Singular States on Maximal Op*-algebras

By

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Abstract

The paper is devoted to the investigation of singular states on topological *-algebras of unbounded operators defined on (F)-domains. There are considered only functionals which are continuous with respect to the so called uniform topology $\tau_{\mathscr{D}}$. Equivalent characterizations of positive singular functionals and a decomposition result for functionals are proved. Analogously to the bounded case positive singular functionals can be given with the help of limits on free ultrafilters. Moreover, the state space of the maximal Op^*-algebra is the w^*-closed convex hull of vector states (pure states)– despite the fact that the state space is not w^*-compact.

§ 1. Introduction

In the last 15 years the theory of topological algebras of unbounded operators has been substantially developed. This concerns the study of the topological and order structure, commutants, ideals, special classes of algebras and representations as well as applications to quantum statistics [10]. As for states on topological operator algebras, satisfying results were obtained first of all for normal states (see [20] for a short summary). The present paper is the first of a series devoted to the investigation of the structure of the state space of Op^*-algebras. We start here with the study of positive singular functionals on the maximal Op^*-algebra $\mathscr{L}^+(\mathscr{D})$ on (F)-domains.

The paper is organized as follows. Section 2 contains the necessary definitions, notations and auxiliary results. In section 3 we describe a restriction-extension procedure which relates classes of $\tau_{\mathscr{D}}$-continuous functionals on $\mathscr{L}^+(\mathscr{D})$ and $\mathscr{B}(\mathscr{H})$. This gives a useful method to

transform results from the bounded case $(\mathscr{B}(\mathscr{H}))$ to the unbounded one $(\mathscr{L}^+(\mathscr{D}))$. As an application we prove that states can be uniquely decomposed into the sum of normal and singular positive functionals. In section 4 a representation theorem for positive singular functionals is given (see also [20]), which has interesting applications. For example one gets that singular states are w^*-limits of vector states. This has the following important conclusion. Despite the fact that the state space of $\mathscr{L}^+(\mathscr{D})$ (i. e. the set of $\tau_{\mathscr{D}}$-continuous states) is not w^*-compact, it is the w^*-closed convex hull of vector states, hence of pure states. A more general result derived by quite other methods is given in [14].

§2. Preliminaries

For a dense linear manifold $\mathscr{D}$ in a separable Hilbert space $\mathscr{H}$ the set of linear operators $\mathscr{L}^+(\mathscr{D}) = \{A: A\mathscr{D} \subset \mathscr{D}, A^*\mathscr{D} \subset \mathscr{D}\}$ is a *-algebra with respect to the usual operations and the involution $A \longrightarrow A^+ = A^* \mid \mathscr{D}$. An Op^*-algebra $\mathscr{A}(\mathscr{D})$ is a *-subalgebra of $\mathscr{L}^+$ $(\mathscr{D})$ containing the identity operator I. The graph topology $t_{\mathscr{A}}$ on $\mathscr{D}$ induced by $\mathscr{A}(\mathscr{D})$ is given by the family of seminorms $\varphi \longrightarrow \|A\varphi\|$ for all $A \in \mathscr{A}(\mathscr{D})$. Denote $t_{\mathscr{L}^+(\mathscr{D})}$ simply by t. This topologization of $\mathscr{D}$ gives rise to a canonical rigged Hilbert space $\mathscr{D}[t] \subset \mathscr{H} \subset \mathscr{D}'[t']$ and a canonical dual pair $(\mathscr{D}, \mathscr{D}')$. Here t' is the strong topology in $\mathscr{D}'$. Let $\sigma = \sigma(\mathscr{D}, \mathscr{D}')$ be the weak topology in $\mathscr{D}$. Remember that a sequence $(\varphi_n) \subset \mathscr{D}$ is σ-convergent to zero $(\varphi_n \xrightarrow{\sigma} 0)$ if and only if $\{\varphi_n\}$ is t-bounded and $\langle \varphi, \varphi_n \rangle \longrightarrow 0$ for all $\varphi \in \mathscr{D}$, hence for all $\varphi \in \mathscr{H}$. An Op^*-algebra $\mathscr{A}(\mathscr{D})$ is called

closed if $\mathscr{D} = \bigcap_{A \in \mathscr{A}} \mathscr{D}(\bar{A})$ or equivalently if $\mathscr{D}[t]$ is complete;

selfadjoint if $\mathscr{D} = \mathscr{D}_* \equiv \bigcap_{A \in \mathscr{A}} \mathscr{D}(A^*)$.

In Op^*-algebras there can be defined a lot of topologies (cf. e. g. [8]-[10], [15], [16]). We mention only those used here: the *uniform topology* $\tau_{\mathscr{D}}$ given by the seminorms

$$A \longrightarrow \|A\|_{\mathscr{U}} = \sup_{\varphi, \psi \in \mathscr{U}} |\langle \varphi, A\psi \rangle|$$

where $\mathscr{U}$ runs over all $t_{\mathscr{A}}$-bounded sets of $\mathscr{D}$; the topology $\tau_{\mathscr{D}}^\circ$ given by the seminorms $\|\cdot\|_{\mathscr{N}}$ where $\mathscr{N}$ now runs over all relatively $t_{\mathscr{A}}$

compact subsets of $\mathcal{D}$. Later on we will use the fact that $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{D}}^c$ are not only defined on $\mathcal{L}^+(\mathcal{D})$ but also on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, hence on $\mathcal{B}(\mathcal{H})$ [10], [6]. The most important domains $\mathcal{D}$ are those where t is a Frechéttopology, i. e. $\mathcal{D}[t]$ is complete and t can be given by $\{\|A_n \cdot\|: n \in N\}$ with $A_n = A_n^+$, $I = A_0 \leq A_1 \leq \cdots$. A special type of such domains is of the form $\mathcal{D} = \mathcal{D}^\infty(T) \equiv \bigcap_n \mathcal{D}(T^n)\, T = T^* \geq I$.

In what follows we allways assume that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint and $\mathcal{D}[t]$ is an (F)-space. Some of the results are valid in more general situations. To simplify notations we denote a bounded operator $A \in \mathcal{L}^+(\mathcal{D})$ and its closure $\bar{A} \in \mathcal{B}(\mathcal{H})$ by the same letter A. The following sets are two-sided *-ideals in $\mathcal{L}^+(\mathcal{D})$ and play an important role in the description of $\tau_{\mathcal{D}}$, $\tau_{\mathcal{D}}^c$ ([5], [9], [19], [20]):

$$\mathcal{B}(\mathcal{D}) = \{T:\ T\mathcal{H} \subset \mathcal{D},\ T^*\mathcal{H} \subset \mathcal{D}\}$$
$$\qquad = \{T:\ AT, AT^* \text{ bounded for all } A \in \mathcal{L}^+(\mathcal{D})\},$$
$$\mathcal{S}_\infty(\mathcal{D}) = \{T:\ T \in \mathcal{S}_\infty(\mathcal{H}) \cap \mathcal{B}(\mathcal{D})\}$$
$$\qquad = \{T:\ AT, AT^* \in \mathcal{S}_\infty(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}.$$

Here $\mathcal{S}_\infty(\mathcal{H})$ denotes the *-ideal of compact operators on $\mathcal{H}$.

Proposition 2.1. *Let $\mathcal{K}$ be the unit ball in $\mathcal{H}$.*

i) *The family $\{B\mathcal{K}:\ B \in \mathcal{B}(\mathcal{D}), B \geq 0, \operatorname{Ker} B = (0), \mathcal{R}(B)$ t-dense in $\mathcal{D}\}$ is a fundamental system of t-bounded sets. Hence $\tau_{\mathcal{D}}$ can be given by the seminorms $A \longrightarrow \|BAB\|$ B as above.*

ii) *The family $\{C\mathcal{K}:\ C \in \mathcal{S}_\infty(\mathcal{D}), C \geq 0, \operatorname{Ker} C = (0), \mathcal{R}(C)$ t-dense in $\mathcal{D}\}$ is a fundamental system of relatively t-compact sets. Hence $\tau_{\mathcal{D}}^c$ can be given by $A \longrightarrow \|CAC\|$ C as above.*

An important consequence is the fact that $\mathcal{B}(\mathcal{D})$ is $\tau_{\mathcal{D}}$-dense in $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, consequently in $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{H})$ [6]. In what follows we need some special assertions along this line and collect them in the next lemma.

Lemma 2.2. *Let $B \in \mathcal{B}(\mathcal{D})$, $B \geq 0$, $B = \int_0^b \lambda dE_\lambda$. For $a > 0$ put $P_a = \int_a^b dE_\lambda$ and $A_a = P_a A P_a$ for $A \in \mathcal{L}^+(\mathcal{D})$. Then*

i) *P_a and A_a are elements of $\mathcal{B}(\mathcal{D})$.*

ii) *$\lim\limits_{a \to 0} \|B^\alpha(A - A_a)B^\alpha\| = 0$ for all $\alpha > 0$.*

Next we consider linear functionals on Op^*-algebras. In contrast to the C^*-case one has to distinguish two notions of positivity. Namely, a linear functional ω on an Op^*-algebra $\mathscr{A}(\mathscr{D})$ is said to be

positive if $\omega(A) \geq 0$ for all $A \in \mathscr{P}(\mathscr{A}(\mathscr{D})) = \{A = \sum_{\text{finite}} A_i^+ A_i, \ A_i \in \mathscr{A}(\mathscr{D})\}$;

strongly positive if $\omega(A) \geq 0$ for all $A \in \mathscr{K}(\mathscr{A}(\mathscr{D})) = \{A \in \mathscr{A}(\mathscr{D}) : \langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathscr{D}\}$.

Remark that these definitions hold for any $*$-subalgebra of $\mathscr{L}^+(\mathscr{D})$, it is not necessary that they contain I. The problem of $\tau_{\mathscr{D}}$-continuity of strongly positive functionals was investigated in [15]. Clearly $\mathscr{P}(\mathscr{A}(\mathscr{D})) \subset \mathscr{K}(\mathscr{A}(\mathscr{D}))$ and the inverse inclusion depends on whether or not $\mathscr{A}(\mathscr{D})$ contains the square root of its positive operators. In [11] it was shown that $B \in \mathscr{B}(\mathscr{D})$ implies $B^\alpha \in \mathscr{B}(\mathscr{D})$ for all $\alpha > 0$. Hence $\mathscr{P}(\mathscr{B}(\mathscr{D})) = \mathscr{K}(\mathscr{B}(\mathscr{D}))$. Moreover, considerations as in Lemma 2.2 show that the following is true.

Lemma 2.3. i) $\mathscr{P}(\mathscr{B}(\mathscr{D}))$ *is $\tau_{\mathscr{D}}$-dense in* $\mathscr{P}(\mathscr{L}^+(\mathscr{D}))$ *and* $\mathscr{K}(\mathscr{L}^+(\mathscr{D}))$.

ii) *On* $\mathscr{L}^+(\mathscr{D})$ *the sets of $\tau_{\mathscr{D}}$-continuous positive and $\tau_{\mathscr{D}}$-continuous strongly positive functionals coincide.*

In view of Lemma 2.3 it is unambiguous to speak about $\tau_{\mathscr{D}}$-continuous states on $\mathscr{L}^+(\mathscr{D})$. This set is denoted by $E = E(\mathscr{L}^+(\mathscr{D})) = \{\omega: \text{linear positive } \tau_{\mathscr{D}}\text{-continuous functional on } \mathscr{L}^+(\mathscr{D}), \omega(I) = 1\}$. Let $\mathscr{L}^+(\mathscr{D})'$ denote the dual space to $\mathscr{L}^+(\mathscr{D})[\tau_{\mathscr{D}}]$ and let $\sigma_0 = \sigma(\mathscr{L}^+(\mathscr{D})', \mathscr{L}^+(\mathscr{D}))$ be the w^*-topology in $\mathscr{L}^+(\mathscr{D})'$. We need the following subsets of E:

vector states: $V_0 = \{\omega \in E : \omega(A) = \langle \varphi, A\varphi \rangle, \ \varphi \in \mathscr{D}, \ \|\varphi\| = 1\}$

pure states: $P_0 = \{\omega \in E : \omega_1 \in \mathscr{L}^+(\mathscr{D})', \ \omega_1 \geq 0, \ \omega_1 \leq \omega$ implies $\omega_1 = \lambda\omega, \ 0 \leq \lambda \leq 1\}$.

Clearly, $\omega_0 \in P_0$ if and only if ω_0 cannot be represented as a nontrivial convex combination of $\omega_1, \omega_2 \in E$. As in the bounded case one defines:

vector state space $V = \sigma_0$-closure of V_0

pure state space $P = \sigma_0$-closure of P_0.

These sets of states will be investigated in detail in [14]. To define normal and singular functionals on $\mathscr{L}^+(\mathscr{D})$ let us introduce the following two-sided *-ideals of $\mathscr{L}^+(\mathscr{D})$:

$$\mathscr{S}_1(\mathscr{D}) = \{T \in \mathscr{L}^+(\mathscr{D}) : AT, AT^* \in \mathscr{S}_1(\mathscr{H}) \text{ for all } A \in \mathscr{L}^+(\mathscr{D})\},$$
$$\mathscr{F}(\mathscr{D}) = \{F \in \mathscr{L}^+(\mathscr{D}) : \dim F\mathscr{D} < \infty\}.$$

Here $\mathscr{S}_1(\mathscr{H})$ stands for the ideal of nuclear operators on $\mathscr{H}$. It appears that the $\tau_\mathscr{D}$-closure of $\mathscr{F}(\mathscr{D})$, $\mathscr{C}(\mathscr{D}) \equiv \overline{\mathscr{F}(\mathscr{D})}^{\tau_\mathscr{D}}$ is a very appropriate generalization of the set of compact operators to the unbounded case. Among other things the following assertions are valid [7], [13]:

Proposition 2.4. i) $A \in \mathscr{C}(\mathscr{D})$ *if and only if* $(A\varphi_n)$ *is t-convergent to zero for any sequence* (φ_n) *which is σ-convergent to zero.*

ii) *If* $\mathscr{D}[t]$ *is not a Montel space, then* $\mathscr{C}(\mathscr{D})$ *is the only nontrivial, $\tau_\mathscr{D}$-closed two-sided *-ideal in* $\mathscr{L}^+(\mathscr{D})$.

iii) $\mathscr{D}[t]$ *is a Montel space if and only if* $\mathscr{C}(\mathscr{D}) = \mathscr{L}^+(\mathscr{D})$.

These properties give rise to the following definition.

Definition 2.5. A linear functional ω on $\mathscr{L}^+(\mathscr{D})$ is said to be *normal* if $\omega(A) = \mathrm{Tr}\ AT$ for some $T \in \mathscr{S}_1(\mathscr{D})$ and all $A \in \mathscr{L}^+(\mathscr{D})$; *singular* if ω is $\tau_\mathscr{D}$-continuous and $\omega(C) = 0$ for all $C \in \mathscr{C}(\mathscr{D})$.

Let us remark that while normal functionals are automatically $\tau_\mathscr{D}$-continuous (even $\tau_\mathscr{D}^c$-continuous) we have included the $\tau_\mathscr{D}$-continuity in the definition of singular functionals. Proposition 2.4 iii) shows that there does not exist nontrivial singular functionals if $\mathscr{D}[t]$ is a Montel space. The proof of a result concerning trace representation of functionals (cf. e.g. [16]) implies immediately the next lemma.

Lemma 2.6. *Every $\omega \in \mathscr{L}^+(\mathscr{D})'$ can be uniquely decomposed into $\omega = \omega_n + \omega_s$, where ω_n, ω_s are respectively normal and singular functionals. Moreover, $\omega \geq 0$ implies $\omega_n \geq 0$.*

In section 3, Theorem 3.4 we give a stronger result.

§3. A Restriction-Extension Procedure

The $\tau_{\mathscr{D}}$-density of $\mathscr{B}(\mathscr{D})$ in $\mathscr{L}^{+}(\mathscr{D})$ and $\mathscr{B}(\mathscr{H})$ leads to the following procedure. Let $\omega \in \mathscr{L}^{+}(\mathscr{D})'$ be given. By restriciton to $\mathscr{B}(\mathscr{D})$ and extension to $\mathscr{B}(\mathscr{H})$ one gets a unique $\tau_{\mathscr{D}}$-continuous linear functional on $\mathscr{B}(\mathscr{H})$; notation: $\omega \to \bar{\omega}$. Conversely, let ω be a $\tau_{\mathscr{D}}$-continuous linear functional on $\mathscr{B}(\mathscr{H})$. Restriciton to $\mathscr{B}(\mathscr{D})$ and extension to $\mathscr{L}^{+}(\mathscr{D})$ gives again a unique $\tau_{\mathscr{D}}$-continuous linear functional on $\mathscr{L}^{+}(\mathscr{D})$; notation: $\omega \to \hat{\omega}$. Obviously: $\omega = \hat{\bar{\omega}} \in \mathscr{L}^{+}(\mathscr{D})'$ and $\omega = \bar{\hat{\omega}} \in \mathscr{B}(\mathscr{H})'$, $\omega - \tau_{\mathscr{D}}$-continuous. This procedure will be frequently used (see also [14]). So we collect some of its properties in the next lemmata. Before doing so, let us mention that in the context of $\mathscr{L}(\mathscr{D}, \mathscr{D}')$ one would call this an extension-restriction procedure (e. g. $\mathscr{L}^{+}(\mathscr{D}) \to \mathscr{L}(\mathscr{D}, \mathscr{D}') \to \mathscr{B}(\mathscr{H})$ and so on). But because we do not want to leave $\mathscr{H}$ we prefer this one. A direct consequence of the definition and of Lemma 2.2 is the following lemma.

Lemma 3.1. i) *If* $\omega \in \mathscr{L}^{+}(\mathscr{D})'$, $|\omega(A)| \leq \|BAB\|$ *for all* $A \in \mathscr{L}^{+}(\mathscr{D})$ *and some* $B \in \mathscr{B}(\mathscr{D})$, *then* $|\bar{\omega}(A)| \leq \|BAB\|$ *for all* $A \in \mathscr{B}(\mathscr{H})$ *and moreover* $\bar{\omega}(A) = \lim\limits_{a \to 0} \omega(P_a A P_a)$;

and conversely

ii) *If* $\omega \in \mathscr{B}(\mathscr{H})'$, $\tau_{\mathscr{D}}$-*continuous*, $|\omega(A)| \leq \|BAB\|$ *for all* $A \in \mathscr{B}(\mathscr{H})$ *and some* $B \in \mathscr{B}(\mathscr{D})$, *then* $|\hat{\omega}(A)| \leq \|BAB\|$ *for all* $A \in \mathscr{L}^{+}(\mathscr{D})$ *and* $\hat{\omega}(A) = \lim\limits_{a \to 0} \omega(P_a A P_a)$.

iii) *The same assertions are true for* $\tau_{\mathscr{D}}^{s}$-*continuous linear functionals on* $\mathscr{L}^{+}(\mathscr{D})$, $\mathscr{B}(\mathscr{H})$ *respectively.*

In the next lemma we will see which sets of $\tau_{\mathscr{D}}$-continuous linear functionals mutually correspond in this procedure.

Lemma 3.2. *In the restriciton-extension procedure described above the following sets of $\tau_{\mathscr{D}}$-continuous linear functionals mutually correspond:*

i)   *positive functionals on* $\mathscr{L}^{+}(\mathscr{D}) \leftrightarrow$ *positive functionals on* $\mathscr{B}(\mathscr{H})$

ii)   *normal functionals on* $\mathscr{L}^{+}(\mathscr{D}) \leftrightarrow$ *normal functionals on* $\mathscr{B}(\mathscr{H})$,

more exactly: $\omega(A) = \mathrm{Tr}\ AT$, $T \in \mathscr{S}_1(\mathscr{D})$, $A \in \mathscr{L}^+(\mathscr{D})$ *implies* $\tilde{\omega}(B) = \mathrm{Tr}$ BT *for all* $B \in \mathscr{B}(\mathscr{H})$ *and conversely,* $\omega(B) = \mathrm{Tr}\ BT$ *for all* $B \in \mathscr{B}(\mathscr{H})$, $T \in \mathscr{S}_1(\mathscr{H})$, $\omega - \tau_{\mathscr{D}}$-*continuous implies* $T \in \mathscr{S}_1(\mathscr{D})$ *and* $\hat{\omega}(A) = \mathrm{Tr} AT$ *for all* $A \in \mathscr{L}^+(\mathscr{D})$.

    iii)   *singular functionals on* $\mathscr{L}^+(\mathscr{D}) \leftrightarrow$ *singular functionals on* $\mathscr{B}(\mathscr{H})$.

    iv)   *pure states on* $\mathscr{L}^+(\mathscr{D}) \leftrightarrow$ *pure states on* $\mathscr{B}(\mathscr{H})$.

Proof. i) follows from Lemma 2.3.

ii) Let $\omega(A) = \mathrm{Tr}\ AT$ for all $A \in \mathscr{L}^+(\mathscr{D})$ and some $T \in \mathscr{S}_1(\mathscr{D})$. Because $T \in \mathscr{S}_1(\mathscr{H})$ it follows immediately that $\tilde{\omega}(A) = \mathrm{Tr}\ AT$ for all $A \in \mathscr{B}(\mathscr{H})$. On the other hand, if ω on $\mathscr{B}(\mathscr{H})$ is $\tau_{\mathscr{D}}$-continuous and $\omega(A) = \mathrm{Tr}\ AT$ for all $A \in \mathscr{B}(\mathscr{H})$ and some $T \in \mathscr{S}_1(\mathscr{H})$, then $\hat{\omega}$ is $\tau_{\mathscr{D}}$-continuous on $\mathscr{L}^+(\mathscr{D})$ and can be decomposed into a linear combination of positive $\tau_{\mathscr{D}}$-continuous functionals which are normal if restricted to $\mathscr{F}(\mathscr{D})$ [16]. Hence $\hat{\omega}(F) = \mathrm{Tr}\ FS$ for all $F \in \mathscr{F}(\mathscr{D})$ and some $S \in \mathscr{S}_1(\mathscr{D})$. Since $\omega(F) = \hat{\omega}(F)$ for all $F \in \mathscr{F}(\mathscr{D})$, we get $S = T$. Thus, $\hat{\omega}$ is normal on $\mathscr{L}^+(\mathscr{D})$ (using the uniqueness of $\omega \to \hat{\omega}$).

    iii) Let ω be singular on $\mathscr{L}^+(\mathscr{D})$, then $\omega(F) = 0$ for all $F \in \mathscr{F}(\mathscr{D})$. But $\mathscr{F}(\mathscr{D})$ is $\tau_{\mathscr{D}}$-dense in $\mathscr{F}(\mathscr{H})$ hence in $\mathscr{S}_\infty(\mathscr{H})$, so $\tilde{\omega}(C) = 0$ for all $C \in \mathscr{S}_\infty(\mathscr{H})$, i.e. $\tilde{\omega}$ is singular. The other direction follows similarly.

    iv) Let ω be pure on $\mathscr{B}(\mathscr{H})$. Suppose $\hat{\omega} = \lambda \omega_1 + (1 - \lambda) \omega_2$, ω_1, ω_2-positive and $\tau_{\mathscr{D}}$-continuous on $\mathscr{L}^+(\mathscr{D})$. This would imply $\omega = \tilde{\hat{\omega}} = \lambda \tilde{\omega}_1 + (1 - \lambda) \tilde{\omega}_2$ with $\tilde{\omega}_i \in \mathscr{B}(\mathscr{H})'$, $\tilde{\omega}_i$-$\tau_{\mathscr{D}}$-continuous and positive according to i). This is a contradiction. In the same way the other direction can be proved.

<div align="right">q. e. d.</div>

It is necessary to add the following remark. The notion "pure state" on $\mathscr{B}(\mathscr{H})$ is here not unambiguous because we can consider decompositions within the set of all states or only within the $\tau_{\mathscr{D}}$-continuous states. We have in mind the $\tau_{\mathscr{D}}$-continuous states alone. To consider pureness in the context of all states and derive the direction→in Lemma 3.2 iv) one would need a result of the following type: if $\omega_1 \leq \omega$, $\omega - \tau_{\mathscr{D}}$-continuous on $\mathscr{B}(\mathscr{H})$ and ω_1 positive, then ω_1 is $\tau_{\mathscr{D}}$-continuous, too. Nevertheless normal pure states (i.e. vector states) on $\mathscr{L}^+(\mathscr{D})$ lead to pure states (vector states) on $\mathscr{B}(\mathscr{H})$ also

in the context of norm-continuity.

Corollary 3.3.   i) *A vector functional* $\omega_{\varphi,\psi}(A) = \langle \varphi, A\psi \rangle$ *on* $\mathscr{B}(\mathscr{H})$ *is* $\tau_{\mathscr{D}}$*-continuous if and only if* $\varphi, \psi \in \mathscr{D}$.

ii) *The vector states* $\omega_{\varphi}(A) = \langle \varphi, A\varphi \rangle$ ($\|\varphi\| = 1$) *on* $\mathscr{L}^+(\mathscr{D})$ *are pure states.*

iii) *Let* ω *be a state on* $\mathscr{L}^+(\mathscr{D})$ *so that* $\omega(C) = \omega_{\varphi}(C)$ *for all* $C \in \mathscr{C}(\mathscr{D})$. *Then* $\omega = \omega_{\varphi}$ *on* $\mathscr{L}^+(\mathscr{D})$.

Proof.   i) follows from Lemma 3.2 ii) and the fact that $P_{\varphi,\psi} = \langle \varphi, \cdot \rangle \psi \in \mathscr{S}_1(\mathscr{D})$ if and only if $\varphi, \psi \in \mathscr{D}$.

ii)   If $\omega = \omega_{\varphi}$ on $\mathscr{L}^+(\mathscr{D})$ then by Lemma 3.2 ii) $\tilde{\omega} = \omega_{\varphi}$ on $\mathscr{B}(\mathscr{H})$. Because vector states are pure on $\mathscr{B}(\mathscr{H})$ the assertion follows from Lemma 3.2 iv).

iii)   It follows that $\tilde{\omega}$ on $\mathscr{B}(\mathscr{H})$ has the property $\tilde{\omega}(C) = \tilde{\omega}_{\varphi}(C) = \omega_{\varphi}(C)$ for all $C \in \mathscr{S}_{\infty}(\mathscr{H})$. Then by [4] $\tilde{\omega} = \tilde{\omega}_{\varphi} = \omega_{\varphi}$ on $\mathscr{B}(\mathscr{H})$. Hence $\omega = \hat{\tilde{\omega}} = \omega_{\varphi}$ again by Lemma 3.2 ii).

<div align="right">q. e. d.</div>

Now we prove the main result of this section, namely the decomposition theorem for positive $\tau_{\mathscr{D}}$-continuous functionals.

Theorem 3.4.   *Let* ω *be a positive* $\tau_{\mathscr{D}}$*-continuous linear functional on* $\mathscr{L}^+(\mathscr{D})$. *Then there is a unique decomposition*

$$\omega = \omega_n + \omega_s$$

such that $\omega_n \geq 0$, *normal,* $\omega_s \geq 0$ *singular.*

Proof.   By Lemma 2.6 we have a unique decomposition $\omega = \omega_n^0 + \omega_s^0$ with $\omega_n^0 \geq 0$ normal and ω_s^0 singular. So it remains to prove that $\omega_s^0 \geq 0$. Let $\tilde{\omega}$, $\tilde{\omega}_n^0$, $\tilde{\omega}_s^0$ be the corresponding functionals on $\mathscr{B}(\mathscr{H})$. Then $\tilde{\omega} = \tilde{\omega}_n^0 + \tilde{\omega}_s^0$. On the other hand $\tilde{\omega}$ can be uniquely decomposed into $\tilde{\omega} = \omega_1 + \omega_2$ with $\omega_1 \geq 0$ normal and $\omega_2 \geq 0$ singular. Since the normal parts coincide on $\mathscr{F}(\mathscr{D})$, the corresponding trace class operators are the same. Hence $\tilde{\omega}_n^0 = \omega_1$, i.e. $\omega_2 = \tilde{\omega}_s^0 \geq 0$. So the assertion follows from Lemma 3.2. i), namely $\omega_s^0 \geq 0$.

<div align="right">q. e. d.</div>

Corollary 3.5. *Let* $\omega \in P_0$, *i. e.* ω *is a pure state. Then either* ω *is a vector state or* ω *is a pure singular state.*

Proof. Let $\omega \in P_0$, then $\omega = \omega_n + \omega_s = \omega_n(I) \cdot (\omega_n/\omega_n(I)) + \omega_s(I) \cdot (\omega_s/\omega_s(I))$ is a convex combination of states. Since ω is pure, either $\omega_n(I) = 0$, i. e. $\omega_n = 0$ and $\omega = \omega_s$ is a pure singular state or $\omega_s(I) = 0$, i. e. $\omega_s = 0$ and $\omega = \omega_n$ is a pure normal state, hence a vector state.

<div align="right">q. e. d.</div>

§4. Singular Positive Functionals on $\mathscr{L}^+(\mathscr{D})$

In this section we start with some simple equivalent characterizations of singular positive functionals which correspond to those of the bounded case [17], [18]. Then there is given a representation theorem for singular states on $\mathscr{L}^+(\mathscr{D})$. This result has important conclusions, for example about the structure of the state space E of $\mathscr{L}^+(\mathscr{D})$, cf. Theorem 4. 8.

Lemma 4.1. *Let* ω *be a positive* $\tau_{\mathscr{D}}$-*continuous linear functional on* $\mathscr{L}^+(\mathscr{D})$. *Then the following statements are equivalent:*

i) ω *is singular.*

ii)   *The only positive normal functional* ρ *with* $\rho \leq \omega$ *is* $\rho = 0$.

iii) *For any projection* $E \in \mathscr{L}^+(\mathscr{D})$, $E \neq 0$ *there is a projection* $F \neq 0$, $F \in \mathscr{L}^+(\mathscr{D})$ *so that* $F \leq E$ *and* $\omega(F) = 0$.

Proof. We show i) $\leftrightarrow$ ii), i) $\leftrightarrow$ iii).

i) $\rightarrow$ ii): Suppose ρ is normal and $0 \leq \rho \leq \omega$, then $\rho(F) = 0$ for all $F \in \mathscr{F}(\mathscr{D})$, i. e. $\rho = 0$.

ii) $\rightarrow$ i): If ω is not singular, then by Theorem 3. 4 $\omega = \omega_n + \omega_s$ with $\omega_n \geq 0$, $\omega_s \geq 0$, $\omega_n \neq 0$. Thus $\omega_n \leq \omega$.

i) $\rightarrow$ iii): If $\omega(E) = 0$, then take $E = F$. If $\omega(E) \neq 0$, then there is a finite dimensional $F \leq E$, $F \neq 0$, so that $F \in \mathscr{L}^+(\mathscr{D})$ and clearly, $\omega(F) = 0$.

iii) $\rightarrow$ i): Take E one-dimensional, $E \in \mathscr{L}^+(\mathscr{D})$. Then $\omega(E) = 0$, so $\omega(F) = 0$ for all $F \in \mathscr{F}(\mathscr{D})$ and consequently $\omega(C) = 0$ for all $C \in \mathscr{C}(\mathscr{D})$. This means ω-singular.

<div align="right">q. e. d.</div>

Lemma 4.2. *Let ω be singular and positive on $\mathscr{L}^+(\mathscr{D})$. Then for every projection $P \in \mathscr{L}^+(\mathscr{D})$, $\omega(P) = 1$ there is a projection $Q \leq P$, $Q \in \mathscr{L}^+$ $(\mathscr{D})$ so that $P - Q$ is infinite dimensional and $\omega(Q) = 1$.*

Proof. It is easy to see that $P\mathscr{D}$ is a t-closed subspace of $\mathscr{D}$ (and again an (F)-space). Moreover one has (cf. e. g. [13], [16]): $P \in \mathscr{C}(\mathscr{D})$ if and only if $P\mathscr{D}$ is a Montel space if and only if $P\mathscr{D}$ does not contain an infinite dimensional Hilbert space $\mathscr{H}_0$ (i. e. $P\mathscr{D}$ is of type I in the sense of [12]). So $\omega(P) = 1$ means $P \notin \mathscr{C}(\mathscr{D})$ and consequently there is such an $\mathscr{H}_0 \subset P\mathscr{D} \subset \mathscr{D}$. Let P_0 be the orthoprojection onto $\mathscr{H}_0$. If $\omega(P_0) = 0$, then we are done taking $Q = P - P_0$ which must be infinite dimensional. Suppose $\omega(P_0) = b \neq 0$, i. e. $\omega(P - P_0) = 1 - b$. Applying the reasoning of [1], p. 305 to $\mathscr{H}_0$ and P_0 one gets a P_1 so that $P_1 \leq P_0$, $P_0 - P_1$ is infinite dimensional and $\omega(P_1) = b$. Putting $Q = P_1 + (P - P_0)$ we get the desired result.

<div align="right">q. e. d.</div>

Our next aim is to describe all positive singular functionals on $\mathscr{L}^+(\mathscr{D})$. Let us remember the situation in the bounded case. It has often been mentioned that $\mathscr{B}(\mathscr{H})$, $\mathscr{S}_1(\mathscr{H})$ and $\mathscr{S}_\infty(\mathscr{H})$ are the non-commutative analogs to 1^∞, 1^1 and c_0 (the zero-sequences). The complex homomorphisms of 1^∞ are given by the elements of βN (the Stone-Čech compactification of N) which can be identified with the ultrafilters on N. There are two types of ultrafilters on N: fixed (consisting of all subsets of N containing a fixed element of N) and free ultrafilters giving the elements of $\beta N \backslash N$. The formula $\omega_{\mathscr{U}}((x_n)) = \lim_{\mathscr{U}} x_n$, $(x_n) \in 1^\infty$, $\mathscr{U}$-ultrafilter gives the complex homomorphisms of 1^∞. If $\mathscr{U}$ is fixed at k, then $\omega_{\mathscr{U}} = \omega_k$ with $\omega_k((x_n)) = x_k$. If $\mathscr{U}$ is free, then $\omega_{\mathscr{U}}((x_n)) = 0$ for all $(x_n) \in c_0$, i. e. free ultrafilters give rise to singular states. This is a guide to construct singular states on $\mathscr{B}(\mathscr{H})$. Let (ψ_n) be a sequence of unit vectors in $\mathscr{H}$ weakly converging to zero. Then $\omega_{\mathscr{U}}$ defined by

$$(1) \qquad \omega_{\mathscr{U}}(A) = \lim_{\mathscr{U}} \langle \psi_n, A\psi_n \rangle$$

gives a state on $\mathscr{B}(\mathscr{H})$ which is singular if and only if $\mathscr{U}$ is free. Moreover, if one takes any sequence (ψ_n) of unit vectors, then exactly those free ultrafilters $\mathscr{U}$ give singular states for which $\lim_{\mathscr{U}} \langle \psi_n, \varphi \rangle = 0$

for all $\varphi \in \mathscr{H}$ [1]. Furthermore, the following theorem was established by Wils [21] and extended to the non–separable case by Anderson [1].

Theorem 4.3. *There is a fixed sequence* (ψ_n) *of unit vectors of* $\mathscr{H}$ *so that any singular state on* $\mathscr{B}(\mathscr{H})$ *has the form*

$$\omega_{\mathscr{U}}(A) = \lim_{\mathscr{U}} \langle \psi_n, A\psi_n \rangle$$

for an appropriate free ultrafilter $\mathscr{U}$ *(namely such one with* $\lim_{\mathscr{U}} \langle \psi_n, \varphi \rangle = 0$ *for all* $\varphi \in \mathscr{H}$*).*

We remark that this sequence (ψ_n) (call it Wils sequence) can be taken to be arbitrary $\| \ \|$-dense in the unit sphere. A little bit more is known. While the only pure normal states on $\mathscr{B}(\mathscr{H})$ are the vector states, pure singular states are obtained if one takes in (1) instead of (ψ_n) an orthonormal system (φ_n). But the question whether or not this gives all pure singular states seems to be open. Now we turn to the unbounded case and start with a simple observation.

Lemma 4.4. *Let* $(\varphi_n) \subset \mathscr{D}$ *be a t-bounded sequence,* $\mathscr{U}$ *a free ultra-filter. Then* $\omega_{\mathscr{U}}$: $\omega_{\mathscr{U}}(A) = \lim_{\mathscr{U}} \langle \varphi_n, A\varphi_n \rangle$ *defines a positive,* $\tau_{\mathscr{D}}$-*continuous functional on* $\mathscr{L}^+(\mathscr{D})$ *which is singular if and only if* $\lim_{\mathscr{U}} \langle \varphi_n, \varphi \rangle = 0$ *for all* $\varphi \in \mathscr{D}$*.*

Proof. The positivity is clear, $\tau_{\mathscr{D}}$-continuity follows by standard estimation. The proof of the remaining part is as in the bounded case. Let $\varphi, \psi \in \mathscr{D}$, $P_{\varphi, \psi} = \langle \varphi, \cdot \rangle \psi$. Then the finite linear combinations of the $P_{\varphi, \psi}$'s are $\tau_{\mathscr{D}}$-dense in $\mathscr{C}(\mathscr{D})$. Thus $\omega_{\mathscr{U}}$ is singular if and only if $\omega_{\mathscr{U}}(P_{\varphi, \psi}) = 0$ for all $\varphi, \psi \in \mathscr{D}$. But this is the case if and only if $\lim_{\mathscr{U}} \langle \varphi, \varphi_n \rangle \langle \varphi_n, \psi \rangle = 0$ for all $\varphi, \psi \in \mathscr{D}$. This implies the assertion.

<div style="text-align: right">q. e. d.</div>

Now let (ψ_n) be a Wils sequence, $B \in \mathscr{B}(\mathscr{D})$, $\mathscr{U}$ a free ultrafilter. Then

$$\omega_{B, \mathscr{U}}: \ \omega_{B, \mathscr{U}}(A) = \lim_{\mathscr{U}} \langle B\psi_n, AB\psi_n \rangle, \ A \in \mathscr{L}^+(\mathscr{D})$$

gives a positive, singular functional on $\mathscr{L}^+(\mathscr{D})$. The positivity is clear, the $\tau_{\mathscr{D}}$-continuity follows from the estimation $|\omega_{B, \mathscr{U}}(A)| = |\lim_{\mathscr{U}} \langle B\psi_n,$

$AB\phi_n\rangle| \leq \sup_{\varphi, \psi \in \mathcal{M}} |\langle \varphi, A\psi \rangle|$ where $\mathcal{M} = \{B\phi_n\}$. The singularity follows from the fact that $S \in \mathscr{C}(\mathscr{D})$ implies $B^*SB \in \mathscr{S}_\infty(\mathscr{H})$ for all $B \in \mathscr{B}(\mathscr{D})$ [13]. To get states on $\mathscr{L}^+(\mathscr{D})$ one takes for example such pairs $(\mathscr{U}, B)$ that $\lim_\mathscr{U} \langle B\phi_n, B\phi_n \rangle = 1$. In the way just described one gets all positive singular functionals.

Theorem 4.5.   *The positive singular functionals on* $\mathscr{L}^+(\mathscr{D})$ *are given by*

$$\omega_{C,\mathscr{U}}: \ \omega_{C,\mathscr{U}}(A) = \lim_\mathscr{U} \langle C\phi_n, AC\phi_n \rangle \ \textit{for all} \ A \in \mathscr{L}^+(\mathscr{D}),$$

where (ϕ_n) *is a Wils sequence,* $\mathscr{U}$ *an appropriate free ultrafilter and* $C \in \mathscr{B}(\mathscr{D})$ *which can be taken to be positive,* $\mathrm{Ker} \ C = (0)$ *and having* t-*dense range.*

According to the considerations above one has to prove only one direction. We will give two different proofs. One uses explicitly the restriction-extension procedure described in section 3.

First proof. Let ω be a positive singular functional on $\mathscr{L}^+(\mathscr{D})$, then $|\omega(A)| \leq \|BAB\|$ for all $A \in \mathscr{L}^+(\mathscr{D})$ and some $B \in \mathscr{B}(\mathscr{D})$ which can assumed to be positive, invertible with t-dense range. The set $\mathscr{A}_1 = \{BAB: A \in \mathscr{L}^+(\mathscr{D})\}$ is a *-algebra which can be considered as a *-subalgebra of $\mathscr{B}(\mathscr{H})$. Consider the linear functional ω_1 on $\mathscr{A}_1$ given by $\omega_1(X) = \omega(A)$ for all $X = BAB \in \mathscr{A}_1$. The properties of B yield that ω_1 is correctly defined, positive and norm-continuous on $\mathscr{A}_1$. Therefore ω_1 can be extended to the C^*-algebra $\bar{\mathscr{A}}_1 \subset \mathscr{B}(\mathscr{H})$ generated by $\mathscr{A}_1$. Denote this extension also by ω_1. The properties of B give also that $\{BFB: F \in \mathscr{F}(\mathscr{D})\}$ is norm-dense in $\mathscr{F}(\mathscr{H})$. Hence $\mathscr{S}_\infty(\mathscr{H}) \subset \bar{\mathscr{A}}_1$. In the usual way ω_1 can be extended to the C^*-algebra $\mathscr{A}$ obtained by adjoining the identity I to $\bar{\mathscr{A}}_1$. This gives again a positive norm-continuous functional, again denoted by ω_1. Finally since $I \in \mathscr{A}$, ω_1 can be extended to a positive linear functional $\tilde{\omega}_1$ on $\mathscr{B}(\mathscr{H})$. This functional is singular because $\tilde{\omega}_1(BFB) = \omega(F) = 0$ for all $F \in \mathscr{F}(\mathscr{D})$, hence by continuity $\tilde{\omega}_1(C) = 0$ for all $C \in \mathscr{S}_\infty(\mathscr{H})$. Put $\tilde{\omega} = \tilde{\omega}_1/\tilde{\omega}_1(I)$ and apply Theorem 4.3. Then

$$\tilde{\omega}(X) = \lim_\mathscr{U} \langle \phi_n, X\phi_n \rangle \ \text{for all} \ X \in \mathscr{B}(\mathscr{H}).$$

If $X = BAB \in \mathscr{A}_1$ we get

$$\tilde{\omega}(X) = \lim_{\mathcal{U}} \langle \psi_n, \ BAB \ \psi_n \rangle.$$

Thus

$$\omega(A) = \tilde{\omega}_1(I) \cdot \lim_{\mathcal{U}} \langle \psi_n, \ BAB\psi_n \rangle = \lim_{\mathcal{U}} \langle C\psi_n, \ AC\psi_n \rangle$$

with $C = (\tilde{\omega}_1(I))^{1/2} \cdot B$.

<div align="right">q. e. d.</div>

Second proof. Let $|\omega(A)| \leq ||BAB||$ with $B = \int_0^b \lambda dE_\lambda$ as above. Then the corresponding functional $\tilde{\omega}$ on $\mathscr{B}(\mathscr{H})$ is positive, singular and fulfills (cf. section 3)

(2) $\tilde{\omega}(X) = \lim_{a \to 0} \omega(P_a X P_a), \quad |\tilde{\omega}(X)| \leq ||BXB||.$

Fix $\alpha \in (0, 1)$ and put $B_a^{-\alpha} = \int_a^b \lambda^\alpha dE_\lambda$, $a > 0$. Then $B \ B_a^{-\alpha} = B^{1-\alpha} P_a$ and $B_a^{-\alpha} \in \mathscr{B}(\mathscr{H})$. From (2) it is easily seen that $\tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha})$ is a Cauchy sequence (having in mind $a \to 0$) for any $X \in \mathscr{B}(\mathscr{H})$. Put

$$\tilde{\omega}_1(X) = \lim_{a \to 0} \tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha}).$$

This gives a positive functional on $\mathscr{B}(\mathscr{H})$. Moreover $\tilde{\omega}_1$ is singular because for $X \in \mathscr{S}_\infty(\mathscr{H})$, $B_a^{-\alpha} X B_a^{-\alpha} \in \mathscr{S}_\infty(\mathscr{H})$, too and $\tilde{\omega}(B_a^{-\alpha} X B_a^{-\alpha}) = 0$ for all $a > 0$. For arbitrary $Y \in \mathscr{B}(\mathscr{H})$ and $X = B^\alpha Y B^\alpha$ one gets

$$\tilde{\omega}_1(B^\alpha Y B^\alpha) = \lim_{a \to 0} \tilde{\omega}(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha}) = \tilde{\omega}(Y).$$

The last equality follows from the estimation

$$|\tilde{\omega}(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha}) - \tilde{\omega}(Y)| \leq ||B(B_a^{-\alpha} B^\alpha Y B^\alpha B_a^{-\alpha} - Y)B||$$
$$= ||B(P_a Y P_a - Y)B|| \to 0 \text{ for } a \to 0 \text{ (see Lemma 2.2 ii)).}$$

Next put $\tilde{\omega}_2 = \tilde{\omega}_1 / \tilde{\omega}_1(I)$, $\tilde{\omega}_2$ is a singular state on $\mathscr{B}(\mathscr{H})$ (the case $\tilde{\omega}_1(I) = 0$ can be excluded since this would imply $\omega = 0$). Applying Theorem 4.3 we get

$$\tilde{\omega}_2(Y) = \lim_{\mathcal{U}} \langle \psi_n, Y\psi_n \rangle \text{ for all } Y \in \mathscr{B}(\mathscr{H}).$$

Consequently,

$$\tilde{\omega}(Y) = \tilde{\omega}_1(B^\alpha Y B^\alpha) = \tilde{\omega}_1(I)\tilde{\omega}_2(B^\alpha Y B^\alpha)$$
$$= \tilde{\omega}_1(I) \lim_{\mathcal{U}} \langle \psi_n, B^\alpha Y B^\alpha \psi_n \rangle = \lim_{\mathcal{U}} \langle C\psi_n, YC\psi_n \rangle$$

with $C = (\tilde{\omega}_1(I))^{1/2} B^\alpha \in \mathscr{B}(\mathscr{D})$. Now we return to $\mathscr{L}^+(\mathscr{D})$ by observing that

$$\hat{\omega}(A) = \omega(A) = \lim_{a \to 0} \tilde{\omega}(P_a A P_a) = \lim_{a \to 0} \lim_{\mathcal{U}} \langle C\psi_n, \ (P_a A P_a) C\psi_n \rangle$$

for all $A \in \mathscr{L}^+(\mathscr{D})$. Then

$$|\lim_{\mathscr{U}} \langle C\phi_n, \ (P_a A P_a) C\phi_n \rangle - \lim_{\mathscr{U}} \langle C\phi_n, \ AC\phi_n \rangle|$$

$$\leq \sup_n |\langle \phi_n, \ C(P_a A P_a - A) C\phi_n \rangle| \leq \tilde{\omega}_1(I) \|B^\alpha (P_a A P_a - A) B^\alpha\|$$

which goes to zero for $a \to 0$. This gives the desired result:

$$\omega(A) = \lim_{\mathscr{U}} \langle C\phi_n, \ AC\phi_n \rangle \quad \text{for all } A \in \mathscr{L}^+(\mathscr{D}).$$

<div align="right">q. e. d.</div>

From this theorem we get a trivial but important corollary.

Corollary 4.6. *Any positive singular functional on $\mathscr{L}^+(\mathscr{D})$ is the weak limit of positive vector functionals. Especially, the singular states are contained in the vector state space of $\mathscr{L}^+(\mathscr{D})$.*

Proof. Let ω be singular and positive on $\mathscr{L}^+(\mathscr{D})$, $A_1, \cdots, A_j \in \mathscr{L}^+$ $(\mathscr{D})$, $\varepsilon > 0$ be given. From Theorem 4.5 we have

$$\omega(A) = \lim_{\mathscr{U}} \langle C\phi_n, \ AC\phi_n \rangle.$$

Hence there are $U_i \in \mathscr{U}$ so that

$$|\omega(A) - \langle C\phi_n, \ A_i C\phi_n \rangle| < \varepsilon \text{ for all } n \in U_i, \ i = 1, \cdots, j$$

i. e.

$$|\omega(A) - \langle C\phi_n, \ A_i C\phi_n \rangle| < \varepsilon \text{ for all } n \in U = \bigcap_i U_i \in \mathscr{U}, \ i = 1, \cdots, j.$$

This proves the first assertion.

Let ω be a state, then if necessary replace ϕ_n by $\phi_n' = \lambda_n \phi_n$ for all $n \in U \in \mathscr{U}$ for some U and $\lim_{\mathscr{U}} \lambda_n = 1$ so that $\langle C\phi_n', C\phi_n' \rangle = 1$ for all $n \in U$. Then the statement follows from the considerations above.

<div align="right">q. e. d.</div>

Finally we derive from Theorem 4.5 an important result about the state space E of $\mathscr{L}^+(\mathscr{D})$. Remember that in the case of C^*-algebras the fact that the unit ball in the dual space is w^*-compact allows to apply the Krein–Milman theorem. This leads to the well-known result that the state space of a C^*-algebra is the w^*-closed convex hull of the pure states. In contrast to this the state space of an Op^*-algebra is not w^*-compact if the algebra contains unbounded operators. But nevertheless for $\mathscr{L}^+(\mathscr{D})$ one can derive the analogous result. Here we will prove this almost constructively using Theorems

3.4 and 4.5. In [14] there is given another proof in the context of more general considerations.

Theorem 4.7. *The state space E of $\mathcal{L}^+(\mathcal{D})$ is the w^*-closed convex hull of the vector states, hence of the pure states.*

Proof. Because of Corollary 3.3 ii) it is enough to prove that E is contained in the w^*-closed convex hull of the vector states. Thus, let $\omega \in E$ and $A_1, \cdots, A_k \in \mathcal{L}^+(\mathcal{D})$, $\varepsilon > 0$ be given. According to Theorem 3.4: $\omega = \omega_n' + \omega_s'$, where $\omega_n' \geq 0$, normal, $\omega_s' \geq 0$, singular. $\omega(I) = \omega_n'(I) + \omega_s'(I) = 1$ implies that

$$(3) \qquad \omega = t\omega_n + (1-t)\omega_s$$

with $t = \omega_n'(I)$, $\omega_n = t^{-1}\omega_n'$, $\omega_s = (1-t)^{-1}\omega_s'$ is a convex combination of states. By Corollary 4.6 there is a vector state ω_φ on $\mathcal{L}^+(\mathcal{D})$ with

$$(4) \qquad |\omega_s(A_i) - \omega_\varphi(A_i)| < \varepsilon \quad \text{for } i=1, \cdots, k.$$

Since $\omega_n(A_i) = \mathrm{Tr}\, A_i T = \sum t_n \langle \varphi_n, A_i \varphi_n \rangle$ for some $T \in \mathcal{S}_1(\mathcal{D})$, $T\varphi_n = t_n \varphi_n$, (φ_n) an orthonomal system in $\mathcal{D}$ and $\sum t_n = 1$, there is an $N \in \mathbf{N}$ so that

$$(5) \qquad |\sum_{n>N} t_n \langle \varphi_n, A_i \varphi_n \rangle| < \varepsilon/2 \quad \text{for } i=1, \cdots, k;$$

and there is a $C > 0$ so that $|\sum_{n=1}^{r} t_n \langle \varphi_n, A_i \varphi_n \rangle| < C$ for all $r \in \mathbf{N}$ and $i=1, \cdots, k$. Then

$$\omega_1 = \sum_{n=1}^{N} s_n \langle \varphi_n, \cdot \varphi_n \rangle \quad \text{with} \quad s_n = t_n \left(\sum_{n=1}^{N} t_n \right)^{-1}$$

is a convex combination of vector states and

$$(6) \qquad |\omega_n(A_i) - \omega_1(A_i)| \leq |\sum_{n=1}^{N} (t_n - s_n) \langle \varphi_n, A_i \varphi_n \rangle|$$
$$+ |\sum_{n>N} t_n \langle \varphi_n, A_i \varphi_n \rangle| \leq C |(1 - 1/\sum_{n \leq N} t_n)| + \varepsilon/2$$
$$\leq \varepsilon \qquad \text{for sufficiently large } N.$$

Combining (3), (4) and (6) we get that

$$\rho = \sum_{n=1}^{N+1} r_n \rho_n \quad \text{with} \quad r_n = t \cdot s_n, \quad \rho_n = \langle \varphi_n, \cdot \varphi_n \rangle, \quad n=1, \cdots, N$$
$$r_{N+1} = (1-t), \quad \rho_{N+1} = \omega_\varphi$$

is a convex combination of vector states satisfying

$$|\omega(A_i) - \rho(A_i)| \leq \varepsilon \quad \text{for} \quad i = 1, \cdots, k.$$

q. e. d.

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