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# On Wall Manifolds with ($\varepsilon$)-Free Involutions

*Dedicated to Professor Nobuo Shimada on his 60th birthday*

By

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## § 0. Introduction

Let $\mathfrak{N}_*$ denote the unoriented cobordism ring. Then $\mathfrak{N}_*$ is the polynomial algebra over $\boldsymbol{Z}_2$ on classes $X_k$ of dimension $k$ for each integer $k$ not of the form $2^i-1$. Let $\boldsymbol{W}_*=\boldsymbol{Z}_2[X_{2k-1}, X_{2k}; k\neq 2^i, (X_{2^i})^2]$ be the polynomial subalgebra of $\mathfrak{N}_*$ defined by Wall [10]. As the bordism theory of free involutions on Wall manifolds, let $\boldsymbol{W}_*^{\varepsilon}(\boldsymbol{Z}_2)$ be the bordism group of Wall manifolds with ($\varepsilon$)-free involutions studied by Komiya in [3]. (Here $\varepsilon$ denotes one of the signs $+$ and $-$.) While let $\mathfrak{N}_*(\boldsymbol{Z}_2)$ denote the unoriented bordism group of free involutions, then there are homomorphisms $F_\varepsilon: \boldsymbol{W}_*^{\varepsilon}(\boldsymbol{Z}_2)\to\mathfrak{N}_*(\boldsymbol{Z}_2)$ which forget a Wall structure. The following theorem is known.

**Theorem** (cf. [3]). *The two homomorphisms $F_\varepsilon$ are monic. Moreover, the image of each $F_\varepsilon$ is a direct summand of $\mathfrak{N}_*(\boldsymbol{Z}_2)$.*

In this paper, we study the image of each $F_\varepsilon$ as a subgroup of $\mathfrak{N}_*(\boldsymbol{Z}_2)$.

In §1, we define the bordism groups $\boldsymbol{W}_*^{\varepsilon}(\boldsymbol{Z}_2)$, and give some examples of Wall manifolds with ($\varepsilon$)-free involutions.

In §2, we state the image of each $F_\varepsilon$ in Theorem 2.8.

In §3, we study Wall manifolds with ($\varepsilon$)-free involutions derived from some generators of $\Omega_*^{\varepsilon}(\boldsymbol{Z}_2)$, the oriented bordism module of all orientation-preserving (or orientation-reversing) free involutions ([5]),

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and calculate their images by the maps $F_\varepsilon$ (Theorems 3. 6 and 3. 8). Finally we refer to the elements in $\mathfrak{N}_*(\boldsymbol{Z}_2)$ which belong to both $\mathrm{Im}F_+$ and $\mathrm{Im}F_-$ (Theorem 3. 12).

## §1. Wall Manifolds with $(\varepsilon)$-Free Involutions

**Definition 1. 1** (cf. [6]). Let $M$ be a compact smooth manifold (with or without boundary), let $\det \tau_M$ be the determinant bundle of the tangent bundle $\tau_M$ of $M$, and let $\alpha: M \to RP(\infty)$ be a classifying map of $\det \tau_M$ where $\xi \to RP(\infty)$ is the universal line bundle. Then $\alpha(M) \subset RP(r)$ for some $r \geqq 0$, since $M$ is compact. Such $\alpha$ is called an $RP(r)$ *-structure* of $M$.

**Definition 1. 2** (cf. [3]). A *Wall manifold with* $(\varepsilon)$ *-free involution* is a triple $(M, \mu, \alpha)$ where:

(i) $M$ is a compact smooth unoriented manifold and $\mu$ is a free involution on $M$;

(ii) $\alpha: M \to RP(1)$ is a $RP(1)$-structure of $M$ which is equivariant with respect to $\mu$, i. e., $\alpha \circ \mu = \alpha$; and

(iii) $\bar{\alpha} \circ \varepsilon \det d_\mu = \bar{\alpha}$ for a bundle map $\bar{\alpha}$ covering $\alpha$ where $\det d_\mu$: $\det \tau_M \to \det \tau_M$ is the map induced by $\mu$ and $\varepsilon \det d_\mu$ means $\det d_\mu$ [resp., $-\det d_\mu$] if $\varepsilon = +$ [resp., $\varepsilon = -$].

The *boundary* $\partial M$ of $M$ is a Wall manifold $(\partial M, \mu|\partial M, \alpha|\partial M)$ since $\det \tau_{\partial M}$ is identified with $(\det \tau_M)|\partial M$ by the inner unit normal vector.

**Definition 1. 3.** Let $\varepsilon$ be one of the signs $+$ and $-$. We say that two closed Wall manifolds with $(\varepsilon)$-free involutions, $(M, \mu, \alpha)$ and $(M', \mu', \alpha')$ are *bordant* if there is a Wall manifold with $(\varepsilon)$-free involution $(W, \nu, \beta)$ such that the disjoint sum $(M, \mu, \alpha) + (M', \mu', \alpha') = (\partial W, \nu|\partial W, \beta|\partial W)$ and an equivariant $RP(1)$-structure $\beta$ satisfies $\bar{\beta}|M = \bar{\alpha}$ and $\bar{\beta}|M' = \bar{\alpha}'$, where $\bar{\alpha}$, $\bar{\alpha}'$ and $\bar{\beta}$ are bundle maps covering $\alpha$, $\alpha'$ and $\beta$, respectively.

**Definition 1. 4.** (cf. [3]). For each $\varepsilon$, we denote the bordism class of $(M, \mu, \alpha)$ by $[M, \mu, \alpha]$, and define

$$W_m^{\varepsilon}(\boldsymbol{Z}_2) = \{[M, \mu, \alpha] \mid \bar{\alpha}\circ\varepsilon \ \det d_{\mu} = \bar{\alpha},\ \dim M = m\}$$

which is an abelian group by the disjoint sum. Thus we have a graded abelian group

$$W_*^{\varepsilon}(\boldsymbol{Z}_2) = \Sigma_m W_m^{\varepsilon}(\boldsymbol{Z}_2),$$

the *bordism group of Wall manifolds with* $(\varepsilon)$ *-free involutions.*

Let $\Omega_*^{\varepsilon}(\boldsymbol{Z}_2) = \Sigma_m \Omega_m^{\varepsilon}(\boldsymbol{Z}_2)$ be the oriented bordism group of all orientation-preserving free involutions $(\varepsilon = +)$ or all orientation-reversing free involutions $(\varepsilon = -)$ (cf. [5]). Then we obtain the exact triangles of the type of Wall.

**Theorem 1.5** (cf. [3]). *For each* $\varepsilon$, *there is an exact triangle:*

$$(W_{\varepsilon}):\quad \begin{array}{ccc} \Omega_*^{\varepsilon}(\boldsymbol{Z}_2) & \xrightarrow{\ 2\ } & \Omega_*^{\varepsilon}(\boldsymbol{Z}_2) \\ {}^{\delta}\nwarrow & & \swarrow{}^{\rho} \\ & W_*^{\varepsilon}(\boldsymbol{Z}_2) & \end{array}$$

*in which* 2 *is the multiplication by the integer* 2, $\rho$ *is defined by considering* $[M, \mu] \in \Omega_*^{\varepsilon}(\boldsymbol{Z}_2)$ *as a Wall manifold with the trivial* $RP(1)$ *-structure* $1: M \to \{1\} \subset S^1 = RP(1)$, *and* $\delta$ *sends* $[M, \mu, \alpha] \in W_m^{\varepsilon}(\boldsymbol{Z}_2)$ *into* $[N, \mu | N] \in \Omega_{m-1}^{\varepsilon}(\boldsymbol{Z}_2)$, *where* $N$ *is the invariant submanifold of* $M$ *dual to* $\det \tau_M$.

To say the fundamental type of Wall manifolds with $(\varepsilon)$ -free involutions, we introduce the following notion.

**Definition 1.6** (cf. [4; p. 88]). Let $(M, \mu)$ be an orientation-preserving involution $\mu$ on an oriented manifold $M$. We say that $(M, \mu)$ is *equivariant reversible* if $M$ admits an orientation-reversing diffeomorphism $R$ such that $\mu\circ R = R\circ\mu$. When $(M, \mu)$ is an orientation-reversing involution, this is always equivariant reversible by considering $R = \mu$.

**Definition 1.7** (cf. [10; §3]). Let $(M, \mu)$ be an orientation-preserving free involution which is equivariant reversible, then $(S^1 \times_R M, id \times \mu, \alpha)$ is a Wall manifold with $(+)$ -free involution where $S^1 \times_R M = S^1 \times M / a \times R$ is the twisted product of the unit circle with antipodal map and $(M, \mu)$, $id$ is the identity map, and $\alpha([z, m]) =$

$[z]\in RP(1)$ for $z\in S^1$, $m\in M$. While let $(M, \mu)$ be an orientation-reversing free involution, then we obtain a Wall manifold with $(-)$-free involution $(S^1\times_{\mu}M, id\times\mu, \alpha)$ in the same way. From now on we denote an involution $(S^1\times_{\mu}M, id\times\mu)$ by $(S^1\otimes M, id\times\mu)$ if no confusion can arise.

**Example 1.8.**

(1.8.1) The antipodal maps on spheres $(S^{2n+1}, a)$ and $(S^{2n}, a)$ are Wall manifolds with $(+)$ and $(-)$-free involutions with the trivial structure 1 respectively.

(1.8.2) Let $R: S^{2n+1}\to S^{2n+1}$ be the reflection defined by $R(x_0, x_1, \ldots, x_{2n+1})=(x_0, -x_1, \ldots, -x_{2n+1})$ for $(x_0, x_1, \ldots, x_{2n+1})\in S^{2n+1}$. Then $(S^1\times_R S^{2n+1}, id\times a, \alpha)$ is a Wall manifold with $(+)$- free involution. While

(1.8.3) $(S^1\otimes S^{2n}, id\times a, \alpha)$ is a Wall manifold with $(-)$-free involution.

## §2. On the Images of $F_\varepsilon$

In this section, we study the homomorphism $F_\varepsilon$: $\boldsymbol{W}_*^\varepsilon(\boldsymbol{Z}_2)\to\mathfrak{N}_*(\boldsymbol{Z}_2)$ defined by $F_\varepsilon([M, \mu, \alpha])=[M, \mu]_2$ as mentioned in Introduction where $[M, \mu]_2$ is the unoriented bordism class of a free involution $(M, \mu)$. First we give some natural basis of $\mathfrak{N}_*(\boldsymbol{Z}_2)$.

**Definition 2.1** (cf. [8]). Let $\{X(n)\,|\deg X(n)=n,\ n\geqq 0\}$ be a homogeneous basis of $\mathfrak{N}_*(\boldsymbol{Z}_2)$ which satisfies:

(i) $X(0)=[S^0, a]_2$,

(ii) $\varepsilon_*(X(n))=0$ for all $n\geqq 1$, where $\varepsilon_*$: $\mathfrak{N}_*(\boldsymbol{Z}_2)\to\mathfrak{N}_*$ is the augmentation map, and

(iii) $\Delta(X(n+1))=X(n)$ for all $n\geqq 0$, where $\Delta$ is the Smith homomorphism.

This basis exists and is unique with respect to satisfying (i)-(iii). It is determined inductively by the following relation:

(2.2) $$[S^n, a]_2=\Sigma_{j=0}^n[RP(n-j)]X(j).$$

This follows that for any $[M^n, \tau]_2\in\mathfrak{N}_n(\boldsymbol{Z}_2)$,

(2.3) $$[M, \tau]_2=\Sigma_{j=0}^n\varepsilon_*(\Delta^j([M, \tau]_2))X(j).$$

*Remark* 2.4. $X(2n+1)=\Sigma_{i=0}^{n}a_{2i}[S^{2n-2i+1}, a]_2$ and $X(2n+2)=\Sigma_{i=0}^{n+1}a_{2i}[S^{2n-2i+2}, a]_2$ for all $n\geqq 0$, where the element $a_{2i}$ in $\mathfrak{N}_{2i}$ is defined by $a_0=1$ and $\Sigma_{i=0}^{k}a_{2i}[RP(2k-2i)]=0$ for all $k\geqq 1$ (cf. [9]).

We first study the images of Wall manifolds in Example 1. 8 by the homomorphisms $F_{\varepsilon}$.

**Theorem 2. 5.** *We have*

(i) $F_-([S^1\otimes S^{2n}, id\times a, \alpha])=X(2n+1)$, *and*

(ii) $F_+([S^1\times_R S^{2n+1}, id\times a, \alpha])=X(2n+2)$ *for each* $n\geqq 0$.

*Proof.* For the proof of (i), see the remark 2. 4 and the corollary 2. 5 in [9]. Next we prove (ii). We see that

$$\Delta([S^1\times_R S^{2n+1}, id\times a]_2)=[S^1\otimes S^{2n}, id\times a]_2=X(2n+1) \tag{2.6}$$

for all $n\geqq 0$. Let $V=\{(z, x_0, x_1, \dots, x_{2n+1})\in S^1\times S^{2n+1}\,|\,x_0\geqq 0\}$ and let $W$ be the image of $V$ in $S^1\times_R S^{2n+1}$. Then $W\cup(id\times a)W=S^1\times_R S^{2n+1}$ and $W\cap(id\times a)W=\partial W=S^1\times S_0^{2n}$ where $S_0^{2n}=\{(x_0, x_1, \dots, x_{2n+1})\in S^{2n+1}\,|\,x_0=0\}$ and the reflection $R$ acts on $S_0^{2n}$ as the antipodal map $a$. Thus we obtain the result (2. 6) by [1; (24. 1)]. Put $X'(2n+2)=[S^1\times_R S^{2n+1}, id\times a]_2$, then it is sufficient to show that $\varepsilon_*(X'(2n+2))=0$ by the definition of $\{X(n)\}$. The manifold $S^1\times_R S^{2n+1}/id\times a$ is diffeomorphic to $V^{2n+2}=S^1\times RP(2n+1)/a\times\bar{R}$ by the natural map where $\bar{R}:RP(2n+1)\to RP(2n+1)$ is defined by $\bar{R}[x_0, x_1, \dots, x_{2n+1}]=[x_0, -x_1, \dots, -x_{2n+1}]$. Let $\tau$ be the involution on $V^{2n+2}$ induced by $T_1(z, x)=(\bar{z}, x)$ for $z\in S^1$ and $x\in RP(2n+1)$ here $\bar{z}$ is the conjugation of $z$. Then the fixed point data of $\tau$ implies that

$$\begin{aligned}\varepsilon_*(X'(2n+2))&=[S^1\times_R S^{2n+1}/id\times a]\\&=[RP(2n+2)]+[RP(\xi\oplus 2R)] \text{ (cf. [1; (21. 8), (22. 2)])}\end{aligned}$$

where $RP(\xi\oplus 2R)$ is the projective space bundle associated to $\xi\oplus 2R\to RP(2n)$, the Whitney sum of the canonical line bundle $\xi\to RP(2n)$ and the trivial 2-plane bundle $2R\to RP(2n)$. In other words, $RP(\xi\oplus 2R)$ is the quotient of the fixed point free involution on $S^{2n}\times RP(2)$ given by $T(x, [x_0, x_1, x_2])=(-x, [-x_0, x_1, x_2])$ for $x\in S^{2n}$ and $[x_0, x_1, x_2]\in RP(2)$. Thus we have $[RP(\xi\oplus 2R)]=[RP(2n+2)]$ by Lemmas (2. 2) and (5. 1) in [2], and $\varepsilon_*(X'(2n+2))=2[RP(2n+2)]=0$. q. e. d.

Considering $RP(1)=S^1$, the groups $W_*^\varepsilon(Z_2)$ is the module over $W_*$ via the multiplication $m: S^1\times S^1\to S^1$ (cf. [6; p. 163] for example.). Since the maps $F_\varepsilon$ are monic as mentioned in Introduction, the following lemma is convenient to explain the images of $F_\varepsilon$.

**Lemma 2.7.** *For each* $\varepsilon$, *let* $\{Y(n)\,|\deg Y(n)=n, n\geqq 0\}$ *be a suitable homogeneous* $\mathfrak{N}_*$*-basis of* $\mathfrak{N}_*(Z_2)$ *and let* $\{M(n)\,|\deg M(n)=n,\ n\geqq 0\}$ *be a set of generators of* $W_*^\varepsilon(Z_2)$ *as the* $W_*$*-module such that* $F_\varepsilon(M(n))=Y(n)$ *for all* $n\geqq 0$, *then we have that*:

(i) $W_*^\varepsilon(Z_2)$ *is a free* $W_*$*-module with basis* $\{M(n)\,|\,n\geqq 0\}$ (*When* $=+$, *see* [6; p. 163] *for example.*), *and*

(ii) $\operatorname{Im} F_\varepsilon = W_*\{\{Y(n)\,|n\geqq 0\}\}$ *in* $\mathfrak{N}_*(Z_2)$, *i.e., a free* $W_*$*-module with basis* $\{Y(n)\}$, *obtained by restricting the coefficient ring* $\mathfrak{N}_*$ *to* $W_*$.

From Example 1. 8 and Theorem 2. 5, we can choose the basis $\{Y(n)\}$ as follows.

**Theorem 2.8.** (i) $W_*^+(Z_2)$ *is the free* $W_*$*-module with basis*:

$$M(0)=[Z_2, \sigma, 1],\ M(2n+2)=[S^1\times_R S^{2n+1},\ id\times a, \alpha] \text{ and }$$
$$M(2n+1)=[S^{2n+1}, a, 1]$$

*for* $n\geqq 0$, *and* $\operatorname{Im} F_+ = W_*\{\{X(2n),\ [S^{2n+1}, a]_2\,|\ n\geqq 0\}\}$, *where* $(Z_2, \sigma)$ *is the action of* $Z_2$ *on itself by the additive homomorphism* $\sigma$.

(ii) $W_*^-(Z_2)$ *is the free* $W_*$*-module with basis*:

$$M(2n)=[S^{2n}, a, 1] \text{ and } M(2n+1)=[S^1\otimes S^{2n},\ id\times a,\ \alpha]$$

*for* $n\geqq 0$, *and* $\operatorname{Im}F_- = W_*\{\{[S^{2n}, a]_2,\ X(2n+1)\,|n\geqq 0\}\}$.

*Proof.* First we note that the above set $\{Y(n)\}$ in each $\operatorname{Im}F_\varepsilon$ forms an $\mathfrak{N}_*$-basis of $\mathfrak{N}_*(Z_2)$. When $\varepsilon=+$, the set $\{M(n)\}$ becomes a $W_*$-basis of $W_*^+(Z_2)$ naturally if it satisfies that $F_+(M(n))=Y(n)$ for the basis $\{Y(n)\}$ of $\mathfrak{N}_*(Z_2)$ as mentioned above (cf. [6; p. 154]). When $\varepsilon=-$, we see that the above set $\{M(n)\}$, in fact, generates $W_*^-(Z_2)$ over $W_*$. (See Remark 3. 9 in the next section.) Hence the theorem follows. q. e. d.

## § 3. Some Examples

In this section, we study the Wall manifolds with $(\varepsilon)$-free involutions derived from a set of generators of $\Omega_*^\varepsilon(\boldsymbol{Z}_2)$.

For each $\varepsilon$, let $\hat{\Omega}_*^\varepsilon(S^N, a)$ be the equivariant singular $\Omega_*$-module of orientation-preserving (or orientation-reversing) free involutions for the antipodal map on $N$-sphere $(S^N, a)$, where $\Omega_*$ is the oriented cobordism ring.

**Definition 3.1** (cf. [5]). Let $(M, \mu, f)$ represent a class in $\hat{\Omega}_k^-(S^N, a)$. For each integer $n\geqq 0$, we define $D^n(M)$, $D^n(\mu)$ and $D^n(f)$ as follows.

(3.2) $D^n(M)=D^n\times M/(s, x)\sim(s, \mu(x))$ for $s\in\partial D^n$ and $x\in M$ ($D^n$: the $n$-disc),

$D^n(\mu):D^n(M)\to D^n(M)$ is defined by $D^n(\mu)([d, x])=[a(d), \mu(x)]$ and $D^n(f):D^n(M)\to E^nS^N=S^{n+N}$ (the $n$-fold unreduced suspension of $S^N$) by $D^n(f)([d, x])=[d, f(x)]$ for $d\in D^n$ and $x\in M$. Then $(D^n(M), D^n(\mu), D^n(f))$ represents a class in $\hat{\Omega}_{k+n}^\varepsilon(S^{n+N}, a)$ $(\varepsilon=(-1)^{n+1})$ and its image in $\hat{\Omega}_{k+n}^\varepsilon(S^\infty, a)=\Omega_{k+n}^\varepsilon(\boldsymbol{Z}_2)$ is denoted by $D^n[M, \mu, f]$. The map $D^n:\hat{\Omega}_*^-(S^N, a)\to\Omega_*^\varepsilon(\boldsymbol{Z}_2)$ $(\varepsilon=(-1)^{n+1})$ sending $[M, \mu, f]$ to $D^n[M, \mu, f]$ is a well-defined $\Omega_*$-homomorphism of degree $n$.

Now let $\pi$ be the set of all partitions $\omega=(p_1, \dots, p_r)$ with unequal parts $p_i$ none of which is a power of 2. And let $|\omega|=r$ be the length of $\omega$. For a partition $\omega=(p_1, \dots, p_r)$, we denote $M_\omega=M_{2p_1}\cdots M_{2p_r}$ the unoriented manifold in [10; §4] representing the generator $X_\omega=X_{2p_1}\cdots X_{2p_r}\in\boldsymbol{W}_*$. And let $[W_\omega, t]\in\hat{\Omega}_{2|\omega|}^-(S^1, a)$ be the bordism class of the orientation-reversing involution $t$ on the orientation bundle $W_\omega$ over $M_\omega$. Then we know that $\hat{\Omega}_*^-(S^1, a)$ is isomorphic to $\boldsymbol{W}_*$ as $\Omega_*$-modules via the map $\eta:\hat{\Omega}_*^-(S^1, a)\to\boldsymbol{W}_*$ sending $[M, \mu, f]$ to $[M/\mu, f/\mu]$ and $\eta^{-1}(X_\omega)=[W_\omega, t]$ by definition, where $\boldsymbol{W}_*$ is regarded as $\Omega_*$-module via the natural map $r:\Omega_*\to\boldsymbol{W}_*$. Thus we see that $\hat{\Omega}_*^-(S^1, a)$ is generated as $\Omega_*$-module by $[S^0, a]$ and $[W_\omega, t]$ $(\omega\in\pi)$.

Using the above maps $D^n:\hat{\Omega}_*^-(S^1, a)\to\Omega_*^\varepsilon(\boldsymbol{Z}_2)$ $(\varepsilon=(-1)^{n+1})$, we obtain some direct sum decompositions of $\Omega_*^\varepsilon(\boldsymbol{Z}_2)$ which are similar to those in [5; Cor. 3.3] by the maps $E^n$. In particular, we have

**Lemma 3.3** (cf. [5]). *As $\Omega_*$-modules,*
(i) $\Omega_*^+(Z_2)$ *is generated by the following elements;*
$[Z_2, \sigma]$, $[S^{2n+1}, a]$, $[D^{2n+1}W_\omega, D^{2n+1}(t)]$ $(n\geqq 0,\ \omega\in\pi)$, *and*
(ii) $\Omega_*^-(Z_2)$ *is generated by the following elements;*
$[S^{2n}, a]$, $[D^{2n}W_\omega, D^{2n}(t)]$ $(n\geqq 0,\ \omega\in\pi)$.

See [5; Th. 4.5] for further structures of $\Omega_*^\varepsilon(Z_2)$, especially the relations among these generators.

For each $n\geqq 0$, let $D_2^n\colon \mathfrak{N}_*(Z_2)\to\mathfrak{N}_*(Z_2)$ be the $\mathfrak{N}_*$-homomorphism of degree $n$ whose definition is entirely analogous to $D^n$ given at (3.2), forgetting the orientations.

**Lemma 3.4.** $D_2^n([S^1, a]_2) = X(n+1)$ *for all* $n\geqq 0$.

*Proof.* First we note $\varepsilon_*(D_2^n[S^1, a]_2)=0$ for all $n\geqq 0$, considering for example the involution $\bar{A}$ on $D^nS^1/D^n(a)$ induced by $A([d, x]) = [a(d), x]$ and its fixed point data (cf. [1; (21.8), (22.2)]). Using the formula (2.3) for $[M, \tau]_2 = D_2^n[S^1, a]_2$, we have the result, since $\Delta\circ D_2^N = D_2^{N-1} (N\geqq 1)$ in general. q.e.d.

Now we return to the study of the maps $F_\varepsilon$. We first consider Wall manifolds with the trivial structure 1 (cf. Theorem 1.5.).

**Lemma 3.5** (cf. [5; Lemma 5.2]). $F_-([W_\omega, t, 1]) = X_\omega[S^0, a]_2 + (\partial_1 X_\omega)[S^1, a]_2$ *in* $\mathfrak{N}_*(Z_2)$, *where* $\partial_1\colon W_*\to W_*$ *is the derivation in* [10].

Using Lemmas 3.4, 3.5 and the fact that $D^N([S^0, a]_2) = [S^N, a]_2$ for each $N\geqq 0$, we have

**Theorem 3.6.** *For each* $n\geqq 0$ *and* $\omega\in\pi$,
(i) $F^+([D^{2n+1}W_\omega, D^{2n+1}(t), 1]) = X_\omega[S^{2n+1}, a]_2 + (\partial_1 X_\omega)X(2n+2)$,
(ii) $F_-([D^{2n}W_\omega, D^{2n}(t), 1]) = X_\omega[S^{2n}, a]_2 + (\partial_1 X_\omega)X(2n+1)$

in $\mathfrak{N}_*(Z_2)$. (cf. Theorem 2.8.)

**Definition 3.7.** Let $(M, \mu)$ be an orientation-reversing free involution. For each $n\geqq 0$, $(D^{2n+1}(M), D^{2n+1}(\mu))$ is equivariant reversible since $D^{2n+1}(M)$ admits an orientation-reversing diffeomorphism $A$ defined by $A([d,x])=[a(d),x]$ for $d\in D^{2n+1}$ and $x\in M$. While the map $(\mu)$ defined by $(\mu)([d,x])=[d,\mu(x)]$ is also an orientation-reversing diffeomorphism on $D^{2n+1}(M)$ (cf. Definition 1. 6). When $(M, \mu)=(S^0, a_0)$, the antipodal map on 0-sphere, the above map $A$ is the reflection $R: S^{2n+1}\to S^{2n+1}$ in Example 1. 8. While $(a_0)$ is another reflection defined by $(a_0)(x_0, x_1, \dots, x_{2n+1})=(-x_0, x_1, \dots, x_{2n+1})$.

Further, putting $(M, \mu)=(W_\omega, t)$, we consider Wall manifolds with the $RP(1)$-structure $\alpha$ as follows (cf. Definition 1. 7).

**Theorem 3. 8.** *For each* $n\geqq 0$ *and* $\omega\in\pi$,

(i-1) $F_+([S^1\times_A D^{2n+1}W,\ id\times D^{2n+1}(t), \alpha])$

$$=(\partial_1 X_\omega)\cdot\Sigma_{i=0}^{[\frac{n+1}{2}]} a_{2i}^2[S^{2n+3-4i}, a]_2+X_\omega X(2n+2),$$

(i-2) $F_+([S^1\times_{(t)}D^{2n+1}W_\omega,\ id\times D^{2n+1}(t),\ \alpha])=X_\omega X(2n+2)$,

(ii) $F_-([S^1\otimes D^{2n}W_\omega,\ id\times D^{2n}(t),\ \alpha])=X_\omega X(2n+1)$,

*where the element* $a_{2i}\in\mathfrak{N}_{2i}$ *is defined in Remark* 2. 4. *We see that* $a_{2i}^2$ *is the homogeneous polynomial in variables* $\beta_j=[CP(2j)]$ $(1\leqq j\leqq i)$.

*Proof.* We first prove (i-1). Using Theorem 2. 5 and Lemma 3. 5, we have

$$\begin{aligned}&F_+([S^1\times_A D^{2n+1}W_\omega,\ id\times D^{2n+1}(t), \alpha])\\&=X_\omega[S^1\times_R S^{2n+1},\ id\times a]_2+(\partial_1 X_\omega)[S^1\times_A D^{2n+1}S^1,\ id\times D^{2n+1}(a)]_2\\&=X_\omega X(2n+2)+(\partial_1 X_\omega)[S^1\times_A D^{2n+1}S^1,\ id\times D^{2n+1}(a)]_2.\end{aligned}$$

Now let

$$f: S^1\times D^{2n+1}S^1\to S^1\times D^{2n+1}S^1$$

be the map defined by $f(z, [d, w])=(z, [d, zw])$ for $z, w\in S^1$ and $d\in D^{2n+1}$. Then $f\circ(a\times A)=(a\times D^{2n+1}(a))\circ f$ and $f\circ(id\times D^{2n+1}(a))=(id\times D^{2n+1}(a))\circ f$. Therefore $f$ induces an equivariant diffeomorphism between $(S^1\times_A D^{2n+1}S^1,\ id\times D^{2n+1}(a))$ and $(S^1\otimes D^{2n+1}S^1,\ id\times D^{2n+1}(a))$. Since $[D^{2n+1}S^1, D^{2n+1}(a)]_2=X(2n+2)$ (cf. Lemma 3. 4), $[S^1\otimes D^{2n+1}S^1,$

$id\times D^{2n+1}(a)]_2=[S^1, a]_2X(2n+2)$ in $\mathfrak{N}_*(\boldsymbol{Z}_2)$ as $\mathfrak{N}_*$-algebra. Thus we have the result (i-1) by the formula for $[S^1, a]_2[S^{2m}, a]_2$ (cf. [9; Cor. 2. 5]) and Remark 2. 4.

We next prove (i-2). Similarly we have

$$F_+([S^1\times_{(t)}D^{2n+1}W, id\times D^{2n+1}(t), \alpha])$$
$$=X_\omega[S^1\times_{R_1}S^{2n+1}, id\times a]_2+(\partial_1X_\omega)[S^1\times_{(a)}D^{2n+1}S^1, id\times D^{2n+1}(a)]_2$$

where $R_1$ is the reflection $(a_0)$ mentioned in Definition 3. 7. It is easy to see that $(S^1\times_{R_1}S^{2n+1}, id\times a)$ is isomorphic to $(S^1\times_R S^{2n+1}, id\times a)$ by a natural diffeomorphism. Thus $[S^1\times_{R_1}S^{2n+1}, id\times a]_2=X(2n+2)$. While the map $f$ in the proof of (i-1) satisfies $f\circ(a\times(a))=(a\times id)\circ f$ in this case. Therefore $[S^1\times_{(a)}D^{2n+1}S^1, id\times D^{2n+1}(a)]_2=[RP(1)]$ $[D^{2n+1}S^1, D^{2n+1}(a)]_2=0$.

The proof of (ii) is entirely analogous to the above one, so we omit it here. q. e. d.

*Remark* 3. 9. Since the map $F_\varepsilon$ is monic, we see the relation among Wall manifolds with $(\varepsilon)$-free involutions via $F_\varepsilon$. In particular, let us consider the natural map $\bar{\delta}:\Omega_*^-(\boldsymbol{Z}_2)\to W_*^-(\boldsymbol{Z}_2)$ defined by $\bar{\delta}([M, \mu])=[S^1\otimes M, id\times\mu, \alpha]$ (cf. Definition 1. 7). Then $\bar{\delta}$ is a splitting map to $\delta$ in the exact triangle $(W_-)$ in Theorem 1. 5. Thus $W_*^-(\boldsymbol{Z}_2)$ is generated as $\Omega_*$-module by $[S^{2n}, a, 1]$, $[D^{2n}W_\omega, D^{2n}(t), 1]$, $[S^1\otimes S^{2n}, id\times a, \alpha]$, $[S^1\otimes D^{2n}W_\omega, id\times D^{2n}(t), \alpha]$ for $n\geqq 0$ and $\omega\in\pi$ by Lemma 3. 3 (ii). Using the map $F_-$, we see that $W_*^-(\boldsymbol{Z}_2)$ is generated as $W_*$-module by $[S^{2n}, a, 1]$, $[S^1\otimes S^{2n}, id\times a, \alpha]$ for $n\geqq 0$ from Theorems 2. 5, 3. 6 and 3. 8.

Let $\mathfrak{N}_*(S^1)$ be the unoriented bordism group of $S^1$, and let $E: W_*^-(\boldsymbol{Z}_2)\to\mathfrak{N}_*(S^1)$ be the map assigning to an element $[M, \mu, \alpha]\in W_*^-(\boldsymbol{Z}_2)$ the induced map $\bar{\alpha}: \bar{M}=M/\mu\to S^1$. Since the manifold $M$ is the double cover associated to $\det\tau_{\bar{M}}\otimes\bar{\alpha}^*(\xi)$ where $\xi$ is the canonical line bundle over $S^1$, the map $E$ is the additive isomorphism (cf. [7]). Let $i$: $\mathfrak{N}_*\to\mathfrak{N}_*(S^1)$ be the natural inclusion defined by $i([M])=[M]$ $[*, 1]$ where $(*, 1)$ is the one-point set $*$ with the trivial map $1:*\to S^1$, and let $L=F_-\circ E^{-1}\circ i$: $\mathfrak{N}_*\to\mathfrak{N}_*(\boldsymbol{Z}_2)$ be the induced injective homomorphism. Then we have

**Lemma 3.10.** *For any Wall manifold* $[M]$ *in* $W_*$ *and integer* $n \geqq 0$,

$$L([M \times RP(2n)]) = [M][S^{2n}, a]_2 + [\partial_1 M] X(2n+1)$$

*in* $\mathfrak{N}_*(Z_2)$.

*Proof.* In case of $[M] = X_\omega$, the above formula is exactly the same as that (ii) of Theorem 3.6 since $E([D^{2n}W_\omega, D^{2n}(t), 1]) = X_\omega[RP(2n)][*, 1]$ by definition. For any element $[M]$, the proof is clear. q. e. d.

*Remark* 3.11. We see that the relation $\Sigma_{i=0}^{n}[M_{2i}][RP(2i)] = 0$ for $[M_{2i}] \in W_*$ if and only if $[M_{2i}] = 0$ for all $i$.

In conclusion of this section, we study the relation between $\mathrm{Im}F_+$ and $\mathrm{Im}F_-$.

**Theorem 3.12.** $\mathrm{Im}F_+ \cap \mathrm{Im}F_- = W_*\{\{[S^0, a]_2, [S^1, a]_2\}\}$.

*Proof.* First we note $[S^0, a]_2 = X(0)$ and $[S^1, a]_2 = X(1)$ by definition. Thus these elements belong to $\mathrm{Im}F_+ \cap \mathrm{Im}F_-$(cf. Theorem 2.8). Next we take an element $x$ in $\mathrm{Im}F_+ \cap \mathrm{Im}F_-$ with $\deg x = 2n$, then

$$(3.13) \qquad \begin{aligned} x &= \Sigma_{i=0}^{n}[M_{2i}]X(2n-2i) + \Sigma_{i=0}^{n-1}[N_{2i+1}][S^{2n-2i-1}, a]_2 \\ &= \Sigma_{j=0}^{n}[M'_{2j}][S^{2n-2j}, a]_2 + \Sigma_{j=0}^{n-1}[N'_{2j+1}]X(2n-2j-1) \end{aligned}$$

by Theorem 2.8, where $[M_s]$, $[M'_s]$, $[N_s]$ and $[N'_s]$ are elements in $W_s$. Consider $\varepsilon_*(x)$, then $[M_{2n}] = [M'_{2n}]$ and $[M'_{2j}] = 0$ for $j = 0, 1, \ldots, n-1$ by Remark 3.11. Substitute this result for the identity (3.13). And we describe it by the basis $\{X(n)\}$ alone, using the relation (2.2). Then a straightforward calculation shows that $x = [M_{2n}]X(0) + [N_{2n-1}][S^1, a]_2 = [M_{2n}][S^0, a]_2 + [N_{2n-1}]X(1)$ from Remark 3.11 again. Thus $x$ belongs to $W_*\{\{[S^0, a]_2, [S^1, a]_2\}\}$. In the case of $\deg x = 2n - 1$, the proof is similar. q. e. d.

Our basis $\{Y(n)\}$ of $\mathfrak{N}_*(Z_2)$ in Theorem 2.8 is essentially unique in the following sense.

**Theorem 3.14.** *Let* $\{C(n) \mid \deg C(n) = n, \ n \geqq 0\}$ *be a homogeneous basis of* $\mathfrak{N}_*(Z_2)$ *with the following properties:*

(i) *For each* $n \geqq 0$, *there is an element* $M(n)$ *in* $W_n^+(Z_2)$ *such that* $F_+(M(n)) = C(n)$, *and*

(ii) $\Delta^2(C(n)) = C(n-2)$ *for all* $n \geqq 2$.

*Then* $C(2n) = X(2n)$ *mod* $\mathscr{I}$ *and* $C(2n+1) = [S^{2n+1}, a]_2$ *mod* $\mathscr{I}$ *where* $\mathscr{I} = \mathrm{Im}F_+ \cap \mathrm{Im}F_-$.

The proof is clear, so we omit it here.

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