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Actions of T-Groups on Lebesgue Spaces and Properties of Full Factors of Type II_1

By

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Abstract

The ergodic actions of TICC groups preserving the finite measure and the II_1-factors constructed on these actions are studied in this paper. To distinguish between the II_1-factors, the properties of a centralizer of such actions are used. For $\mathrm{SL}(n, \mathbb{Z})\ominus\mathbb{Z}^n$, $n\geq3$, a continuum of orbit (weakly) nonequivalent actions is constructed. Full II_1-factors having the properties opposite to the known properties of the hyperfinite factor are constructed. A full II_1-factor is presented, whose all tensor powers are non-isomorphic in pairs. It is shown that the full factor can contain a non-isomorphic factor as a finite index subfactor and possess externally non-conjugated periodic automorphisms. Similar results are valid for ergodic equivalency relations.

The Supplement presents the principal points of the proof of the fact that the group $\mathrm{SL}(n, \mathbb{Z})$ for each $n\geq3$ has at least a countable number of orbit-nonequivalent actions preserving the finite measure.

§1. Introduction

1. The recent progress in the ergodic theory [1-3] and in the theory of factors [4-9] is connected with consideration of T-groups [10-12]. The ergodic systems and von Neumann algebras related to T-groups have interesting properties different from those of similar objects related to amenable groups.

In this paper ergodic actions of T-groups on the Lebesgue spaces and the factors of II_1-type constructed by these actions are considered. It is proved that the group $\mathrm{SL}(n, \mathbb{Z})\ominus\mathbb{Z}^n$, $n\geq3$, can have a continuum of orbit (weakly)-nonequivalent ergodic free actions on the Lebesgue space and the corresponding II_1-factors are non-isomorphic in pairs (see Theorem 7.1 and its Remark). Full II_1 factors having the properties opposite to the known properties of the hyperfinite II_1-

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factor are also constructed. Thus, a full II_1 factor is constructed whose all tensor powers are in pairs nonisomorphic (see Theorem 7. 2). It is proved that the full II_1-factor can contain a non-isomorphic subfactor of the finite index (see Theorem 8. 3 (ii)) and have externally non-conjugated periodic automorphisms (Theorem 8. 3 (i)). It is proved that there exists a full II_1-factor whose group Out $N=\mathrm{Aut}\,N/\mathrm{Int}\,N$ is a continuous locally compact group (see Theorem 8. 4). The case of a discrete Out N is described in [4]. Similar facts take place also for ergodic dynamic systems.

Our invariants employed to distinguish between the ergodic actions of TICC-group (and the II_1-factors constructed by these actions) are based on the properties of centralizer of such actions (see Prop. 2. 5). We also use for this purpose the known results of irreducible lattice of connected semisimple Lie groups [13-15].

The Supplement presents the principal points of the proof of the fact that the group $\mathrm{SL}(n, \boldsymbol{Z})$ for each $n\geq 3$ has at least a countable number of orbit-nonequivalent actions preserving the finite measure.

The question on the existence of such actions arises in particular in connection with [1], where it is proved that the ergodic actions of the group $\mathrm{SL}(n, \boldsymbol{Z})$, $n\geq 3$, with various n are always orbit-nonequivalent.

§2. General Properties of II_1-Factors Constructed by Actions of T-Groups on Lebesgue Space

Let G be a countable group for which all classes of conjugated elements, except for the trivial one, are infinite (an ICC group). Assume that G is a T-group, i. e. there is a finite subset $F\subset G$ and $\varepsilon>0$ such that for any unitary representation π of G the existence of a vector $\xi\in H_\pi$ such that

$$||\xi||=1,\ ||\pi(g)\xi-\xi||<\varepsilon,\qquad g\in F,$$

implies the existence of a vector $\xi'\in H_\pi$, $||\xi'||=1$, satisfying $\pi(g)\xi'=\xi'$, $g\in G$, [11].

Let (X, μ) be a Lebesgue space, α a free ergodic action of G on (X, μ). M denotes the factor of II_1 type, which is a crossed product of $A=L^\infty(X, \mu)$ on G with respect to α, i. e. $M=W^*(A, \alpha, G)$. Then the operators from M act in the space $H=l^2(G, H_1)$ of the vector functions $\xi=(\xi(g))$ on G with the values in $H_1=L^2(X,\mu)$, for which $\|\xi\|^2=\sum\|\xi(g)\|^2<\infty$. The factor M is generated by the operators $\pi(a)$, $a\in A$, and λ_g, $g\in G$:

$$(\pi(a)\xi)(g)=\alpha_{g^{-1}}(a)\xi(g), \quad \text{where } a\in L^\infty(X,\mu),$$
$$(\lambda_h\xi)(g)=\xi(h^{-1}g), \quad h,g\in G.$$

Lemma 2.1. *If* $R(G)=(\lambda_g, g\in G)$, *then* $R(G)'\cap M=\mathbf{C}$ and M *is a full factor*, *i.e.* Int M *is a closed subgroup* Aut M.

Proof. $R(G)'\cap M=\mathbf{C}$ is a result of the fact G that G is an ICC group, the closedness of Int M is proved in [7]. □

Theorem 2.2. *Let* A_G^M *be a subgroup of* Aut M *generated algebraically by the subgroup* $A_G=\{\theta\in\text{Aut } M: \theta(\lambda_g)=\lambda_g,\ g\in G\}$ *and* Int M. *Then* A_G^M *is a closed and open subgroup of* Aut M.

Proof. Let $\xi_0=(\xi_0(g))$, where $\xi_0(g)=1$ at $g=e$ and $\xi(g)=0$ at $g\neq e$. Then ξ_0 is a cyclic separating vector for M in H. The corresponding unitary involution is denoted as J: $Jx\xi_0=x^*\xi_0$, $x\in M$. Assume that

$$V=\{\theta\in\text{Aut } M:\ \|\theta(\lambda_g)J\lambda_gJ\xi_0-\xi_0\|<\varepsilon,\ \forall g\in F\}.$$

Then V is an open neighbourhood of unity in Aut M. Let $\theta\in V$. Since $\pi_\theta(g)=\theta(\lambda_g)J\lambda_gJ$ is a unitary representation of G in H, then it follows from the T-property that there exists $\xi\in H$ such that

$$\theta(\lambda_g)J\lambda_gJ\xi=\xi, \quad \forall g\in G,$$
$$\theta(\lambda_g)\xi=J\lambda_g^*J\xi, \quad \forall g\in G.$$

Since M is a II_1-factor, there exists a closed operator a, joined to M [16] such that $a\xi_0=\xi$. Let $a=u|a|$ be its polar decomposition. Then from the previous equality we obtain

$$\theta(\lambda_g)u=J\lambda_g^*Ju=u\lambda_g, \quad \forall g\in G. \tag{2.1}$$

Hence by Lemma 2.1 it follows that $u^*u=uu^*\in\mathbb{R}_+$. But then one may assume that in (2.1) u is the unitary operator, which means that Ad u^* $\theta(\lambda_g)=\lambda_g$, $g\in G$, i.e. $\theta\in A_G^M$. The closedness of A_G^M is proved in a similar way. □

Lemma 2.3. *Let the conditions of Theorem* 2.2 *be obeyed. Then* A_G *is a closed subgroup of* Aut M; $A_G \cap \text{Int}\, M = \text{id}$. *There exists an algebraic homomorphism* π_G *mapping* A_G^M *on* A_G, *with* $\text{Ker}\, \pi_G = \text{Int}\, M$.

Proof. The closedness of A_G follows from the definition of A_G, and the relation $A_G \cap \text{Int}\, M = \text{id}$ from Lemma 2.1. Because of Theorem 2.2 there exists an algebraic homomorphism π_G of A_G^M on A_G, which brings the point g into accordance with the co-set $(\text{Int}\, M) g$, where $g \in A_G$. The relation $A_G \cap \text{Int}\, M$ shows that $\ker \pi_G = \text{Int}\, M$. □

Lemma 2.4. *The homomorphism* π_G *is continuous in the following sense. If* θ_n $(n \in N)$, $\theta \in \text{Aut}\ M$ *and* $\lim_{n\to\infty} \theta_n = \theta$, *then* $\lim_n \pi_G(\theta_n) = \pi_G(\theta)$ *with respect to the topology in* Aut M.

Proof. By Lemma 2.3 there exists the following realization $\theta_n = \text{Ad}\, u_n \cdot \pi_G(\theta_n)$ and $\theta = \text{Ad}\, u \cdot \pi_G(\theta)$, where u_n $(n \in N)$ and u are unitary operators in M. Since $\gamma(\lambda_g) = \lambda_g$ for $\gamma \in A_G$ and $\forall g \in G$, we obtain

$$s^* - \lim_n (\text{Ad}\, u_n(\lambda_g) - \text{Ad}\, u(\lambda_g)) = s^* - \lim_n (\theta_n(\lambda_g) - \theta(\lambda_g)) = 0$$

for any $g \in G$. But then $(u^* u_n)_{n=1}^{\infty}$ is a G-asymptotically invariant sequence and therefore it follows from the results of [8] that $(u^* u_n)_{n=1}^{\infty}$ is equivalent to the G-invariant sequence. Thus, by Lemma 2.1 $(u^* u_n)_{n=1}^{\infty}$ is equivalent to the trivial sequence $(\lambda_n I)_{n=1}^{\infty}$ where $\lambda_n \in \boldsymbol{C}$ i.e. $s^* - \lim_n (u_n - \lambda_n u) = 0$. From this and the assumption that $\lim_n \theta_n = \theta$, the relation $\lim_{n\to\infty} \pi_G(\theta_n) = \pi(\theta)$ follows. □

Proposition 2.5. *Let* $A_i = L^\infty(X, \mu_i)$, G_i, α_i *and* $M_i = W^*(A_i, \alpha_i, G_i)$, $i = 1, 2$ *be such as in Theorem* 2.2. *If* $M_1 \sim M_2$, *then* A_{G_i}, *assumed to be continuous groups having the topology induced with* Aut M, *contains closed algebraically isomorphic subgroups* H_i *with respect to the countable index in* A_{G_i}. *Besides, if* φ *is an isomorphism of* H_1 *on* H_2, *then it follows from* $\lim_{n\to\infty} \theta_n = \theta$ *where* θ_n, $\theta \in H_1$, *that* $\lim_{n\to\infty} \varphi(\theta_n) = \varphi(\theta)$.

Proof. Let M_1 be identified with $M_2 = M$ and assume $A_{G_1 G_2}^M = A_{G_1}^M \cap A_{G_2}^M$. By Theorem 2.2 $A_{G_1 G_2}^M$ is a closed and open subgroup of Aut M, and therefore $A^M_{G_i}$, i=1, 2. Thus, $A^M_{G_1G_2}$ has a countable index in $A^M_{G_i}$ and hence

$$A^M_{G_i} = \sum_{j\in N} g^i_j A^M_{G_1G_2}$$

where $g^i_j \in A^M_{G_i}$. Thus, $A_{G_i} = \pi_{G_i}(A^M_{G_i}) = \sum_j \pi_{G_i}(g^i_j)\pi_{G_i}(A^M_{G_1G_2})$ and the index $H_i = \pi_{G_i}(A^M_{G_1G_2})$ in A_{G_i} is not more than countable.

Now we prove the closedness of H_i in A_{G_i}: If $\vartheta_n = \pi_{G_i}(\theta_n) \in H_i$ and $\vartheta_n \to \vartheta$ in Aut M, then $\vartheta \in A_{G_i}$ because of the closedness of A_{G_i}. On the other hand, $\vartheta_n \in A^M_{G_1G_2}$ and therefore $\vartheta \in A^M_{G_1G_2}$ and hence $\pi_{G_i}(\vartheta) = \vartheta \in H_i$. Then $H_1 \sim H_2$ in the algebraic sense since $H_i \sim A^M_{G_1G_2}/\text{Int } M$. This isomorphism φ of H_1 on H_2 obviously coincides with the restriction π_{G_2} on H_1. Hence by Lemma 2.4 the continuity φ follows. □

§3. The Centralizers of Actions of Groups $G(n)$, $n \geq 2$, on Finite Dimensional Tori

Before proceeding to the construction of examples of full factors of II_1-type, we prove some auxiliary statements which can be of independent interest.

Lemma 3.1. *The centralizer* $C\alpha\,\mathrm{SL}(n, \mathbb{Z})$ *of the natural action* α *of the group* $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 2$, *on the torus* $\mathbb{T}^n$ *is a group of the order of* 2. (The Proof is obtained jointly with S. L. Gelter).

Proof. Let us consider the case $n=2$, the general case being considered in a similar way. Assume that $\mathbb{T}^2 = (\lambda_1, \lambda_2)$ where $\lambda_i \in \mathbb{T}$, $i=1, 2$. If $g = (n_{ij})$ $(i, j=1, 2) \in \mathrm{SL}(2, \mathbb{Z})$ then the standard action is found as

$$\alpha(g)(\lambda_1, \lambda_2) = (\lambda_1^{n_{11}}\lambda_2^{n_{12}}, \lambda_1^{n_{21}}\lambda_2^{n_{22}}).$$

Let $\theta \in C\alpha\mathrm{SL}(2, \mathbb{Z})$. Then θ: $(\lambda_1, \lambda_2) \to (\theta_1(\lambda_1, \lambda_2), \theta_2(\lambda_1, \lambda_2))$, where $\theta_i(\lambda_1, \lambda_2)$, $i=1, 2$, are Borel mappings from $\mathbb{T}^2$ in $\mathbb{T}$, satisfying for almost all (λ_1, λ_2) the relation

$$\begin{aligned}(\theta_1^{n_{11}}\theta_2^{n_{12}}, \theta_1^{n_{21}}\theta_2^{n_{22}}) &= (\theta_1(\alpha(g)(\lambda_1, \lambda_2)), \theta_2(\alpha(g)(\lambda_1, \lambda_2))) \\ &= (\theta_1(\lambda_1^{n_{11}}\lambda_2^{n_{12}}, \lambda_1^{n_{21}}\lambda_2^{n_{22}}), \theta_2(\lambda_1^{n_{11}}\lambda_2^{n_{12}}, \lambda_1^{n_{21}}\lambda_2^{n_{22}})).\end{aligned}$$

Then for $g=\begin{pmatrix}1 & n_{12}\\ 0 & 1\end{pmatrix}$, the equality

$$\theta_2(\lambda_1, \lambda_2) = \theta_2(\lambda_1\lambda_2^{n_{12}}, \lambda_2) \tag{3. 1}$$

is valid for almost all (λ_1, λ_2). If $1/2\pi i \log \lambda_2$ is an irrational number, then it follows from the ergodicity of the automorphism $T\lambda = \lambda\lambda_2$ on $\mathbb{T}$ that θ_2 is independent of λ_2 for almost all $\lambda_2 \in \mathbb{T}$. Similarly, θ_1 is independent of λ_2. Thus, $\theta(\lambda_1, \lambda_2) = (\theta_1(\lambda_1), \theta_2(\lambda_2))$. From the commutativity of θ with $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ we find that for almost all (λ_1, λ_2) the equality

$$(\theta_2(\lambda_2), \theta_1^{-1}(\lambda_1)) = (\theta_1(\lambda_2), \theta_2(\lambda_1^{-1}))$$

holds, that is,

$$\begin{aligned} &\theta_2(\lambda_2) = \theta_1(\lambda_2) \quad \text{for almost all } \lambda_2, \\ &\theta_1(\lambda_1)^{-1} = \theta_2(\lambda_1^{-1}) \quad \text{for almost all } \lambda_1. \end{aligned} \tag{3. 2}$$

Then from Eq. (3. 1) we obtain for $g=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$$\theta_1(\lambda_1\lambda_2) = \theta_1(\lambda_1)\theta_1(\lambda_2) \tag{3. 3}$$

for almost all (λ_1, λ_2).

Now it follows from Eqs. (3. 2) and (3. 3) that θ_1 is a measurable character of $\mathbb{T}$ and therefore is, as known, continuous. But then $\theta: (\lambda_1, \lambda_2) \to (\lambda_1^n, \lambda_2^n)$ for some $n \in \mathbb{Z}$, and since θ is the automorphism of $(\mathbb{T}^2, m_2)$, where m_2 is the Haar measure of $\mathbb{T}^2$, we have $n = \pm 1$. □

Now consider the group $G(n) = \mathrm{SL}(n, \mathbb{Z}) \ominus \mathbb{Z}^n$, i. e. the semidirect product of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{Z}^n$, where $\mathrm{SL}(n, \mathbb{Z})$ naturally acts on $\mathbb{Z}^n$. It is known [12], that $G(n)$ is a TICC group for $n \geq 3$. Then the action of $G(n)$ on $\mathbb{T}^n$ is defined. Let $\gamma \in \mathbb{T}$ and $\bar{\gamma} = (\gamma_j)_{j=1}^n$, where $\gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma$. The automorphism $\alpha_1(\bar{1}_i)$ of the space $(\mathbb{T}^n, m_n)$ with m_n the Haar measure of $\mathbb{T}^n$, multiplying by γ the i-th component of the vector $(\lambda(j))_{j=1}^n \in \mathbb{T}^n$ and leaving the rest of the components unchanged, is set in correspondence to each vector $\bar{1}_i$ from $\mathbb{Z}^n$, whose i-th coordinate is equal to 1 and the rest of them are zero. The action α_1 of the group $1 \ominus \mathbb{Z}^n$ is thereby defined on $\mathbb{T}^n$. The action α_1 of the group $\mathrm{SL}(n, \mathbb{Z})$ on $(\mathbb{T}^n, m_n)$ is defined in the natural way. Thus, the ergodic action α_1 of $G(n)$ on $(\mathbb{T}^n, m_n)$ is constructed.

Lemma 3.2. *The constructed action α_1 of $G(n)$ on $(\mathbf{T}^n, m_n)$ is such that its centralizer is trivial $(n\geq 2)$.*

The proof follows from the previous Lemma.

We shall construct the action α_m of the group $G(n)$, $n\geq 2$, on $(\mathbf{T}^n, m_n)^m$, $m\in \mathbf{N}$. Let $\bar{\lambda}_k=(\lambda_k(1),\dots, \lambda_k(n))\in \mathbf{T}^n$ and $(\bar{\lambda}_k)_{k=1}^m\in(\mathbf{T}^n)^m$. For $g\in \mathrm{SL}(n, \mathbf{Z})$ it is defined

$$\alpha_m(g):(\bar{\lambda}_k)_{k=1}^m \longrightarrow (\alpha_1(g)\bar{\lambda}_k)_{k=1}^m .$$

For $\bar{1}_j\in 1\ominus \mathbf{Z}^n$, $1\leq j\leq n$, it is defined

$$\alpha_m(\bar{1}_j):\ (\bar{\lambda}_k)_{k=1}^m \longrightarrow (\bar{\mu}_k)_{k=1}^m ,$$

where $\mu_k(s)=\lambda_k(s)$ at $s\neq j$ and $\mu_k(j)=\lambda_k(j)\gamma$ at $s=j$, $1\leq k\leq m$. It is clear that the action α_m is ergodic.

Lemma 3.3. *If $\theta\in C\alpha_m G(n)$, then θ corresponds to the integral matrix (n_{ij}) $i,j=1,\dots, m$, for which $\sum_{j=1}^{m} n_{ij}=1$, $1\leq i\leq m$, and* $\det(n_{ij})=\pm 1$. (Below, the group of all such matrices will be referred to as $S(m)$).

Proof. Let $\theta\in C\alpha_m G(n)$. Then θ corresponds to the transformation

$$\theta:\ (\bar{\lambda}_j)_{j=1}^m \longrightarrow (\theta_k^i[(\bar{\lambda}_j)_{j=1}^m]),\qquad 1\leq i\leq n,\ 1\leq k\leq m .$$

To simplify the notation, it is assumed that $n=2$. The general case is considered similarly. Then from the commutativity of θ with $\alpha_m(g)$, $g\in \mathrm{SL}(2, \mathbf{Z})$, we find that

$$\{(\theta_j^1)^{n_{11}}(\theta_j^2)^{n_{12}},\ (\theta_j^1)^{n_{21}}(\theta_j^2)^{n_{22}}\}_{j=1}^m=$$
$$\{\theta_j^1[(\lambda_k(1)^{n_{11}}\lambda_k(2)^{n_{12}},\ \lambda_k(1)^{n_{21}}\lambda_k(2)^{n_{22}})_{k=1}^m],$$
$$\theta_j^2[(\lambda_k(1)^{n_{11}}\lambda_k(2)^{n_{12}},\ \lambda_k(1)^{n_{21}}\lambda_k(2)^{n_{22}})_{k=1}^m]_{j=1}^m\}.$$

As in the proof of Lemma 3.1, we have that at $j=1,\dots, m$ for $\forall n_{12}\in \mathbf{Z}$ for almost all $(\bar{\lambda}_k)_{k=1}^m$

$$\theta_j^2[(\lambda_k(1),\ \lambda_k(2))_{k=1}^m]=\theta_j^2[(\lambda_k(1)\lambda_k(2)^{n_{12}},\ \lambda_k(2))_{k=1}^m]$$

As earlier, this means that θ_j^2 is independent of $(\lambda_k(1))_{k=1}^m$. Then we find that $\theta_j^2=\theta_j^1=\theta_j$, $1\leq j\leq m$, and θ_j is a continuous character on $\mathbf{T}^m$, i.e.

$$\theta_j^1[(\lambda_k(1))_{k=1}^m]=\prod_{i=1}^{m}\lambda_i(1)^{n_{ji}},\ n_{ji}\in \mathbf{Z},$$

$j=1,\ldots,m$. Thus, $\theta=(\theta_1,\ldots,\theta_m)$ corresponds to the integral matrix (n_{ij}), $i,j=1,\ldots,m$, which defines the transformation of $\mathcal{T}^m$. From the commutativity of θ with $\alpha_m(\bar{1}_i)$, $1\leq i\leq 2$, we have that

$$\theta_j^1[(\lambda_k(1)\gamma)_{k=1}^m]=\theta_j^1[(\lambda_k(1))_{k=1}^m]\gamma.$$

But $\theta_j^1[(\lambda_k(1)\gamma)_{k=1}^m]=\prod_{i=1}^m(\lambda_i(1)\gamma)^{n_{ji}}$ i.e. $\gamma^{\sum_{i=1}^m n_{ji}}=\gamma$, or $\sum_{i=1}^m n_{ji}=1$, $1\leq j\leq m$. As θ is the automorphism of $(\mathcal{T}^m, m_m)$ which conserves the measure, so det $(n_{ij})=\pm 1$. □

Lemma 3.4. *The group $S(m)$ introduced in Lemma 3.3 for $m\geq 2$ is noncommutative and conjugated to the matrix subgroup $m\times m$ over $\mathbf{Z}$ with a determinant equal to ± 1 of the form*

$$\begin{pmatrix} 1 & \\ 0 & \\ \vdots & \times \\ 0 & \end{pmatrix}$$

Proof. Indeed, let $(n_{ij})_{i,j=1,\ldots,m}\in S(m)$. Then since $\sum n_{ij}=1$ the vector $(1,1,\ldots,1)$ is invariant for $\forall g\in S(m)$. But the vector $(1,1,\ldots,1)$ transforms into the vector $(1,0,\ldots,0)$ under the action of the matrix

$$\begin{pmatrix} 1 & & & \\ -1 & 1 & & 0 \\ -2 & 1 & 1 & \\ & & & \\ -(m-1) & 1 & 1 & \ldots 1 \end{pmatrix}.$$

From this Lemma follows. □

§4. Centralizers of Actions of Groups $G(n)$, $n\geq 2$, on Infinite Dimensional Tori

Now we proceed to construction of the desired ergodic actions of $G(n)$. As earlier, let m_n be the Haar measure of $\mathcal{T}^n$. We put $(T(n), m(n))=(\mathcal{T}^n, m_n)^N$, then $T(n)$ is a commutative compact group, which is a product of a countable number of copies of $\mathcal{T}^n$ groups, and $m(n)$ is a Haar measure on $T(n)$. The elements of $T(n)$ are various sequences of the vectors $(\bar{\lambda}_k)_{k=1}^\infty$, where $\bar{\lambda}_k=(\lambda_k(1),\ldots,\lambda_k(n))\in\mathcal{T}^n$. The multiplication in $T(n)$ is found as

$$(\bar{\lambda}_k)_{k=1}^{\infty}\cdot(\bar{\mu}_k)_{k=1}^{\infty}=(\overline{\lambda_k\mu_k})_{k=1}^{\infty},$$

where

$$\overline{\lambda_k\mu_k}=(\lambda_k(1)\mu_k(1),\dots,\lambda_k(n)\mu_k(n))$$

Note that $T(n)=T(1)^n$, $T(1)=\mathbb{T}^N$.

Now we define the action α of the group $G(n)=\mathrm{SL}(n,\mathbb{Z})\ominus\mathbb{Z}^n$ on $(T(n),m(n))$. If $(\lambda_k)_{k=1}^{\infty}\in T(n)$ and $g\in\mathrm{SL}(n,\mathbb{Z})$ then

$$\alpha(g):\ (\lambda_k)_{k=1}^{\infty}\longrightarrow(\alpha_1(g)\bar{\lambda}_k)_{k=1}^{\infty} \tag{4. 1}$$

Let $(\gamma_k)_{k=1}^{\infty}\in\mathbb{T}^N$. The element $\bar{1}_i$, $1\leq i\leq n$, from $1\ominus\mathbb{Z}^n$ is set in correspondence with the action

$$\alpha(\bar{1}_i):\ (\bar{\lambda}_k)_{k=1}^{\infty}\longrightarrow(\alpha_1(\bar{1}_i)\bar{\lambda}_k)_{k=1}^{\infty}. \tag{4. 2}$$

It is clear that because of Eq. (4. 1) the constructed action α of $G(n)$ is ergodic.

Proposition 4.1. *Let $(\gamma'_j)_{j=1}^{\infty}$ be a sequence of numbers of $\mathbb{T}$, the numbers $(1/2\pi i\log\gamma'_j)_{j=1}^{\infty}$ being rationally incommensurable. Put $\gamma_k=\gamma'_j$ for $n_j\leq k<n_{j+1}$, where $n_1=1$ and n_j is an increasing sequence of natural numbers. Construct from $(\gamma_k)_{k=1}^{\infty}$ the action α of the group $G(n)$, $n\geq 2$, on $(T(n), m(n))$ in the way just described. If $\theta\in C\alpha G(n)$, then $\theta=\bigotimes_{i=1}^{\infty}\theta_i$, where θ_i is the automorphism $(\mathbb{T}^n, m_n)^{m_i}$, $m_i=n_{i+1}-n_i$, and $\theta_i\in S(m_i)$* (see Lemma 3. 3).

First prove a subsidiary Lemma.

Lemma 4.2. *Let I be a unit interval, m the Lebesgue measure on I. Consider I as an additive group mod. 1, then I^N is also a compact commutative group. Let $(\alpha_i)_{i=1}^{\infty}\in I^N$, where $0<\alpha_i<1$ and the numbers 1 and (α_i) are rationally incommensurable. Then the automorphism T: $(x_i)_{i=1}^{\infty}\rightarrow(x_i+\alpha_i)_{i=1}^{\infty}$ of the space $(I,m)^N$ is ergodic.*

Proof. Let us consider a complete orthogonal basis in $L^2((I, m)^N)$ of the form

$$f_{n_1,\dots,n_k}(\bar{x})=\prod_{j=1}^{k}e^{2\pi i n_j x_j}$$

where $\bar{x}=(x_j)_{j=1}^{\infty}\in I^N$. Then $\{f_{n_1,\cdots,n_k}\}$ is a complete system of characters of the group I^N. If $f\in L^{\infty}((I, m)^N)$ and $f(T\bar{x})=f(\bar{x})$ for almost all $\bar{x}\in I^N$, then it is necessary to consider the orthogonal series $f(\bar{x})\sim\sum a_{n_1,\cdots,n_k}f_{n_1,\cdots,n_k}$ and compare the Fourier coefficients of the function $F(\bar{x})=f(T\bar{x})$ and $f(\bar{x})$ taking into account the equality $f(\bar{x})=f(T\bar{x})$ for almost all $\bar{x}\in I^N$, from which we have that $f(\bar{x})=a_0$. □

Proof of Proposition 4. 1. Note that $(T(n), m(n))=\bigotimes_{i=1}^{\infty}(\mathbf{T}^n, m_n)^{m_i}$ and the action α transforms the component $(\mathbf{T}^n, m_n)^{m_i}$ in itself. The restriction of α on $(\mathbf{T}^n, m_n)^{m_i}$ coincides with α_{m_i} according to the notation of Lemma 3. 3.

Again we put $n=2$ to simplify the notation. Then

$$\theta:\ (\bar{\lambda}_q)_{q=1}^{\infty}\longrightarrow(\theta_k^i[(\bar{\lambda}_q)_{q=1}^{\infty}]),\quad 1\leq i\leq 2,\ k=1,\ 2,\ldots$$

where

$$\bar{\lambda}_q=(\lambda_q(1),\ \lambda_q(2)),\ \lambda_q(i)\in\mathbf{T},\ i=1,\ 2,\ q\in\mathbf{N}.$$

As in the proof of Lemma 3. 3, we have that for each $k\in\mathbf{N}$ and $\forall n\in\mathbf{Z}$

$$\theta_k^2[(\lambda_q(1),\ \lambda_q(2))_{q=1}^{\infty}]=\theta_k^2[(\lambda_q(1)\lambda_q(2)^n,\ \lambda_q(2))_{q=1}^{\infty}].$$

for almost all $(\bar{\lambda}_q)_{q=1}^{\infty}$. Because of Lemma 4. 2 it follows that θ_k^2, $k\in\mathbf{N}$, is independent of $(\lambda_q(1))_{q=1}^{\infty}$. Then, as before, $\theta_k^2[(\lambda_q(2))_{q=1}^{\infty}]=\theta_k^1[(\lambda_q(2)_{q=1}^{\infty}]$ and $\theta_k^i[(\lambda_q(i))_{q=1}^{\infty}]$ is a continuous character on $T(1)$. Using the commutativity θ with $\alpha(\bar{1}_i)$, $1\leq i\leq 2$, we have

$$\theta_k^j[(\lambda_q(j)\gamma_q)_{q=1}^{\infty}]=\theta_k^j[(\lambda_q(j)_{q=1}^{\infty}]\gamma_k.$$

But since θ_k^j is a character, then

$$\theta_k^j((\gamma_q)_{q=1}^{\infty})=\gamma_k. \tag{4. 3}$$

Let $k\in[n_i,\ n_{i+1})$. Since θ_k^j is a continuous character on $T(1)$, we have

$$\theta_k^j[(\lambda_q(j))_{q=1}^{\infty}]=\prod_{i=1}^{N}\lambda_{q_i}(j)^{n_{k,q_i}},$$

where $N\in\mathbf{N}$, $n_{k,q_i}\in\mathbf{Z}$. From (4. 3) follows

$$\prod_{i=1}^{N}(\gamma_{q_i})^{n_{k,q_i}}=\gamma_k.$$

It follows from the incommensurability relation of the numbers

$1/2\pi i(\log\gamma'_j)_{j=1}^{\infty}$ that $n_{k,q_i}=0$ for $q_i\notin[n_i, n_{i+1})$, that is,

$$\theta_k^j[(\lambda_q(j))_{q=1}^{\infty}]=\prod_{q=n_i}^{n_{i+1}-1}\lambda_q(j)^{n_{k,q}},$$

where $\sum_{q=n_i}^{n_{i+1}-1} n_{k,q}=1$. Thus, $(\theta_k^j)_{k=n_i}^{n_{i+1}-1}$, $j=1,2$, determines the automorphism of $(\mathbb{T}^2, m_2)$ from $C\alpha_{m_i}G(2)$. □

§5. Properties of II_1-Factors Constructed by the Actions of Groups $G(n), n\geq 2$, on Tori

We study the factors $M=W^*(A, \alpha, G(n))$ of type II_1 constructed by the ergodic action α of $G(n)$ according to Proposition 4.1. We shall recall that M acts in the space H of the vector-functions $\xi(g)$, $g\in G(n)$, whose values are in $L^2(T(n), m(n))$ and is generated by the operators $\pi(\varphi)$, where $\varphi\in L^{\infty}(T(n), m(n))$, and λ_g, $g\in G(n)$,

$$(\pi(\varphi)\xi)(g)=\varphi(\alpha(g^{-1})x)\xi(g), \quad x\in T(n),$$
$$(\lambda_h\xi)(g)=\xi(h^{-1}g), \quad h, g\in G(n).$$

Proposition 5.1. *Let $M=W^*(A, \alpha, G(n))$ be a factor constructed by the ergodic action α of the group $G(n)$, $n\geq 2$, according to Proposition 4.1. If $\theta\in A_{G(n)}$, then $\theta(A)=A$ and $\theta|_A\in C\alpha G(n)$.*

First we prove the subsidiary Lemma.

Lemma 5.2. *Let α_m be an action of $G(n)$, $n\geq 2$, considered in Lemma 3.3. $M_m=W^*(A_m, \alpha_m, G(n))$ where $A_m=L^{\infty}((\mathbb{T}^n, m_n)^m)$. If $\theta\in A_{G(n)}$, then $\theta(A_m)=A_m$.*

Proof. To simplify the notation, put $n=2$. The general case is considered similarly. Let $B=\theta(A_m)$, the character $\chi_k(1)$ on $\mathbb{T}^{2m}$ is defined by $\chi_k(1)[(\bar{\lambda}_i)_{i=1}^m]=\lambda_k(1)$ where we use the notation of Lemma 3.3. Put $a_k(1)=\pi(\chi_k(1))$, $k=1,2,\dots, m$, and consider $\theta(a_k(1))\in B$. Then $\theta(a_k(1))$ corresponds to the orthogonal series

$$\theta(a_k(1))\sim\sum a(g, \delta, \chi)\lambda_g\lambda_\delta\pi(\chi), \tag{5.1}$$

where $\sum|a(g, \delta, \chi)|^2<\infty$, and the summation is over $g\in\mathrm{SL}(2, \mathbb{Z})$, $\delta\in\mathbb{Z}^2$, $\chi\in\mathbb{Z}^{2m}$.

Since $\theta \in A_{G(n)}$, then the equalities

$$\begin{aligned}&\lambda_1\theta(a_k(1))\lambda_1^{-1}=\theta(\lambda_1 a_k(1)\lambda_1^{-1})=\gamma\theta(a_k(1)),\\&\lambda_2\theta(a_k(1))\lambda_2^{-1}=\theta(a_k(1))\end{aligned} \tag{5.2}$$

are valid, where we put $\lambda_i=\lambda_{\bar{1}_i}$, $i=1, 2$.

Note also the relations

$$\begin{aligned}&\lambda_g\lambda_\delta=\lambda_{g\delta}\lambda_g,\ \delta\in \boldsymbol{Z}^2,\ g\in \mathrm{SL}(2, \boldsymbol{Z}),\\&\lambda_i\pi(\chi)\lambda_i^{-1}=\chi_i(\bar{\gamma})\pi(\chi),\ i=1, 2,\end{aligned} \tag{5.3}$$

where $\bar{\gamma}=(\gamma, \dots, \gamma)\in \boldsymbol{T}^m$, $\chi=(\chi_1, \chi_2)$ and $\chi_i\in \boldsymbol{Z}^m$, $i=1, 2$.

Now let $a(g, \delta, \chi)\neq 0$ for $g\neq e$. Then by taking into account (5.3) it follows from (5.2) that

$$|a(g, \delta, \chi)|=|a(g, \delta+g^{-1}\bar{1}_i-\bar{1}_i, \chi)|,\ i=1, 2,\ \forall\delta\in \boldsymbol{Z}^2,\ \chi\in \boldsymbol{Z}^{2m}.$$

But the equality contradicts the condition $\sum|a(g, \delta, \chi)|^2<\infty$, since $g^{-1}\bar{1}_i=\bar{1}_i$, $i=1, 2$, for $g\neq e$, is impossible. Hence, $a(g, \delta, \chi)=0$ for $g\neq e$ and

$$\theta(a_k(1))\sim\sum a(\delta, \chi)\lambda_\delta\pi(\chi), \tag{5.4}$$

where $\delta\in \boldsymbol{Z}^2$, $\chi\in \boldsymbol{Z}^{2m}$.

Now we find from (5.2) that the summation in (5.4) extends over $\delta\in \boldsymbol{Z}^2$ and those $\chi\in \boldsymbol{Z}^{2m}$ for which $\chi_1(\bar{\gamma})=\gamma$, $\chi_2(\bar{\gamma})=1$.

Note further that if $g_1=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, then

$$\begin{aligned}&\lambda_{g_1}\pi(\chi_k(1))\lambda_{g_1}^{-1}=\pi(\chi_k(1)),\\&\lambda_{g_1}\pi(\chi_k(2))\lambda_{g_1}^{-1}=\pi(\chi_k(1)\chi_k(2)).\end{aligned} \tag{5.5}$$

Using the above consideration we have from (5.3) and these equalities that

$$\theta(a_k(1))\sim\sum a(t, n_1, \dots, n_m)\lambda_2^t\prod_{j=1}^m\pi(\chi_j(1)^{n_j}), \tag{5.6}$$

where t, n_i, $1\leq i\leq m$, $\in \boldsymbol{Z}$ and $\sum_{j=1} n_j=1$.

Since $g_2\chi_k(1)g_2^{-1}=\chi_k(2)$ for $g_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, we have

$$\theta(a_k(2))\sim\sum a(t, n_1, \dots, n_m)\lambda_1^t\prod_{j=1}^m\pi(\chi_j(2)^{n_j}). \tag{5.7}$$

For $g_3=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, we have $g_3\chi_k(1)g_3=\chi_k(1)\chi_k(2)$, and hence

$$\lambda_{g_3}\theta(a_k(1))\lambda_{g_3}^{-1}=\theta(a_k(1))\theta(a_k(2)). \tag{5.8}$$

On the other hand, we have

$$\lambda_{g_3}\theta(a_k(1))\lambda_{g_3}^{-1}\sim\sum a(t,n_1,\dots,n_m)(\lambda_1\lambda_2)^t\prod_{j=1}^{m}\pi(\chi_j(1)\chi_j(2))^{n_j}. \quad (5.9)$$

Then, substituting (5.6) and (5.7) into the right-hand side of (5.8) and comparing it with (5.9), we find that

$$\theta(a_k(1))=a\prod_{j=1}^{m}\pi(\chi_j(1)^{n_j}),\ \sum_{j=1}^{m}n_j=1,\ a\in\mathbb{T}.$$

But this means that $\theta(a_k(1))\in A$, $\quad k\in\mathbb{Z}$. □

Now we complete the proof of Propositions 5. 1. As above, it may be assumed that $\theta(a_k(1))$ corresponds to the orthogonal series of type (5. 4). For $k\in[n_i, n_{i+1})$, we have because of (5. 2) that

$$\sum a(\delta,\chi)\lambda_\delta\chi_1(\bar{\gamma})\pi(\chi)=\gamma_k\sum a(\delta,\chi)\lambda_\delta\pi(\chi),$$

i. e. $\chi_1(\bar{\gamma})=\gamma_k$, where χ_1 is a character on $\mathbb{T}^N$ and $\bar{\gamma}=(\gamma_i)$. But since the numbers $1/2\pi i\ \log\gamma_j'$, $j=1,2,\dots$ (see Proposition 4. 1) are rationally incommensurable, then χ_1 is a character dependent only on the numbers $\lambda_k(1)$, where $k\in[n_i, n_{i+1})$. Then taking into account the relations (5. 5), we have that $\theta(a_k(1))$ corresponds to the series

$$\theta(a_k(1))\sim\sum a(t,n_0',\dots,n_{m-1}')\lambda_2^t\prod_{j=0}^{m-1}\pi(\chi_{n_i+j}(1)^{n_j'}),\ \sum n_j'=1,$$

where m $=(n_{i+1}-n_i)$. Thus we arrive at the sitnation considered in Lemma 5. 2. □

Corollary 5. 3. *Let $M=W^*(A,\alpha,G(n))$ be a* II$_1$*-factor considered in Proposition* 5. 1. *Then the following statements hold*: (i) *In the algebraic sense the $A_{G(n)}$ group is isomorphic to $\prod_{i=1}^{\infty}S(m_i)$*. (ii) *The topology on $A_{G(n)}$, induced by the topology on* Aut *M, coincides with the product-topology on $\prod_{i=1}^{\infty}S(m_i)$*. (iii) *If the factors $M_1=W^*(A,\alpha_1,G(n_1))$ and $M_2=W^*(A,\alpha_2,G(n_2))$, where $n_i\geq3$, $i=1,2$, are isomorphic, then $C\alpha_1G(n_1)\sim\prod_{i=1}^{\infty}S(m_i^1)$ and $C\alpha_2G(n_2)\sim\prod_{i=1}^{\infty}S(m_i^2)$ contain isomorphic closed subgroups H_1 and H_2 with respect to the countable index in $\prod_{i=1}^{\infty}S(m_i^1)$ and $\prod_{i=1}^{\infty}S(m_i^2)$.*

Proof. (i) follows directly from Proposition 5. 1. To prove (ii), remember that the topology on Aut M is given by the system of unit neighbourhoods $V(\varepsilon, x_1, \ldots, x_n) = \{\theta \in \mathrm{Aut}\, M;\ \|\theta(x_i) - x_i\|_2 < \varepsilon,\ x_i \in M,\ 1 \leq i \leq n\}$. Now (ii) follows directly from the construction of the factor M. (iii) follows from (i), (ii) and Proposition 2.4. □

§ 6. Properties of Centralizers of Actions $G(n), n \geq 2$, on Tori

In paragraph 5 we constructed the II_1-factors $M = W^*(A, \alpha, G(n))$ for which $A_{G(n)} \sim \prod_{i=1}^{\infty} S(m_i)$. We study the properties of the groups $\prod_i S(m_i)$.

Let G_i, $i \in N$ be countable discrete groups. $G = \prod_i G_i$ denotes the group which is a direct product of the groups G_i. The functions on N with values in G_i are the elements of $\prod_i G_i$: For $f(i) \in G_i$, let $f \sim (f(i))$, $g \sim (g(i)) \in \prod_i G_i$, then $f^{-1} \sim (f(i)^{-1})$, $fg \sim (f(i)g(i))$.

We find on $\prod_i G_i$ the weak (product) topology which is given by the system of the unit neighobourhood

$$U_I = \{f \in \prod_i G_i,\ f(n) = e_n,\ n \in I\}$$

where I is a finite subset N, and e_i is the unit of the group G_i. As is well known, $\prod_i G_i$ is a topological group with respect to this topology.

Let I be a subset N and put $G(I) = \prod_{i \in I} G_i$. If $f \in \prod_i G_i$ then the mapping $\pi(I)f = \chi_I f$, where χ_I is a characteristic function of I, determines the canonical homomorphism of $\prod_i G_i$ on $G(I)$. In particular $\pi(j)G = G_j$.

Proposition 6.1. *If H is a closed subgroup of $G = \prod_i G_i$ of the countable index in G, then there exists $n \in N$ such that*

$$\pi([n+1, \infty))H = G([n+1, \infty)).$$

Lemma 6.2. *If, for any $n \in N$, $\pi([1, n])H = G([1, n])$ then $H = G(= \prod_i G_i)$.*

Proof. Let $f \in G$. If $f_n \in H$ and $f_n = f$ on $[1, n]$, then $\lim_{n \to \infty} f_n = f$ with respect to the product-topology in G and hence $H = G$. □

Proof of Proposition 6.1. Assume that the theorem is not correct. Since $H \subset G$, then by Lemma 6.2 there exists $n_1 \in N$ such that

$\pi([1, n_1])H\subset G([1,n_1])$ and $\pi([n_1+1, \infty))H\subset G([n_1+1, \infty))$. Since $\pi([1,n_1])H\subset G([1,n_1])$, then there exists $h_1\in G([1,n_1])$ and $h_1\notin\pi([1,n_1])H$. Since $\pi([n_1+1,\infty))H\subset G([n_1+1,\infty))$, then by the Lemma there exists $n_2\in N$ such that $\pi([n_1+1, n_2])H\subset G([n_1+1, n_2])$, where $n_2>n_1+1$ and h_2 of $G([n_1+1, n_2])$ not belonging to $\pi([n_1+1, n_2])H$. Similarly we obtain partition of N into the intervals $I_{i+1}=[n_i+1, n_{i+1}]$, where $n_0=0$, $n_i+1<n_{i+1}$, $\pi(I_{i+1})H\subset G(I_{i+1})$ and $\exists h_{i+1}\in G(I_{i+1})$, $h_{i+1}\notin\pi(I_{i+1})H$.

It is obvious that $G=\prod_i G(I_i)$ and $(h_i^{\alpha_i})_{i=1}^{\infty}\in\prod_i G(I_i)$ where $\alpha_i=0$ or 1. We state that $(h_i^{\alpha_i})_{i=1}^{\infty}\notin H$, if not all $\alpha_i=0$, and if $(\alpha_i)\neq(\beta_i)$, then $(h_i^{\alpha_i-\beta_i})_{i=1}^{\infty}\notin H$. Indeed, if $\alpha_{j+1}=1$ and $\beta_{j+1}=0$ and if $(h_i^{\alpha_i-\beta_i})_{i=1}^{\infty}\in H$, then $\pi(I_{j+1})(h_i^{\alpha_i-\beta_i})_{i=1}^{\infty}=h_{j+1}\subset\pi(I_{j+1})H$ which is impossible due to the choice of h_{j+1}. Because there are different elements $(h_i^{\alpha_i})_{i=1}^{\infty}$ of the cardinality of continuum due to the continuum cardinality of the choice $\alpha_i=0$ or $\alpha_i=1$, the power of the set $[G: H]$ is equal to the continuum. The obtained discrepancy proves Proposition. □

Denote the group of all integral matrices having the determinant ±1 as $\mathrm{GL}(n, \mathbb{Z})$. According to Lemma 3. 4, $S(n)\sim\mathrm{GL}(n-1, \mathbb{Z})\circledS\mathbb{Z}^{n-1}$.

Lemma 6.3. (i) *It is impossible to represent the group $S(n)$, $n\geq4$, as a direct product of two groups.* (ii) *There is no homomorphism of $S(m)$ on $S(n)$, where $m\neq n\geq4$, m, $n\in N$.*

The proof of the Lemma uses Theorem 3 [14] from which it follows that any normal subgroup of $\mathrm{SL}(n, \mathbb{Z})$, $n\geq3$, for which the factor-group is non-amenable, belongs to the center $Z\,\mathrm{SL}(n, \mathbb{Z})$ of $\mathrm{SL}(n, \mathbb{Z})$. The proof is also based on the Margulis rigidity theorem (see Theorem 3 [13]) and some simple results on lattices [15]. These theorems were also formulated in the survey article [1].

We present the proof of (ii) assuming the oddness of m and n. Let (ii) be incorrect in this case. The homomorphism of $S(m)$ on $S(n)$ is denoted as φ_1. N_m and N_n denote the subgroups of $\mathrm{id}\times\mathbb{Z}^{m-1}$ and $\mathrm{id}\times\mathbb{Z}^{n-1}$, respectively, and ϕ_1 denotes the homomorphism of $S(n)$ on $\mathrm{GL}(n-1, \mathbb{Z})$. Then $\varphi=\phi_1\varphi_1$ is the homomorphism of $S(m)$ on $\mathrm{GL}(n-1, \mathbb{Z})$. We prove that $\varphi(N_m)\subseteq Z\mathrm{GL}(n-1, \mathbb{Z})$. Let N be the normal subgroup of $\mathrm{GL}(n-1, \mathbb{Z})$ generated by $\varphi(N_m)$ and $Z\mathrm{GL}(n-1,\mathbb{Z})$.

Then N is a commutative normal subgroup of $\mathrm{GL}(n-1, \boldsymbol{Z})$. Since $\mathrm{GL}(n-1, \boldsymbol{Z})$ is a non-amenable group, the group $\mathrm{GL}(n-1, \boldsymbol{Z})/N$ is also non-amenable. ψ denotes the homomorphism of $\mathrm{GL}(n-1, \boldsymbol{Z})$ on $\mathrm{GL}(n-1, \boldsymbol{Z})/N$. Since $\mathrm{SL}(n-1, \boldsymbol{Z})$ has index 2 in $\mathrm{GL}(n-1, \boldsymbol{Z})$, $\psi(\mathrm{SL}(n-1, \boldsymbol{Z}))$ is also a non-amenable subgroup of $\mathrm{GL}(n-1, \boldsymbol{Z})/N$ and ψ denotes the homomorphism of $\mathrm{SL}(n-1, \boldsymbol{Z})$ on $\psi(\mathrm{SL}(n-1, \boldsymbol{Z}))$. According to Theorem 3 [14], the kernel of this homomorphism is contained in $Z = Z\mathrm{SL}(n-1, \boldsymbol{Z})$ and in $\psi(\mathrm{SL}(n-1, \boldsymbol{Z})) \approx (\mathrm{SL}(n-1, \boldsymbol{Z})/K$ where $K = \mathrm{SL}(n-1, \boldsymbol{Z}) \cap N \subset Z$. Let $g = (n_{ij})$ be a diagonal matrix for which $n_{11} = -1$ and $n_{ii} = 1, i = 2, \ldots, n-1$. Obviously g_1 and $\mathrm{SL}(n-1, \boldsymbol{Z})$ generate $\mathrm{GL}(n-1, \boldsymbol{Z})$. We prove that $g_i h \notin N$ for $\forall h \in \mathrm{SL}(n-1, \boldsymbol{Z})$. Indeed, if $g_1 h = n \in N$, then $g_1 h a h^{-1} g_1^{-1} = n a n^{-1} = a n_1 \in \mathrm{SL}(n-1, \boldsymbol{Z})$, where $n_1 \in N$ for $\forall a \in \mathrm{SL}\ (n-1, \boldsymbol{Z})$. Thus $g_1 h a h^{-1} g_1^{-1} = a\varepsilon$ where $\varepsilon \in Z$. Evidently, $\varepsilon \neq -I$, since $\mathrm{Ad} g_1 h$ is the automorphism of $\mathrm{SL}(n-1, \boldsymbol{Z})$. But then $\mathrm{Ad} g_1 h(a) = a$ for $\forall a \in \mathrm{SL}(n-1, \boldsymbol{Z})$ that is also impossible because $\mathrm{Ad} g_1$ is the outer automorphism of $\mathrm{SL}(n-1, \boldsymbol{Z})$. Hence, we have $g_1 h \notin N$ for $\forall h \in \mathrm{SL}(n-1, \boldsymbol{Z})$, and herefore $\varphi(N_m) \subseteq N \subseteq Z\mathrm{GL}(n-1, \boldsymbol{Z})$.

Thus, $\varphi(N_m) \subseteq Z\mathrm{GL}(n-1, \boldsymbol{Z})$ and hence $\varphi(N_m) = \mathrm{id}$ hold, Therefore φ determines the homomorphism of $\mathrm{GL}(m-1, \mathrm{Z})$ on $\mathrm{GL}(n-1, \boldsymbol{Z})$ which is denoted as $\tilde{\varphi}$. If $\tilde{N} = \mathrm{Ker}\, \tilde{\varphi}$ then repeating the above arguments we have that $N \subseteq Z\mathrm{GL}(m-1, \boldsymbol{Z})$. Then a simple test shows that the centre of $\mathrm{GL}(m-1, \boldsymbol{Z})/Z\mathrm{GL}(m-1, \boldsymbol{Z})$ is trivial but $Z\mathrm{GL}(n-1, \boldsymbol{Z}) \neq \mathrm{id}$. Hence $\tilde{N} = \mathrm{id}$ and $\varphi(Z\mathrm{GL}(m-1, \boldsymbol{Z})) = Z\mathrm{GL}(n-1, \boldsymbol{Z})$. Thus $\tilde{\varphi}$ is an isomorphism on $\mathrm{GL}(n-1, \boldsymbol{Z})$.

Now let π be a natural homomorphism of $\mathrm{GL}(n-1, \boldsymbol{Z})$ on $\mathrm{GL}(n-1, \boldsymbol{Z})/\mathrm{SL}(n-1, \boldsymbol{Z})$ then $\pi(\mathrm{SL}(n-1, \boldsymbol{Z})) = 0 \pmod 2$ and $\pi(g_1) = 1 \pmod 2$. Let K_n denote the maximum subgroup of $\tilde{\varphi}(\mathrm{SL}(m-1, \boldsymbol{Z}))$ such that $\pi(K_n) = 0$. Then K_n is a normal subgroup of $\tilde{\varphi}(\mathrm{SL}(m-1, \boldsymbol{Z}))$ having an index not more than 2, that means that K_n is a normal subgroup of $\mathrm{GL}(n-1, \boldsymbol{Z})$ with an even index not more than 4. Since $K_n \subset \mathrm{SL}(n-1, \boldsymbol{Z})$ then K_n is a normal subgroup of $\mathrm{SL}(n-1, \boldsymbol{Z})$ with an index not more than 2. But then according to 10.5, 1.6 and 5.2 [15] the group K_n is an irreducible lattice in $\mathrm{SL}(n-1, \boldsymbol{Z})$ and its isomorphic subgroup $K_m = \tilde{\varphi}^{-1} K_n$ is by the same reason an irreducible lattice in $\mathrm{SL}(\mathrm{m}-1, \boldsymbol{Z})$. By the Margulis rigidity theorem [13] this is excluded and hence the homomorphism φ of $S(m)$ on $S(n)$ does not exists. This proves (ii) for odd n, m. The other cases are considered in a simpler way. □

If $\{m_i\}_{i=1}^{\infty}$ is a sequence of natural numbers, then $S(\{m_i\}_{i=1}^{\infty})$ will denote the group, which is a product of the groups $\{S(m_i)\}_{i=1}^{\infty}$ with the topology of a direct product.

Proposition 6.4. *Let $m\in N$, $m\notin\{m_i\}_{i=1}^{\infty}$. Then $S=S(\{m_i\}_{i=1}^{\infty})$ cannot be represented an $S=S(m)\otimes N$, where N is a subgroup of S.*

Proof. Assume the opposite, i.e. $S=S(m)\otimes N$. Let $S_1=(f_1(i)_{i=1}^{\infty})$ and $S_2=(f_2(i)_{i=1}^{\infty})$ be non-commuting elements of $S(m)\otimes\mathrm{id}$. There exists $j\in N$ such that $f_1(j)$ and $f_2(j)$ do not commute either. Consider $\pi(j)S=\pi(j)S(m)\otimes\pi(j)N$. But $\pi(j)S=S(m_j)$ and $S(m_j)$ cannot be represented as a direct product of two groups. Since $f_i(j)\neq\mathrm{id}$, $i=1, 2$, and do not commute, therefore $\pi(j)(S(m)\otimes\mathrm{id})=S(m_j)$, which is impossible for $m\neq m_j$ due to the results given in Lemma 6.3. □

§7. A Continuum of Nonequivalent Actions of Groups $G(n)$, $n\geq 3$

Now we can proceed to the proof of the main statements of the paper.

Theorem 7.1. *There exists a continuum of orbit-nonequivalent free actions of the group $G(n)$, $n\geq 3$, on the Lebesgue space, which preserve the finite measure. The factors constructed by these actions are non-isomorphic full factors of II_1 type.*

Proof. The sequence of the groups $(S(2i+\alpha_i)_{i=1}^{\infty})$ is corresponding to any $(\alpha_i)_{i=1}^{\infty}$, where $\alpha_i=0$ or 1. Because of Proposition 4.1 an ergodic free action α of the group $G(n)$ on $(T(n), m(n))$ can be constructed so that $C\alpha G(n)=\prod_{i=1}^{\infty}S(2i+\alpha_i)$. Let $(\beta_i)_{i=1}^{\infty}$ be another binary sequence, such that the set $\{i: \alpha_i\neq\beta_i\}$ is infinite. Let β denote the action of $G(n)$ on $(T(n), m(n))$, corresponding to $(\beta_i)_{i=1}^{\infty}$. We state that the II_1-factors $W^*(A, \alpha, G(n))$ and $W^*(A, \beta, G(n))$ are not isomorphic. Assume the opposite. By Corollary 5.3 there exist closed isomorphic subgroups $H_\alpha\subset C\alpha G(n)$ and $H_\beta\subset C\beta G(n)$ having a countable index in $C\alpha G(n)$ and $C\beta G(n)$, respectively. Because of Proposition 6. 1 $H_\alpha=H_1\otimes\prod_{i=N+1}^{\infty}S(2i+\alpha_i)$, $H_\beta=H_2\otimes\prod_{i=N+1}^{\infty}S(2i+\beta_i)$, where H_1 is a subgroup of $\prod_{i=1}^{N}S(2i+\alpha_i)$ and H_2 is a subgroup of $\prod_{i=1}^{N}S(2i+\beta_i)$. Let $\pi(1)$ denote the homomorphism of H_β on H_2: $\pi(1)H_\beta=H_2$, and $\pi(j)$ denote the homomorphism of H_β on $S(2j+\beta_j)$ for $j>N$. If φ is an isomorphism of H_α on H_β then by Lemma 6. 3 $\pi(j)\varphi(S(2i+\alpha_i))=\mathrm{id}$ for $i\in K=(\alpha_t\neq\beta_t,\ t>N)$ and $j>N$. But then $\varphi(S(2i+\alpha_i))\subseteq H_2$ for $i\in K$. Hence $\varphi(\bigoplus_{i\in K}S(2i+\alpha_i))\subset H_2$, and from the continuity of φ (see Proposition 2. 5) we find that $\varphi(\prod_{i\in K}S(2i+\alpha_i))\subseteq H_2$ since H_2 is a discrete closed subgroup of $A_{\beta G(n)}$. But φ is an isomorphism, the group H_2 is a countable and $\prod_{i\in K}S(2i+\alpha_i)$ is continual, which is impossible. The obtained contradiction shows that the factors $W^*(A,\alpha,G(n))$ and $W^*(A,\beta,G(n))$ are not isomorphic, and the actions α and β of the group $G(n)$ are orbit-non-equivalent. It remains to note that the set of all binary sequences $(\alpha_i)_{i=1}^{\infty}$, where $\alpha_i=0$ or 1, every two of which have an infinite number of different components, has the power of the continuum since such a set is isomorphic to the factor-group with respect to the subgroup of all sequences with a finite number of components other than zero. □

Remark. If α is an ergodic action of the group $G(n)=\mathrm{SL}(n,\boldsymbol{Z})\ominus\boldsymbol{Z}^n$, $n\geq 3$, constructed in Theorem 7. 1, then using an inducing construction one can construct an ergodic action $\tilde{\alpha}$ of the group $\mathrm{SL}(n,\boldsymbol{R})\ominus\boldsymbol{R}^n$ preserving the finite measure. But then from Theorem 7. 1 and the inducing construction it follows that $\mathrm{SL}(n,\boldsymbol{R})\ominus\boldsymbol{R}^n$, $n\geq 3$, has a continuum of orbit-non-equivalent free ergodic actions preserving the finite measure.

Theorem 7.2. *There exists a full* II_1*-factor* M *whose all tensor degrees* $M, M\otimes M,\dots$ *are non-isomorphic by pairs and the dynamical systems* $(T(n),\alpha,G(n))^p$, $p\in N$ *are orbit-non-equivalent for different* p.

Proof. We construct an action α of the group $G(n)$, $n\geq 3$, on $(T(n),m(n))$ so that $C\alpha G(n)\sim\prod_{i=1}^{\infty}S(2i)$ according to Proposition 4. 1. Let $M=W^*(A,\alpha,G(n))$. Then $M\otimes M=W^*(A\otimes A,\alpha^2,G(n)^2)$ where $\alpha^2=\alpha\otimes\alpha$

and $G(n)^2=G(n)\times G(n)$. Describe $C\alpha^2 G(n)^2$: If $\tilde{G}(n)=(g\times g,\ g\in G(n))$ is a diagonal subgroup of $G(n)^2$, then evidently $G(n)\sim\tilde{G}(n)$. Consider the action α^2 of the group $\tilde{G}(n)$ on $T(n)\times T(n)$. Because of Proposition 4.1 $C\alpha^2\tilde{G}(n)\sim\prod_{i=1}^{\infty}S(4i)$. Taking into account the commutativity of the elements of $C\alpha^2\tilde{G}(n)$ with $\alpha(g)\times\mathrm{id}$ and $\mathrm{id}\times\alpha(g)$, where $g\in G(n)$, we have that $C\alpha^2 G(n)^2\sim\prod_{i=1}^{\infty}(S(2i)\oplus S(2i))\sim\prod_{i=1}^{\infty}(S(2i))^2$.

Now by repeating the same arguments as in the proof of Proposition 5.1 it can be shown that if $\theta\in A_{G(n)^2}$, then $\theta(A\otimes A)=A\otimes A$ and hence $A_{G(n)^2}\sim C\alpha^2 G(n)^2\sim(\prod_{i=1}^{\infty}S(2i))^2$.

We prove a subsidiary Lemma.

Lemma 7.3. *Let* $H_1=K_1\otimes\prod_{i=N+1}^{\infty}S(2i)$ *and* $H_2=K_2\otimes\prod_{i=N+1}^{\infty}(S(2i)\oplus S(2i))$ *be groups where* K_1 *is a subgroup of* $\prod_{i=1}^{N}S(2i)$ *and* K_2 *is a subgroup of* $\prod_{i=1}^{N}(S(2i)\oplus S(2i))$. *Then the groups* H_1 *and* H_2 *are non-isomorphic.*

Proof. Assume the opposite. Let φ denote the isomorphism of Lemma 2.5 of H_2 on H_1. Put $S_1(2i)=S(2i)\otimes\mathrm{id}$ and $S_2(2i)=\mathrm{id}\otimes S(2i)$. Then for any $i>N$ there exists $j(=1,2)$ such that $\varphi(S_j(2i))\subset K_1$. Indeed, if $\pi(1)$ denotes the homomorphism of H_1 on K_1 and $\pi(i)$ of H_1 on $S(2i)$ for $i>N$, then from Lemma 6.3 it follows that $\pi(p)\varphi S_j(2i)=\mathrm{id}$ for $p\neq i$ and hence $\varphi(S_j(2i))\subset K_1\otimes S(2i)$. If for example $\pi(i)\varphi S_1(2i)\neq\mathrm{id}$, then because of Lemma 6.3 we have that $\pi(i)\varphi S_1(2i)=S(2i)$. But then $\varphi S_2(2i)\subset K_1$. Thus, the group K_1 contains a subgroup isomorphic to $\prod_{i>N}S(2i)$, which is excluded. □

Completion of the proof of Theorem 7.2. Let H_1 and H_2 be closed subgroups of a countable index in $A_{G(n)}$ and $A_{G(n)^2}$, respectively. Since $A_{G(n)}\sim\prod_{i=1}^{\infty}S(2i)$ and $A_{G(n)^2}\sim\prod_{i=1}^{\infty}(S(2i)\oplus S(2i))$, then according to Proposition 6.1 H_1 and H_2 have the form such as in the formulation of Lemma 7.3. According to this Lemma the groups H_1 and H_2 are non-isomorphic. Therefore from Proposition 2.5 the nonisomorphism between M and $M\otimes M$ follows. The general case is considered similarly. □

§ 8. Full II_1-Factors with Non-Isomorphic Subfactors of Finite Index

We continue the discussion of the main results of the paper. For $n=2k+1$, $G(n)$ has the automorphism γ of the order of two:

$$\gamma(g)=g \quad \text{at} \quad g\in \mathrm{SL}(n, \boldsymbol{Z}),$$
$$\gamma(\bar{1}_i)=-\bar{1}_i \quad \text{at} \quad \bar{1}_i\in \boldsymbol{Z}^n,\ 1\leq i\leq n.$$

$G_2(n)$ denotes a semidirect product of the group $\boldsymbol{Z}_2$ on $G(n)$. Evidently, $G_2(n)$ for $n=2k+1$, $k=1, 2, \ldots$, is a TICC-group. Construct an ergodic action α of $G(n)$ on $(T(n), m(n))$ according to Proposition 4. 1 putting $n_{j+1}-n_j=2$ for $\forall j$. Then $M=W^*(A, \alpha, G(n))$, where $A=L^\infty(T(n), m(n))$ is a full II_1-factor. Construct the automorphism γ_1 of the space $(T(n), m(n))$, putting

$$\gamma_1[(\bar{\lambda}_q)_{q=1}^\infty]=[(\overline{\lambda_q^{-1}})_{q=1}^\infty],$$

where $(\bar{\lambda}_q)_{q=1}^\infty\in T(n)$. A direct test shows that

$$\gamma_1\alpha(g)\gamma_1^{-1}=\alpha(\gamma(g)), \qquad \forall g\in G(n). \tag{8. 1}$$

Thus, we construct a free ergodic action of the group $G_2(n)$ on $(T(n), m(n))$, which is denoted as α_1. Let $N_1=W^*(A, \alpha_1, G_2(n))$ be a II_1 type factor corresponding to this action. From (8. 1) it follows that $\gamma_1\in N[\alpha(G(n))]$, where $[\alpha G(n)]$ denotes a full group of automorphisms $(T(n), m(n))$ created by $\alpha G(n)$ [17]. As known in this case, γ_1 extends to the automorphism of M which is denoted again as γ_1.

Lemma 8. 1. (i) *$W^*(M, \gamma_1)\sim N_1$ where $N_1=W^*(A, \alpha_1, G_2(n))$.* (ii) *$C\alpha_1G_2(n)\sim S(2)^N$, where $S(2)^N$ denotes a direct product of the countable number of copies of the group $S(2)$.*

Proof. (i) is evident. To prove (ii), note that according to Proposition 4. 1, $C\alpha G(n)\sim S(2)^N$. From the definition of γ_1 it follows that γ_1 commutes with the automorphisms of $C\alpha G(n)$ and hence $C\alpha_1G_2(n)=C\alpha G(n)$. □

Construct another action of $G_2(n)$ on $(T(n), m(n))$. Let δ be an automorphism of $S(2)^N$ which corresponds to the element $(u_i)_{i=1}^\infty$ of $S(2)^N$, where $u_i=u=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. It is clear that $\delta^2=\mathrm{id}$. Put $\gamma_2=\gamma_1\delta=\delta\gamma_1$.

Since $\alpha(g)\delta=\delta\alpha(g)$, $g\in G(n)$, We have

$$\gamma_2\alpha(g)\gamma_2^{-1}=\alpha(\gamma_2(g)), \ \forall g\in G(n). \tag{8.2}$$

Thus, the actions of $G_2(n)$ is constructed, which is denoted as α_2.

Lemma 8.2. (i) *$W^*(M,\gamma_2)\sim N_2$, where $N_2=W^*(A,\alpha_2,G_2(n))$.*
(ii) $C\alpha_2G_2(n)\sim(\mathbb{Z}_2)^N$.

Proof. Let Cu be a centralizer u in $S(2)$. It is easy to check that $Cu=(1, u)$ and the centre $S(2)$ is trivial. Since $C\alpha G(n)\sim S(2)^N$ the automorphism of $C\alpha_2G_2(n)$ corresponds to the matrix sequence $(v_i)_{i=1}^{\infty}$, where $v_i\in Cu$ i.e. $v_i=u$ or 1. Hence $C\alpha_2G_2(n)\sim(\mathbb{Z}_2)^N$. □

Theorem 8.3. (i) *There exists a full II_1-factor M possessing outer automorphisms γ_i, $i=1,2$ such that $\gamma_i^2=\mathrm{id}$ and the factors $N_i=W^*(M,\gamma_i)$, which are crossed products of γ_i on M, are non-isomorphic and hence γ_1 and γ_2 are not outer conjugate.*

(ii) *The factors $N_2=W^*(M,\gamma_2)$ and M are non-isomorphic, i.e. a full factor of type II_1 can contain a non-isomorphic subfactor of a finite index (see [18]).*

Proof. Remember that $M=W^*(A, \alpha, G(n))$ and $N_2=W^*(A, \alpha_2, G_2(n))$ where $G(n)$ and $G_2(n)$, $n=2k+1$, $k=1,2,\dots$, are TICC-groups. Use Proposition 6.1 taking into account that $C\alpha G(n)\sim S(2)^N$ and $C\alpha_2G_2(n)\sim(\mathbb{Z}_2)^N$ (see Lemma 8.2). According to this Proposition, $C\alpha G(n)$ and $C\alpha_2G_2(n)$ cannot contain closed isomorphic subgroups of a countable index. Therefore by Corollary 5.3 (iii) the factors M and N_2 are non-isomorphic. It is similarly proved that N_1 and N_2 are non-isomorphic. □

Theorem 8.4. *There exists a full factor N_2 of type II_1, in which* Out N_2=Aut N_2/Int N_2 *is a continuous locally compact totally disconnected group. (The case when* Out N *is discrete is described in [4].)*

Proof. Since $N_2=W^*(A, \alpha_2, G_2(n))$ is a full II_1-factor, Int N_2 is closed in Aut N_2 and on the group Out N_2 a factor-topology τ_1 is induced, with respect to which all the points Out N_2 are closed. According to Theorem 2.2 $A_{G_2(n)}^{N_2}$ is an open subgroup Aut N_2, but since (Int N_2). $A_{G_2(n)} \subset A^{N_2}_{G_2(n)}$ then $A^{N_2}_{G_2(n)}/\mathrm{Int}\, N_2 \sim A_{G_2(n)}$ is also an open and closed subgroup of Out N_2. But on $A_{G_2(n)}$ one can consider the topology τ_2 induced directly with Aut N_2. By Corollary 8. 2 (ii) with respect to τ_2 the group $A_{G_2(n)}$ is compact (and isomorphic $(\mathbb{Z}_2)^N$). Hence since τ_2 is stronger than τ_1, we find through the standard considerations that τ_2 and τ_1 coincide on $A_{G_2(n)}$. Thus $A^{N_2}_{G_2(n)}/\mathrm{Int}\, N_2$ is an open compact subgroup Out N_2 isomorphic to $(\mathbb{Z}_2)^N$ and hence the group Out N_2 itself is locally compact. □

Supplement

On Ergodic Actions of Groups $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$.

In this Supplement we illustrate the concept of the proof of the following theorem.

Theorem S.1. *The group $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, has at least a countable number of orbit-non-equivalent ergodic actions preserving the finite measure.*

Let Γ_m be the normal subgroup of Γ such that $\Gamma/\Gamma_m \sim \mathrm{SL}(n, \mathbb{Z}/p^m)$ where $\mathbb{Z}/p^m$ is the residue ring modulo p^m. In what follows p is a simple number. Let $K_p = \lim\limits_{\infty \leftarrow m} \Gamma/\Gamma_m$ be the projective limit of the group Γ/Γ_m, then, as is well known, K_p is a compact group including Γ as a dense subgroup.

Lemma S.2. *The group K_p and K_q are nonisomorphic for different simple numbers p and q and do not contain isomorphic subgroups of a finite index.*

The proof is based on the following fact. Let $K_p^m = \lim\limits_{\infty \leftarrow t} \Gamma_m/\Gamma_{m+t}$ then K_p^m is a normal subgroup of K_p of a finite index, in this case $K_p/K_p^m \approx \mathrm{SL}(n, \mathbb{Z}/p^m)$. Besides, $\bigcap\limits_m K_p^m = \{e\}$.

Lemma S.3. *Let μ_p be a the Haar measure of the group K_p. Consider the right-hand α_2 and left-hand α_e actions of the group K_p on (K_p, μ_p): $\alpha_2(k)h = hk$, $\alpha_e(k)h = kh$ where k, $h \in K_p$. Then $C\alpha_e(\Gamma) = \alpha_r(K_p)$, where $C\alpha_e(\Gamma)$ is a centralizer of $\alpha_e(\Gamma)$ in* Aut (K_p, μ_p).

The proof uses simple properties of the matrix elements of irreducible representations of the compact groups.

We shall agree to denote the action $\alpha_e(\gamma)$, $\gamma\in\Gamma$, on (K_p, μ_p) as $\alpha_p(\gamma)$.

Lemma S. 4. *Let* $M_p=W^*(\beta_p, \alpha_p, \Gamma)$ *be a factor, where* $B_p=L^\infty(K_p, \mu_p)$. *Then* $A_{\alpha_p\Gamma}\sim C\alpha_p\Gamma$ (see the notation of Theorem 2. 2). *Thus,* M_p *and* M_q *at* $p\neq q$ *are not isomorphic.*

It is readily tested that if $\gamma\in A_{\alpha_p\Gamma}$, then $\gamma(B_p)=B_p$ and $\gamma|_{B_p}\in C\alpha_p\Gamma$. Therefore $A_{\alpha_p\Gamma}\sim K_p$ and, according to Lemma S. 2, as well as to the arguments given in the proof of Theorem 8. 4 and Proposition 2. 5, the factors M_p and M_q are not isomorphic for simple $p\neq q$.

The statements given are true for the case when $\Gamma=\mathrm{SL}(n, \mathbb{Z})$ has a trivial centre, i. e. $n=2k+1$, $k=1, 2, \ldots$. If $n=2k$, $k=2, 3, \ldots$ then the problem is readily reduced to the case, when $\Gamma=P\mathrm{SL}(n, \mathbb{Z})$.

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