. Let $X\in\mathfrak{X}(E)$. X is said to be fibration preserving if $(dp)_{e_1}(X_{e_1})=(dp)_{e_2}(X_{e_2})$ for any $b\in M$ and $e_1, e_2\in p^{-1}(b)$. Let $\mathfrak{X}(E;p)=\{X\in\mathfrak{X}(E);X$

is fibration preserving}. Let $p':E'\to M'$ be another smooth fibration over a smooth orbifold M' with generic fibre F'. We say a map $\sigma:E\to E'$ is smooth if $f\circ\sigma\in C^\infty(E)$ for any $f\in C^\infty(E')$, and σ is diffeomorphic if σ and σ^{-1} are smooth. For a diffeomorphism $\sigma:E\to E'$, let $\sigma_*:\mathfrak{X}(E)\to\mathfrak{X}(E')$ be a Lie algebra isomorphism defined by $\sigma_*(X)(f)(e)=X(f\circ\sigma)(\sigma^{-1}(e))$ for $X\in\mathfrak{X}(E)$, $f\in C^\infty(E')$, $e\in E'$.

§ 2. Preliminaries

Let Γ be a finite group and let $\Gamma\times\tilde{U}\to\tilde{U}$ be a smooth action on an open set $\tilde{U}$ of R^n. Let $U=\tilde{U}/\Gamma$ be the orbit space of $\tilde{U}$ and π: $\tilde{U}\to U$ the natural projection. In this section, we shall consider that $p:E\to U$ is a smooth fibration with generic fibre F, a closed connected smooth m-manifold, and $E=\tilde{U}\times F/\Gamma$. Let $\mathfrak{X}_\Gamma(\tilde{U})$ be the Lie algebra of all Γ-invariant smooth vector fields on $\tilde{U}$ with compact support. Let $\tilde{E}=\tilde{U}\times F$. Let $\pi:\tilde{E}\to E$ and $\tilde{p}:\tilde{E}\to\tilde{U}$ be the natural projections.

Lemma 2. 1. *The induced map $\tilde{\pi}_*:\mathfrak{X}_\Gamma(\tilde{U})\to\mathfrak{X}(U)$ is a Lie algebra isomorphism.*

Proof. By Bierstone [2] and Schwarz [8], $\tilde{\pi}_*$ is epimorphic. For $a\in\tilde{U}$, the isotropy subgroup Γ_a at a acts on the tangent space $\tau_a(\tilde{U})$. Let $\tau_a(\tilde{U})^{\Gamma_a}$ denote the set of Γ_a-invaraint vectors of $\tau_a(\tilde{U})$. Let $X\in\mathfrak{X}_\Gamma(\tilde{U})$ such that $\tilde{\pi}_*(X)=0$. Then $X_a\in\tau_a(\tilde{U})^{\Gamma_a}$ for any $a\in\tilde{U}$. Since $(d\pi)_a:\tau_a(\tilde{U})\to\tau_a(U)$ is monomorphic on $\tau_a(\tilde{U})^{\Gamma_a}$, we have $X_a=0$, and this completes the proof of Lemma 2. 1.

Let $\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})=\{X\in\mathfrak{X}_\Gamma(\tilde{E})\,;X$ is fibration preserving$\}$.

Lemma 2. 2. *The induced map $\tilde{\pi}_*:\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})\to\mathfrak{X}(E;p)$ is a Lie algebra isomorphism.*

Proof. By Lemma 2. 1, the map $\tilde{\pi}_*$ is monomorphic. Let $X\in\mathfrak{X}(E;p)$. By Lemma 2. 1, there exists a vector field $Y\in\mathfrak{X}_\Gamma(\tilde{E})$ such that $\tilde{\pi}_*(Y)=X$. We shall prove that Y is fibration preserving. Let $\tilde{e}_1,\tilde{e}_2\in\tilde{E}$ such that $\tilde{p}(\tilde{e}_1)=\tilde{p}(\tilde{e}_2)=\tilde{b}\in\tilde{U}$. Let $b=\pi(\tilde{b})$ and $e_i=\tilde{\pi}(\tilde{e}_i)$ for $i=1,2$. Then $(d\pi)_{\tilde{b}}((d\tilde{p})_{\tilde{e}_i}(Y_{\tilde{e}_i}))=(dp)_{e_i}(X_{e_i})$ for $i=1,2$. Since X is fibration preserving, $(d\pi)_{\tilde{b}}((d\tilde{p})_{\tilde{e}_1}(Y_{\tilde{e}_1}))=(d\pi)_{\tilde{b}}((d\tilde{p})_{\tilde{e}_2}(Y_{\tilde{e}_2}))$. Note that, if $\Gamma_{\tilde{b}}$ is a principal isotopy group, then $(d\pi)_{\tilde{b}}:\tau_{\tilde{b}}(\tilde{U})^{\Gamma_{\tilde{b}}}\to\tau_b(U)$ is isomorphic, and $(d\tilde{p})_{\tilde{e}_1}(Y_{\tilde{e}_1})=(d\tilde{p})_{\tilde{e}_2}(Y_{\tilde{e}_2})$. Since the set of principal orbits of $\tilde{U}$ is open dense in $\tilde{U}$, we see that Y is fibration preserving. This completes the proof of Lemma 2.2.

For $\tilde{e}\in\tilde{E}$, let $(V;x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a local coordinate at $\tilde{e}$ such that $(x_1,\ldots,x_n)$ is a local coordinate of R^n and $(y_1,\ldots,y_m)$ is a local coordinate of the fibre F. We can assume that V is a $\Gamma_{\tilde{e}}$-invariant neighborhood of $\tilde{e}$. Then we see that each $X\in\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})$ is described on V as follows;

$$X=\sum_{i=1}^n a_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}+\sum_{j=1}^m b_j(x_1,\ldots,x_n,y_1,\ldots,y_m)\frac{\partial}{\partial y_j},$$

where a_i and b_j are smooth functions on V.

Lemma 2.3. *If $X\in\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})$ satisfies $(d\tilde{p})_{\tilde{e}}X_{\tilde{e}}\neq 0$, then there exists a local coordinate $(W;x_1,\ldots,x_n,y_1,\ldots,y_m)$ at $\tilde{e}$ such that*

(1) *$(x_1,\ldots,x_n)$ is a local coordinate of $\tilde{U}$ satisfying $x_i(\tilde{e}_1)=x_i(\tilde{e}_2)$ for $\tilde{b}\in\tilde{U}$ and $\tilde{e}_1,\ \tilde{e}_2\in p^{-1}(\tilde{b})\cap W$.*

(2) *x_1 is a $\Gamma_{\tilde{e}}$-invariant smooth function on W such that $X=\dfrac{\partial}{\partial x_1}$.*

(3) *$(y_1,\ldots,y_m)$ is a local coordinate of the fibre F.*

Proof. We can prove by easy computations.

Lemma 2.4. *Let $X\in\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})$ satisfying $(d\tilde{p})_{\tilde{e}}X_{\tilde{e}}\neq 0$. Then for any $Y\in\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})$ with $supp(Y)\subset W$, there are a neighborhood $W_1\subset W$ of $\tilde{e}$ and a vector field $Z\in\mathfrak{X}_\Gamma(\tilde{E};\tilde{p})$ such that $[X,Z]=Y$ on W_1, where W is a neighborhood of $\tilde{e}$ as in Lemma 2.3.*

Proof. Let $(W;x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a local coordinate at $\tilde{e}$ as Lemma 2.3. Let $Y=\sum_{i=1}^n a_i(x_1,\ldots,x_n)\frac{\partial}{\partial x_i}+\sum_{j=1}^m b_j(x_1,\ldots,x_n,y_1,\ldots,y_m)\frac{\partial}{\partial y_j}$ *on W. Put* $Z_1=\sum_{i=1}^n(\int_{-\infty}^{x_1}a_i(x_1,\ldots,x_n)dx_1)\frac{\partial}{\partial x_i}+\sum_{j=1}^m(\int_{-\infty}^{x_1}b_j(x_1,\ldots,x_n,y_1,\ldots,y_m)dx_1)\frac{\partial}{\partial y_j}$. Then Z_1 is a fibration preserving $\Gamma_{\tilde{e}}$-invariant smooth vector field on W. By using a Γ-invariant partition of unity on $\tilde{E}$, we can extend Z_1 to a Γ-invariant smooth vector field on $\tilde{E}$ with compact support such that $Z=Z_1$ on a neighborhood $W_1 \subset W$ of $\tilde{e}$. Obviously, $[X,Z]=Y$ on W_1, and this completes the proof of Lemma 2.4.

Lemma 2.5. (cf. Omori [6], 10.7.1) *If X satisfies $X_{\tilde{e}} \neq 0$ and $(dp)_{\tilde{e}} X_{\tilde{e}}=0$, then there exists a local coordinate $(W; x_1, \ldots, x_n, y_1, \ldots, y_m)$ such that $(y_1, \ldots, y_m)$ is a local coordinate of the fibre F and*

$$X=\sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_1} \text{ with } a_i(0, \ldots, 0)=0,$$

where the origin of the coordinate corresponds to the point $\tilde{e}$. Moreover, the coordinate can be chosen such that y_1 is $\Gamma_{\tilde{e}}$-invariant.

Poof. We can show by easy computations.

§ 3. Maximal Ideals of $\mathfrak{X}(B)$ and $\mathfrak{X}(E;p)$

Let B be a connected smooth orbifold. Let $p: E \to B$ be a fibration over B with generic fibre F, a connected closed manifold. In this section we shall determine the maximal ideals of $\mathfrak{X}(E;p)$.

Let $\mathfrak{S}_b(B)=\{X \in \mathfrak{X}(B); X=0$ on a neighborhood of b for $b \in B\}$.

Lemma 3.1 (cf. Koriyama, Maeda, Omori [4], Lemma 4.3) *Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{X}(B)$ such that $\mathfrak{M} \not\supset [\mathfrak{X}(B), \mathfrak{X}(B)]$. Then there exists a unique point $b \in B$ such that $\mathfrak{M} \supset \mathfrak{S}_b(B)$.*

Proof. We give a metric on B. Let $\mathfrak{S}_b(\varepsilon)=\{X \in \mathfrak{X}(B); X=0$ on an ε-neighborhood of $b\}$. Assume that $\mathfrak{M}+\mathfrak{S}_b(\varepsilon)=\mathfrak{X}(B)$ for any $b \in B$. As in the proof of Koriyama, Maeda and Omori [4], Lemma 4.1, we can prove that $\mathfrak{M} \supset [\mathfrak{X}(B), \mathfrak{X}(B)]$, contradicting the assumption for $\mathfrak{M}$. Thus there exists a point $b \in B$ with $\mathfrak{M} \supset \mathfrak{S}_b(\varepsilon)$.

Let $A_\varepsilon=\{b \in B; \mathfrak{S}_b(\varepsilon) \subset \mathfrak{M}\}$. It is easy to see that, if $\varepsilon>\delta$, then $A_\varepsilon \supset cl(A_\delta)$ (the closure of A). Then there exists $b \in \bigcap_{\varepsilon>0} A_\varepsilon$, and we see that $\mathfrak{M} \supset \mathfrak{S}_b(B)$. If $b \neq b'$, then we can prove that $\mathfrak{S}_b(B)+\mathfrak{S}_{b'}(B)=\mathfrak{X}(B)$. Therefore such a point b must be unique and this completes the proof of Lemma 3.1.

Let $B_0=\{b\in B; X_b=0 \text{ for any } X\in\mathfrak{X}(B)\}$ and let $B_1=B-B_0$. From [1], Lemma 3.9 and Proposition 3.10 we have the following.

Proposition 3.2. (1) *For any point* $b\in B_1$, *there exists a unique maximal ideal* $\bar{\mathfrak{J}}(b)$ *such that* $\bar{\mathfrak{J}}(b)\supset\mathfrak{S}_b(B)$, $\bar{\mathfrak{J}}(b)\not\supset[\mathfrak{X}(B), \mathfrak{X}(B)]$ *and codim* $\bar{\mathfrak{J}}(b)=\infty$.

(2) *Let* $\bar{\mathfrak{M}}$ *be a maximal ideal of* $\mathfrak{X}(B)$ *such that* $\bar{\mathfrak{M}}\supset\mathfrak{S}_b(B)$ *for* $b\in B_0$. *Then codim* $\bar{\mathfrak{M}}<\infty$.

Let B^* be the space of all maximal ideals $\bar{\mathfrak{M}}$ of $\mathfrak{X}(B)$ such that $\bar{\mathfrak{M}}\not\supset[\mathfrak{X}(B), \mathfrak{X}(B)]$, with Stone topology (see [1], Definition 4.1). Using Lemma 3.1, we define a map $\bar{\tau}:B^*\to B$ such that $\bar{\tau}(\bar{\mathfrak{M}})=b$ if $\bar{\mathfrak{M}}\supset\mathfrak{S}_b(B)$. Let $B_1^*=\{\bar{\mathfrak{M}}\in B^*; \text{codim } \bar{\mathfrak{M}}=\infty\}$.

From [1], Lemma 4.2 we have the following.

Proposition 3.3. $\bar{\tau}:B_1^*\to B_1$ *is homeomorphic.*

Let $\mathfrak{S}_e(E;p)=\{X\in\mathfrak{X}(E;p); X=0$ on a neighborhood of e for $e\in E\}$. Then we have the following useful lemma.

Lemma 3.4. *Let* $\mathfrak{A}$ *be a subalgebra of* $\mathfrak{X}(E;p)$ *such that* $fX\in\mathfrak{A}$ *for any* $f\in C^\infty(E)$ *and* $X\in\mathfrak{A}\cap(\text{Ker } p_*)$. *Let* $\mathfrak{M}$ *be a maximal ideal of* $\mathfrak{A}$ *such that* $\mathfrak{M}\not\supset[\mathfrak{A},\mathfrak{A}]$ *and* $p_*(\mathfrak{M})=p_*(\mathfrak{A})$. *Then there exists a unique point* $e\in E$ *such that* $\mathfrak{M}\supset\mathfrak{S}_e(\mathfrak{A})$, *where* $\mathfrak{S}_e(\mathfrak{A})=\mathfrak{S}_e(E;p)\cap\mathfrak{A}$.

Proof. Suppose that $\mathfrak{A}=\mathfrak{M}+\mathfrak{S}_e(\varepsilon;\mathfrak{A})$ for any $e\in E$, where $\mathfrak{S}_e(\varepsilon;\mathfrak{A}) =\{X\in\mathfrak{A}; X=0$ on an ε-neighborhood of $e\}$. For any vector field $X\in\mathfrak{A}$, there exists a vector field $Y\in\mathfrak{M}$ such that $Z=X-Y\in \text{Ker } p_*$. For each positive number $\varepsilon>0$, Z can be written as finite sums $Z=\sum Z_i$ such that $Z_i\in\mathfrak{A}\cap(\text{Ker } p_*)$ and diam $(\text{supp } Z_i)<\varepsilon$. Then as in the proof of Koriyama, Maede and Omori [4], Lemma 4.1, $[\mathfrak{A}\cap(\text{Ker } p_*), \mathfrak{A}\cap(\text{Ker } p_*)]$ is contained in $\mathfrak{M}$. Then $[\mathfrak{A},\mathfrak{A}]$ is contained in $\mathfrak{M}$, contradicting the assumption for $\mathfrak{M}$. Therefore there exists a point $e\in E$ such that $\mathfrak{M}\supset\mathfrak{S}_e(\varepsilon;\mathfrak{A})$. As in the proof of Lemma 3.1, there exists a unique point $e\in E$ such that $\mathfrak{M}\supset\mathfrak{S}_e(\varepsilon;\mathfrak{A})$, and Lemma 3.4 follows.

Let $F_b=p^{-1}(b)$ for $b\in B_0$. Put $F_{b,0}=\{e\in F_b; X_e=0$ for any $X\in\mathfrak{X}(E;p)\}$ and put $F_{b,1}=F_b-F_{b,0}$. Let $\mathfrak{X}_F(E;p)=\{X\in\mathfrak{X}(E;p); p_*(X)=0\}$. For $b\in B_0$, $e\in F_b$, put $\mathfrak{J}(e)=\{X\in\mathfrak{X}(E;p); ((ad\, Y_1)\dots(ad\, Y_k)X)_e=0$ for any $Y_i\in\mathfrak{X}_F(E;p)$ and any integer $k\geqq 0\}$, where $(ad\ Y)(Z)=[Y,Z]$ for $Y,Z\in\mathfrak{X}(E;p)$.

Lemma 3.5. *If $b\in B_0$ and $e\in F_{b,1}$, then $\mathfrak{J}(e)$ is an infinite codimensional maximal ideal of $\mathfrak{X}(E;p)$. Moreover $\mathfrak{J}(e)$ is a unique maximal ideal containing $\mathfrak{S}_e(E;p)$, and $\mathfrak{J}(e)\not\supset[\mathfrak{X}_F(E;p), \mathfrak{X}_F(E;p)]$.*

Proof. There exist a finite group Γ, a smooth action of Γ on an open set $\tilde{U}$ of R^n such that $\tilde{U}/\Gamma=U$ is a neighborhood of b, and there exists a smooth Γ-action on the fibre F such that $p^{-1}(U)=\tilde{U}\times F/\Gamma$. Put $E_U=\tilde{U}\times F/\Gamma$ and $\tilde{E}_U=\tilde{U}\times F$. Let $\tilde{\pi}:\tilde{E}_U\to E_U$ and $\tilde{p}:\tilde{E}_U\to\tilde{U}$ be the projections. By Lemma 2.2 $\tilde{\pi}_*:\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})\to\mathfrak{X}(E_U;p)$ is a Lie algebra isomorphism. Let $\tilde{e}$ be a point of $\tilde{E}_U$ such that $\pi(\tilde{e})=e$, and let $\tilde{\mathfrak{J}}(\tilde{e})=\{X\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p}); ((ad\, Y_1)\dots(ad\, Y_k)X)_{\tilde{e}}=0$ for any $Y_i\in\mathfrak{X}_{\Gamma,F}(\tilde{E}_U;\tilde{p})$ and any integer $k\geqq 0\}$. Note that $\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$ can be identified with $\mathfrak{X}_\Gamma(\tilde{U})\times\mathfrak{X}_{\Gamma,F}(\tilde{E}_U;\tilde{p})$ as an R-algebra, where $\mathfrak{X}_{\Gamma,F}(\tilde{E}_U;\tilde{p})=\{X\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p}); \tilde{p}_*X=0\}$. By easy computations we can see that $\tilde{\mathfrak{J}}(\tilde{e})$ is an ideal of $\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$. Since $\tilde{\pi}_*^{-1}(\mathfrak{J}(e)\cap\mathfrak{X}(E_U;p))=\tilde{\mathfrak{J}}(\tilde{e})$, $\mathfrak{J}(e)$ is an ideal of $\mathfrak{X}(E;\mathrm{p})$.

Let $\mathfrak{M}$ be a maximal ideal containing $\mathfrak{S}_e(E;p)$. Assume that there exists a vector field $X\in\mathfrak{M}$ with $X_e\neq 0$. There exists $\tilde{X}\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$ such that $\pi_*(\tilde{X})=X$ on a neighborhood V of e. Since $b\in B_0$, $(d\tilde{p})_{\tilde{e}}\tilde{X}_{\tilde{e}}=0$. By Lemma 2.5, there exists a local coordinate $(W;x_1,\dots,x_n,y_1,\dots,y_m)$ around e such that $(y_1,\dots,y_m)$ is a local coordinate for a fibre F and $\tilde{X}=\sum_{i=1}^n a_i(x_1,\dots,x_n)\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_1}$ with $a_i(0,\dots,0)=0$, where the origin of the coordinate corresponds to the point $\tilde{e}$. There is a vector fields $\tilde{X}_1\in\mathfrak{X}_{\Gamma,F}(\tilde{E}_U;\tilde{p})$ such that $\tilde{X}_1=y_1\frac{\partial}{\partial y_1}$ on a neighborhood $W_1\subset W$ or $\tilde{e}$. Note that $\tilde{X}_2=[\tilde{X},\tilde{X}_1]=\frac{\partial}{\partial y_1}$ on W_1. Put $X_2=[X,\tilde{\pi}_*(\tilde{X}_1)]$. Then $X_2\in\mathfrak{M}$. Let $Y\in\mathfrak{X}(E_U;p)$ such that $supp(Y)\subset V$. Let $\tilde{Y}\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$ such that $\tilde{\pi}_*(\tilde{Y})=Y$. As in the proof of Lemma 2.4, we can prove that there are a neighborhood W_1 of e and a vector field $\tilde{Z}\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$ such that $[\tilde{X}_2,\tilde{Z}]=\tilde{Y}$ on $W_1\subset W$. Put $Z=\tilde{\pi}_*(\tilde{Z})$.

Then $[X_2, Z]\in\mathfrak{M}$ and $[X_2, Z]=Y$ on a neighborhood of e. Since $\mathfrak{M}$ contains $\mathfrak{G}_e(E;p)$, $\mathfrak{M}=\mathfrak{X}(E;p)$. This is a contradiction. Thus $X_e=0$ for any $X\in\mathfrak{M}$, and we have $\mathfrak{M}=\mathfrak{J}(e)$. Therefore $\mathfrak{J}(e)$ is a unique maximal ideal containing $\mathfrak{G}_e(E;p)$. There are vector fields $\tilde{Y}_i\in\mathfrak{X}_\Gamma(\tilde{E}_U;\tilde{p})$ $(i=0,1,2,\dots)$ such that $\tilde{Y}_i=y_1^i\frac{\partial}{\partial y_1}$ on a neighborhood of $\tilde{e}$. Put $Y_i=\tilde{\pi}_*(\tilde{Y}_i)$. Then $Y_i\notin\mathfrak{J}(e)$, and $\mathfrak{J}(e)$ is infinite codimensional. Since $[Y_0, Y_1]\notin\mathfrak{J}(e)$, $\mathfrak{J}(e)\not\supset[\mathfrak{X}_F(E;p),\ \mathfrak{X}_F(E;p)]$. This completes the proof of Lemma 3.5.

Let $b\in B_0$ and $e\in F_{b,0}$. Let $U=\tilde{U}/\Gamma$ be a neighborhood of b as in § 2. For $X\in\mathfrak{X}(E;p)$, there is $Y\in\mathfrak{X}_\Gamma(\tilde{U}\times F;\tilde{p})$ such that $\tilde{\pi}_*(Y)=X$ on a neighborhood W of e, where $\tilde{\pi}:\tilde{U}\times F\to\tilde{U}\times F/\Gamma$ is the natural projection.

Choose a point $\tilde{e}\in\tilde{U}\times F$ with $\tilde{\pi}(\tilde{e})=e$. Let $j^1_{\tilde{e}}(Y)$ denote the 1-jet of Y at $\tilde{e}$. Note that $j^1_{\tilde{e}}(Y)$ defines an element of $\mathfrak{gl}_{\Gamma_{\tilde{e}}}(\tau_{\tilde{e}}(\tilde{U}\times F))$, where $\mathfrak{gl}_{\Gamma_{\tilde{e}}}(\tau_{\tilde{e}}(\tilde{U}\times F))$ denotes the set of all $\Gamma_{\tilde{e}}$-invariant endmorphism of the tangent space $\tau_{\tilde{e}}((\tilde{U}\times F))$ at $\tilde{e}$. Let $J^1_e(E;p)=\{\mathrm{A}\in\mathfrak{gl}_{\Gamma_{\tilde{e}}}(\tau_{\tilde{e}}(\tilde{U}\times F))\,;\,(d\tilde{p})_{\tilde{e}}(\mathrm{A}(v))=0$ for any $v\in\tau_{\tilde{e}}(\tilde{U}\times F)$ with $(d\tilde{p})_{\tilde{e}}(v)=0\}$, where $\tilde{p}:\tilde{U}\times F\to\tilde{U}$ is the natural projection. Since Y is fibration preserving, $j^1_{\tilde{e}}(Y)\in J^1_e(E;p)$. Let $j^1_e:\mathfrak{X}(E;p)\to J^1_e(E;p)$ be a map defined by $j^1_e(X)=j^1_{\tilde{e}}(Y)$. Then it is easy to see the following.

Lemma 3.6. *$j^1_e:\mathfrak{X}(E;p)\to J^1_e(E;p)$ is an onto Lie algebra homomorphism.*

Lemma 3.7. *Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{X}(E;p)$ such that $p_*(\mathfrak{M})=\mathfrak{X}(B)$ and $\mathfrak{M}\supset\mathfrak{G}_e(E;p)$ for some point $e\in F_{b,0}$ with $b\in B_0$. Then $\mathfrak{M}=(j^1_e)^{-1}(\mathfrak{L})$ for some maximal ideal $\mathfrak{L}$ of $J^1_e(E;p)$, and $\mathfrak{M}$ is finite codimensional.*

Proof. Assume that $j^1_e(\mathfrak{M})=J^1_e(E;p)$. We use the same notations as in the proof of Lemma 3.8. Take a vector field $X\in\mathfrak{M}$ such that $j^1_e(X)$ is a unit matrix. There exists $Y\in\mathfrak{X}_\Gamma(\tilde{U}\times F;\tilde{p})$ such that $\tilde{\pi}_*(Y)=X$ on a neighborhood of e. By Sternberg's linearization theorem [9], there exists a local coordinate $(W;z_1,\dots,z_{n+m})$ at e such that $Y=\sum_{i=1}^{n+m}z_i\frac{\partial}{\partial z_i}$ on W.

Let $Z\in\mathfrak{X}(E;p)$ with $j_e^1(Z)=0$. There exists $\tilde{Z}\in\mathfrak{X}_\Gamma(\tilde{U}\times F;\tilde{p})$ such that $\tilde{\pi}_*(\tilde{Z})=Z$ on a neighborhood of e. Then $j_{\tilde{e}}^1(\tilde{Z})=0$. Let $\tilde{Z}_1=\int_0^\infty Ad(exp\, tY)\,\tilde{Z}dt$, where $(Ad(exp\, tY)\tilde{Z})_x=(d\phi_t)_{\phi_t^{-1}(x)}\tilde{Z}_{\phi_t^{-1}(x)}$, $\phi_t=exp\, tY$ (see Koriyama, Maeda and Omori [5], § 1). $\tilde{Z}$ can be written in the form $\tilde{Z}=\sum_{i=1}^{n+m}a_i(z_1,\dots,z_{n+m})\frac{\partial}{\partial z_i}$ on W with $\frac{\partial a_i}{\partial z_j}(0,\dots,0)=0$ for $i,j=1,\dots,n+m$. Then $Ad(exp\, tY)\tilde{Z}=\sum_{i=1}^{n+m}a_i(e^{-t}z_1,\dots,e^{-t}z_{n+m})e^t\frac{\partial}{\partial z_i}$ on W. Since $\frac{\partial a_i}{\partial z_j}(0,\dots,0)=0$, it is clear that $\int_0^\infty a_i(e^{-t}z_1,\dots,e^{-t}z_{n+m})e^t dt$ exists. Hence $\tilde{Z}_1\in\mathfrak{X}_\Gamma(\tilde{U}\times F)$. We can see that $exp\, s(Ad(exp\, tY)\tilde{Z})=\phi_t\circ\psi_s\circ\phi_t^{-1}$, where $\psi_s=exp\, s\tilde{Z}$. Thus $\tilde{Z}_1$ is fibration preserving, and $\tilde{Z}_1\in\mathfrak{X}_\Gamma(\tilde{U}\times F;\tilde{p})$. Note that $[Y,\tilde{Z}_1]=\tilde{Z}$, and $[X,\tilde{\pi}_*(\tilde{Z}_1)]=Z$ on a neighborhood of e. Since $\mathfrak{M}$ contains $\mathfrak{G}_e(E;p)$, $\mathfrak{M}$ contains the ideal Ker j_e^1 of $\mathfrak{X}(E;p)$. But $j_e^1(\mathfrak{M})=J_e^1(E;p)$, hence $\mathfrak{M}=\mathfrak{X}(E;p)$. This is a contradiction. Thus $j_e^1(\mathfrak{M})$ must be a proper ideal $\mathfrak{L}$ of $J_e^1(E;p)$, and $\mathfrak{M}=(j_e^1)^{-1}(\mathfrak{L})$. Since $J_e^1(E;p)$ is a finite dimensional Lie algebra, $\mathfrak{M}$ is finite codimensional. This completes the proof of Lemma 3.7.

Let E^* be the space of all maximal ideals $\mathfrak{M}$ of $\mathfrak{X}(E;p)$ such that $\mathfrak{M}\not\supset[\mathfrak{X}(E;p),\mathfrak{X}(E;p)]$, with Stone topology. Let $E_1^*=\{\mathfrak{M}\in E^*;\ codim\ \mathfrak{M}=\infty\}$. Combining Proposition 3.2, Lemmas 3.4, 3.5 and 3.7, we have the following.

Proposition 3.8. *If* $\mathfrak{M}\in E_1^*$, *then*
(1) $\mathfrak{M}=p_*^{-1}(\bar{\mathfrak{M}})$ *for some* $\bar{\mathfrak{M}}\in B_1^*$, *or*
(2) $\mathfrak{M}=\mathfrak{J}(e)$ *for a point* $e\in F_{b,1}$ *with* $b\in B_0$.

Let $E_1=B_1\cup(\bigcup_{b\in B_0}F_{b,1})$ be a subspace of the disjoint union $B\cup E$. Using Proposition 3.8 we can define a map $\tau: E_1^*\to E_1$ as follows.
(1) If $\mathfrak{M}=p_*^{-1}(\bar{\mathfrak{M}})$ for some $\bar{\mathfrak{M}}\in B_1^*$, then $\tau(\mathfrak{M})=\bar{\tau}(\bar{\mathfrak{M}})$.
(2) If $\mathfrak{M}=\mathfrak{J}(e)$ for $e\in F_{b,1}$ with $b\in B_0$, then $\tau(\mathfrak{M})=e$.

Proposition 3.9. $\tau: E_1^*\to E_1$ *is homeomorphic.*

Proof. By Proposition 3.8, τ is a bijection. It is enough to show that $\tau(CL(S))=cl(\tau(S))$ for any subset S of E_1^*. Here CL and cl are closure operators of E_1^* and E_1, respectively. Assume that

$\tau(\mathfrak{M}) \notin cl(\tau(S))$ for some $\mathfrak{M} \in CL(S)$. In the case that $\tau(\mathfrak{M}) = b \in B_1$, we can find $X \in \mathfrak{X}(E;p)$ such that $p_*(X)_b \neq 0$ and $X=0$ on a neighborhood of $cl(\tau(S))$. Then we see that $X \in \bigcap_{\mathfrak{N} \in S} \mathfrak{N} \subset \mathfrak{M}$ (cf. [1], §4). Since $p_*(\mathfrak{M}) = \mathfrak{J}(b)$, $p_*(X)_b = 0$. This is a contradiction. In the case that $\tau(\mathfrak{M}) = e \in F_{b,1} (b \in B_0)$, we can find $X \in \mathfrak{X}(E;p)$ such that $p_*(X) = 0$, $X_e \neq 0$ and $X=0$ on a neighborgood of $\tau(S)$. Then $X \in \bigcap_{\mathfrak{N} \in S} \mathfrak{N} \subset \mathfrak{M} = \mathfrak{J}(e)$, and $X_e = 0$. This is a contradiction. Therefore $\tau(CL(S))$ is contained in $cl(\tau(S))$.

If $X \in \bigcap_{\mathfrak{M} \in S \cap \tau^{-1}(B_1)} \mathfrak{M}$, then $p_*(X) = 0$ on $\tau(S) \cap B_1$, so $p_*(X) = 0$ on $cl(\tau(S) \cap B_1)$. Hence, for any $X \in \bigcap_{\mathfrak{M} \in S} \mathfrak{M}$, $p_*(X) = 0$ on $cl(\tau(S) \cap B_1)$. Then $\bigcap_{\mathfrak{M} \in S} \mathfrak{M}$ is an ideal of $\mathfrak{X}(E;p)$ contained in $p_*^{-1}(\overline{\mathfrak{J}}(b))$ for any $b \in cl(\tau(S) \cap B_1)$. Similarly, $\bigcap_{\mathfrak{M} \in S} \mathfrak{M}$ is contained in $\mathfrak{J}(e)$ for any $e \in cl(\tau(S) \cap p^{-1}(B_0))$. Thus $\bigcap_{\mathfrak{M} \in S} \mathfrak{M}$ is contained in $\bigcap_{\tau(\mathfrak{M}) \in cl(\tau(S))} \mathfrak{M}$, and $\tau(CL(S))$ contains $cl(\tau(S))$. This completes the proof of Proposition 3.9.

§4. Maximal Ideals of Some Subalgebras of $\mathfrak{X}(E;p)$

Let $\mathfrak{X}(B)_b = \{X \in \mathfrak{X}(B); X_b = 0\}$ for $b \in B$, and $\mathfrak{A}_b = p_*^{-1}(\mathfrak{X}(B)_b)$ be a subalgebra of $\mathfrak{X}(E;p)$. For b, $b' \in B$, let $\hat{F}^b_{b',0} = \{e \in F_{b'}; X_e = 0$ for $X \in \mathfrak{A}_b\}$ and let $\hat{F}^b_{b',1} = F_{b'} - \hat{F}^b_{b',0}$. Let $\mathfrak{J}(e)_b = \mathfrak{A}_b \cap \mathfrak{J}(e)$ and let $\mathfrak{S}_e(\mathfrak{A}_b) = \mathfrak{A}_b \cap \mathfrak{S}_e(E;p)$ for $e \in E$. As in the proofs of Lemmas 3.5 and 3.7, we prove the following lemmas respectively.

Lemma 4.1. *If $e \in \hat{F}^b_{b',1}$ for $b' \in B_0$ or $b' = b$, $\mathfrak{J}(e)_b$ is an infinite codimensional maximal ideal of $\mathfrak{A}_b$. Moreover $\mathfrak{J}(e)_b$ is a unique maximal ideal containing $\mathfrak{S}_e(\mathfrak{A}_b)$.*

Lemma 4.2. *Let $\mathfrak{M}$ be a maximal ideal of $\mathfrak{A}_b$ such that $p_*(\mathfrak{M}) = \mathfrak{X}(B)_b$ and $\mathfrak{M} \supset \mathfrak{S}_e(\mathfrak{A}_b)$ for some $e \in \hat{F}^b_{b',0}$ with $b' \in B_0$ or $b' = b$. Then $\mathfrak{M}$ is finite codimensional.*

Let A^*_b be the set of all maximal ideals $\mathfrak{M}$ of $\mathfrak{A}_b$ such that $\mathfrak{M} \not\supset [\mathfrak{A}_b, \mathfrak{A}_b]$ and $p_*(\mathfrak{M}) = \mathfrak{X}(B)_b$. Let $A^* = \{\mathfrak{M}; \mathfrak{M} \in A^*_b$ for some $b \in B\}$. We give the Stone topology on A^*. Combining Lemmas 3.4, 4.1 and 4.2 we have the following.

Proposition 4.3. *If $\mathfrak{M} \in A^*_b$, then we have the following cases.*

(1) *There exists a point* $e\in p^{-1}(B_0)$ *such that* $\mathfrak{M}\supset\mathfrak{S}_e(\mathfrak{A}_b)$, *and* $\mathfrak{M}=\hat{\mathfrak{M}}\cap\mathfrak{A}_b$ *for some* $\hat{\mathfrak{M}}\in E^*$.

(2) *There exists a point* $e\in F_b(b\in B_1)$ *such that* $\mathfrak{M}\supset\mathfrak{S}_e(\mathfrak{A}_b)$, *and* $\mathfrak{M}\not\subset\hat{\mathfrak{M}}$ *for any* $\hat{\mathfrak{M}}\in E^*$.

Using Lemma 3.4, we have a map $t:A^*\to E$ such that $\mathfrak{M}\supset\mathfrak{S}_{t(\mathfrak{M})}(\mathfrak{A}_b)$ for $\mathfrak{M}\in\mathfrak{A}_b^*$. Let $\hat{A}_{b,1}^*=\{\mathfrak{M}\in A_b^*;codim\ \mathfrak{M}=\infty$ and $\mathfrak{M}\not\subset\hat{\mathfrak{M}}$ for any $\hat{\mathfrak{M}}\in E^*\}$ for $b\in B_1$ and $\hat{A}_{b,1}^*=\{\mathfrak{M}\in A^*;codim\ \mathfrak{M}=\infty,\ p(t(\mathfrak{M}))=b\}$ for $b\in B_0$. Let $\hat{A}_1^*=\bigcup_{b\in B}A_{b,1}^*$. Setting $\hat{E}_1=\bigcup_{b\in B}\hat{F}_{b,1}^b$, we have a bijective map $t:\hat{A}_1^*\to\hat{E}_1$ by Lemma 4.1 and Proposition 4.3.

Proposition 4.4. *The map* $t:\hat{A}_1^*\to\hat{E}_1$ *is homeomorphic.*

Proof. It is enough to show that $t(CL(S))=cl(t(S))$ for any subset S of $\hat{A}_1^*$, where CL and cl are closure operators of $\hat{A}_1^*$ and $\hat{E}_1$ respectively. As in the proof of Proposition 3.9, we see that $t(CL(S))$ is contained in $cl(t(S))$.

Note that, for $X\in\mathfrak{X}(E;p)$ and $e\in\hat{E}_1$, $X\in\mathfrak{J}(e)_{p(e)}$ if $((ad\,Y_1)\dots(ad\,Y_k)X)_e=0$ for any $Y_i\in\mathfrak{X}_F(E;p)$ and any integer $k\geqq0$. Then we see that $\bigcap_{\mathfrak{m}\in S}\mathfrak{M}=\bigcap_{e\in t(s)}\mathfrak{J}(e)_{p(e)}=\bigcap_{e\in cl(t(s))}\mathfrak{J}(e)_{p(e)}$. Thus $cl(t(S))$ is contained in $t(CL(S))$. This completes the proof of Proposition 4.4.

§5. Theorem 0.2 (2) Implies (3)

Let B and B' be connected smooth orbifolds without boundary. Let $p:E\to B$ (resp. $p':E'\to B'$) be a fibration over B (resp. B') with generic fibre F (resp. F'), a connected closed manifold. Suppose that there exists a Lie algebra isomorphism $\Phi:\mathfrak{X}(E;p)\to\mathfrak{X}(E';p')$.

Lemma 5.1. $\Phi(\mathfrak{X}_F(E;p))=\mathfrak{X}_{F'}(E';p')$.

Proof. From Proposition 3.9, there exists a homeomorphism ϕ: $B_1\cup(\bigcup_{b\in B_0}F_{b,1})\xrightarrow{\tau^{-1}}E_1^*\xrightarrow{\Phi}E_1'^*\xrightarrow{\tau'}B_1'\cup(\bigcup_{b'\in B'_0}F_{b',1})$. Note that B_1, B_1', $F_{b,1}$ and $F_{b',1}$ are connected smooth orbifolds. Assume that $\phi(B_1)=F'_{b',1}$ for some $b'\in B_0$. Put $\mathfrak{A}_{b',0}=\{X\in\mathfrak{X}(E';p');\ X=0$ on $p'^{-1}(b')\}$. We see that $\bigcap_{b\in B_1}\tau^{-1}(b)=\mathfrak{X}_F(E;p)$ and $\bigcap_{e'\in F'_{b',1}}\tau'^{-1}(e')=\mathfrak{A}_{b',0}$. Let $E_{1,F}=\{e\in E;$ $X_e\neq0$ for some $X\in\mathfrak{X}_F(E;p)\}$. Using the method of the proof of

[1], Proposition 3. 2, we see that $E_{1,F}$ is dense in E. As in the proof of Lemma 3. 5, $\mathfrak{J}(e)\cap\mathfrak{X}_F(E;p)$ is an infinite codimensional maximal ideal of $\mathfrak{X}_F(E;p)$ and we see that $\mathfrak{J}(e)\cap\mathfrak{X}_F(E;p)\supset[\mathfrak{X}_F(E;p),\mathfrak{X}_F(E;p)]$. Hence the intersection of all maximal ideals $\mathfrak{M}$ of $\mathfrak{X}_F(E;p)$ with $\mathfrak{M}\supset[\mathfrak{X}_F(E;p),\ \mathfrak{X}_F(E;p)]$ is zero.

Let $\mathfrak{M}'$ be a maximal ideal of $\mathfrak{A}'_{b',0}$ with $\mathfrak{M}\not\supset[\mathfrak{A}'_{b',0},\mathfrak{A}'_{b',0}]$. If $p'_*(\mathfrak{M}')$ is a proper ideal of $\mathfrak{X}(B')$, then $\mathfrak{M}'=(p'_*)^{-1}(\overline{\mathfrak{M}}')$ for some $\overline{\mathfrak{M}}'\in B'^*$. If $p'_*(\mathfrak{M}')=\mathfrak{X}(B')$, then, by Lemma 3. 4, there exists a unique point $e'\in E'$ such that $\mathfrak{M}'\supset\mathfrak{S}_{e'}(\mathfrak{A}'_{b',0})$ for $e'\in p'^{-1}(B'_0)$ or $p'(e')=b'$, where $\mathfrak{S}_{e'}(\mathfrak{A}'_{b',0})=\mathfrak{S}_{e'}(E';p')\cap\mathfrak{A}'_{b',0}$. Choose a point $e'_1\in E'$ with $p'(e'_1)\in B'_1$. There exists $X'\in\mathfrak{A}'_{b',0}\cap(Ker\,p'_*)$ such that $X'_{e'_1}\neq 0$ and $X'=0$ on a neighborhood of $p'^{-1}(B'_0)$. Then X' is contained in any maximal ideal of $\mathfrak{A}'_{b',0}$. But, for each maximal ideal $\mathfrak{M}'$ of $\mathfrak{A}'_{b',0}$ with $\mathfrak{M}'\not\supset[\mathfrak{A}'_{b',0},\mathfrak{A}'_{b',0}]$, there exists a maximal ideal $\mathfrak{M}$ of $\mathfrak{X}_F(E;p)$ with $\mathfrak{M}\not\supset[\mathfrak{X}_F(E;p),\mathfrak{X}(E;\mathrm{p})]$ such that $\Phi(\mathfrak{M})=\mathfrak{M}'$. Thus the intersection of these maximal ideals $\mathfrak{M}'$ of $\mathfrak{A}'_{b',0}$ must be zero, and this is a contradiction. Therefore $\phi(B_1)=B'_1$ and we have $\Phi(\mathfrak{X}_F(E;p))=\mathfrak{X}_{F'}(E';p')$. This completes the proof of Lemma 5. 1.

Proposition 5. 2. *If there exists a Lie algebra isomorphism $\Phi:\mathfrak{X}(E;p)\to\mathfrak{X}(E';p')$, then there exists a Lie algebra isomorphism $\bar{\Phi}:\mathfrak{X}(B)\to\mathfrak{X}(B')$ and a diffeomorphism $\bar{\sigma}:B\to B'$ such that $p'_*\circ\Phi=\bar{\Phi}\circ p_*$ and $\bar{\Phi}=\bar{\sigma}_*$.*

Proof. By Lemma 5. 1, the Lie algebra isomorphism Φ induces a Lie algebra isomorphism $\bar{\Phi}:\mathfrak{X}(B)\to\mathfrak{X}(B')$. Using a partition of unity, we can reduce the proof of Proposition 5. 2 to the special case when the orbifolds B and B' are orbit spaces of representation spaces of finite groups. Thus Proposition 5. 2 follows from Theorem of [1].

We shall use the notations given in §4. From Proposition 5. 2, we have a Lie algebra isomorphism $\Phi:\mathfrak{A}_b\to\mathfrak{A}'_{\bar{\sigma}(b)}$ for each $\mathrm{b}\in B$. By Proposition 4. 4, there exists a homeomorphism $\sigma_1:\hat{E}_1\xrightarrow{t^{-1}}\hat{A}_1^*\xrightarrow{\Phi}\hat{A}_1'^*\xrightarrow{t'}\hat{E}'_1$.

Proposition 5. 3. *The map $\sigma_1:\hat{E}_1\to\hat{E}'_1$ is extended to a homeomorphism $\sigma:E\to E'$ such that $p'\circ\sigma=\bar{\sigma}\circ p$.*

Let $\hat{A}_b^*=\{\mathfrak{M}\in A_b^*;\mathfrak{M}\not\subset\hat{\mathfrak{M}}$ for any $\hat{\mathfrak{M}}\in E^*\}$ for $\mathrm{b}\in B_1$ and $\hat{A}_b^*=\{\mathfrak{M}$

$\in A^*; p(t(\mathfrak{M}))=b\}$ for $b\in B_0$. Let $\hat{A}^*=\{\mathfrak{M}; \mathfrak{M}\in\hat{A}_b^*$ for some $b\in B\}$. Let CL and cl be the closure operators of $\hat{A}^*$ and E, respectively. To prove Proposition 5. 3, we need the following.

Lemma 5. 4. *$t(CL(S))=cl(t(S))$ for any subset S of $\hat{A}^*$.*

Proof. As in the proof of Proposition 3. 9, we see that $t(CL(S))$ is contained in $cl(t(S))$. Let $e\in cl(t(S))$. By the similar argument to Proposition 4. 4, we see that $\bigcap_{\mathfrak{m}\in S}\mathfrak{M}$ is contained in $\mathfrak{J}(e)_{p(e)}$ for $e\in cl(t(S))\cap\hat{E}_1$. Thus $cl(t(S))\cap\hat{E}_1$ is contained in $t(CL(S))$.

Now assume that $j_e^1(\mathfrak{N})=J_e^1(\mathfrak{A}_b)$ for some $e\in cl(t(S))\cap\hat{F}_{b,0}^b$, where $\mathfrak{N}=\bigcap_{\mathfrak{m}\in S}\mathfrak{M}$, $b=p(e)$. As in the proof of Lemma 3. 7, we can prove that, for any $X\in\mathfrak{A}_b\cap(Ker\, p_*)$, there exists $Y\in\mathfrak{N}$ such that $X=Y$ on an open neighborhood W of e. There exists $\mathfrak{M}\in S$ with $t(\mathfrak{M})\in W$. Since $\mathfrak{M}$ contains $\mathfrak{G}_{e'}(\mathfrak{A}_{b'})$, we have that $\mathfrak{M}=\mathfrak{A}_{b'}$, where $e'=t(\mathfrak{M})$ and $b'=p(e')$. This is a contradiction. Therefore $j_e^1(\mathfrak{N})$ must be a proper ideal of $J_e^1(\mathfrak{A}_b)$ for any $e\in cl(t(S))\cap\hat{F}_{b,0}^b(b=p(e))$. Hence, for any $e\in cl(t(S))\cap\hat{F}_{b,0}^b$, there exists a maximal ideal $\mathfrak{M}\in CL(S)$ such that $t(\mathfrak{M})=e$. This completes the proof of Lemma 5. 4.

Proof of Proposition 5. 3. Note that the map $t:\hat{A}^*\to E$ is surjective. We define $\sigma:E\to E'$ by $\sigma(t(\mathfrak{M}))=t'(\Phi(\mathfrak{M}))$. First we shall prove that σ is well defined. Note that $\sigma=\sigma_1$ on $\hat{E}_1$ by the definition, and $p'(\sigma(t(\mathfrak{M}))=\bar{\sigma}(p(t(\mathfrak{M})))$ for any $\mathfrak{M}\in A^*$. By the same argument as the proof of [1], Proposition 3. 2, we see that $\hat{F}_{b,0}^b$ is discrete in F_b for each $b\in B$. Thus, for any $e\in E-\hat{E}_1$, there exists a closed neighborhood D of e in F_b such that $D\cap\hat{F}_{b,0}^b=\{e\}$, where $b=p(e)$. Let $D_0=D-\{e\}$ and let $D_0^*=t^{-1}(D_0)$, $D^*=t^{-1}(D)$. Since $\bigcap_{\mathfrak{m}\in D_0^*}\mathfrak{M}\subset\mathfrak{G}_e(\mathfrak{A}_b)$, $CL(D_0^*)=D^*$.

Together with Lemma 5. 4, we have that $cl(\sigma_1(D_0))=cl(\sigma_1(t(D_0^*)))$ $=cl(t'(\Phi(D_0^*)))=t'(CL(\Phi(D_0^*)))=t'(\Phi(CL(D_0^*)))=t'(\Phi(D^*))$. Since σ_1 is homeomorphic, $cl(\sigma(D_0))\cap\hat{F}_{\bar{\sigma}(b),0}^{\bar{\sigma}(b)}$ consist of a point e'. Thus $t'(\Phi(D^*))\cap\hat{F}_{\bar{\sigma}(b),0}^{\bar{\sigma}(b)}=\{e'\}$. Therefore σ is a well defined bijective map. It is clear by the definition that $p'\circ\sigma=\bar{\sigma}\circ p$. By Lemma 5. 4, σ is homeomorphic, and this completes the proof of Proposition 5. 3.

Lemma 5. 5. (Cf. Koriyama, Maeda and Omori [4], Lemma 5. 1).

Let $X\in\mathfrak{X}(E;p)$ *and* $e\in E_1$, $b=p(e)$. *Then* $X_e\neq 0$ *if and only if* $[p_*(X), \mathfrak{X}(B)]+\bar{\mathfrak{J}}(b)=\mathfrak{X}(B)$ *or* $[X,\mathfrak{A}_b]+\mathfrak{J}(e)_b=\mathfrak{A}_b$.

Proof. From [1], Lemma 5.2, $p_*(X)_b\neq 0$ if and only if $[p_*(X), \mathfrak{X}(B)]+\bar{\mathfrak{J}}(b)=\mathfrak{X}(B)$. We assume that $p_*(X)_b=0$. By the same argument as the proof of Lemma 3.5, we see that if $X_e\neq 0$, then $[X,\mathfrak{A}_b]+\mathfrak{J}(e)_b=\mathfrak{A}_b$. Now assume that $[X,\mathfrak{A}_b]+\mathfrak{J}(e)_b=\mathfrak{A}_b$ but $X_e=0$. Let $\mathfrak{J}(e)^1_b=\{Z\in\mathfrak{A}_b;((Ad\,Y)Z)_e=0$ for $Y\in\mathfrak{X}_F(E;p)\}$ be a subalgebra of $\mathfrak{A}_b$. It is easy to see that $[X,Z]\in\mathfrak{J}(e)^1_b$ for any $Z\in\mathfrak{J}(e)^1_b$. Note that $\mathfrak{J}(e)^1_b$ is finite codimensional in $\mathfrak{A}_b$.

Let $ad\ X:\mathfrak{A}_b\to\mathfrak{A}_b$ be a map defined by $(ad\ X)(Z)=[X, Z]$ for $Z\in\mathfrak{A}_b$. Then the map $ad\ X$ induces a map $A(X)_*:\mathfrak{A}_b/\mathfrak{J}(e)^1_b\to\mathfrak{A}_b/\mathfrak{J}(e)^1_b$. Since $[X,\mathfrak{A}_b]+\mathfrak{J}(e)_b=\mathfrak{A}_b$, $A(X)_*$ is epimorphic. Since $A(X)_*$ is an endmorphism of the finite dimensional vector space $\mathfrak{A}_b/\mathfrak{J}(e)^1_b$, $A(X)_*$ is isomorphic. Let $\{X\}$ be the equivalence class of X in $\mathfrak{A}_b/\mathfrak{J}(e)^1_b$. Then $A(X)_*\{X\}=0$, hence $X\in\mathfrak{J}(e)^1_b$. Therefore $[X, (\mathfrak{A}_b)_e]\subset\mathfrak{J}(e)^1_b$, where $(\mathfrak{A}_b)_e=\{Y\in\mathfrak{A}_b;Y_e=0\}$. This means that $A(X)_*((\mathfrak{A}_b)_e/\mathfrak{J}(e)^1_b)=0$, and $(\mathfrak{A}_b)_e=\mathfrak{J}(e)^1_b$. Since $(\mathfrak{A}_b)_e\supsetneqq\mathfrak{J}(e)^1_b$, this is a contradiction. This completes the proof of Lemma 5.5.

Let $\mathfrak{X}(E;p)_e=\{X\in\mathfrak{X}(E;p);X_e=0\}$ for each $e\in E$. From Propositions 5.2, 5.3 and Lemma 5.5, we have the following.

Proposition 5.6. *The Lie algebra isomorphism* $\Phi:\mathfrak{X}(E;p)\to\mathfrak{X}(E';p')$ *induces a Lie algebra isomorphism* $\Phi:\mathfrak{X}(E;p)_e\to\mathfrak{X}(E';p')_{\sigma(e)}$ for each $e\in E$.

Let V be a product Γ-module $V_1\times V_2$ of a finite group Γ such that $V^\Gamma=\{0\}$, and $p_1:V\to V_1$ the natural projection. Let $\{\eta_1,\dots,\eta_s\}$ be a minimal set of homogeneous generators for $R[V_2]^\Gamma_0$. Here $R[V_2]^\Gamma_0$ is the algebra of Γ-invariant polynomials which vanish at 0. Let $\{y_1,\dots,y_m\}$ be a canonical coordinate of V_2 such that Γ acts orthogonally on this coordinate. We can assume $\eta_1=y_1^2+\cdots+y_m^2$. Let $C^\infty_\Gamma(V)$ denote the set of all Γ-invariant smooth functions on M.

Lemma 5.7. *Let g be a Γ-invariant continuous function on V such that $\eta_i g\in C^\infty_\Gamma(V)$ for $i=1,\dots,s$. Then g is a Γ-invariant smooth function.*

Proof. If $s=1$, by using Taylor expansion, it is easy to prove that g is a smooth function. We consider the case that $s\geqq 2$. Put $h_i=\eta_i g$. Then we see that $h_1\eta_2=h_2\eta_1$. Put $G=h_1\eta_2$. Let T_aG be a formal Taylor expansion at $a\in V$. Then $T_aG=\eta_2T_a(h_1)=\eta_1T_a(h_2)$. Since $\{\eta_1,\dots,\eta_s\}$ is a minimal set of homogeneous generators for $R[V_2]_0^{\Gamma}$, there exists a smooth function h_1' on V such that $h_1=\eta_1h_1'$. Then g is a smooth function.

Theorem 5.8. *There exists a Lie algebra isomorphism $\Phi:\mathfrak{X}(E;p)\to\mathfrak{X}(E';p')$ if and only if there exists a fibration preserving diffeomorphism $\sigma:E\to E'$ such that $\Phi=\sigma_*$.*

Proof. Using Proposition 5.6 and Lemma 5.7, we can prove Theorem 5.8 by the same argument as in [1], §5.

§6. Reflection Groups

Let V be an n-dimensional product Γ-module $V_1\times V_2$ of a finite group Γ. Let Γ_v denote the isotropy subgroup of Γ at $v\in V$. Let $V^{(1)}=\{v\in V;\Gamma_v$ is a cyclic group of order two generated by a reflection of $V\}$. Let Γ_1 be the subgroup of Γ generated by reflections $\{\gamma\in\Gamma_v;v\in V^{(1)}\}$. Γ_1 is a normal subgroup of Γ which is a reflection group. Let $\Gamma_1^i (i=1,2)$ be the subgroup of Γ_1 generated by reflections $\{\gamma\in\Gamma_v;v\in V^{(1)}\cap V_i\}$. Then $\Gamma_1=\Gamma_1^1\times\Gamma_1^2$. Since Γ_1 is a reflection subgroup, there is a homogeneous minimal set of generators $\{\theta_1,\dots,\theta_n\}$ for $R[V]_0^{\Gamma_1}$ (see Bourbaki [3], Chapitre V, §5, Théorème 3]). Let $n_i=dim\, V_i$ for $i=1,2$. We can assume that $\{\theta_1,\dots,\theta_{n_1}\}$ and $\{\theta_{n_1+1},\dots,\theta_n\}$ are homogeneous minimal sets of generators for $R[V_1]_0^{\Gamma_1^1}$ and $R[V_2]_0^{\Gamma_1^2}$ respectively. Let $\theta^1=(\theta_1,\dots,\theta_{n_1}):V_1\to R^{n_1}$, $\theta^2=(\theta_{n_1+1},\dots,\theta_n):V_2\to R^{n_2}$ and $\theta=(\theta^1,\theta^2):V\to R^n$ be polynomial maps. Let $\bar\theta^1;V_1/\Gamma_1^1\to R^{n_1}$, $\bar\theta^2:V_2/\Gamma_1^2\to R^{n_2}$ and $\bar\theta:V/\Gamma_1\to R^n$ be the induced orbit maps which are embeddings. Since Γ_1 is a normal subgroup of Γ, the Γ-action on V induces an action $\Psi_0:\bar\Gamma\times V/\Gamma_1\to V/\Gamma_1$, where $\bar\Gamma$ is a factor group Γ/Γ_1.

Lemma 6.1. *There exists a linear action $\Psi:\bar\Gamma\times R^n\to R^n$ such that*

(1) $\Psi(\gamma, \bar{\theta}(X)) = \bar{\theta}(\Psi_0(\gamma, x))$ *for* $\gamma \in \bar{\Gamma}$, $x \in V/\Gamma_1$, and $\bar{\theta}: V/\Gamma_1 \to R^n$ *is a* $\bar{\Gamma}$*-equivariant embedding.*

(2) $\Psi(\gamma, \bar{\theta}^i(x)) = \bar{\theta}^i(\Psi_0(\gamma, x))$ *for* $\gamma \in \bar{\Gamma}, x \in V_i/\Gamma_1^i (i=1,2)$, *and* $\bar{\theta}^i$: $V_i/\Gamma_1^i \to R^{n_i}$ *is a* $\bar{\Gamma}$*-equivariant embedding.*

Proof. Since Γ_1 is a normal subgroup of $\Gamma, \bar{\Gamma}$ acts on $R[V]_0^{\Gamma_1}$ by $(\gamma \cdot f)(v) = f(\gamma^{-1} \cdot v)$ for $f \in R\{V\}_0^{\Gamma_1}$, $\gamma \in \bar{\Gamma}$, $v \in V$. We can choose a homogeneous minimal set of generators for $R[V]_0^{\Gamma_1}$ such that $\gamma \cdot \theta_i = \Sigma_{deg\,\theta_i = deg\,\theta_j} a_{ij}(\gamma)\theta_j$ with $(a_{ij}(\gamma)) \in GL(n, R)$ for $\gamma \in \bar{\Gamma}$(cf. Schwarz [8], Lemma 8.1). We can assume that $a_{ij}=0$ if $1 \leqq i \leqq n_1$, $n_1+1 \leqq j \leqq n$ or $1 \leqq j \leqq n_1$, $n_1+1 \leqq i \leqq n$. Let $\{e_1, \ldots, e_n\}$ be a canonical basis of R^n. We define a linear action $\Psi: \bar{\Gamma} \times R^n \to R^n$ by $\Psi(\gamma, \Sigma_{j=1}^n x_j e_j) = \Sigma_{i,j=1}^n a_{ij}(\gamma^{-1}) x_j e_i$. Then we can check that the action satisfies (1) and (2). This completes the proof of Lemma 6.1.

Let $p: V/\Gamma \to V_1/\Gamma$ and $\rho: R^n/\bar{\Gamma} \to R^{n_1}/\bar{\Gamma}$ be maps induced from the natural projections $V \to V_1$ and $R^n \to R^{n_1}$ respectively. We can regard p and ρ as fibrations over smooth orbifolds. Since $\bar{\theta}: V/\Gamma_1 \to R^n$ is a Γ-equivariant embedding by Lemma 6.1, we have an induced embedding $\bar{\bar{\theta}}: V/\Gamma \to R^n/\bar{\Gamma}$. It is easy to see that the induced map $\bar{\bar{\theta}}^*: C^\infty(R^n/\bar{\Gamma}) \to C^\infty(V/\Gamma)$ is epimorphic. We define $\bar{\bar{\theta}}^*: \mathfrak{D}(R^n/\bar{\Gamma}) \to \mathfrak{D}(V/\Gamma)$ by $\bar{\bar{\theta}}^*(X)(\bar{\bar{\theta}}^*(f)) = X(f) \circ \bar{\bar{\theta}}$ for $X \in \mathfrak{D}(R^n/\bar{\Gamma})$, $f \in C^\infty(R^n/\bar{\Gamma})$. Since $\bar{\bar{\theta}}(V/\Gamma)$ is the closure of an open set in $R^n/\bar{\Gamma}$, we see that $\bar{\bar{\theta}}^*$ is a well defined Lie algebra homomorphism. Clearly $\bar{\bar{\theta}}^*$ induces a Lie algebra homomorphism $\bar{\bar{\theta}}^*: \mathfrak{D}(R^n/\bar{\Gamma}; \rho) \to \mathfrak{D}(V/\Gamma; p)$.

Let $\{\eta_1, \ldots, \eta_k\}$ be a minimal set of generators for $R[R^n]_0^{\bar{\Gamma}}$ such that $\{\eta_1, \ldots, \eta_{k_1}\}$ $(k_1 \leqq k)$ is a minimal set of generators for $R[R^{n_1}]_0^{\bar{\Gamma}}$. Let $\bar{\eta}: R^n/\bar{\Gamma} \to R^k$ be the orbit map of a polynomial map $\eta = (\eta_1, \ldots, \eta_k)$: $R^n \to R^k$. Let $\nu = (\nu_1, \ldots, \nu_k): V \to R^k$, where $\nu_i = \eta_i \circ \theta$. Since $\bar{\bar{\theta}}$ and $\bar{\eta}$ are smooth embeddings, $\nu(V) = \bar{\eta}(\bar{\bar{\theta}}(V/\Gamma))$ is diffeomorphic to V/Γ. Let $r_i \in R[y_1, \ldots, y_k]$, $i=1, \ldots, l$, be the generators of the ideal I of algebraic relations among $\eta_1, \ldots, \eta_k$. Let S be a subset of R^n and $I^\infty(S)$ denote the ideal in $C^\infty(R^n)$ which vanish on S. For a smooth vector field X on R^n, we say that X is tangent to S if X preserves the ideal $I^\infty(S)$.

Lemma 6.2. *Let X be a polynomial vector field on R^k. Then*
(1) *X is tangent to $\eta(R^n)$ if and only if $X(r_i)\in I$, $i=1,\dots,l$.*
(2) *X is tangent to $\eta(R^n)$ if and only if X is tangent to $\nu(V)$.*

Proof. Since the ring of polynomial functions on $\eta(R^n)$ can be identified with $R[y_1,\dots,y_k]/I$, we see (1). Since $\nu(V)$ contains an open set in R^n, we can prove that I is also the ideal of algebraic relations among $\nu_1,\dots,\nu_k$, and (2) follows.

Let $D(\eta(R^n))$ be the Lie algebra of polynomial vector fields on R^k which tangent to $\eta(R^n)$. By the same argument as in Schwarz [8, Proposition 6.14], we see the following.

Proposition 6.3. (1) $\mathfrak{D}(\eta(R^n))=C^\infty(\eta(R^n))\cdot D(\eta(R^n))$.
(2) $\mathfrak{D}(\nu(V))=C^\infty(\nu(V))\cdot D(\nu(V))$.

Proposition 6.4. *The Lie algebra homomorphism $\bar{\theta}^*:\mathfrak{D}(R^n/\bar{\Gamma})\to\mathfrak{D}(V/\Gamma)$ is epimorphic.*

Proof. By Lemma 6.2 $D(\eta(R^n))=D(\nu(V))$. Let $j:\nu(V)\subset\eta(R^n)$ be the inclusion. Then the induced map $j^*:C^\infty(\eta(R^n))\to C^\infty(\nu(V))$ is epimorphic. Since $\bar{\theta}^*$ is identified with a homomorphism $\mathfrak{D}(R^n/\bar{\Gamma})=\mathfrak{D}(\eta(R^n))\xrightarrow{j^*}\mathfrak{D}(\nu(V))=\mathfrak{D}(V/\Gamma)$, $\bar{\theta}^*$ is epimorphic from Proposition 6.3. This completes the proof of Proposition 6.4.

Note that the maps $p:V/\Gamma\to V_1/\Gamma$ and $\rho:R^n/\bar{\Gamma}\to R^{n_1}/\bar{\Gamma}$ induce Lie algebra homomorphisms $p_*:\mathfrak{D}(V/\Gamma;p)\to\mathfrak{D}(V_1/\Gamma)$ and $\rho_*:\mathfrak{D}(R^n/\bar{\Gamma};\rho)\to\mathfrak{D}(R^{n_1}/\bar{\Gamma})$. We set $\mathfrak{D}_F(V/\Gamma;p)=\ker p_*$ and $\mathfrak{D}_F(R^n/\bar{\Gamma};\rho)=\ker\rho_*$. Then $\mathfrak{D}_F(V/\Gamma;p)$ is a $C^\infty(V/\Gamma)$-module and $\mathfrak{D}_F(R^n/\bar{\Gamma};\rho)$ is a $C^\infty(R^n/\bar{\Gamma})$-module respectively. Let $q:R^k\to R^{k_1}$ be the natural projection q induces a Lie algebra homomorphism $q_*:\mathfrak{D}(R^k;q)\to\mathfrak{D}(R^k)$. Let $D_F(\eta(R^n);q)=\ker q_*\cap D(\eta(R^n))$ and $D_F(\nu(V);q)=\ker q_*\cap D(\nu(V))$. By the same argument as in Proposition 6.3 and Proposition 6.4, we have the following.

Proposition 6.5. (1) $\mathfrak{D}_F(\eta(V);q)=C^\infty(\eta(V))\cdot D_F(\eta(V);q)$.
(2) $\mathfrak{D}_F(\nu(V);q)=C^\infty(\nu(V))\cdot D_F(\nu(V);q)$.

(3) *The Lie algebra homomorphism* $\bar{\theta}^*: \mathfrak{D}_F(R^n/\bar{\Gamma};\rho)\to\mathfrak{D}_F(V/\Gamma;p)$ *is epimorphic.*

Combining Propositions 6.4 and 6.5, we have.

Corollary 6.6. *The Lie algebra homomorphism* $\bar{\theta}^*: \mathfrak{D}(R^n/\bar{\Gamma};\rho)\to \mathfrak{D}(V/\Gamma;p)$ *is epimorphic.*

Lemma 6.7 *If* $V^{(1)}$ *is not empty, then there exists a smooth vector field* $X\in\mathfrak{D}(V/\Gamma;p)$ *such that* $X_{\pi(0)}\neq 0$, *where* $\pi: V\to V/\Gamma$ *is the natural projection.*

Proof. V is an orthogomal representation with respect to a suitable basis of V. Let W be the orthogonal complement of V^Γ. Then W is a Γ-module. Since $V^{(1)}$ is not empty, $dim\ W>0$. Let $\{x_1,\dots,x_p\}$ be a canonical coordinate of W. If $dim\ W^\Gamma>0$, we see that there exists a Γ-invariant smooth vector field Y on V such that $X=\pi_*(Y)\in \mathfrak{D}(V/\Gamma;p)$ and $X_{\pi(0)}\neq 0$. If $dim\ W^\Gamma=0$, then we can choose a homogeneous minimal set of generators $\{\theta_1,\dots,\theta_n\}$ for $R[V]_0^{\Gamma_1}$ such that $\theta_i=x_1^2+\cdots+x_p^2$ for some i. Since $\gamma\circ\theta_i=\theta_i$ for any $\gamma\in\Gamma$, we see that $dim(R^n)\bar{\Gamma}>0$. Then we have a $\bar{\Gamma}$-invariant smooth vector field Z on R^n such that $X=\pi'_*(Z)\in\mathfrak{D}(R^n/\bar{\Gamma};p)$ and $X_{\pi'(0)}\neq 0$, where $\pi': R^n\to R^n/\bar{\Gamma}$ is the natural projection. Since $(d\bar{\theta})_{\pi(0)}:\tau_{\pi(0)}(V/\Gamma)\to\tau_{\pi'(0)}(R^n/\bar{\Gamma})$ is isomorphic, $\theta_*(X)_{\pi(0)}\neq 0$. This completes the proof of Lemma 6.7.

Let $\mathfrak{D}(V/\Gamma;p)(x)=\{Y_x; Y\in\mathfrak{D}(V/\Gamma;p)\}$ for $x\in V/\Gamma$.

Corollary 6.8. $\mathfrak{D}(V/\Gamma;p)(\pi(0))=\{0\}$ *if and only if* $dim\ V^\Gamma=0$ *and* $V^{(1)}=\phi$.

Proof. If $V^{(1)}=\phi$, then $\mathfrak{D}(V/\Gamma;p)=\mathfrak{X}(V/\Gamma;p)$ by Schwarz [8, Chapter I, Proposition 3.5]. By Lemma 2.2 $\mathfrak{X}(V/\Gamma;p)=\mathfrak{X}_\Gamma(V;\tilde{p})$, where $\tilde{p}: V\to V_1$ is the natural projection. Thus if $V^{(1)}=\phi$ and $dim\ V^\Gamma=0$, $\mathfrak{D}(V/\Gamma;p)(\pi(0))=\{0\}$ from Lemmas 2.3 and 2.5. If $V^{(1)}\neq\phi$, then $D(V/\Gamma;p)(\pi(0))\neq\{0\}$ by Lemma 6.7, and Corollary 6.8 follows.

§7. Proof of Theorem 0.1 and Theorem 0.2

Let $p:E\to B$ be a fibration over a connected smooth orbifold B with generic fibre F, a connected smooth manifold. We set $B_0=\{b\in B; X_b=0$ for any $X\in\mathfrak{D}(B)\}$, $B_1=B-B_0$ and $\mathfrak{D}(B)_b=\{X\in\mathfrak{D}(B); X_b=0\}$.

Proposition 7.1. *For $b\in B_1$, there exists a unique infinite codimensional maximal ideal $\bar{\mathfrak{J}}_b(B)$ of $\mathfrak{D}(B)$ which is contained in $\mathfrak{D}(B)_b$.*

Proof. For $b\in B$, there exist a finite group Γ and a linear action $\Gamma\times V\to V$ on an n-dimensional vector space V such that V/Γ is diffeomorphic to an open neighborhood U of b. By Lemma 6.1 and Proposition 6.4, we have a linear action $\bar{\Gamma}\times R^n\to R^n$ such that there exists an embedding $\bar{\theta}: V/\Gamma\to R^n/\bar{\Gamma}$ and $\bar{\theta}^*:\mathfrak{D}(R^n/\bar{\Gamma})\to\mathfrak{D}(V/\Gamma)$ is a Lie algebra epimorphism. Using the property of the reflection group Γ_1, the natural group homomorphism $\Gamma\to\bar{\Gamma}$ has a right inverse. Then it is easy to see that $\bar{\Gamma}$ has no reflection subgroups. Then $\mathfrak{D}(R^n/\bar{\Gamma})=\mathfrak{X}(R^n/\bar{\Gamma})$. By Corollary 6.6 and Lemma 6.7, there exists a smooth vector field $X\in\mathfrak{D}(R^n/\bar{\Gamma})$ such that $X_{\pi(0)}\neq 0$.

From [1], Lemma 3.9, there exists a unique infinite codimensional maximal ideal $\mathfrak{N}_1$ of $\mathfrak{X}(R^n/\bar{\Gamma})$ which is contained in $\mathfrak{X}(R^n/\bar{\Gamma})_b$. Then $\mathfrak{N}_2=\bar{\theta}^*(\mathfrak{N}_1)$ is an infinite codimensional maximal ideal of $\mathfrak{D}(V/\Gamma)$ which is contained in $\mathfrak{D}(V/\Gamma)_b$. $\bar{\mathfrak{J}}_b(B)=\{X\in\mathfrak{D}(B); X=Y$ on a neighborhood of b in B for some $Y\in\mathfrak{N}_2\}$. As in the proof of [1], Proposition 3.8, we can prove that $\bar{\mathfrak{J}}_b(B)$ is an infinite codimensional maximal ideal of $\mathfrak{D}(B)$. This completes the proof of Proposition 7.1.

Lemma 7.2. *Let $b\in B_1$. Then, for $X\in\mathfrak{D}(B)$, $X(b)\neq 0$ if and only if $[X,\mathfrak{D}(B)]+\bar{\mathfrak{J}}_b(B)=\mathfrak{D}(B)$.*

Proof. We use the same notation as in the proof of Proposition 7.1. To prove Lemma 7.2, it is sufficient to prove that, for $X\in\mathfrak{D}(V/\Gamma)$, $X_{\pi(0)}\neq 0$ if and only if $[X,\mathfrak{D}(V/\Gamma)]+\mathfrak{N}_2=\mathfrak{D}(V/\Gamma)$ where $\pi: V\to V/\Gamma$ is the natural projection. Note that $\mathfrak{D}(R^n/\bar{\Gamma})=\mathfrak{X}(R^n/\bar{\Gamma})$. It follows from [1], Lemma 5.2 that, for $Y\in\mathfrak{D}(R^n/\bar{\Gamma})$, $Y_{\rho(0)}\neq 0$ if and only if $[Y,\mathfrak{D}(R^n/\bar{\Gamma})]+\mathfrak{N}_1=\mathfrak{D}(R^n/\bar{\Gamma})$, where $\rho: R^n\to R^n/\bar{\Gamma}$ is the natural projection. Let $X\in\mathfrak{D}(V/\Gamma)$ with $X_{\pi(0)}\neq 0$. There exists $Y\in\mathfrak{D}(R^n/\bar{\Gamma})$ such that $\bar{\bar{\theta}}^*(Y)=X$. Then $Y_{\rho(0)}\neq 0$ and hence $[Y, \mathfrak{D}(R^n/\bar{\Gamma})]+\mathfrak{N}_1=\mathfrak{D}(R^n/\bar{\Gamma})$. Since $\bar{\bar{\theta}}^*$ is epimorphic, $[X, \mathfrak{D}(V/\Gamma)]+\mathfrak{N}_2=\mathfrak{D}(V/\Gamma)$. Conversely, suppose that $[X, \mathfrak{D}(V/\Gamma)]+\mathfrak{N}_2=\mathfrak{D}(V/\Gamma)$. Let $Y\in\mathfrak{D}(R^n/\bar{\Gamma})$ such that $\bar{\bar{\theta}}^*(Y)=X$. It is easy to see that $\mathfrak{D}(R^n/\bar{\Gamma})=[Y, \mathfrak{D}(R^n/\bar{\Gamma})]+\mathfrak{N}_1$. Then $Y_{\rho(0)}\neq 0$. Hence $X_{\pi(0)}\neq 0$, and this completes the proof of Lemma 7.2.

Proof of Theorem 0.1. Using the result of [1], we see that (2) implies (3). By Schwarz [8], Corollary 1.7, if $\sigma: B\to B'$ is diffeomorphic, then σ is strata preserving. Then we see that (3) implies (2). Assume that there exists a Lie algebra isomorphism $\Phi:\mathfrak{D}(B)\to\mathfrak{D}(B')$. As in [1], § 5, using Proposition 7.1, and Lemma 7.2, we can prove that there exists a diffeomorphism $\sigma: B\to B'$ such that $\Phi=\sigma_*$. This completes the proof of Theorem 0.1.

Proof of Theorem 0.2. By Theorem 5.8, (2) implies (3). By Schwarz [8], Corollary 1.7, (3) implies (2). Assume that there exists a Lie algebra isomorphism $\Phi:\mathfrak{D}(E;p)\to\mathfrak{D}(E';p')$. As the proof of Theorem 5.7, using Corollary 6.6 and Theorem 0.1, we can prove that there exists a fibration preserving diffeomorphism $\sigma: E\to E'$ such that $\Phi=\sigma_*$. This complete the proof of Theorem 0.2.

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