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A Characterization of Weak Radon-Nikodym Sets in Dual Banach Spaces

By

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§ 1. Introduction

Throughout this paper X and Y denote real Banach spaces with topological duals X^* and Y^* respectively. The closed unit ball in X is denoted by B_X. In the following, (Ω, Σ, μ) always denotes a complete finite measure space and $([0, 1], \Lambda, \lambda)$ is the Lebesgue measure space on $[0, 1]$. For each (Ω, Σ, μ), a function $f : \Omega \to X$ is said to be weakly measurable if for each $x^* \in X^*$ the real-valued function $(x^*, f(\omega))$ is μ-measurable. We say that a weakly measurable function $f : \Omega \to X$ is Pettis integrable if $(x^*, f(\omega)) \in L_1(\Omega, \Sigma, \mu)$ for every $x^* \in X^*$ and moreover for each $E \in \Sigma$ there exists an element $x_E \in X$ that satisfies

$$(x^*, x_E) = \int_E (x^*, f(\omega)) d\mu(\omega)$$

for every $x^* \in X^*$.

A subset C of X is said to have the weak Radon-Nikodym property if for any (Ω, Σ, μ) and any measure $\alpha : \Sigma \to X$ for which $\alpha(E) \in \mu(E) \cdot C$ for every $E \in \Sigma$, there exists a Pettis integrable function $g : \Omega \to C$ such that

$$(x^*, \alpha(E)) = \int_E (x^*, g(\omega)) d\mu(\omega)$$

for each $E \in \Sigma$ and $x^* \in X^*$. Such a set C is called a weak Radon-Nikodym set. A Banach space X is said to have the weak Radon-Nikodym property if B_X is a weak Radon-Nikodym set.

A sequence $\{x_n\}_{n \geq 1}$ in X is called a tree if $x_n = (x_{2n} + x_{2n+1})/2$ for

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all $n\geqq 1$. Following Riddle and Uhl [6], we say that a tree $\{x_n\}_{n\geqq 1}$ is a δ-Rademacher tree if there exists a $\delta>0$ such that

$$||x_1||\geqq\delta,\quad ||x_2-x_3||\geqq 2\delta,\quad ||x_4-x_5+x_6-x_7||\geqq 4\delta,$$

and, in general,

$$||\sum_{n=2^m}^{2^{m+1}-1}(-1)^n x_n||\quad (=||\sum_{i=0}^{2^m-1}(-1)^i x_{2^m+i}||)\geqq 2^m\delta$$

for all $m\geqq 0$. Further it is said that a tree $\{x_n\}_{n\geqq 1}$ is a δ-tree if there exists a $\delta>0$ such that $||x_n-x_{2n}||\geqq\delta$ and $||x_n-x_{2n+1}||\geqq\delta$ for all $n\geqq 1$.

Corresponding to that of dual Banach spaces with the Radon-Nikodym property in terms of δ-trees, a characterization of such spaces with the weak Radon-Nikodym property as well as Banach spaces not containing a copy of l_1 in terms of δ-Rademacher trees has been obtained by Riddle and Uhl [6]. That is,

Theorem A (Theorem B in § 3 of [6]). *A Banach space X contains no copy of l_1 if and only if X^* contains no bounded δ-Rademacher tree. That is, the set B_{X^*} is a weak Radon-Nikodym set if and only if it contains no δ-Rademacher tree.*

Succeedingly, Riddle [4] made an attempt to localize Theorem A and obtained the following:

Theorem B (Theorem 2 in [4]). *For a bounded linear operator $T: X\rightarrow Y$, the set $T^*(B_{Y^*})$ is a weak Radon-Nikodym set if and only if it contains no δ-Rademacher tree. Consequently, a weak*-compact absolutely convex subset of X^* is a weak Radon-Nikodym set if and only if it contains no δ-Rademacher tree.*

Recall that a subset C of X is weakly precompact if every bounded sequence in C has a weakly Cauchy subsequence. In his fundamental paper [7], Rosenthal proved that a bounded subset of X is weakly precompact if and only if it contains no copy of the l_1-basis. Their proofs of Theorems A and B heavily depend on this Rosenthal's signal theorem, and moreover the proof of Theorem B depends on a factorization lemma (Theorem 1 in [4] or Theorem 3 in [5]) which has been shown by a method based on the factorization construction of Davis, Figiel, Johnson and Pelczynski [1].

By the way, we have noticed in [3] that the effective use of the following Theorems I and II instead of these results is very powerful and slightly direct (in the meaning that we can do without any factorization theorem and absolutely convex argument) for investigating some properties (for instance, the weak Radon-Nikodym property) of weak*-compact convex subsets of dual Banach spaces. Before stating Theorem I, we need the

Definition. Let $(A_n, B_n)_{n\geqq 1}$ be a sequence of pairs of some set S with $A_n \cap B_n = \phi$ for all $n \geqq 1$. We say that $(A_n, B_n)_{n\geqq 1}$ converges if every point $s \in S$ belongs to at most finitely many A_n's or finitely many B_n's.

Theorem I (Theorem 3.17 in [8]). *Let Z be a Polish space and $(A_n, B_n)_{n\geqq 1}$ a sequence of pairs of subsets of Z with A_n, B_n closed and $A_n \cap B_n = \phi$. Assume that $(A_n, B_n)_{n\geqq 1}$ has no convergent subsequence. Then there exist a compact subset Γ of Z homeomorphic to Δ $(=\{0, 1\}^N$, the Cantor set) a homeomorphism σ from Γ onto Δ, and a sequence $n(1) < n(2) < \ldots$ such that $A_{n(k)} \cap \Gamma = \sigma^{-1}(U_k)$ and $B_{n(k)} \cap \Gamma = \sigma^{-1}(U_k^c)$ for all $k \geqq 1$. Here $U_k = \{s = \{s_n\}_{n\geqq 1} \in \Delta : s_k = 0\}$.*

Theorem II. *Let $T : X \to Y$ be a bounded linear operator. If K is a weak*-compact convex subset of Y^*, then for every measure $\alpha : \Sigma \to X^*$ such that $\alpha(E) \in \mu(E) \cdot T^*(K)$ for each $E \in \Sigma$, there exists a measure $\beta : \Sigma \to Y^*$ such that*

(1) $\beta(E) \in \mu(E) \cdot K$ *for each* $E \in \Sigma$,

(2) $T^*\beta(E) = \alpha(E)$ *for each* $E \in \Sigma$.

The cornerstone for our result in this paper is these Theorems I, II and Theorem III below. Making the best use of Theorems I, II and III (that is, adopting an argument similar to one employed in [3]), we can present a following characterization of weak*-compact convex (not necessarily absolutely convex) sets with the weak Radon-Nikodym property in terms of δ-Rademacher trees. This is the aim of our paper. Clearly, our characterization theorem is a generalization of Theorems A and B above.

Theorem 1. *Let K be a weak*-compact convex subset of X^*. Then the set K is a weak Radon-Nikodym set if and only if it contains no δ-Rademacher tree.*

§ 2. Proof of Theorem 1

Before proving Theorem 1, let us recall some facts that are needed in the process of our proof of Theorem 1. First, in [9] (or [3]), the following has been obtained.

Theorem III. *Let K be a weak*-compact convex subset of X^*. Then each of the following statements about K implies all the others.*

(1) *The set K is a weak Radon-Nikodym set.*

(2) *The set B_X is weakly precompact with respect to K (namely, every sequence $\{x_n\}_{n\geqq 1}$ in B_X has a subsequence $\{x_{n(k)}\}_{k\geqq 1}$ such that for every $x^*\in K$, $\lim_{k\to\infty}(x^*, x_{n(k)})$ exists).*

(3) *The set K is a set of complete continuity (namely, for every (Ω, Σ, μ), every bounded linear operator $S: L_1(\Omega, \Sigma, \mu)\to X^*$ for which $S(\chi_E/\mu(E))$ belongs to K for each non null set $E\in\Sigma$ is a Dunford-Pettis operator).*

In [3], Theorem III has been proved in our way different from Riddle, Saab and Uhl [5] and Saab [9].

Next we note some fundamental facts concerning the connection between trees and martingales (cf. [6]). Let $\{x_n\}_{n\geqq 1}$ be a tree in X. Then we can define the usual martingale $\{f_n\}_{n\geqq 1}$ in $L_1([0, 1], \Lambda, \lambda, X)$ associated with $\{x_n\}_{n\geqq 1}$. That is, let

$$f_1=x_1\chi_{[0,1]},\quad f_2=x_2\chi_{[0,1/2)}+x_3\chi_{[1/2,1]}$$

and, in general, let

$$f_n=\sum_{i=2^{n-1}}^{2^n-1}x_i\chi_{I_{n,i}}$$

where $I_{n,i}=[(i-2^{n-1})/2^{n-1}, (i-2^{n-1}+1)/2^{n-1})$ for $i=2^{n-1}, 2^{n-1}+1, \dots, 2^n-2$ and $n\geqq 1$ and $I_{n,2^n-1}=[(2^{n-1}-1)/2^{n-1}, 1]$ for $n\geqq 1$. If Λ_0 is the σ-field consisting of ϕ and $[0, 1]$ and Λ_n is the finite σ-field generated by $\{I_{n,i}: i=2^{n-1}, 2^{n-1}+1, \dots, 2^n-1\}$, $n\geqq 1$, then $(f_n, \Lambda_n, n\geqq 1)$ is a martingale in $L_1([0, 1], \Lambda, \lambda, X)$. Let $\{x_n\}_{n\geqq 1}$ be bounded. Then, by virtue of the crucial property of the martingale $(f_n, \Lambda_n, n \geqq 1)$ in $L_1([0, 1], \Lambda, \lambda, X)$ and the boundedness of $\{x_n\}_{n\geqq 1}$, we can define a measure $\alpha : \Lambda \to X$ that satisfies $\alpha(E) = \lim_{n\to\infty} \int_E f_n d\lambda$ for each $E \in \Lambda$. Moreover the measure α induces a bounded linear operator $S : L_1([0, 1], \Lambda, \lambda) \to X$ such that

$$S(g) = \lim_{n\to\infty} \int_{[0,1]} f_n(t) g(t) d\lambda(t)$$

for each $g \in L_1([0, 1], \Lambda, \lambda)$.

In the following, we always understand that for every weak*-compact convex subset K of a dual Banach space X^*, K is equipped with the weak*-topology.

Under these preparations, we are ready to prove Theorem 1.

Proof of Theorem 1. (Necessity). Although the proof of this part can be given by the same argument as in the necessity part of Theorem 2 in [4], we dare give it for the sake of completeness. Suppose that K contains a δ-Rademacher tree $\{x_n^*\}_{n\geqq 1}$. Then, by the remark stated above, we have a bounded linear operator $S : L_1([0, 1], \Lambda, \lambda) \to X^*$ such that $S(g) = \lim_{n\to\infty} \int f_n g d\lambda$ for each $g \in L_1([0, 1], \Lambda, \lambda)$, where $\{f_n\}_{n\geqq 1}$ is the usual martingale associated with the tree $\{x_n^*\}_{n\geqq 1}$. Now we easily have $S(\chi_E) = \lim_{n\to\infty} \int_E f_n d\lambda \in \lambda(E) \cdot K$ for each $E \in \bigcup_{n\geqq 1} \Lambda_n$ by the property of the martingale. So it follows that $S(\chi_E) \in \lambda(E) \cdot K$ for each $E \in \Lambda$ by the routine calculation. Moreover we get that if r_n is the nth Rademacher function, then

$$\|S(r_j)\| = \lim_{n\to\infty} \|\int f_n r_j d\lambda\| \geqq \delta$$

for all $j \geqq 1$. Since the sequence $\{r_n\}_{n\geqq 1}$ converges to 0 weakly in $L_1([0, 1], \Lambda, \lambda)$, we know that S is not a Dunford-Pettis operator. So, in virtue of Theorem III, we have that K is not a weak Radon-Nikodym set.

(Sufficiency). This is the crucial part of our proof. Suppose that K is not a weak Radon-Nikodym set. Invoke Theorem III to conclude that there exists a sequence $\{x_n\}_{n\geqq 1}$ in B_X having no pointwise convergent subsequence on K. Let Y denote the closed linear span of the set $\{x_n : n \geqq 1\}$, and let $j : Y \to X$ be the inclusion map.

Then a sequence $\{x_n\}_{n\geqq 1}$ in B_Y has no point-wise convergent subsequence on $j^*(K)$. Hence, by the same argument as in the proof of Lemma 3 in [3], we have a subsequence $\{x_{n(k)}\}_{k\geqq 1}$ of $\{x_n\}_{n\geqq 1}$ and real numbers r and η with $\eta>0$ such that putting $A_k=\{y^*\in j^*(K) : (y^*, x_{n(k)})\geqq r+2\eta\}$ and $B_k=\{y^*\in j^*(K) : (y^*, x_{n(k)})\leqq r\}$ for all $k\geqq 1$, then $(A_k, B_k)_{k\geqq 1}$ is a sequence of pairs of disjoint closed subsets of $j^*(K)$ having no convergent subsequence. Since $j^*(K)$ is a compact metrizable space, in virtue of Theorem I, there exist a compact subset Γ of $j^*(K)$ homeomorphic to Δ, a homeomorphism σ from Γ onto Δ, and a sequence $k(1)<k(2)<\dots$ such that $A_{k(i)}\cap\Gamma=\sigma^{-1}(U_i)$ and $B_{k(i)}\cap\Gamma=\sigma^{-1}(U_i^c)$ for all $i\geqq 1$. Let $\psi=\sigma^{-1} : \Delta\to\Gamma$. Then $\psi(U_i)=A_{k(i)}\cap\Gamma$ and $\psi(U_i^c)=B_{k(i)}\cap\Gamma$. Let $\phi : [0, 1]\to\Delta$ be the Sierpinski's function (see, for instance, § 1 in [2]). Consider a measure $\alpha : \Lambda\to Y^*$ defined for each $E\in\Lambda$ by putting

$$(\alpha(E), y)=\int_E (\psi(\phi(t)), y)\,d\lambda(t)$$

for every $y\in Y$. Then we have $\alpha(E)\in\lambda(E)\cdot j^*(K)$ for each $E\in\Lambda$. Furthermore we have

$$\begin{aligned}(\alpha(\phi^{-1}(B)), y)&=\int_{\phi^{-1}(B)} (\psi(\phi(t)), y)\,d\lambda(t)\\&=\int_B (\psi(s), y)\,d\phi(\lambda)(s)=\int_B (\psi(s), y)\,d\nu(s)\end{aligned}$$

for each $B\in\mathfrak{B}(\Delta)$ (the Borel σ-field of Δ), making use of the change-of-variables formula. Here ν denotes the normalized Haar measure on Δ. In virtue of Theorem II, there exists a measure $\beta : \Lambda\to X^*$ such that

(1) $\beta(E)\in\lambda(E)\cdot K$ for each $E\in\Lambda$,

(2) $j^*\beta(E)=\alpha(E)$ for each $E\in\Lambda$.

Define a sequence $\{x_n^*\}_{n\geqq 1}$ in X^* by

$$\begin{gathered}x_1^*=\beta(\phi^{-1}(\Delta)),\\ x_2^*=2\beta(\phi^{-1}(U_1)),\quad x_3^*=2\beta(\phi^{-1}(U_1^c)),\\ x_4^*=2^2\beta(\phi^{-1}(U_1\cap U_2)),\quad x_5^*=2^2\beta(\phi^{-1}(U_1\cap U_2^c)),\\ x_6^*=2^2\beta(\phi^{-1}(U_1^c\cap U_2)),\quad x_7^*=2^2\beta(\phi^{-1}(U_1^c\cap U_2^c)),\ \text{etc.}\end{gathered}$$

That is, $x_1^*=\beta(\phi^{-1}(\Delta))$ and if $x^*_{2^m+i}=2^m\beta(\phi^{-1}(U_1^{\varepsilon^1(i)}\cap\dots\cap U_m^{\varepsilon^m(i)}))$ $(0\leqq i\leqq 2^m-1,\ m\geqq 1)$, then

$$x^*_{2^{m+1}+2i}=2^{m+1}\beta(\phi^{-1}(U_1^{\varepsilon^1(i)}\cap\dots\cap U_m^{\varepsilon^m(i)}\cap U_{m+1}))$$

and

$$x^*_{2^{m+1}+2i+1} = 2^{m+1}\beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)} \cap U_{m+1}^c)).$$

Here $\{\varepsilon^j(i)\}_{1 \leqq j \leqq m}$ denotes a sequence consisting of 1 or c (complement) and $\{(\varepsilon^1(i), \dots, \varepsilon^m(i)) : 0 \leqq i \leqq 2^m - 1\} = \{(\varepsilon^1, \dots, \varepsilon^m) : \varepsilon^j = 1 \text{ or } c\}$ for all $m \geqq 1$. Then we have

$$\begin{aligned}
&(x^*_{2^{m+1}+2i} + x^*_{2^{m+1}+2i+1})/2 \\
&= 2^m \{\beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)} \cap U_{m+1})) \\
&\qquad + \beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)} \cap U_{m+1}^c))\} \\
&= 2^m \beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)})) = x^*_{2^m+i}.
\end{aligned}$$

Clearly $(x_2^* + x_3^*)/2 = x_1^*$. Hence it holds that $x_n^* = (x_{2n}^* + x_{2n+1}^*)/2$ for all $n \geqq 1$. Further, by virtue of (1), we have

$$\begin{aligned}
x^*_{2^m+i} &= 2^m \beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)})) \\
&\in 2^m \lambda(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)})) \cdot K \\
&= 2^m \nu(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)}) \cdot K = K
\end{aligned}$$

for all $m \geqq 1$ and all i with $0 \leqq i \leqq 2^m - 1$. Clearly $x_1^* \in K$. So we get that $x_n^* \in K$ for all $n \geqq 1$. Finally, we have for all $m \geqq 0$

$$\begin{aligned}
&\Big(\sum_{i=0}^{2^{m+1}-1} (-1)^i x^*_{2^{m+1}+i}, jx_{n(k(m+1))}\Big) \\
&= \sum_{i=0}^{2^m-1} (x^*_{2^{m+1}+2i}, jx_{n(k(m+1))}) - \sum_{i=0}^{2^m-1} (x^*_{2^{m+1}+2i+1}, jx_{n(k(m+1))}) \\
&= 2^{m+1} \cdot \sum_{i=0}^{2^m-1} (\beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)} \cap U_{m+1})) \\
&\qquad - \beta(\phi^{-1}(U_1^{\varepsilon^1(i)} \cap \dots \cap U_m^{\varepsilon^m(i)} \cap U_{m+1}^c)), jx_{n(k(m+1))}) \\
&= 2^{m+1}(\beta(\phi^{-1}(U_{m+1})) - \beta(\phi^{-1}(U_{m+1}^c)), jx_{n(k(m+1))}) \\
&= 2^{m+1}(j^*\beta(\phi^{-1}(U_{m+1})) - j^*\beta(\phi^{-1}(U_{m+1}^c)), x_{n(k(m+1))}) \\
&= 2^{m+1}(\alpha(\phi^{-1}(U_{m+1})) - \alpha(\phi^{-1}(U_{m+1}^c)), x_{n(k(m+1))}) \quad \text{(by (2))} \\
&= 2^{m+1} \cdot \Big\{\int_{U_{m+1}} (\psi(s), x_{n(k(m+1))}) d\nu(s) \\
&\qquad - \int_{U_{m+1}^c} (\psi(s), x_{n(k(m+1))}) d\nu(s)\Big\}.
\end{aligned}$$

Let $s_1 \in U_{m+1}$ and $s_2 \in U_{m+1}^c$. Then we have $(\psi(s_1), x_{n(k(m+1))}) \geqq r + 2\eta$ and $-(\psi(s_2), x_{n(k(m+1))}) \geqq -r$. Thus we have for all $m \geqq 0$,

$$\begin{aligned}
&\|\sum_{i=0}^{2^{m+1}-1}(-1)^i x^*_{2^{m+1}+i}\| \\
&\geqq\Big(\sum_{i=0}^{2^{m+1}-1}(-1)^i x^*_{2^{m+1}+i},\ jx_{n(k(m+1))}\Big) \\
&\geqq 2^{m+1}\cdot\{(r+2\eta)/2-r/2\}=2^{m+1}\eta.
\end{aligned}$$

In the case where $x_1^*\neq 0$, setting $\delta=\min\{\|x_1^*\|,\ \eta\}\,(>0)$, then we obtain a δ-Rademacher tree $\{x_n^*\}_{n\geqq 1}$ contained in K. Next consider the case where $x_1^*=0$. Then, choose a non-zero element a^* of K, since the sequence $\{x_n\}_{n\geqq 1}$ has no point-wise convergent subsequence on K. Setting $\delta=\min\{\|a^*\|/2,\ \eta/2\}\,(>0)$ and $a_n^*=(x_n^*+a^*)/2$, then we obtain a δ-Rademacher tree $\{a_n^*\}_{n\geqq 1}$ contained in K. Thus the proof is completed.

As an immediate result of our Theorem 1, we have:

Corollary. *Let $T:X\rightarrow Y$ be a bounded linear operator and let K be a weak*-compact convex subset of Y^*. Then the set $T^*(K)$ is a weak Radon-Nikodym set if and only if it contains no δ-Rademacher tree.*

Setting $K=B_{Y^*}$ in Corollary, we have Theorem B stated above. Furthermore, our Theorem 1 combined with Theorem III yields the following:

Theorem 2. *Let K be a weak*-compact convex subset of X^*. Then each of the following statements about K implies all the others.*

(1) *The set K is a weak Radon-Nikodym set.*

(2) *The set B_X is weakly precompact with respect to K.*

(3) *The set K is a set of complete continuity.*

(4) *The set K contains no δ-Rademacher tree.*

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