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Domains of Holomorphy in Segre Cones

By

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Introduction

Let X be a normal Stein space and D a domain (open set) in X. If D is Stein, then it is a domain of holomorphy. The converse is valid when X is a manifold (Docquier-Grauert [2]). However, this is not the case in general, as was pointed out by Grauert-Remmert [6], [7]. They gave an example of a non-Stein domain of holomorphy in a Stein space (Segre cone).

This problem is naturally related with the Levi problem, which asks whether a domain in X is Stein if it is locally Stein (at all boundary points). Concerning some results on the Levi problem for Stein spaces, see Andreotti-Narasimhan [1], Fornaess-Narasimhan [4], and Fornaess [3] particularly for Segre cones.

A domain of holomorphy is locally Stein at the boundary points which are non-singular points of X. So we pose the problem : Suppose that D is locally Stein at the boundary points which are non-singular points of X. Under what additional condition is D a domain of holomorphy, or a Stein domain?

In the present note we will give an answer to this problem for the case where X is a Segre cone. The method used here is the same as the one in the previous note of the author [11], i. e., to go over to a domain in an affine space and to apply Oka's theorem.

§ 1. Segre Cones

Let r, s be integers $\geqq 1$. We identify the complex affine space $\mathbf{C}^{(r+1)(s+1)}$ with the set of all $(r+1, s+1)$ matrices $z=(z_{ij})$, $i=0, 1, \dots,$

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r, $j=0, 1, \ldots, s$. The Segre cone $Z=Z_{r,s}$ is the algebraic set

$$Z:=\{z\in \boldsymbol{C}^{(r+1)(s+1)}\,|\,\mathrm{rank}\, z\leqq 1\},$$

which is naturally regarded as a normal and irreducible Stein space of dimension $r+s+1$. The origin 0 is the only singular point of Z. (See Grauert-Remmert [8, Chap. 7, § 5].)

We will describe three kinds of desingularizations of Z. Let $\boldsymbol{P}^r$, $\boldsymbol{P}^s$ be the projective spaces with homogeneous coordinate systems $[\xi]=[\xi_0, \xi_1, \ldots, \xi_r]$, $[\eta]=[\eta_0, \eta_1, \ldots, \eta_s]$ respectively. We set

$$Z_0:=\{(z, [\xi], [\eta])\in Z\times \boldsymbol{P}^r\times \boldsymbol{P}^s\,|\, \text{there is a constant } \lambda\in \boldsymbol{C} \text{ such that } z_{ij}=\lambda\xi_i\eta_j \text{ for all } i, j\},$$

$$Z_1:=\{(z, [\xi])\in Z\times \boldsymbol{P}^r\,|\, \text{there are constants } \nu_0, \nu_1, \ldots, \nu_s\in \boldsymbol{C} \text{ such that } z_{ij}=\xi_i\nu_j \text{ for all } i, j\},$$

$$Z_2:=\{(z, [\eta])\in Z\times \boldsymbol{P}^s\,|\, \text{there are constants } \mu_0, \mu_1, \ldots, \mu_r\in \boldsymbol{C} \text{ such that } z_{ij}=\mu_i\eta_j \text{ for all } i, j\},$$

The projection $Z_0\to \boldsymbol{P}^r\times \boldsymbol{P}^s$ defines a holomorphic line bundle over $\boldsymbol{P}^r\times \boldsymbol{P}^s$, whose zero section O_0 is canonically identified with $\boldsymbol{P}^r\times \boldsymbol{P}^s$. Let σ_0 denote the projection $Z_0\to Z$. Then $\sigma_0^{-1}(0)=O_0$, and $\sigma_0|Z_0\backslash O_0$ is a biholomorphic map of $Z\backslash O_0$ onto $Z\backslash\{0\}$. Thus Z is obtaind from Z_0 by contracting O_0. The fibers of the line bundle $Z_0\to \boldsymbol{P}^r\times \boldsymbol{P}^s$ correspond by σ_0 to lines in $\boldsymbol{C}^{(r+1)(s+1)}$ which pass through 0.

The projection $Z_1\to \boldsymbol{P}^r$ defines a holomorphic vector bundle of rank $s+1$ over $\boldsymbol{P}^r$, whose zero section O_1 is canonically identified with $\boldsymbol{P}^r$. Let σ_1 denote the projection $Z_1\to Z$. Then $\sigma_1^{-1}(0)=O_1$ and $\sigma_1|Z_1\backslash O_1$ is a biholomorphic map of $Z_1\backslash O_1$ onto $Z\backslash\{0\}$. Thus Z is obtained from Z_1 by contracting O_1. The fibers of the vector bundle $Z_1\to \boldsymbol{P}^r$ correspond by σ_1 to vector subspaces of dimension $s+1$ in $\boldsymbol{C}^{(r+1)(s+1)}$.

Now we define a holomorphic mapping τ_1: $Z_0\to Z_1$ by $(z, [\xi], [\eta])\to(z, [\xi])$. O_0 is mapped by τ_1 onto O_1 correspondingly to the projection $\boldsymbol{P}^r\times \boldsymbol{P}^s\to \boldsymbol{P}^r$. The restriction $\tau_1|Z_0\backslash O_0$ is a biholomorphic map of $Z_0\backslash O_0$ onto $Z_1\backslash O_1$. We have obviously $\sigma_1\circ\tau_1=\sigma_0$.

The above observations for Z_1 is applied analogously to Z_2. We obtain the commutative diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{\tau_1} & Z_1 \\ {\scriptstyle\tau_2}\downarrow & & \downarrow{\scriptstyle\sigma_1} \\ Z_2 & \xrightarrow[\sigma_2]{} & Z \end{array}$$

with $\sigma_0=\sigma_1\circ\tau_1=\sigma_2\circ\tau_2$.

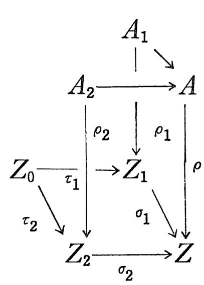
Now we will represent the Segre cone as the quotient of an affine space. Let $A:=\boldsymbol{C}^{(r+1)+(s+1)}$ be the affine space with the coordinates $(x, y)=(x_0, x_1, \dots, x_r; y_0, y_1, \dots, y_s)$. We set

$$L_1:=\{x=0\},\ L_2:=\{y=0\},\ L_0:=L_1\cup L_2,$$
$$A_1:=A\backslash L_1,\ A_2:=A\backslash L_2,\ A_0:=A\backslash L_0.$$

We define the holomorphic mapping $\rho: A\to\boldsymbol{C}^{(r+1)(s+1)}$ by $(x,y)\to(x_iy_j)$. We have $\rho(A)=Z$ and $\rho^{-1}(0)=L_0$. For the points $z\in Z\backslash\{0\}$, the fibers are biholomorphic to $\boldsymbol{C}^*:=\boldsymbol{C}\backslash\{0\}$. Hence $Z\backslash\{0\}$ is the quotient space of A_0 by the action $A_0\times\boldsymbol{C}^*\to A_0$, $((x,y), c)\to(cx, c^{-1}y)$. In other words, $\rho|A_0$: $A_0\to Z\backslash\{0\}$ is regarded as a holomorphic principal $\boldsymbol{C}^*$-bundle.

We define a holomorphic mapping ρ_1: $A_1\to Z_1$ by $(x, y)\to((x_iy_j), [x])$. ρ_1 is surjective and Z_1 is the quotient space of A_1 by the action $A_1\times\boldsymbol{C}^*\to A_1$, $((x, y), c)\mapsto(cx, c^{-1}y)$. Thus ρ_1 defines a holomorphic principal $\boldsymbol{C}^*$-bundle over Z_1. We have $\sigma_1\circ\rho_1=\rho|A_1$ and the pull-back of the bundle $A_0\to Z\backslash\{0\}$ by σ_1 coincides with the restriction of the bundle $A_1\to Z_1$ to $Z_1\backslash O_1$.

Analogously we define a holomorphic mapping ρ_2: $A_2\to Z_2$ by $(x, y)\mapsto((x_iy_j), [y])$. The commutative diagram can be augmented in the following way:



where $A_1\to A$, $A_2\to A$ are inclusion mappings.

Remark. The map ρ is used also by Fornaess [3].

§2. Boundary Points

Let us recall some properties of boundary points of domains in complex spaces. Let E be a domain (open set) in a complex space

X, and let ∂E denote the set of all boundary points of E in X. The domain E is said to be locally Stein at $q \in \partial E$ if there is a neighborhood U of q such that $U \cap E$ is a Stein space. When E is locally Stein at a non-singular point q of X, we say that E is pseudoconvex at q.

Now let S be an analytic set of positive codimension in X. A point $q \in \partial E \cap S$ is said to be removable along S if there is a neighborhood U of q such that $U \backslash S \subset E$. We denote by R the set of the boundary points of E that are removable along S. We set $E^* = E \cup R$. Then E^* is a domain in X, which will be called the extension of E along S. The following lemma is essentially due to Grauert-Remmert [6].

Lemma. (Ueda [11]) *Let E be a domain in a complex manifold X and S be an analytic set of positive codimension in X. Suppose that E is pseudoconvex at every point $q \in \partial E \backslash S$.* (1) *If there is no boundary point removable along S, then E is pseudoconvex (at every boundary point.)* (2) *The extension E^* of E along S is pseudoconvex.*

Remark. When X is a complex space, this lemma (with the word "pseudoconvex" replaced by "locally Stein") is false in general, as is shown by the example in [6].

§ 3. Domains in a Segre Cone

Let D be a domain in a Segre cone Z. For $k=0, 1, 2$, we set $D_k := \sigma_k^{-1}(D)$. D_k is a domain in Z_k. Since $\sigma_k | Z_k \backslash O_k$ is a biholomorphic map of $Z_k \backslash O_k$ onto $Z \backslash \{0\}$, the domain D_k is biholomorphic to D if $0 \notin D$. Let R_k be the set of all boundary points of D_k that are removable along O_k, and $D_k^* := D_k \cup R_k$ be the extension of D_k along O_k.

Assume that D satisfies the condition

(*) *D is pseudoconvex at every boundary point in $Z \backslash \{0\}$.*

Then D_k is pseudoconvex at every boundary point in $Z_k \backslash O_k$. By the lemma, D_k^* is pseudoconvex at every boundary point. Hence the set $R_k = D_k^* \cap O_k$ is empty or pseudoconvex, considered as a domain (not necessarily connected) in O_k.

The sets O_0, O_1, O_2 are naturally identified with $\boldsymbol{P}^r \times \boldsymbol{P}^s, \boldsymbol{P}^r, \boldsymbol{P}^s$, and the maps $\tau_1|O_0: O_0 \to O_1$, $\tau_2|O_0: O_0 \to O_2$ with the projections $\boldsymbol{P}^r \times \boldsymbol{P}^s \to \boldsymbol{P}^r$, $\boldsymbol{P}^r \times \boldsymbol{P}^s \to \boldsymbol{P}^s$. The following proposition is an immediate consequence of a theorem of Fujita [5] on the Levi problem for the product of projective spaces.

Proposition 1. *If D satisfies the condition* (*), *then one of the following four cases occurs*:

(i) *R_0 is empty or a Stein domain in O_0.*

(ii) *$R_0 = \tau_1^{-1}(E_1) \cong E_1 \times \boldsymbol{P}^s$, where E_1 is a Stein domain in O_1.*

(iii) *$R_0 = \tau_2^{-1}(E_2) \cong \boldsymbol{P}^r \times E_2$, where E_2 is a Stein domain in O_2.*

(iv) *$R_0 = O_0 \cong \boldsymbol{P}^r \times \boldsymbol{P}^s$.*

Now we observe how R_0 is related with R_1, R_2.

Suppose that $R_1 \neq \phi$, and take a point $q \in R_1$. We will show that $\tau_1^{-1}(q) \subset R_0$. Let U be a neighborhood of q such that $U \backslash O_1 \subset D_1$. Then

$$\tau_1^{-1}(U) \backslash O_0 = \tau_1^{-1}(U \backslash O_1) \subset \tau_1^{-1}(D_1) = D_0.$$

Here, $\tau_1^{-1}(U)$ is an open set containing $\tau_1^{-1}(q)$. Therefore all points in $\tau_1^{-1}(q)$ are removable boundary points of D_0 along O_0, i.e., $\tau_1^{-1}(q) \subset R_0$.

Conversely, suppose that R_0 contains a set of the form $\tau_1^{-1}(q)$, $q \in O_1$. We will show that $q \in R_1$. Since $\tau_1^{-1}(q)$ is compact, we can take an open set V in Z_0 containing $\tau_1^{-1}(q)$ such that $V \backslash O_0 \subset D_0$. Since τ_1 is a proper map, we can take a neighborhood U of q in Z_1 such that $\tau_1^{-1}(U) \subset V$. Then

$$U \backslash O_1 \subset \tau_1(V \backslash O_0) \subset \tau_1(D_0) = D_1$$

Therefore $q \in R_1$.

These observations are valid also for R_2.

Combining these with Proposition 1, we obtain

Proposition 2. *Suppose that D satisfies the condition* (*). *We have $R_1 = \phi$, $R_2 = \phi$ in the case* (i) *of Proposition* 1; *$R_1 = E_1$, $R_2 = \phi$ in the case* (ii); *$R_1 = \phi$, $R_2 = E_2$ in the case* (iii); *and $R_1 = O_1$, $R_2 = O_2$ in the case* (iv).

Now we can state the main result.

Theorem. *Let D be a domain in the Segre cone Z satisfying the* *condition* (*). *In the case* (i) *of Proposition* 1, D *is Stein. In the case* (ii), D_1^* *is Stein, and* D *is a non-Stein domain of holomorphy in* Z. *In the case* (iii), D_2^* *is Stein and* D *is a non-Stein domain of holomorphy. In the case* (iv), 0 *is an isolated boundary point of* D, D *is not a domain of holomorphy and* $D \cup \{0\}$ *is a Stein domain.*

The form of R_0 depends only on the form of D in the vicinity of 0. Hence we have

Corollary 1. *A domain in the Segre cone is Stein if it is locally Stein at every boundary point.*

This is also an immediate consequence of a result of Andreotti-Narasimhan [1, Corollary 1 to Theorem 4].

In the case (ii), D is biholomorphic to $D_1 = D_1^* \backslash R_1$, where R_1 is an analytic set of codimension $s+1$ in D_1^*. The situation is similar in the case (iii). Hence we have

Corollary 2. *If* D *is a domain in the Segre cone satisfying the condition* (*), *the set of all holomorphic functions on* D *constitutes a Stein algebra.*

§ 4. Proof of Theorem

Consider the domain $\tilde{D} := \rho^{-1}(D)$ in A. We notice that $\tilde{D} \subseteq A_0$ if $0 \notin D$, and $L_0 \subset \tilde{D}$ if $0 \in D$. Further we have $\rho_k^{-1}(D_k) = \tilde{D} \cap A_k$, $k = 1, 2$. Let $\tilde{R}$ be the set of all boundary points of $\tilde{D}$ that are removable along L_0, and let $\tilde{D}^* = \tilde{D} \cup \tilde{R}$ be the extension of $\tilde{D}$ along L_0. The set $\tilde{R}$ is related with R_1, R_2 in the following way.

Proposition 3. *We have* $\rho_k^{-1}(R_k) = \tilde{R} \cap A_k$ *and* $\rho_k^{-1}(D_k^*) = \tilde{D}^* \cap A_k$, $k = 1, 2$.

Proof. First we remark that, if $0 \in D$, then the sets $\tilde{R}$, R_1, R_2 are empty and the proposition is trivially true.

Suppose that $q \in R_1$. We choose a neighborhood U of q in Z_1 such that $U \backslash O_1 \subset D_1$. Then $\rho_1^{-1}(U)$ is an open set in A_1 such that

$$\rho_1^{-1}(U) \backslash L_2 = \rho_1^{-1}(U \backslash O_1) \subset \rho_1^{-1}(D_1) = \tilde{D}.$$

This shows that all points in $\rho_1^{-1}(q)$ are removable along L_2. Hence $\rho_1^{-1}(R_1) \subseteq R \cap A_1$.

Conversely suppose that $(x, 0) \in \tilde{R} \cap A_1$. We choose a neighborhood W of $(x, 0)$ such that $W \backslash L_0 \subset \tilde{D}$. By shrinking W, we can suppose that $W \subset A_1$ and hence $W \backslash L_2 \subset \tilde{D}$. Then we have

$$\rho_1(W) \backslash O_1 = \rho_1(W \backslash L_2) \subset \rho_1(\tilde{D}) = D_1.$$

Here $\rho_1(W)$ is an open set in Z_1, since the projection ρ_1 of the vector bundle $A_1 \to Z_1$ is an open map. Thus $\rho_1(x, 0) = (0, [x])$ is a removable boundary point of D_1 along O_1. Hence $\rho_1^{-1}(R_1) \supseteq \tilde{R} \cap A_1$.

Thus we have shown $\rho_1^{-1}(R_1) = \tilde{R} \cap A_1$. From this follows immediately that $\rho_1^{-1}(D_1^*) = \tilde{D}^* \cap A_1$. The proof for $k=2$ is similar.

q. e. d.

Proposition 4. *Suppose that D satisfies the condition* (*). *Then $\tilde{D}^*$ is Stein. $\tilde{D}$ is Stein if and only if $\tilde{R} = \phi$.*

Proof. The domain $\tilde{D}$ is pseudoconvex at every boundary point in $A_0 = A \backslash L_0$. Hence, by Lemma, $\tilde{D}^*$ is pseudoconvex at every boundary point. $\tilde{D}^*$ is Stein by Oka's theorem. We have $\tilde{D} = \tilde{D}^* \backslash \tilde{R}$, where $\tilde{R}$ is an analytic set of codimension $\geqq 2$ if it is not empty. The second assertion follows from this fact. q. e. d.

Let us consider the case (i) of Proposition 1, i. e., the case where R_0 is empty or Stein. We have $R_1 = \phi$, $R_2 = \phi$ by Proposition 2, and hence $\tilde{R} \cap A_1 = \phi$, $\tilde{R} \cap A_2 = \phi$ by Proposition 3. Since $\tilde{R}$ is an open subset of L_0, we have $\tilde{R} = \phi$. Therefore $\tilde{D}$ is Stein by Proposition 4. The theorem is proved for the case (i) if we show the following

Proposition 5. *D is Stein if $\tilde{D}$ is Stein.*

Proof. If $0 \notin D$, the map $\rho | \tilde{D}: \tilde{D} \to D$ defines a holomorphic principal bundle with structure group $\boldsymbol{C}^*$. For this case, the proposition follows from a theorem of Matsushima-Morimoto [9, Théorème 5].

To cover the general case, we will go back to the construction of holomorphic functions on D. The domain D is Stein if it has the following property: For any sequence $\{z_\kappa\}_\kappa$ of points in D which has no accumulation point in D and for any sequence $\{c_\kappa\}_\kappa$ of complex numbers, there exists a holomorphic function f no D such that $f(z_\kappa)=c_\kappa$.

Consider the analytic set $\rho^{-1}(\{z_\kappa\})$ in $\tilde{D}$. If $\tilde{D}$ is Stein, there is a holomorphic function F on $\tilde{D}$ such that $F|\rho^{-1}(z_\kappa)\equiv c_\kappa$ for all κ. When $z\in D\setminus\{0\}$, the fiber $\rho^{-1}(z)$ is biholomorphic to $\boldsymbol{C}^*$. Choose a fiber coordinate w on $\rho^{-1}(z)$ so that $\rho^{-1}(z)\cong\{w\in\boldsymbol{C}\mid w\neq 0\}$, and a smooth Jordan curve γ_z equipped with orientation such that $\frac{1}{2\pi i}\int_{\gamma_z}\frac{dw}{w}=1$. We define

$$f(z):=\frac{1}{2\pi i}\int_{\gamma_z}(F|\rho^{-1}(z))(w)\frac{dw}{w}.$$

In other words, $f(z)$ is the constant term of the Laurent expansion of $(F|\rho^{-1}(z))(w)$ in w. Clearly $f(z)$ is defined independently of the choice of w and γ_z. The function f is holomorphic on $D\setminus\{0\}$, and $f(z_\kappa)=c_\kappa$ if $z_\kappa\neq 0$.

When $0\in D$, we define $f(0)=F(0,0)$, Then f is holomorphic at 0. To show this we specify γ_z as follows: Let

$$\Gamma:=\{(x,y)\in A\mid |x_0|^2+\ldots+|x_r|^2=|y_0|^2+\ldots+|y_s|^2\}$$

and $\gamma_z:=\Gamma\cap\rho^{-1}(z)$, with an appropriate orientation. When z tends to $0\in Z$, the set γ_z tends to $(0,0)\in A$; hence $f(z)$ tends to $F(0,0)=f(0)$. Thus f is continuous and hence holomorphic at 0, because Z is a normal space. Clearly $f(z_\kappa)=c_\kappa$ for $z_\kappa=0$, too.

Thus f is a holomorphic function with the desired property.

q. e. d.

Let us next consider the case (ii) of Proposition 1, i. e., $R_0=\tau_1^{-1}(R_1)$, $R_1\neq\phi$ and $R_2=\phi$. We have $\tilde{R}\cap A_2=\phi$ by Proposition 3 and hence $\tilde{D}^*\subseteq A_1$. Therefore we have $\rho_1^{-1}(D_1^*)=\tilde{D}^*$. The domain $\tilde{D}^*$ is Stein by Proposition 4. The map $\rho_1|\tilde{D}^*$ defines a holomorphic principal bundle with structure group $\boldsymbol{C}^*$ over D_1^*. By the theorem of Matsushima-Morimoto, mentioned above, D_1^* is Stein.

D is biholomorphic to $D_1=D_1^*\setminus R_1$, where R_1 is a nonempty analytic set in D_1^* of codimension $s+1$. Therefore D is not Stein. Since D_1^* is Stein, it is a domain of holomorphy in Z_1, i. e., there is a holomorphic function g_1 on D_1^* which is singular at all points in ∂D_1^*. We set $g=g_1\circ(\sigma_1|D_1)^{-1}$. Then g is holomorphic on D and singular at all points in $\partial D \backslash \{0\}$. Consequently it is singular also at 0. Thus D is a domain of holomorphy.

We have thus proved the theorem for the case (ii). The case (iii) is treated in the same way.

Finally consider the case (iv). There is an open set V containing O_0 such that $V \backslash O_0 \subset D_0$. Hence $\sigma_0(V) \backslash \{0\} = \sigma_0(V \backslash O_0) \subset \sigma_0(D_0) = D$. This implies that 0 is an isolated boundary point of D. $D \cup \{0\}$ is a domain in Z containing 0, which falls upon the case (i).

This completes the proof of the theorem.

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