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Modules with Regular Singularities over Filtered Rings

By

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Introduction

In [9] and [10] an impressive theory of $\mathscr{D}$ and $\mathscr{E}$-modules with regular singularities is developed. Many of the results are proved using complex analysis or better micro-local analysis. In this paper we develop a purely algebraic theory of modules with regular singularities over a large class of filtered rings (including the rings of differential operators considered in [9], [10] and [1], in which case we have the same notion of regular singularities). The main result of this paper (Theorem 7.3) gives several equivalent descriptions of the notion of a holonomic A-module M with regular singularities (A is a filtered ring). One of them is the existence of a so-called very good filtration on M, which makes the link with the results of [9]. An equivalent description asserts that $\mathscr{E}_{\mathfrak{p}}(M)$ (the algebraic micro-localization of M at $\mathfrak{p}$) is an $\mathscr{E}_{\mathfrak{p}}(A)$-module with regular singularities for every minimal prime component $\mathfrak{p}$ of the characteristic ideal $J(M)$.

To prove these results we use the algebraic micro-localization developed in [5] and a theorem of Gabber (cf. Theorem 4.9).

The algebraic micro-localization enables us to generalize the ideas and results of [4], replacing the usual localization used there by the micro-localization (compare Theorem 1.26 in [4], with Theorem 7.3 below).

Now we give a detailed description of the contents.

In §1 we recall some well-known facts on filtrations and establish some useful facts. A filtration FA on a ring A is called Artin-Rees if all finitely generated A-modules satisfy the Artin-Rees property (cf. Definition 1.6). We

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also introduce the notion of a noetherian filtration on A and show that both concepts coincide if $gr(A)$ is left noetherian. A discrete filtration is noetherian iff $gr(A)$ is left noetherian. An important result is Theorem 1.11 which shows that filtrations equivalent to a good filtration are good, if FA is noetherian. At the end of §1 we give some results on involutive ideals and show that the notion of an involutive ideal is stable under extension and contraction (cf. Proposition 1.19).

In §2 we recall some of the basic results on micro-localization obtained in [5] (universal property, the graded ring of a micro-localization etc.). Following ideas of commutative algebra we micro-localize in prime ideals of $gr(A)$. More precisely, if M is a filtered A-module and $\mathfrak{p}$ a prime ideal in $gr(A)$ we define a ring $\mathscr{E}_{\mathfrak{p}}(A)$ and an $\mathscr{E}_{\mathfrak{p}}(A)$-module $\mathscr{E}_{\mathfrak{p}}(M)$. Furthermore we introduce the notion of an I-good filtration on a filtered A-module M (I is an ideal in $gr(A)$) and show that this notion is preserved under micro-localization (cf. Proposition 2.9). If $I=(0)$ an I-good filtration is simply a good filtration and if $I=J(M)$ (the characteristic ideal of M) an I-good filtration is called very good. Modules possessing a very good filtration are said to have regular singularities (Definition 7.1).

In §3 we introduce holonomic A-modules for filtered rings A such that $gr(A)$ is a commutative $\mathbb{Q}$-algebra and FA is noetherian. We show that "holonomicity" is stable under micro-localization in prime ideals of $gr(A)$ (cf. Proposition 3.4).

In §4 we define a special class of filtered rings R, the so-called E-rings. These rings possess an invertible element of order one, which makes it possible to reduce many problems to problems over the subring R_0 and its quotient $gr_0(R)$. Furthermore we formulate an involutivity theorem of Gabber (Theorem 4.9) and derive a micro-local criterion to decide when an R_0-submodule of a holonomic R-module is of finite type (Proposition 4.12). This criterion is used during the construction of very good filtrations.

In §5 we develop the theory of modules with regular singularities over E-rings. Following [4] and [9] we give several equivalent descriptions of R-modules M with regular singularities (Proposition 5.3). If furthermore M is a holonomic R-module, then we prove (Theorem 5.7) that M has regular singularities iff $\mathscr{E}_{\mathfrak{p}}(M)$ has regular singularities for all minimal prime components of the characteristic ideal $J(M)$ iff $\mathscr{E}_{\mathfrak{p}}(M)$ has regular singularities for all $\mathfrak{p}\in Spec\,(gr(A))$.

In §6 we develop a formalism which makes it possible to obtain results for arbitrary filtered rings from results over E-rings. Following [9] we introduce a dummy variable and associate to a filtered ring A an E-ring denoted by $\mathscr{E}_X(A[X])$. Similarly to an A-module M we associate an $\mathscr{E}_X(A[X])$-module $\mathscr{E}_X(M[X])$. We define two maps $\mathscr{L}$ resp. $\mathscr{G}$ going from good filtrations on M resp. $\mathscr{E}_X(M[X])$ to filtrations on $\mathscr{E}_X(M[X])$ resp. M and show that $\mathscr{L}$ and $\mathscr{G}$ preserve good and very good filtrations (Proposition 6.8).

In §7 we define A-modules with regular singularities when A is a filtered ring satisfying $gr(A)$ is a commutative Q-algebra, FA is noetherian and $\mu_A = \nu_A$ (cf. §7). An A-module M is said to have regular singularities if it possesses a very good filtration Γ i.e. $Ann\ gr^{\Gamma}(M) = J(M)$. Using the material from §6 the main theorem (Theorem 7.3) will be derived from the analogous result for E-rings (Theorem 5.7) by micro-localizing several times.

In §8 we study rings of differential operators, denoted by D, and we show that they satisfy the condition $\mu_D = \nu_D$. Also we prove that the notion of a holonomic D-module as introduced in §3 coincides with the usual concept of holonomic D-modules (cf. [1]).

Finally §9 is a kind of appendix collecting some elementary results of commutative algebra which we need in the proofs. I would like to thank Professor Springer for his stimulating discussions and advice.

Throughout this paper we use the following notations.

$\mathbb{N}$ is the set of positive integers, $\bar{\mathbb{N}} := \mathbb{N} \cup \{0\}$, Z is the set of integers and $\mathbb{Q}$ the set of rational numbers.

If R is an arbitrary ring (always having identity) then $\underline{\underline{M}}(R)$ denotes the category of left R-modules of finite type.

All modules considered will be left modules.

If I is an ideal in a commutative ring, $r(I)$ denotes the radical of I. Finally, "iff" means if and only if.

§1. Generalities on Filtered Rings

1.1. Filtrations

Definition 1.2. *i) Let $\mathscr{G}$ be an additive group. A filtration on $\mathscr{G}$ is an ascending sequence of subgroups $\{G_n\}_{n \in Z}$ such that $\cup \mathscr{G}_n = \mathscr{G}$. The group $\mathscr{G}$ equipped with such a filtration is called a filtered group.*

ii) *Let A be a ring. A filtration* $(A_n)_{n\in Z}$ *on A is compatible with the ringstructure if* $A_nA_m\subset A_{n+m}$, *all* n, $m\in Z$ *and* $1\in A_0$. *The ring A equipped with such a filtration is called a filtered ring.*

iii) *Let A be a filtered ring, with filtration* $(A_n)_{n\in Z}$, *M an A-module. A filtration* $(M_n)_{n\in Z}$ *on M is compatible with the module structure on the filtered ring A if* $A_mM_n\subset M_{n+m}$, *all* n, $m\in Z$. *The A-module M equipped with such a filtration is called a filtered A-module.*

The subgroups $\mathscr{G}_n$ in i) above will be denoted by $F_n\mathscr{G}$ and their family $(F_n\mathscr{G})_{n\in Z}$ as $F\mathscr{G}$.

Example 1.3. Let $\mathscr{G}'$ be a subgroup of $\mathscr{G}$. Then $F_n\mathscr{G}':=\mathscr{G}'\cap F_n\mathscr{G}$, all $n\in Z$ define *the induced filtration on* $\mathscr{G}'$. If $\mathscr{G}$ is commutative then $F_n\mathscr{G}'':=(F_n\mathscr{G}+\mathscr{G}')/\mathscr{G}'$, all $n\in Z$ define the *image filtration* on $\mathscr{G}'':=\mathscr{G}/\mathscr{G}'$.

Let $\mathscr{G}$ be an additive commutative group. Put

$$gr_n(\mathscr{G}):=\mathscr{G}_n/\mathscr{G}_{n-1},\quad \text{all}\quad n\in Z;\quad gr(\mathscr{G}):=\bigoplus_{n\in Z} gr_n(\mathscr{G}).$$

The commutative group $gr(\mathscr{G})$ is called the *associated graded group* (to the filtered group $\mathscr{G}$). In case ii) above $gr(A)$ becomes a graded ring called the *associated graded ring* by defining

$$(a+A_{p-1})(b+A_{q-1})=ab+A_{p+q-1},\quad \text{all}\quad a\in A_p,\quad \text{all}\quad b\in A_q.$$

In case iii) above $gr(M)$ becomes a graded $gr(A)$-module by defining

$$(a+A_{p-1})(m+M_{q-1})=am+M_{p+q-1},\quad \text{all}\quad a\in A_p,\quad \text{all}\quad m\in M_q.$$

To indicate that $gr_n(\mathscr{G})$ and $gr(\mathscr{G})$ depend on the filtration F we sometimes write $gr_n^{F\mathscr{G}}(\mathscr{G})$ resp. $gr^{F\mathscr{G}}(\mathscr{G})$.

Let M be a filtered A-module with filtration $F=FM=(F_nM)_{n\in Z}$. If $m\in F_nM$ we put $\sigma_n(m):=m+F_{n-1}M$. Furthermore we define an order function v on M as follows: put $v(m)=-\infty$ if $m\in\bigcap F_nM$ and $v(m)=n$ if $m\in F_nM\backslash F_{n-1}M$. The *symbol map* $\sigma\colon M\to gr(M)$ is defined by $\sigma(m)=0$ if $v(m)=-\infty$ and $\sigma(m)=\sigma_n(m)$ if $v(m)=n\in Z$. To indicate that we work with the filtration F we sometimes write v^F and σ^F.

1.4. Good Filtrations

Definition 1.5. *Let M be a filtered A-module. A filtration FM on M is called good if there exist* $m_1,\dots, m_q\in M$ *and* $v_1,\dots, v_q\in Z$ *such that*

$$F_nM=\sum A_{n-v_i}m_i,\quad \textit{all}\quad n\in Z.$$

Observe that M possesses a good filtration iff $M \in \underline{\underline{M}}(A)$ and that all good filtrations are equivalent (two filtrations $F'M$ and FM on M are called *equivalent* if there exists some $c \in N$ such that

$$F_{n-c}M \subset F'_nM \subset F_{n+c}M, \quad \text{all} \quad n \in Z).$$

Definition 1.6. *A filtration FA on A is called Artin-Rees if for every A-module $M \in \underline{\underline{M}}(A)$ each good filtration on M is separated and all its induced filtrations on submodules of M are again good.*

Lemma 1.7. *Suppose FA is Artin-Rees. Then*

i) A is left noetherian.

Let $M \in \underline{\underline{M}}(A)$ and FM is good on M, then

ii) $M' = \cap (M' + F_nM)$ for every A-submodule M' of M.

iii) $S = \cap (S + F_nM)$ for every $S = \sum_{i=1}^{q} A_{n'-v_i}m_i$, with $q \in N$, $v_1, \dots, v_q \in Z$, $n' \in Z$ and $m_1, \dots, m_q \in M$.

Proof. i) Let I be a left ideal in A. Since FA is good on A, $FA \cap I$ is good on I. So in particular $I \in \underline{\underline{M}}(A)$.

ii) The image filtration of FM on M/M' is good, hence separated, which proves ii).

iii) Put $M' = \sum Am_i$. Then $M' \cap FM$ is good on M'. Since all good filtrations on M' are equivalent there exist $c \in N$ with $M' \cap F_nM \subset \sum A_{n+c}m_i$, all $n \in Z$. Let $n_0 \in Z$ satisfy $n_0 + c \leq n' - v_i$, all i. By ii) $\cap S + F_nM \subset \cap M' + F_nM \subset M'$. So if $m \in \cap S + F_nM$, then $m \in M'$. Also $m \in S + F_{n_0}M$, say $m = s + f$ with $s \in S \subset M'$ and $f \in F_{n_0}M$. Since $m, s \in M'$ we have $f \in M' \cap F_{n_0}M \subset \sum A_{n_0+c}m_i \subset \sum A_{n'-v_i}m_i \subset S$. So $f \in S$, implying $m = s + f \in S$. Hence $\cap S + F_nM \subset S$, which implies iii).

Now we recall some results of [5], §6.

Let $t \in N$, $w_1, \dots, w_t \in N$. On A^t we define the filtration $F = F^{(w)}A^t$ by $F_n^{(w)}A^t := \sum_{i=1}^{t} A_{n-w_i}e_i$, where e_i denotes the i-th standard basis vector of A^t. If M is an A-submodule of A^t, $\sigma^F(M)$ denotes the $gr(A)$-submodule of $gr^F(A^t)$ generated by the elements $\sigma^F(m)$, $m \in M$.

The filtration FA *is called $\sum$-noetherian* if $gr(A)$ is left noetherian and FA satisfies the following condition, $\sum$:

For every $t \in N$, $w_1, \dots, w_t \in Z$ and every A-submodule M of A^t we have: if

$\sigma^F(m_1),\ldots,\sigma^F(m_q)$ generate $\sigma^F(M)$, then $M \cap F_n^{(w)}A^t = \sum A_{n-v_i}m_i$, where $v_i = v^F(m_i)$.

In the remainder of this paper we write FA is *noetherian* (instead of FA is $\sum$-noetherian).

Proposition 1.8. *Let $gr(A)$ be left noetherian. Then are equivalent.*

i) FA is noetherian

ii) FA is Artin-Rees.

Proof. i)→ii) follows from [5], Propositions 6.16 and 6.19.

ii)→i) Let $t \in N$, $w_1,\ldots,w_t \in Z$ and $0 \neq M$ an A-submodule of A^t. Put $F:=F^{(w)}A^t$. So F is good on A^t, hence separated. Put $\sigma:=\sigma^F$. Then $\sigma(m)=0$ iff $m=0$, all $m \in M \subset A^t$. Suppose $\sigma(m_1),\ldots,\sigma(m_q)$ generate $\sigma(M)$. Then we can assume $m_i \neq 0$, all i. Put $v_i = v(m_i)$ and $F_nM := M \cap F_n^{(w)}A^t$ all $n \in Z$. Let $m \in M$. Then $\sigma(m) = \sum \sigma(a_{n-v_i})\sigma(m_i)$ for some $a_{n-v_i} \in A_{n-v_i}$. Consequently $m \in J_n + F_nM$, where $J_n := \sum A_{n-v_i}m_i$. So $F_nM \subset J_n + F_{n-1}M$, all $n \in Z$. Iterating this formula gives $F_nM \subset J_n + F_kM$, all n, $k \in Z$. So $F_nM \subset \bigcap_k J_n + F_kM = J_n$ (where the equality follows from Lemma 1.7 iii) and the fact that FM is good, because FA is Artin-Rees). Since obviously $J_n \subset F_nM$ we get $F_nM = J_n$, all $n \in Z$, as desired.

Corollary 1.9. *Let FA satisfy: all subsets $\sum_{i=1}^q A_{n-v_i}a_i$ of A are FA closed. Then*

i) If $gr(A)$ is left noetherian, then A is left noetherian.

ii) Let $FA_0 := A_0 \cap FA$. If $gr(A_0)$ is left noetherian, then A_0 is left noetherian.

Proof. Since $A_0 0$ is closed, FA is separated.

i) Let M be an ideal of A. Since $gr(A)$ is left noetherian there exists a finite number of elements $m_1,\ldots,m_q \in M$ such that $\sigma(m_1),\ldots,\sigma(m_q)$ generate $\sigma(M)$. Then arguing as in the proof of Proposition 1.8 (with $w=0$ and $t=1$) we find $F_nM \subset \bigcap_k J_n + F_kM$, all $n \in Z$, where $J_n = \sum A_{n-v_i}m_i$, $m_i \in M$, $v_i = v(m_i)$ and $F_kM = A_k \cap M \subset A_k$. So by the hypothesis we get $F_nM \subset J_n$, whence $F_nM = J_n$, all $n \in Z$. Hence FM is good on M. So $M \in \underline{\underline{M}}(A)$, implying that A is left noetherian.

ii) Repeat the proof of i) for A_0.

Proposition 1.10. *If FA is complete and separated and $gr(A)$ is left noetherian, then FA is noetherian.*

Proof. This follows from [5], Corollary 6.11.

Without proof we mention if $gr(A)$ is left noetherian, then FA is Artin-Rees iff for every $t \in N$ the subsets $\sum A_{n-v_i}m_i$ of A^t are closed with respect to the filtration $F^{(0)}A^t$ (FA is then called *Zariskian*). Consequently, if for every $t \in N$ each A_0-submodule of A_0^t is closed, with respect to $F^{(0)}A^t$ (FA is then called *strong noetherian*), then FA is Artin-Rees.

As observed before all good filtrations on an A-module M are equivalent. Now we will give a kind of inverse, which will be an important tool in the study of A-modules with regular singularities (cf. Proposition 6.8).

Theorem 1.11. *Assume FA is noetherian. Let FM be good on M. If $F'M$ is equivalent to FM then $F'M$ is good on M.*

We need the following lemma, the proof of it is due to Professor T. A. Springer:

Lemma 1.12. *Assume $gr(A)$ is left noetherian. Let $F'M$, FM be equivalent filtrations on M. If $gr^F(M) \in \underline{\underline{M}}(gr(A))$, then $gr^{F'}(M) \in \underline{\underline{M}}(gr(A))$.*

Proof. i) There exists $d \in N$ with $F_{n-d}M \subset F'_nM \subset F_{n+d}M$, all $n \in Z$. Put $\Lambda_n := F_{n-d}M$, $\Gamma_n := F'_nM$, all $n \in Z$ and $c := 2d$. Then $\Lambda_n \subset \Gamma_n \subset \Lambda_{n+c}$, all $n \in Z$. Observe that $gr^\Lambda(M) \in \underline{\underline{M}}(gr(A))$. We must derive $gr^\Gamma(M) \in \underline{\underline{M}}(gr(A))$.

ii) Define $T_i = \oplus \Gamma_n \cap \Lambda_{n+i}/\Gamma_{n-1} \cap \Lambda_{n+i}$, for all $0 \le i \le c$. Observe $T_c = \oplus \Gamma_n/\Gamma_{n-1} = gr^\Gamma(M)$. We have to prove $T_c \in \underline{\underline{M}}(gr(A))$. First consider T_0. Observe $T_0 = \oplus \Lambda_n/\Gamma_{n-1} \cap \Lambda_n$. Since $\Lambda_{n-1} \subset \Gamma_{n-1} \cap \Lambda_n$ we get $gr^\Lambda(M) \to T_0 \to 0$ is exact. Since $gr^\Lambda(M) \in \underline{\underline{M}}(gr(A))$ also $T_0 \in \underline{\underline{M}}(gr(A))$. Using induction on i we prove $T_i \in \underline{\underline{M}}(gr(A))$ for all $0 \le i \le c$. Hence $T_c \in \underline{\underline{M}}(gr(A))$ follows as desired. Consider the canonical map ϕ_i: $T_i \to T_{i+1}$ i.e.

$$\phi_i:\quad \oplus \Gamma_n \cap \Lambda_{n+i}/\Gamma_{n-1} \cap \Lambda_{n+i} \longrightarrow \oplus \Gamma_n \cap \Lambda_{n+i+1}/\Gamma_{n-1} \cap \Lambda_{n+i+1}.$$

Then

$$\begin{aligned} \operatorname{Ker} \phi_i &= \oplus \Gamma_n \cap \Lambda_{n+i} \cap \Gamma_{n-1} \cap \Lambda_{n+i+1}/\Gamma_{n-1} \cap \Lambda_{n+i} \\ &= \oplus \Gamma_{n-1} \cap \Lambda_{n+i}/\Gamma_{n-1} \cap \Lambda_{n+i} = 0 \end{aligned}$$

and

$$\begin{aligned} \operatorname{Coker} \phi_i &= \oplus \Gamma_n \cap \Lambda_{n+i+1}/\Gamma_{n-1} \cap \Lambda_{n+i+1}/\Gamma_n \cap \Lambda_{n+i} \\ &\qquad + \Gamma_{n-1} \cap \Lambda_{n+i+1}/\Gamma_{n-1} \cap \Lambda_{n+i+1} \\ &\simeq \Gamma_n \cap \Lambda_{n+i+1}/\Gamma_n \cap \Lambda_{n+i} + \Gamma_{n-1} \cap \Lambda_{n+i+1}. \end{aligned}$$

Using the exact sequence

$$U_i := \oplus \Gamma_n \cap \Lambda_{n+i+1}/\Gamma_n \cap \Lambda_{n+i} \longrightarrow \oplus \Gamma_n \cap \Lambda_{n+i+1}/\Gamma_n \cap \Lambda_{n+i} + \Gamma_{n-1} \cap \Lambda_{n+i+1} \longrightarrow 0$$

we find that $\operatorname{coker} \phi_i \in \underline{\underline{M}}(gr(A))$ if $U_i \in \underline{\underline{M}}(gr(A))$. However $U_i \hookrightarrow V_i := \oplus \Lambda_{n+i+1}/\Lambda_{n+i}$. Observe that V_i is a $gr(A)$-module isomorphic with $gr^A(M)$. So $V_i \in \underline{\underline{M}}(gr(A))$. Hence $U_i \in \underline{\underline{M}}(gr(A))$, implying $\operatorname{coker} \phi_i \in \underline{\underline{M}}(gr(A))$. Consider finally the exact sequence

$$0 \longrightarrow T_i \xrightarrow{\phi_i} T_{i+1} \longrightarrow \operatorname{coker} \phi_i \longrightarrow 0.$$

Since by induction $T_i \in \underline{\underline{M}}(gr(A))$ we find $T_{i+1} \in \underline{\underline{M}}(gr(A))$, which completes the proof.

Proof of Theorem 1.11. We can assume $M \neq 0$. Since $F = FM$ is good $gr^F(M) \in \underline{\underline{M}}(gr(A))$. So by Lemma 1.12 $gr^{F'}(M) \in \underline{\underline{M}}(gr(A))$. Put $\sigma := \sigma^{F'}$. Then $gr^{F'}(M) = \sum_{i=1}^{q} gr(A)\sigma(m_i)$ for some $m_i \in M$. Since FA is noetherian FA is Artin-Rees (Proposition 1.8), so FM is separated. Consequently $F'M$ is separated. So we may assume $\sigma(m_i) \neq 0$ all i, say $v_i := v^{F'}(m_i)$. Similarly as in the proof of Proposition 1.8 we derive $F'_nM \subset J_n + F'_kM$, all n, $k \in Z$ where $J_n = \sum A_{n-v_i}m_i$. Since FM and $F'M$ are equivalent there exists $c \in N$ with $F'_kM \subset F_{k+c}M$, all $k \in Z$, whence $F'_nM \subset \bigcap_k J_n + F_{k+c}M = J_n$ by Lemma 1.7 iii). Obviously $J_n \subset F'_nM$, all $n \in Z$. So $F'_nM = J_n$, all $n \in Z$, i.e. $F'M$ is good.

Let $\mathscr{G}$, $\mathscr{G}'$ be two commutative (additively written) filtered groups with filtrations $(\mathscr{G}_n)_{n\in Z}$ resp. $(\mathscr{G}'_n)_{n\in Z}$. A group homomorphism $h: \mathscr{G} \to \mathscr{G}'$ is called a *morphism of filtered groups* if it respects the filtrations i.e. $h(\mathscr{G}_n) \subset \mathscr{G}'_n$, all $n \in Z$. In the obvious way such an h induces a groups homomorphism of the associated graded groups, denoted $gr(h): gr(\mathscr{G}) \to gr(\mathscr{G}')$, sometimes written as $\bar{h}$. If $\mathscr{G} = A$, $\mathscr{G}' = A'$ are filtered rings, then a morphism h is called a *morphism of filtered rings* if $h: A \to A'$ is a ring homomorphism. Then $gr(h)$ becomes a ring homomorphism. Finally, if $\mathscr{G} = M$, $\mathscr{G} = M'$ are filtered A-modules, a morphism $h: M \to M'$ is called a *morphism of filtered A-modules* if h is an A-module homomorphism.

Proposition 1.12. *Let $h: \mathscr{G} \to \mathscr{G}'$ be a morphism of filtered groups.*

i) $gr(h)$ is injective iff $h^{-1}(\mathscr{G}'_n) = \mathscr{G}_n$, all $n \in Z$.

ii) Let $\mathscr{G}$ be complete and $\mathscr{G}'$ separable. Then $gr(h)$ is surjective iff $\mathscr{G}'_n = h(\mathscr{G}_n)$, all $n \in Z$.

iii) Let $\mathscr{G}$, $\mathscr{G}'$ be separated and $\mathscr{G}$ complete. If $gr(h)$ is bijective then h is bijective.

Proof. See [3], Chap. III, §2, no. 8, Theorem 1 and Corollary 3.

Corollary 1.14. *If $\mathscr{G}$, $\mathscr{G}'$ are separated, $\mathscr{G}$ complete and $gr(h)$ bijective, then h is an isomorphism of filtered groups.*

1.15. The Poisson Product, Involutive Ideals and the Characteristic Ideal

Let again A be a filtered ring. In the remainder of the section assume: *$gr(A)$ is commutative.* So if $a \in A_n$, $b \in A_m$, then $[a, b] := ab - ba \in A_{n+m-1}$, all $n, m \in Z$. Put $f := a + A_{n-1}$, $g := b + A_{m-1}$ and define $\{f, g\} := [a, b] + A_{n+m-2} \in gr_{n+m-1}(A)$. One checks that $\{f, g\}$ is independent of the choice of a and b. So for every $n, m \in Z$ we get a Z-bilinear map $\{\ ,\ \}: gr_n(A) \times gr_m(A) \to gr_{n+m-1}(A)$. Therefore we can extend these maps to a Z-bilinear map $\{\ ,\ \}: gr(A) \times gr(A) \to gr(A)$. It is easy to verify that $\{\ ,\ \}$ is a bi-derivation (cf. Definition 9.1) called the *Poisson-product* on $gr(A)$. An ideal I in $gr(A)$ is called *involutive* if $\{a, b\} \in I$ for all $a, b \in I$ i.e. I is $\{\ ,\ \}$-stable (cf. Definition 9.1).

Let $M \in \underline{\underline{M}}(A)$ and let $F = FM$ be a good filtration on M. Put

$$I^F = Ann\, gr^F(M), \quad J^F = r(I^F).$$

Both I^F and J^F are homogeneous ideals in $gr(A)$ and it is well-known that J^F does not depend on the choice of the good filtration F. We denote this ideal by $J(M)$ and call it *the characteristic ideal of M* or *the ideal of the characteristic variety of M.*

Theorem 1.16 (Gabber). *If $gr(A)$ is a noetherian $\underline{Q}$-algebra and $M \in \underline{\underline{M}}(A)$, then $J(M)$ is involutive.*

Proof. See [6], Theorem I.

Let $\mathscr{G}(J(M))$ denote the set of minimal prime components of $J(M)$ (cf. §9). Since $J(M)$ is homogeneous and involutive (by Theorem 1.16) it is easy to verify that all its minimal prime components are so. Hence

Corollary 1.17. *Assumptions as in Theorem* 1.16. *If $\mathfrak{p} \in \mathscr{G}(J(M))$,* then *$\mathfrak{p}$ is involutive and homogeneous.*

Finally we study the behaviour of involutive ideals under extensions and contractions. Therefore let B be a filtered ring with $gr(B)$ commutative and $\phi: A \to B$ a morphism of filtered rings and let $\{\ ,\ \}$ resp. $\{\ ,\ \}'$ denote the Poisson-products on $gr(A)$ resp. $gr(B)$. Put $\bar{\phi} = gr(\phi)$.

Lemma 1.18. $\bar{\phi}(\{f, g\}) = \{\bar{\phi}(f), \bar{\phi}(g)\}'$, all $f, g \in gr(A)$.

Proof. Since $\{\ ,\ \}$ and $\{\ ,\ \}'$ are both Z-bilinear and $\bar{\phi}$ is additive, we may assume $f = a + A_{n-1}$, $g = b + A_{m-1}$, $a \in A_n$, $b \in A_m$, in which case the formula readily follows.

Proposition 1.19. *If I is an involutive ideal in $gr(A)$, then $I^e := gr(A)\bar{\phi}(I)$ is involutive in $gr(B)$. If J is an involutive ideal in $gr(B)$, then $J^c := \bar{\phi}^{-1}(J)$ is involutive in $gr(A)$.*

Proof. Apply Lemma 1.18 and Proposition 9.2.

§ 2. Algebraic Micro-localization

Throughout this section A will be a filtered ring with filtration $FA = (A_n)_{n \in Z}$ and M denotes a filtered A-module with filtration FM. In §1.1 we have associated an order function v to the filtration FM. Now define

$$|m|_M = 2^{v(m)}, \quad \text{all} \quad m \in M \quad (\text{where } 2^{-\infty} := 0).$$

In particular taking $M = A$ we get $|\ |_A$ on A. We often write $|\ |$ instead of $|\ |_M$. It is easy to verify that $|\ |$ defines a non-archimedean norm on M. i.e. $|am| \le |a|_A |m|$, $|m + m'| \le \max(|m|, |m'|)$, all $a \in A$, all $m, m' \in M$, called the *associated pseudo-norm* (to the filtration FM of M). The strong triangle inequality implies that $|m + m'| = \max(|m|, |m'|)$ if $|m| \ne |m'|$. Furthermore $F_n M$ is the set of $m \in M$ satisfying $|m| \le 2^n$, so $|\ |$ is a norm on M iff FM is separated.

The following two theorems are proved in [5].

Theorem 2.1. *Let A be a filtered ring with associated pseudo-norm $|\ |$. Let S be a multiplicatively closed subset of A such that $\sigma(S)$ is a multiplicatively closed subset of $gr(A)$ satisfying the two left Ore conditions and $0 \notin \sigma(S)$. Then there exists a complete separated filtered ring R with norm $\|\ \|$ and a morphism $\phi: A \to R$ of filtered rings satisfying*

i) $\phi(s)$ is invertible in R, all $s \in S$.

ii) $\|\phi(s)^{-1}\| \le |s|^{-1}$, all $s \in S$.

iii) For every morphism $h: A \to B$ of filtered rings, where B is a complete *and separated filtered ring with norm* $|\ |_B$, *such that* $h(s)$ *is invertible in* B *and* $|h(s)^{-1}|_B \leq |s|^{-1}$ *all* $s \in S$, *there exists a unique morphism of filtered rings* χ: $R \to B$ *satisfying* $\chi \circ \phi = h$.

Moreover, if (R, ϕ) *and* (R', ϕ') *are two such pairs, then there exists a unique isomorphism* γ: $R \to R'$ *of filtered rings satisfying* $\gamma \circ \phi = \phi'$.

Theorem 2.2. *Let* A, S *be as in Theorem* 2.1 *and let* M *be a filtered* A-*module with associated pseudo-norm* $|\ |_M$. *Let* (R, ϕ) *be a solution of Theorem* 2.1. *Then there exists a complete separated* R-*module* M' *and a morphism* ϕ': $M \to M'$ *of filtered* A-*modules which satisfy*: *for every morphism of filtered* A-*modules* h: $M \to N$ *where* N *is a complete separated filtered* R-*module there exists a unique morphism of filtered* R-*modules* χ: $M' \to N$ *such that* $\chi \circ \phi' = h$.

Moreover, if (M', ϕ') *and* (M'', ϕ'') *are two such pairs, then there exists a unique isomorphism of filtered modules* γ: $M' \to M''$ *satisfying* $\gamma \circ \phi' = \phi''$.

The solution of Theorem 2.1 constructed explicitely in [5] will be denoted by $(\mathscr{E}_S(A), \phi_S^A)$. The ring $\mathscr{E}_S(A)$ is called *the left algebraic micro-localization of* A *with respect to* S.

If F denotes the filtration on M, then the solution of Theorem 2.2 constructed in [5] to the pair $(\mathscr{E}_S(A), \phi_S^A)$ will be denoted by $(\mathscr{E}_S(M, F), \phi_S^M)$. The left $\mathscr{E}_S(A)$-module $\mathscr{E}_S(M, F)$ is called *the left algebraic micro-localization of* M *with respect to* S. If there is no confusion possible we write ϕ and ϕ_S instead of ϕ_S^M.

During the construction of $\mathscr{E}_S(M, F)$ in [5] we obtained the following results ([5], Lemma 5.17).

(2.3) *The elements* $\phi(s)^{-1}\phi(m)$ *with* $(s, m) \in S \times M$ *form a dense subset of* $\mathscr{E}_S(M, F)$ *in the* $\|\ \|$-*topology, where* $\|\ \|$ *denotes the norm associated to the filtration on* $\mathscr{E}_S(M, F)$.

Furthermore, the norm of these special elements can be calculated as follows. To the pseudo-norm $|\ |_M$ on M we define its *localized pseudo-norm*, denoted by $|\ |_{M,S}$ or simply $|\ |_S$:

$$|m|_{M,S} := \inf_{\rho \in S} |\rho|_A^{-1} |\rho m|_M, \quad \text{all} \quad m \in M.$$

It is proved in [5], Proposition 3.2 and Corollary 5.20 that $|\ |_S$ is a pseudo-norm satisfying

(2.4) $|sm|_S = |s|_{A,S} |m|_S$, $|s|_{A,S} = |s|_A$, $|m|_S \leq |m|$, all $s \in S$, $m \in M$.

(2.5) $\|\phi(s)^{-1}\phi(m)\|=|s|^{-1}|m|_S$, all $(s, m)\in S\times M$.

Corollary 2.5.1. *i) If $|m|_S\neq 0$, then $|\rho m|=|\rho m|_S$ for some $\rho\in S$.*

ii) If $|m|=|m|_S$, then $|tm|_S=|tm|$ all $t\in S$.

iii) If $\phi(t)^{-1}\phi(m)\neq 0$ then there exist $(\tilde{t}, \tilde{m})\in S\times M$ with

$$\phi(t)^{-1}\phi(m)=\phi(\tilde{t})^{-1}\phi(\tilde{m}) \quad \text{and} \quad |\tilde{m}|=|\tilde{m}|_S=|\phi(\tilde{m})|.$$

Proof. i) Since $|m|_S\neq 0$ there exists $\rho\in S$ with $|m|_S=|\rho|^{-1}|\rho m|$. So $|\rho m|=|\rho m|_S$ by (2.4).

ii) Let $t\in S$. Then $|tm|\leq|t|\,|m|=|t|\,|m|_S=|tm|_S$ (by (2.4)) $\leq|tm|$.

iii) By the hypothesis and (2.5) $|m|_S\neq 0$. So $|\rho m|=|\rho m|_S$ for some $\rho\in S$ (by i)). Then $\tilde{t}:=\rho t$ and $\tilde{m}:=\rho m$ are as desired, since $|\tilde{m}|_S=|\phi(\tilde{m})|$ (by (2.5)).

Consider the filtered $\mathscr{E}_S(A)$-module $\mathscr{E}_S(M, F)$. So the n-th "layer" of the filtration on $\mathscr{E}_S(M, F)$, which we denote by $\mathscr{E}_S^{(n)}(M, F)$, consists of the elements $\mu\in\mathscr{E}_S(M, F)$ with $\|\mu\|\leq 2^n$. We want to describe $gr(\mathscr{E}_S(M, F))$. First observe: since $\sigma(S)$ satisfies the two left Ore conditions $\sigma(S)^{-1}gr(A)$, the left localization of $gr(A)$ with respect to $\sigma(S)$ exists. In fact it is a graded ring: for $n\in Z$ the n-th homogeneous component of $\sigma(S)^{-1}gr(A)$ is the set of elements $\sigma(s)^{-1}\sigma(a)$ with $\sigma(a)\in gr_l(A)$, $\sigma(s)\in\sigma(S)\cap gr_k(A)$ and $1-k=n$. Similarly $\sigma(S)^{-1}gr(M)$ is a graded $\sigma(S)^{-1}gr(A)$-module (cf. [5], Proposition 5.22).

Theorem 2.6 ([5], Proposition 5.24). *There exists an isomorphism ψ_A of graded rings from $\sigma(S)^{-1}gr(A)$ to $gr(\mathscr{E}_S(A))$ defined by*

$$\psi_A(\sigma(s)^{-1}\sigma(a))=\phi(s)^{-1}\phi(a)+\mathscr{E}_S^{(n-1)}(A), \quad \text{all} \quad \sigma(s)^{-1}\sigma(a)\in\sigma(S)^{-1}gr(A)(n).$$

More generally: there exists an isomorphism ψ_M of graded modules over $\sigma(S)^{-1}gr(A)$ from $\sigma(S)^{-1}gr(M)$ to $gr(\mathscr{E}_S(M, F))$ defined by

$$\psi_M(\sigma(s)^{-1}\sigma(m))=\phi(s)^{-1}\phi(m)+\mathscr{E}_S^{(n-1)}(M, F), \quad \text{all} \quad \sigma(s)^{-1}\sigma(m)\in\sigma(S)^{-1}gr(M)(n).$$

2.7. Some Consequences of Theorem 2.6

In the remainder of this section we assume: *$gr(A)$ is commutative.*

Theorem 2.6, the proof of which is a consequence of (2.3), (2.4) and (2.5) plays a fundamental role in the study of micro-localizations. We derive some consequences.

If F and F' are equivalent filtrations on M, then it follows from the construction of algebraic micro-localizations that $\mathscr{E}_S(M, F)=\mathscr{E}_S(M, F')$ ([5], Proposition 6.7). In particular if $M \in \underline{\underline{M}}(A)$ then $\mathscr{E}_S(M, F)$ does not depend on the choice of the good filtration F on M. Instead of $\mathscr{E}_S(M, F)$ we therefore write $\mathscr{E}_S(M)$. In fact, if FA is noetherian it is proved in [5], Corollary 6.25 that $M \to \mathscr{E}_S(M)$ gives an exact functor from $\underline{\underline{M}}(A)$ to $\underline{\underline{M}}(\mathscr{E}_S(A))$. Using this result it is shown that there exists a canonical isomorphism of $\mathscr{E}_S(A)$-modules between $\mathscr{E}_S(A) \otimes_A M$ and $\mathscr{E}_S(M)$ ([5], Proposition 6.26). Consequently, we obtain that $\mathscr{E}_S(A)$ is a flat right A-module ([5], Corollary 6.27).

Let $M \in \underline{\underline{M}}(A)$ and FM a good filtration on M. The filtration $(\mathscr{E}_S^{(n)}(M, FM))_{n \in Z}$ on $\mathscr{E}_S(M)$ we denote by $L(FM)$ or $L_S(FM)$ and its n-th layer by $L^{(n)}(FM)$. Let I be an ideal in $gr(A)$. Put $I^e := gr(\mathscr{E}_S(A))\bar{\phi}_S(I)$. So $I^e = \psi_A(\sigma(S)^{-1} I)$ where ψ_A is as in Theorem 2.6.

Definition 2.8. *A good filtration FM on M is called I-good if $I \subset Ann\, gr^{FM}(M)$. If $I = J(M)$ (see 1.15) FM is called very good.*

Proposition 2.9. *If FM is I-good on M, then $L(FM)$ is I^e-good on $\mathscr{E}_S(M)$. In particular L preserves good and very good filtrations and $\mathscr{E}_S(M) \in \underline{\underline{M}}(\mathscr{E}_S(A))$.*

Proof. If FM is good on M then $L(FM)$ is good on $\mathscr{E}_S(M)$ ([5], Corollary 6.23), hence $\mathscr{E}_S(M) \in \underline{\underline{M}}(\mathscr{E}_S(A))$. Furthermore, if $I \subset Ann\, gr^{FM}(M)$, then $I^e \subset (Ann\, gr(M))^e = \psi_A(\sigma(S)^{-1} Ann\, gr(M)) = Ann\ \ gr^{L(FM)}(\mathscr{E}_S(M))$ by Theorem 2.6. So $L(FM)$ is I^e-good. Finally, taking radicals of the last two equalities we get

$$J(M)^e = \psi_A(\sigma(S)^{-1} J(M)) = J(\mathscr{E}_S(M)) \tag{2.10}$$

which shows that L preserves very good filtrations.

2.11. Micro-Localizations in Prime Idea of $gr(A)$

Let $\not{p} \in Spec\,(gr(A))$. Put $S_{\not{p}}$ is the set of all $a \in A$ with $\sigma(a) \notin \not{p}$. It is easy to verify that $S_{\not{p}}$ is a multiplicatively closed subset of A with $0 \notin \sigma(S_{\not{p}})$ and that $\sigma(S_{\not{p}})$ is a multiplicatively closed subset of $gr(A)$ satisfying the two left Ore conditions, since $gr(A)$ is commutative. So by Theorem 2.1 the left micro-localizations of A with respect to $S_{\not{p}}$ exists. Instead of $\mathscr{E}_{S_{\not{p}}}(A)$ we write $\mathscr{E}_{\not{p}}(A)$. Similarly, if M is a filtered A-module with filtration F on M the micro-localization of M with respect to $S_{\not{p}}$ exists and we write $\mathscr{E}_{\not{p}}(M, F)$ (resp. $\mathscr{E}_{\not{p}}^{(n)}(M, F)$) instead of $\mathscr{E}_{S_{\not{p}}}(M, F)$ (resp. $\mathscr{E}_{S_{\not{p}}}^{(n)}(M, F)$). Now assume $M \in \underline{\underline{M}}(A)$ and F is good on M. Then we can write $\mathscr{E}_{\not{p}}(M)$ instead of $\mathscr{E}_{\not{p}}(M, F)$ by (2.7).

Warning. If F' is another good filtration on M the filtrations

$(\mathscr{E}_{\wp}^{(n)}(M, F))_{n\in Z}$ and $(\mathscr{E}_{\wp}^{(n)}(M, F'))_{n\in Z}$ are not equal. However, by Proposition 2.9 they are good filtrations on $\mathscr{E}_{\wp}(M)$. So they are equivalent and $\mathscr{E}_{\wp}(M)\in \underline{\underline{M}}(\mathscr{E}_{\wp}(A))$.

Lemma 2.12. *Let F be a separated filtration on M. Then $M=0$ iff $gr^F(M)=0$.*

Proof. If $gr^F(M)=0$, then $F_nM=F_{n+1}M$, all $n\in Z$. So $M=\cup F_nM=\cap F_nM=0$.

Proposition 2.14. $\mathscr{E}_{\wp}(M)\neq 0$ *iff* $\wp\supset J(M)$.

Proof. Let F be a good filtration on M. Then $F\mathscr{E}_{\wp}(M)$ is good by Proposition 2.9 and separated. So by (2.12) $\mathscr{E}_{\wp}(M)=0$ iff $gr(\mathscr{E}_{\wp}(M))=0$ iff $\sigma(S_{\wp})^{-1}gr^F(M)=0$ (by Theorem 2.6) iff $\sigma(S_{\wp})\cap Ann\, gr^F(M)\neq\emptyset$ iff $\sigma(S_{\wp})\cap J(M)\neq\emptyset$ iff $\wp\not\supset J(M)$.

Proposition 2.15. *If $gr(A)$ is noetherian, then $F\mathscr{E}_{\wp}(A)$ is noetherian.*

Proof. Since $gr(A)$ is noetherian, so is $\sigma(S)^{-1}gr(A)$. Hence $gr(\mathscr{E}_{\wp}(A))$ is noetherian by Theorem 2.6. Then apply Corollary 1.10.

2.16. Morphisms between Micro-Localized Rings

We consider the following situation. Let B be a filtered ring with $gr(B)$ commutative and ϕ: $A\to B$ is a morphism of filtered rings. Furthermore $\bar{\phi}:=gr(\phi)$: $gr(A)\to gr(B)$. If I is an ideal in $gr(A)$ (resp. J an ideal in $gr(B)$) then $I^e:=gr(B)\bar{\phi}(I)$ (resp. $J^c:=\bar{\phi}^{-1}(J)$). If $\wp\in Spec\,(gr(A))$ then $\phi_{\wp}$ denotes the canonical map $\phi^A_{S_{\wp}}$ from A to $\mathscr{E}_{\wp}(A)$ and $\|\ \|_{\wp}$ denotes the norm on $\mathscr{E}_{\wp}(A)$. Similar notations we use for $q\in Spec\,(gr(B))$.

Lemma 2.17. *Let $\wp\in Spec\,(gr(A))$. Assume $\wp^e\in Spec\,(gr(B))$ and $(\wp^e)^c=\wp$. Then*

i) $|\phi(s)|=|s|$, all $s\in S_{\wp}$ and $\phi(S_{\wp})\subset S_{\wp^e}$.

ii) Put $u:=\phi^B_{\wp^e}\circ\phi$. Then $u(s)$ is invertible in $\mathscr{E}_{\wp^e}(B)$ for all $s\in S_{\wp}$.

iii) $\|u(s)^{-1}\|_{\wp^e}=|s|^{-1}$, all $s\in S_{\wp}$.

Proof. i) Let $s\in S_{\wp}$, say $v(s)=n$ ($\sigma(s)\neq 0$ since $\sigma(s)\notin\wp$). Then $\sigma(s)\notin\wp=\bar{\phi}^{-1}(\wp^e)$ i.e. $\bar{\phi}(\sigma(s))\notin\wp^e$. In particular $\bar{\phi}(\sigma(s))\neq 0$ i.e. $\phi(s)+B_{n-1}\neq 0$. So $|\phi(s)|=2^n=|s|$. Hence $\sigma_B(\phi(s))=\phi(s)+B_{n-1}=\bar{\phi}(\sigma(s))\notin\wp^e$ i.e. $\phi(s)\in S_{\wp^e}$.

ii) follows from i) since $\phi_{\wp}(t)$ is invertible in $\mathscr{E}_{\wp^e}(B)$ for all $t\in S_{\wp^e}$.

iii) Let $s\in S_{\wp}$. Then $\|u(s)^{-1}\|^B_{\wp^e}=\|\phi^B_{\wp^e}(\phi(s))^{-1}\|_{\wp^e}=|\phi(s)|^{-1}$ (by i) and

(2.5) with $M=A$ and $m=1=|s|^{-1}$ by i).

By Lemma 2.17 ii) and iii) and Theorem 2.1 we obtain

Corollary 2.18. *There exists a unique morphism of filtered rings*

$$\tilde{\phi}: \mathscr{E}_{p}(A) \longrightarrow \mathscr{E}_{p^e}(B) \quad \text{with} \quad \tilde{\phi}\circ\phi_{p}=\phi_{p^e}\circ\phi .$$

Let M be a filtered A-module and N a filtered B-module with filtration FM resp. FN. By means of ϕ N becomes a filtered A-module. Let $\psi: M \to N$ be a morphism of filtered A-modules. Put $h:=\phi^N_{p^e}\circ\psi$. So h is a morphism of filtered A-modules from M to $\mathscr{E}_{p^e}(N, FN)$. The $\mathscr{E}_{p^e}(B)$-module $\mathscr{E}_{p^e}(N, FN)$ is a filtered $\mathscr{E}_{p}(A)$-module by means of $\tilde{\phi}$ from Corollary 2.18. Then Theorem 2.2 gives

Corollary 2.19. *There exists a unique morphism* $\tilde{\psi}: \mathscr{E}_{p}(M, FM) \to \mathscr{E}_{p^e}(N, FN)$ *of filtered* $\mathscr{E}_{p}(A)$*-modules with* $\tilde{\psi}\circ\phi^M_{p}=\phi^N_{p^e}\circ\psi$.

2.20. The Set Spec $^\circ(gr(A))$

Let X be a complex analytic manifold. In the micro-analytic study of sheaves of $\mathscr{D}$-modules the interesting points to consider are the points $(z, \zeta) \in T^*X$ outside the zero-section i.e. points with $\zeta \neq 0$. The set $T^*X\backslash$zero-section is often denoted by $\mathring{T}^*X$. Let $\mathscr{D}_n=\mathscr{O}_n[\partial/\partial z_1,\dots, \partial/\partial z_n]$ with $\mathscr{O}_n=\mathbf{C}\{z_1,\dots, z_n\}$ the ring of convergent power series. Then $gr(\mathscr{D}_n)\simeq\mathscr{O}[\zeta_1,\dots, \zeta_n]$ as usual. So we want to consider primes p in $gr(\mathscr{D}_n)$ not containing all ζ_i i.e. primes p with the property that there exists a homogeneous element of degree one in $gr(\mathscr{D}_n)$ which does not belong to p. These considerations lead us to the following definition. Let A be a filtered ring.

Definition 2.21. *$Spec^\circ(gr(A))$ is the set of $p \in Spec(gr(A))$ such that $\sigma(t) \notin p$ for some $t \in A_1\backslash A_0$.*

Lemma 2.22. *If $p \in Spec^\circ(gr(A))$ then there exists $s \in \mathscr{E}^{(1)}_{p}(A)\backslash\mathscr{E}^{(0)}_{p}(A)$ invertible in $\mathscr{E}_{p}(A)$ with $s^{-1} \in \mathscr{E}^{(-1)}_{p}(A)$.*

Proof. Let $\sigma(t) \notin p$ with $t \in A_1\backslash A_0$. So $t \in S_{p}$. Put $s=\phi_{p}(t)$. Then $\|s\|=\|\phi_{p}(t)\|=|t|_{p}$ (by (2.5))$=|t|$ (by (2.4) since $t \in S_{p}$)$=2^1$. Finally $\|s^{-1}\|=\|\phi_{p}(t)^{-1}\|=|t|^{-1}$ (by 2.5))$=2^{-1}$.

Corollary 2.23. *If $p \in Spec^\circ(gr(A))$ and $gr(A)$ is a noetherian $\mathbf{Q}$-algebra then the filtered ring $R:=\mathscr{E}_{p}(A)$ satisfies*

i) *There exists an element $s \in R_1\backslash R_0$ invertible in R with $s^{-1} \in R_{-1}$.*

ii) $gr(R)$ is a commutative $\mathbf{Q}$-algebra.

iii) FR is noetherian.

Proof. i) follows from Lemma 2.22., ii) from Theorem 2.6 and iii) from Proposition 2.15.

§ 3. Holonomic A-Modules

In this section A will be a filtered ring with filtration FA satisfying

a) $gr(A)$ is a commutative $\mathbf{Q}$-algebra.

b) FA is noetherian.

Lemma 3.1. *If $0 \neq M \in \underline{\underline{M}}(A)$, then $J(M) \neq gr(A)$. So $\mathscr{G}(J(M)) \neq \emptyset$.*

Proof. Let F be a good filtration on M. By Proposition 1.8 F is separated, so $gr(M) \neq 0$ by Lemma 2.12. This implies $1 \notin Ann\, gr^F(M)$, so $1 \notin J(M)$.

Put $\mathscr{I}_h :=$ the set of involutive homogeneous prime ideals in $gr(A)$

$\mu_A := \sup_{\not p \in \mathscr{I}_h} \mathrm{ht} \not p$ (where $\mathrm{ht} \not p = \mathrm{height}\ \not p$).

By Corollary 1.17 $\mathscr{G}(J(M)) \subset \mathscr{I}_h$, so $\mathrm{ht} \not p \leq \mu_A$ for all $\not p \in \mathscr{G}(J(M))$.

Definition 3.2. *$0 \neq M \in \underline{\underline{M}}(A)$ is called holonomic if $\mathrm{ht} \not p = \mu_A$ for all $\not p \in \mathscr{G}(J(M))$. Also $M = 0$ is holonomic.*

Remark 3.3. If there exists a non-zero holonomic A-module M, then μ_A is finite since $gr(A)$ is noetherian and $\mu_A = \mathrm{ht} \not p$ for some $\not p \in Spec\,(gr(A))$.

Let $\not q \in Spec\,(gr(A))$. By Theorem 2.6 and Proposition 2.15 it follows that $\mathscr{E}_{\not q}(A)$ with its filtration $F\mathscr{E}_{\not q}(A)$ also satisfies the conditions a) and b) above. So we also have the notion of a holonomic $\mathscr{E}_{\not q}(A)$-module.

Proposition 3.4. *Let M be a holonomic A-module and $\not q \in Spec\,(gr(A))$. Then $\mathscr{E}_{\not q}(M)$ is a holonomic $\mathscr{E}_{\not q}(A)$-module.*

Proof. i) We can assume $\mathscr{E}_{\not q}(M) \neq 0$, whence by Lemma 3.1 $\mathscr{G}(J(\mathscr{E}_{\not q}(M))$ $\neq \emptyset$. So we can choose $\mathscr{P} \in \mathscr{G}(J(\mathscr{E}_{\not q}(M)))$. Since by (2.10) $J(\mathscr{E}_{\not q}(M))$ $= \psi_A(\sigma(S_{\not q})^{-1} J(M))$ $\mathscr{P} = \psi_A(\sigma(S_{\not q})^{-1} \not p)$ for some $\not p \in \mathscr{G}(J(M))$ with $\not p \cap \sigma(S_{\not q}) = \emptyset$ (Corollary 9.3 ii)). Then Corollary 9.3 i) gives $\mathrm{ht}\mathscr{P} = \mathrm{ht} \not p = \mu_A$ since M is holonomic. By Corollary 1.17 $\mathscr{P}$ is involutive and homogeneous, hence $\mu_{\mathscr{E}_{\not q}(A)} \geq \mu_A$. Furthermore the argument above gives that $\mathrm{ht}\mathscr{P} = \mu_A$ for all $\mathscr{P} \in \mathscr{G}(J(\mathscr{E}_{\not q}(M)))$. So $\mathscr{E}_{\not q}(M)$ is holonomic if we can prove that $\mu_A = \mu_{\mathscr{E}_{\not q}(A)}$. It therefore remains to prove that $\mu_{\mathscr{E}_{\not q}(A)} \leq \mu_A$.

ii) Let $\mathscr{P}$ be an arbitrary involutive homogeneous prime ideal in $gr(\mathscr{E}_{\wp}(A))$. By Proposition 1.19 (applied to A and $\mathscr{E}_{\wp}(A)$) $\wp := \mathscr{P}^c$ is an involutive and homogeneous prime ideal of $gr(A)$. Since $\mathscr{P} = \psi_A(\sigma(S_{\wp})^{-1}\wp)$, Corollary 9.3 i) implies $\mathrm{ht}\mathscr{P} = \mathrm{ht}\wp \leq \mu_A$, whence $\mu_{\mathscr{E}_{\wp}(A)} \leq \mu_A$ as desired.

§ 4. E-Rings and Their Properties

Let A be a filtered ring such that $gr(A)$ is a commutative noetherian Q-algebra. Just as in the micro-analytic theory of $\mathscr{D}$-modules the rings $\mathscr{E}_{\wp}(A)$ with $\wp \in Spec^{\circ}(gr(A))$ play a very important role. As shown in §2 these rings have the properties of Corollary 2.23. Filtered rings having these properties will be studied in this and the next section.

Definition 4.1. *A filtered ring R will be called an E-ring if the following conditions are satisfied*

i) There exists an element $s \in R_1 \backslash R_0$ invertible in R with $s^{-1} \in R_{-1}$.

ii) $gr(R)$ is a commutative Q-algebra.

iii) FR is noetherian.

Lemma 4.2. *Let R be a filtered ring satisfying i) of Definition 4.1 and M a filtered R-module with filtration $FM = (M_n)_{n \in Z}$. Then*

i) $s^{-1} \notin R_{-2}$ and $v := s + R_0$ is a unit in $gr(R)$ with inverse $v^{-1} = s^{-1} + R_{-2}$.

ii) $R_n = s^n R_0 = R_0 s^n$, $M_n = s^n M_0$, all $n \in Z$.

iii) FM is good iff $M_0 \in \underline{\underline{M}}(R_0)$.

Proof. i) Since $s \in R_1 \backslash R_0$ $gr_1(R) \neq 0$. So $gr(R) \neq 0$ whence $1 + R_{-1} \neq 0$. Consequently, $(s^{-1} + R_{-2})(s + R_0) = (s + R_0)(s^{-1} + R_{-2}) = 1 + R_{-1} \neq 0$ which implies i).

ii) If $r \in R_n$ then $(s^{-1})^n r \in R_{-n} R_n \subset R_0$, so $r \in s^n R_0$ implying $R_n = s^n R_0$. Similarly, $M_n = s^n R_0$ and $R_n = R_0 s^n$.

iii) If FM is good then $M_0 = \sum_{i=1}^{q} R_{-v_i} m_i = \sum R_0 s^{-v_i} m_i$ (by i)) for some $m_i \in M$, $v_i \in Z$. So $M_0 \in \underline{\underline{M}}(R_0)$. The converse follows from ii).

So if we define $\mathscr{F}(M_0) := (R_n M_0)_{n \in Z}$ for each R_0-submodule of M satisfying $RM_0 = M$ we get a one-to-one correspondence between these R_0-submodules of M and the set of filtrations on M. Furthermore, if $M \in \underline{\underline{M}}(R)$ then iii) shows that the restriction of $\mathscr{F}$ to the set of finitely generated R_0-submodules M_0 of

M satisfying $RM_0=M$ gives a one-to-one correspondence with the good filtrations on M.

Proposition 4.3. *Notations as in Lemma* 4.2. *Then* $\psi: gr_0(R)[X, X^{-1}] \to gr(R)$ *defined by* $\psi(\sum f_i X^i)=\sum f_i v^i$ *is an isomorphism of graded rings.*

Proof. Left to the reader (cf. §9 for the graded ring structure of $gr_0(R)\cdot[X, X^{-1}]$).

Corollary 4.4. *Extension and contraction give a one-to-one correspondence between the prime ideals of* $gr_0(R)$ *and the homogeneous prime ideals of* $gr(R)$.

Proof. Apply Proposition 4.3 and Proposition 9.9.

Corollary 4.5. *If* R *is an E-ring then* R *and* R_0 *are left noetherian.*

Proof. By Proposition 1.8 Lemma 1.7 ii) (with $M=A$) the hypothesis of Corollary 1.9 is satisfied. So by Corollary 1.9 i) R is left noetherian. Finally by Proposition 9.10 $gr_0(R)$ is noetherian, whence $gr(R_0)$ is noetherian since by the description of Proposition 4.3 $gr(R_0)$ is isomorphic to $gr_0(R)[X^{-1}]$.

4.6. Gabbers Theorem and R_0-Modules of Finite Type

From now on we assume: *R is an E-ring.* So on $gr(R)$ we have a Poisson product (see 1.15). Let $f, g \in gr_0(R)$. Then $\{f, g\} \in gr_{-1}(R)$, whence $v\{f, g\} \in gr_0(R)$ ($v=s+R_0$ as above). So putting $P(f, g):=v\{f, g\}$ all $f, g \in gr_0(R)$ we get a Poisson product on $gr_0(R)$ which (as one easily checks) is a bi-derivation on $gr_0(R)$ (cf. §9). An ideal I in $gr_0(R)$ is called *involutive if it is P-stable* (cf. Definition 9.1).

Proposition 4.7. *If I is an involutive ideal in* $gr_0(R)$ *then* $I^e:=gr(R)I$ *is an involutive ideal in* $gr(R)$. *If J is an involutive ideal in* $gr(R)$ *then* $J^c:=gr_0(R)\cap J$ *is an involutive ideal in* $gr_0(R)$.

Proof. Apply Proposition 9.2 to $A=gr_0(R), B=gr(R)$ and ϕ the inclusion map.

Let $M \in \underline{\underline{M}}(R)$, FM a good filtration on M and $N \subset M$ an arbitrary R_0-submodule of M. We want to find out if $N \in \underline{\underline{M}}(R_0)$. Since by Corollary 4.5 R_0 is left noetherian we get $N \in \underline{\underline{M}}(R_0)$ iff $N \in M_{n_0}$ for some $n_0 \in N$ iff $M_n \cap N/M_{n-1} \cap N=0$ for all $n \geq n_0$ and some $n_0 \in N$. We therefore put

$$Q(n, N):=M_n \cap N/M_{n-1} \cap N, \quad \text{all} \quad n \in Z.$$

Observe that $s^{-1}(M_n \cap N) \subset M_{n-1} \cap N$. So $Q(n, N)$ is a $gr_0(R)$-module. In fact it is isomorphic to a $gr_0(R)$-submodule of $gr_n(M)$. Since $M_n = s^n M_0$ and $M_0 \in \underline{\underline{M}}(R_0)$ by Lemma 4.2 ii) and iii) we get $M_n \in M(R_0)$, implying $gr_n(M) \in \underline{\underline{M}}(gr_0(R))$. Consequently $Q(n, N) \in \underline{\underline{M}}(gr_0(R))$. Put

$$I(n) := Ann\, Q(n, N) \subset gr_0(R),\ J(n) := r(I(n)), \quad \text{all} \quad n \in Z.$$

Lemma 4.8. *Left multiplication by s^{-1} induces an injective $gr_0(R)$-linear map from $Q(n+1, N)$ into $Q(n, N)$, all $n \in Z$.*

Proof. Straightforward.

So we get an increasing sequence of ideals $I(1) \subset I(2) \subset \cdots$ in $gr_0(R)$. Since $gr_0(R)$ is noetherian there exists $n_0 \in N$ with $I(n) = I(n_0)$ for all $n \geq n_0$. Hence $J(1) \subset J(2) \subset \cdots$ and $J(n) = J(n_0)$ for all $n \geq n_0$. Put

$$J := J(N) := J(n_0).$$

So $J = \cup J(n)$.

Theorem 4.9 (Gabber). *J is an involutive ideal in $gr_0(R)$.*

At the end of this section we give a very simply proof of this important result, in fact we use algebraic micro-localization, to make a reduction to [6], Theorem II. We also refer to [7] and [2].

For $p \in Spec(gr(R))$ define $N(p)$ as the $\mathscr{E}_p^{(0)}(R)$-submodule of $\mathscr{E}_p(M)$ generated by the elements $\phi_p(m)$, with $m \in N$. Put

$$Q(n, N(p)) := \mathscr{E}_p^{(n)}(M) \cap N(p) / \mathscr{E}_p^{(n-1)}(M) \cap N(p)$$

where $\mathscr{E}_p^{(n)}(M) := \mathscr{E}_p^{(n)}(M, FM)$. Let $p_0 \in Spec(gr_0(R))$ and $p := p_0^e$ $(= gr(R) p_0)$. So p is a homogeneous prime ideal in $gr(R)$ by Corollary 4.4. Let $r_0 + R_{-1} \in gr_0(R) \backslash p_0$. Then $\phi_p(r_0) \in S_p$ and $|\phi_p(r_0)| = |r_0| = 1$, so $\phi_p(r_0) + \mathscr{E}_p^{(-1)}(R)$ is invertible in $gr_0(\mathscr{E}_p(R))$. Hence the canonical map $gr(\phi_p)$: $gr_0(R) \to gr_0(\mathscr{E}_p(R))$ extends to a ring-homomorphism ψ: $gr_0(R)_{p_0} \to gr_0(\mathscr{E}_p(R))$. Fix $n \in N$ and put $B := gr_0(R)$. The canonical map χ: $Q(n, N) \to Q(n, N(p))$ is a B-module homomorphism ($Q(n, N(p))$ is a B-module by means of $gr(\phi_p)$). Since by ψ $Q(n, N(p))$ is a left B_{p_0}-module, χ extends to a B_{p_0}-module homomorphism $\tilde{\chi}$: $Q(n, N)_{p_0} \to Q(n, N(p))$.

Lemma 4.10. *$\tilde{\chi}$ is an isomorphism of B_{p_0}-modules.*

Proof. By Lemma 4.2 i) $\sigma(s)$ is a unit in $gr(R)$ with inverse $\sigma(s^{-1})$. So s

and s^{-1} belong to $S_{\mathfrak{p}}$. Hence $s^r \in S_{\mathfrak{p}}$, all $r \in Z$. Using 2.5.1. i), ii) this gives

(4.11) if $m \in M$, then $|\rho m|_{\mathfrak{p}} = |\rho m|$, for some $\rho \in S_{\mathfrak{p}}$ with $v(\rho)=0$.

i) $\tilde{\chi}$ is injective: let $m \in M_n \cap N$. Suppose $\phi_{\mathfrak{p}}(m) \in \mathscr{E}_{\mathfrak{p}}^{(n-1)}(M)$ i.e. $|m|_{\mathfrak{p}} \leq 2^{n-1}$. Apply (4.11). Then $|\rho m| = |\rho m|_{\mathfrak{p}} \leq 2^{n-1}$. So $\rho m \in M_{n-1} \cap N$, whence $\sigma(\rho)\overline{m}=0$ in $Q(n, N)$ implying $\overline{m}=0$ in $Q(n, N)_{\mathfrak{p}_0}$ since $\sigma(\rho) \in B \backslash \mathfrak{p}_0$.

ii) $\tilde{\chi}$ is surjective: put $\phi = \phi_{\mathfrak{p}}$. Every element of $\mathscr{E}_{\mathfrak{p}}^{(n)}(M) \cap N(\mathfrak{p})$ is a finite sum of elements of the form $\alpha\phi(m)$ with $\alpha \in \mathscr{E}_{\mathfrak{p}}^{(0)}(R)$ and $m \in N$. Therefore it suffices to show that all these elements $\alpha\phi(m)$ belong to the image of $\tilde{\chi}$. Take such an element $\alpha\phi(m)$. Then $\phi(m) \in \mathscr{E}_{\mathfrak{p}}^{(n_0)}(M)$ for some $n_0 \geq n-1$. By (2.3) choose $(t, a) \in S_{\mathfrak{p}} \times R$ with $\alpha - \phi(t)^{-1}\phi(a) \in \mathscr{E}_{\mathfrak{p}}^{(-n_0+n-1)}(R)$. Then $\alpha\phi(m) = \phi(t)^{-1} \cdot \phi(am) \bmod (\mathscr{E}_{\mathfrak{p}}^{(n-1)}(M) \cap N(\mathfrak{p}))$ and $\phi(t)^{-1}\phi(a) \in \mathscr{E}_{\mathfrak{p}}^{(0)}(R)$. We may assume $|am|_{\mathfrak{p}} = |am|$ (use (4.11) and replace t by ρt and am by ρam). Similarly we can assume $|a|_{\mathfrak{p}} = |a|$ and $v(t)=0$. So $\sigma(t) \in B \backslash \mathfrak{p}_0$ and $a \in R_0$. Hence $am \in N$ and $|am| = |\phi(am)| \leq 2^n$ i.e. $am \in M_n \cap N$. So $\alpha\phi(m) + \mathscr{E}_{\mathfrak{p}}^{(n-1)}(M) \cap N(\mathfrak{p}) = \tilde{\chi}(\sigma(t)^{-1}(am + M_{n-1} \cap N)) \in \tilde{\chi}(Q(n, N)_{\mathfrak{p}_0})$.

Corollary 4.12. *If* $N \notin \underline{\underline{M}}(R_0)$ *then*

i) $\mathscr{G}(J(N)) \neq \emptyset$.

ii) If $\mathfrak{p}_0 \in \mathscr{G}(J(N))$, *then* $\mathfrak{p} := \mathfrak{p}_0^e$ *satisfies* $\operatorname{ht} \mathfrak{p} \leq \mu_R$ *and* $N(\mathfrak{p}) \notin \underline{\underline{M}}(\mathscr{E}_{\mathfrak{p}}^{(0)}(R))$.

Proof. i) Since $N \notin \underline{\underline{M}}(R_0)$ $Q(n, N) \neq 0$ for all $n \in \boldsymbol{N}$ (if $Q(n_0, N)=0$ for some $n_0 \in \boldsymbol{N}$ then $Q(n, N)=0$ for all $n \geq n_0$ by Lemma 4.8 implying $N \in \underline{\underline{M}}(R_0)$). So $1 \notin I(n)$ for all $n \in \boldsymbol{N}$ i.e. $1 \notin J$, which proves i).

ii) Let $\mathfrak{p}_0 \in \mathscr{G}(J(N))$. Then $\mathfrak{p}_0$ is involutive (by Theorem 4.9). Hence $\mathfrak{p} := \mathfrak{p}_0^e$ is a homogeneous involutive prime ideal in $gr(R)$ by Corollary 4.4 and Proposition 4.7. So $\operatorname{ht} \mathfrak{p} \leq \mu_R$. Now suppose $N(\mathfrak{p}) \in \underline{\underline{M}}(\mathscr{E}_{\mathfrak{p}}^{(0)}(R))$. Then $Q(n, N(\mathfrak{p}))=0$ for all $n \geq n_0$ (some $n_0 \in \boldsymbol{N}$). So $Q(n, N)_{\mathfrak{p}_0}=0$ for all $n \geq n_0$ (Lemma 4.10). However $\mathfrak{p}_0 \supset J(N) \supset J(n)$, all $n \in \boldsymbol{N}$, so $\mathfrak{p}_0 \supset I(n)$. This gives $Q(n, N)_{\mathfrak{p}_0} \neq 0$, a contradiction.

Corollary 4.14. *Let* M *be holonomic and* $N \notin \underline{\underline{M}}(R_0)$. *If* $\mathfrak{p}_0 \in \mathscr{G}(J(N))$, *then* $\mathfrak{p} := \mathfrak{p}_0^e \in \mathscr{G}(J(M))$.

Proof. By Corollary 4.12 ii) $Q(n, N(\mathfrak{p})) \neq 0$, all $n \in \boldsymbol{N}$, whence $\mathscr{E}_{\mathfrak{p}}(M) \neq 0$. So $\mathfrak{p} \supset J(M)$ by Proposition 2.16. Hence $\mathfrak{p} \supset \mathfrak{p}'$ for some $\mathfrak{p}' \in \mathscr{G}(J(M))$. Since $\operatorname{ht} \mathfrak{p}' = \mu_R$ (M is holonomic) $\operatorname{ht} \mathfrak{p} \geq \mu_R$. Then Corollary 4.12 ii) implies $\operatorname{ht} \mathfrak{p} = \mu_R = \operatorname{ht} \mathfrak{p}'$. So $\mathfrak{p} = \mathfrak{p}' \in \mathscr{G}(J(M))$ (since μ_R is finite by Remark 3.3).

Proposition 4.15. *Let* M *be a holonomic* R-*module. Then are equivalent* *i)* $N \in \underline{\underline{M}}(R_0)$.

ii) $N(\not{p}) \in \underline{\underline{M}}(\mathscr{E}_{\not{p}}^{(0)}(R))$ *for all* $\not{p} \in \mathscr{G}(J(M))$.

Proof. i)→ii). If $n_1, \ldots, n_q$ generate N as R_0-module, then $\phi_{\not{p}}(n_1), \ldots, \phi_{\not{p}}(n_q)$ generate $N(\not{p})$ as $\mathscr{E}_{\not{p}}^{(0)}(R)$-module.

ii)→i). Suppose $N \notin \underline{\underline{M}}(R_0)$. Then $\mathscr{G}(J(N)) \neq \emptyset$ by Corollary 4.12 i). Let $\not{p}_0 \in \mathscr{G}(J(N))$. Put $\not{p} := \not{p}_0^e$. Then $N(\not{p}) \notin \underline{\underline{M}}(\mathscr{E}_{\not{p}}^{(0)}(R))$ by Corollary 4.12 and $\not{p} \in \mathscr{G}(J(M))$ by Corollary 4.14. So by ii) $N(\not{p}) \in \underline{\underline{M}}(\mathscr{E}_{\not{p}}^{(0)}(R))$ a contradiction. Hence $N \in \underline{\underline{M}}(R_0)$.

Proof of Theorem 4.9. i) Obviously it suffices to show that every element of $\mathscr{G}(J)$ is involutive. So let $\not{p}_0 \in \mathscr{G}(J)$. Put $\not{p} = gr(R)\not{p}_0$. By Lemma 4.10 the rings $gr_0(R)_{\not{p}_0}$ and $gr_0(\mathscr{E}_{\not{p}}(R))$ are isomorphic. We identify them. The Poisson product on $gr_0(R)$ can be extended to $gr_0(R)_{\not{p}_0}$ and equals the Poisson product on $gr_0(\mathscr{E}_{\not{p}}(R))$. If we can show that $\not{p}_0 gr_0(R)_{\not{p}_0}$ is involutive in $gr_0(R)_{\not{p}_0}$ it readily follows that the contracted ideal in $gr_0(R)$ i.e. $\not{p}_0$ is involutive.

Let $n_0 \in N$ be such that $J = J(n)$ for all $n \geq n_0$. By Lemma 4.10 $J(n)_{\not{p}_0} = J(Q(n, N(\not{p})))$ whence $J(Q(n, N(\not{p})) = J_{\not{p}_0} = \not{p}_0 gr_0(R)_{\not{p}_0}$. Therefore we may replace the triple $(N, M, (M_n)_n)$ by $(N(\not{p}), \mathscr{E}_{\not{p}}(M), (\mathscr{E}_{\not{p}}^{(n)}(M))_n)$ and we are reduced to a micro-local case. However since $\not{p}_0 \in \mathscr{G}(J(n))$ if $n \geq n_0$, $Q(n, N)_{\not{p}_0}$ is a $gr_0(R)_{\not{p}_0}$-module of finite length so $Q(n, N(\not{p}))$ is a $gr_0(\mathscr{E}_{\not{p}}(R))$-module of finite length. Hence we can assume:

ii) $Q(n, N)$ is a $gr_0(R)$-module of finite length for all $n \geq n_0$. Then Lemma 4.8 implies: there exists $n_1 \geq n_0$ such that for every $n \geq n_1$ the left multiplication by s^{-1} gives an isomorphism from $Q(n+1, N)$ onto $Q(n, N)$. So $N(n) = s^{-1}N(n+1) + N(n-1)$ (where $N(k) := M_k \cap N$ for all $k \in Z$). Put $A := R_0/R_{-2}$, $u := s^{-1} + R_{-2} \in A$, $M' := N(n+1)/N(n-1)$. Then u is a central element in A with $u^2 = 0$, $A/uA \simeq R_0/R_{-1}$, $Q(n+1, N) \simeq M'/uM'$ and $uM' = \mathrm{Ker}_u M'$. Finally $uA = \mathrm{Ker}_u A$, whence $J = J(n+1) = J(M'/uM')$ is involutive by [6], Theorem II.

§5. Modules with Regular Singularities over E-Rings

In this section R denotes an E-ring, s its special element (Definition 4.1 i)) and $v = \sigma(s)$ which is a unit in $gr(R)$ with $\sigma(s^{-1})$ as inverse (Lemma 4.2 i)). By Corollary 4.5 R and R_0 are left noetherian. If $r \in R$ put $\sigma_1(r) = r + R_0 \in gr_1(R)$ (cf. §1). Furthermore, I always (in this section) denotes a homogeneous involutive radical ideal in $gr(R)$. Put

$$\mathscr{g}(I) := \{\tau \in R_1 \mid \sigma_1(\tau) \in I\},$$

$$R(I) := \textit{the subring of } R \textit{ generated by } \mathscr{g}(I) \textit{ over } R_0.$$

Lemma 5.1. *$\mathscr{g}(I)$ is an R_0-module of finite type and a Lie-algebra.*

Proof. It is easy to verify that $\mathscr{g}(I)$ is an R_0-submodule of $R_1 = R_0 s$. Since R_0 is left noetherian $\mathscr{g}(I) \in \underline{\underline{M}}(R_0)$. Let $\tau, \tau' \in \mathscr{g}(I)$. Put $[\tau, \tau'] := \tau\tau' - \tau'\tau$. Then $[\tau, \tau'] \in R_1$. If $[\tau, \tau'] \in R_0$ then $\sigma_1([\tau, \tau']) = 0$. If $[\tau, \tau'] \in R_1 \backslash R_0$ then $\sigma_1([\tau, \tau']) = \{\sigma(\tau), \sigma(\tau')\} \in I$ since I is involutive, which proves the lemma.

Definition 5.2. *Let $M \in \underline{\underline{M}}(R)$. We say that M has regular singularities along I (M has R.S. along I) if there exists an R_0-submodule M_0 of M of finite type such that $RM_0 = M$ and $\mathscr{g}(I) M_0 \subset M_0$.*

Proposition 5.3. *Let $M \in \underline{\underline{M}}(R)$. There is equivalence between*

i) *M has regular singularities along I.*
ii) *If N is an R_0-submodule of M of finite type, then $R(I)N \in \underline{\underline{M}}(R)$.*
iii) *If N is an $R(I)$-submodule of finite type of M, then $N \in \underline{\underline{M}}(R_0)$.*
iv) *$E_\tau(R_0 m) := \sum_{i=0}^{\infty} R_0 \tau^i m \in \underline{\underline{M}}(R_0)$ for all $\tau \in \mathscr{g}(I)$, all $m \in M$.*

Proof. i)→ii). Since $N \in \underline{\underline{M}}(R_0)$ and $RM_0 = M$ there exists $k \in N$ with $N \subset R_k M_0$. Let $\tau \in \mathscr{g}(I)$, $r \in R_k$, $m \in M_0$. Then $\tau r m = r\tau m + [\tau, r] m \in R_k M_0$ since $\tau m \in M_0$ and $[\tau, r] \in R_k$. So $R(I) R_k M_0 \subset R_k M_0$, implying $R(I)N \subset R_k M_0 = s^k M_0 \in \underline{\underline{M}}(R_0)$. Hence $R(I)N \in \underline{\underline{M}}(R_0)$ since R_0 is left noetherian.

ii)→iii). Let $N := \sum_{i=1}^{q} R(I) n_i$. Put $N_0 := \sum R_0 n_i$. Then $N = R(I) N_0$ and we can apply ii).

iii)→iv). Since $E_\tau(R_0 m) \subset R(I) m \in \underline{\underline{M}}(R_0)$ (by iii)) we get iv).

iv)→i). By Lemma 5.1 $\mathscr{g}(I) = \sum_{i=1}^{d} R_0 \tau_i$ for some $\tau_i \in \mathscr{g}(I)$. Let $m \in M$. By iv) there exists $k \in N$ with

$$\tau_i^k m \in \sum_{j=0}^{k-1} R_0 \tau_i^j m, \quad \text{all} \quad 1 \leq i \leq d.$$

Since by Lemma 5.1 $\mathscr{g}(I)$ is a Lie-algebra it follows that $R(I)m$ is generated as an R_0-module by the elements $\tau_1^{i_1} \cdots \tau_d^{i_d} m$ with $0 \leq j \leq k-1$, all $j \in \{1, 2, \dots, d\}$. So $R(I)m \in \underline{\underline{M}}(R_0)$ all $m \in M$. Since $M \in \underline{\underline{M}}(R)$, say $M = \sum_{i=1}^{t} R m_i$ we put $M_0 := \sum_{i=1}^{t} R(I) m_i$. Then M_0 has the properties of Definition 5.2, which concludes the proof.

Let $M \in \underline{\underline{M}}(R)$ and let $\mathscr{F}$, denote the filtration map introduced in §4 (cf. Lemma 4.2).

Proposition 5.4. *The map $\mathscr{F}$ restricted to the set of R_0-submodules of M satisfying the conditions of Definition 5.2, gives a one-to-one correspondence with the I-good filtrations on M. In particular the very good filtrations on M correspond one-to-one with the R_0-submodules of M satisfying the conditions of Definition 5.2 with $I=J(M)$.*

Proof. Let M_0 be as in Definition 5.2. Put $F:=\mathscr{F}(M_0)$. Then F is good on M (Lemma 4.2). It remains to prove that $I\subset Ann\, gr^F(M)$. Let $\sigma(r)\in I$ with $v(r)=k\in Z$. Then $s^{-(k-1)}r\in \mathscr{J}(I)$, so $s^{-(k-1)}rM_0\subset M_0$ i.e. $rM_0\subset R_{k-1}M_0$. Hence $rR_nM_0\subset R_{n+k-1}M_0$ all $n\in Z$ (since $ram=arm+[r,a]m$, all $a\in R_n$, $m\in M_0$). So $\sigma(r)gr_n(M)=0$ all $n\in Z$. Consequently $I\subset Ann\, gr(M)$, since I is homogeneous. Since $\mathscr{F}$ is injective it suffices to show that $\mathscr{F}$ is surjective. Let F' be an I-good filtration on M. By Lemma 4.2 $F'=\mathscr{F}(M_0)$ for some R_0-submodule M_0 of M satisfying $M_0\in \underline{\underline{M}}(R_0)$ and $RM_0=M$. It remains to prove $\mathscr{J}(I)M_0\subset M_0$. Let $r\in R_1$ with $\sigma_1(r)\in I$. We must show $rM_0\subset M_0$. If $r\in R_0$ we are done. So assume $r\notin R_0$. Then $\sigma(r)\in gr_1(R)$ and $\sigma(r)=\sigma_1(r)$ $\in I\subset Ann\, gr^{F'}(M_0)$. In particular $\sigma(r)M_0/R_{-1}M_0=0$ in R_1M_0/M_0 i.e. rM_0 $\subset M_0$ as desired.

Corollary 5.5. *Let $M\in\underline{\underline{M}}(R)$. Then M has R.S. along I iff M possesses an I-good filtration.*

Definition 5.6. *Let $M\in\underline{\underline{M}}(R)$. We say that M has regular singularities (M has R.S.) if M possesses a very good filtration.*

So by Corollary 5.5 M has R.S. iff M has R.S. along $J(M)$.

Since $\sigma(s)$ is a unit in $gr(R)$ $\sigma(s)\notin \mathfrak{p}$ for all $\mathfrak{p}\in Spec\,(gr(R))$ i.e. $s\in S_{\mathfrak{p}}$ all $\mathfrak{p}$. Since $s\in R_1\backslash R_0$ this means that $Spec\,(gr(R))=Spec^{\circ}(gr(R))$ (see Definition 2.21). So by Corollary 2.23 $\mathscr{E}_{\mathfrak{p}}(R)$ is an E-ring for every $\mathfrak{p}\in Spec\,(gr(R))$. The main result of this section is

Theorem 5.7. *Let M be a holonomic R-module. There is equivalence between*

i) *M is an R-module with R.S.*

ii) *$\mathscr{E}_{\mathfrak{p}}(M)$ is an $\mathscr{E}_{\mathfrak{p}}(R)$-module with R.S. for all $\mathfrak{p}\in\mathscr{G}(J(M))$.*

iii) *$\mathscr{E}_{\mathfrak{p}}(M)$ is an $\mathscr{E}_{\mathfrak{p}}(R)$-module with R.S. for all $\mathfrak{p}\in Spec\,(gr(R))$.*

Proof. i)→iii). Apply Proposition 2.9 with $S:=S_{\mathfrak{p}}$. iii)→ii) is obvious. So it remains to prove ii)→i). Let $m\in M$ and $\tau\in\mathscr{J}(J(M))$. By Proposition 5.3 it suffices to show that $N:=\sum R_0\tau^i m\in\underline{\underline{M}}(R_0)$. We want to apply Proposition

4.12. So let $\mathfrak{p} \in \mathscr{G}(J(M))$. Then $N(\mathfrak{p}) = \sum \mathscr{E}_{\mathfrak{p}}^{(0)} \phi_{\mathfrak{p}}(\tau)^i \phi_{\mathfrak{p}}(m)$. Since $\phi_{\mathfrak{p}}(\tau) \in \mathfrak{g}(J(\mathscr{E}_{\mathfrak{p}}(M)))$ by (2.10) the hypothesis and Proposition 5.3 imply $N(\mathfrak{p}) \in \underline{\underline{M}}(\mathscr{E}_{\mathfrak{p}}^{(0)}(R))$. So $N \in \underline{\underline{M}}(R_0)$ by Proposition 4.12, as desired.

Remark 5.8. The assumption M is holonomic in Theorem 5.7 is only used to prove the implication ii)→i).

5.9. Some Consequences of Theorem 5.7.

From Corollary 5.5 and Definition 5.6 we deduce

(5.10) *If M has R.S. along I, then $I \subset J(M)$.*

(5.11) *If $I \subset J(M)$ and M has R.S., then M has R.S. along I.*

Proposition 5.12. *Let M be holonomic and* $\mathrm{ht} I = \mu_R$. *Then M has R.S. along I iff M has R.S. and $I \subset J(M)$.*

Proof. "if" follows from (5.11). Conversely, let M have R.S. along I. So $I \subset J(M)$ by (5.10). Since $\mathrm{ht} I = \mu_R$, $\mathrm{ht}\,\mathfrak{q} \geq \mu_R$ for all $\mathfrak{q} \in \mathscr{G}(I)$. Let $\mathfrak{p} \in \mathscr{G}(J(M))$ then $\mathfrak{p} \supset \mathfrak{q}$ for some $\mathfrak{q} \in \mathscr{G}(I)$. So $\mathrm{ht}\,\mathfrak{q} \leq \mathrm{ht}\,\mathfrak{p} = \mu_R$ i.e. $\mathrm{ht}\,\mathfrak{q} = \mu_R$ implying $\mathfrak{p} = \mathfrak{q} \in \mathscr{G}(I)$. By Proposition 2.9 and Corollary 5.5 $\mathscr{E}_{\mathfrak{p}}(M)$ has R.S. along $\psi_R(\sigma(S_{\mathfrak{p}})^{-1} I) = \psi_R(\sigma(S_{\mathfrak{p}})^{-1} \mathfrak{p}) = \psi_R(\sigma(S_{\mathfrak{p}})^{-1} J(M)) = J(\mathscr{E}_{\mathfrak{p}}(M))$ (by (2.10)). So M has R.S. by Theorem 5.7 ii)→i).

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules of finite type and $\tau \in R_1$. Then (cf. Proposition 5.3) $E_\tau(R_0 m) \in \underline{\underline{M}}(R_0)$ all $m \in M$ iff $E_\tau(R_0 m') \in \underline{\underline{M}}(R_0)$ and $E_\tau(R_0 m'') \in \underline{\underline{M}}(R_0)$ all $m' \in M'$, all $m'' \in M''$ (left to the reader). Consequently, using Proposition 5.3 iv) we obtain.

Lemma 5.14. *M has R.S. along I iff M' and M'' have R.S. along I.*

Corollary 5.15. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of holonomic R-modules. Then M has R.S. iff M' and M'' have R.S.*

Proof. Observe $J(M) = J(M') \cap J(M'')$. Then apply Proposition 5.12 and Lemma 5.14 with $I = J(M)$.

§ 6. A Dictionary between E-Rings and Noetherian Filtered Rings

Let A be a filtered ring with filtration FA and order function v. The ring of polynomials $A[X]$ can be made into a filtered ring with filtration $FA[X]$ and order function v_X by putting

$$v_X(\sum a_i X^i) = \sup_i v(a_i) + i \quad \text{and} \quad FA[X](n) := \{a(X) \in A[X] \mid v_X(a(X)) \leq n\}.$$

Similarly starting from an A-module M we can consider the module of polynomials $M[X]$. It is easily checked that $M[X]$ is an $A[X]$-module isomorphic to $A[X] \otimes_A M$. If FM is a filtration on M with order function v, we can define a filtration $FM[X]$ on M with order function v_X by

$$v_X(\sum m_i X^i) = \sup v(m_i) + i \quad \text{and} \quad FM[X](n) := \{m(X) \mid v_X(m(X)) \leq n\}.$$

Write v instead of v_X. Then obviously

$$(6.1) \qquad v(X^n m(X)) = v(X^n) v(m(X)), \quad \text{all} \quad n \in \bar{N}, \quad \text{all} \quad m(X) \in M[X].$$

Let $i\colon M \to M[X]$ be the inclusion map. Since $i(F_n M) \subset F_n M[X]$ we have the induced map $\bar{\imath}\colon gr(M) \to gr(M[X])$. Obviously $\bar{\imath}$ is injective. Put $\bar{X} := X + F_0 A[X]$. Let $gr(M)[Y]$ be the external homogenization introduced in 9.11. Extend $\bar{\imath}$ to a map $\bar{\imath}\colon gr(M)[Y] \to gr(M[X])$ by putting $\bar{\imath}(\sum \tilde{m}_j Y^j) = \sum_j \tilde{m}_j \bar{X}^j$, $\tilde{m}_j \in gr(M)$ all j.

Lemma 6.2. *i) $\bar{\imath}\colon gr(A)[Y] \to gr(A[X])$ is an isomorphism of graded rings.*
ii) $\bar{\imath}\colon gr(M)[Y] \to gr(M[X])$ is an isomorphism of $gr(A)[Y]$-modules.

Proof. Left to the reader.

From now on we identify $gr(M)[Y]$ with $gr(M[X])$ by $\bar{\imath}$. So we write $gr(M[X]) = gr(M)[\bar{X}]$. Instead of $F_n A$ and $F_n A[X]$ we write $A(n)$ resp. $A[X](n)$.

Let I be an ideal in $gr(A)$. Then $I^e := gr(A)[\bar{X}]I$ in $gr(A)[\bar{X}]$.

Lemma 6.3. *If FM is I-good on M, then $FM[X]$ is I^e-good on $M[X]$. If FM is very good, then $FM[X]$ is very good.*

Proof. In particular FM is good on M. So $F_n M = \sum A(n - v_i) m_i$ for some $v_i \in Z$, $m_i \in M$ all $n \in Z$. We claim: $F_n M[X] = \sum A[X](n - v_i) m_i$. Obviously "$\supset$" holds. Conversely if $m \in M$ and $mX^j \in F_n M[X]$ then $v(m) + j \leq n$ i.e. $m \in F_{n-j} M = \sum A(n - j - v_i) m_i$. So $mX^j \in \sum A[X](n - v_i) m_i$. Since all elements of $F_n M[X]$ are finite sums of elements of the form mX^j with $v(m) + j \leq n$ it follows that $FM[X]$ is good. By Lemma 9.5 v) and vi) $Ann\, gr(M[X]) = (Ann\, gr(M))^e$ and $J(M[X]) = J(M)^e$. So $I \subset Ann\, gr(M)$ implies $I^e \subset Ann\, gr(M[X])$ i.e. $FM[X]$ is I^e good. Finally $J(M[X]) = J(M)^e$ implies that $FM[X]$ is very good if FM is very good.

Now we introduce the main objects of this section. Put $S = \{X^n | n \in \bar{N}\}$. Then S is a multiplicatively closed subset of $A[X]$ satisfying the conditions of

Theorem 2.1. We put

$$\mathscr{E}_X := \mathscr{E}_X(A[X]) := \mathscr{E}_S(A[X], FA[X]).$$

Instead of $\mathscr{E}_S^{(n)}(A[X], FA[X])$ we write $\mathscr{E}_X^{(n)}$ and ϕ_S we denote by ϕ_X. Since $|\phi_X(X)| = |X| = 2^1$ and $|\phi_X(X)^{-1}| = |X|^{-1} = 2^{-1}$ we get

(6.4) $s := \phi_X(X)$ satisfies the conditions of Definition 4.1.

Corollary 6.5. *If $gr(A)$ is a commutative noetherian Q-algebra, then $\mathscr{E}_X$ is an E-ring.*

Proof. By Lemma 6.2 and Theorem 2.6 $gr(\mathscr{E}_X) \simeq gr(A)[\bar{X}, \bar{X}^{-1}]$. Consequently since $gr(A)$ is a commutative noetherian Q-algebra, so is $gr(\mathscr{E}_X)$. By Proposition 1.10 it follows that $F\mathscr{E}_X$ is noetherian.

If M is an A-module with filtration FM we similarly have the $\mathscr{E}_X$-module $\mathscr{E}_X(M[X], FM[X])$ with filtration $\mathscr{E}_X^{(n)}(M[X], FM[X]))_{n \in Z}$. From this filtration we can recover the filtration FM as follows. Put $j := \phi_X \circ i$ where i: $M \to M[X]$ is the inclusion map and ϕ_X: $M[X] \to \mathscr{E}_X(M[X], FM[X])$ the canonical map.

Lemma 6.6. $F_n M = j^{-1}(\mathscr{E}_X^{(n)}(M[X], FM[X]))$, all $n \in Z$.

Proof. "$\subset$" is obvious. Conversely, let $n \in Z$, $m \in M$ and suppose $j(m) \in \mathscr{E}_X^{(n)}(M[X], FM[X])$. Then $|\phi_X(m)| \leq 2^n$. So by (6.1) and (2.5) $|m| = |m|_X = |\phi_X(m)| \leq 2^n$ i.e. $m \in F_n M$.

6.7. Filtrations on M and $\mathscr{E}_X$ $(M[X])$

From now on we assume: *$gr(A)$ is commutative.*

Let $M \in \underline{\underline{M}}(A)$ and FM a good filtration on M. Then $FM[X]$ is good on $M[X]$ (by Lemma 6.3). Hence $\mathscr{E}_X(M[X], FM[X])$ does not depend on the choice of the good filtration FM on M (by (2.7)). We denote this $\mathscr{E}_X$-module by M_X or $\mathscr{E}_X(M[X])$. However, the filtration

$$\mathscr{L}(FM) := (\mathscr{E}_X(M[X], FM[X]))_{n \in Z}$$

on M_X does depend on FM. With the notations of (2.7) we have $\mathscr{L}(FM) = L(FM[X])$. So by Proposition 2.9 and Lemma 6.3 $\mathscr{L}(FM)$ is good. Hence $M_X \in \underline{\underline{M}}(\mathscr{E}_X)$. Conversely, let F be a good filtration on M_X. Put

$$\mathscr{G}(F) := (j^{-1}(F_n))_{n \in Z}$$

which is a filtration on M. So we have maps $\mathscr{L}$ resp. $\mathscr{G}$ going from good filtrations on M resp. M_X to filtrations on M_X resp. M. Let I be an ideal in $gr(A)$, J an ideal in $gr(\mathscr{E}_X)$ and $\bar{j}=gr(j)$. Put $I^e:=gr(\mathscr{E}_X)\bar{j}(I)$ and $J^c:=\bar{j}^{-1}(J)$.

Proposition 6.8. *i) If FM is I-good on M, then $\mathscr{L}(FM)$ is I^e-good on M_X.*

ii) Suppose FA is noetherian. If F is J-good on M_X, then $\mathscr{G}(F)$ is J^c-good on M. The same conclusions hold for every good filtrations.

Proof. i) follows From Lemma 6.3 and Proposition 2.9 with $S=\{X^n | n\in\bar{N}\}$ since $\mathscr{L}(FM)=L(F(M[X])$.

ii) Since F is good on M_X, $F=\mathscr{F}(\mathscr{M}_0)$ where $\mathscr{M}_0$ is an $\mathscr{E}_X^{(0)}$-submodule of M_X of finite type with $\mathscr{E}_X\mathscr{M}_0=M_X$ (Lemma 4.2). Choose a good filtration $F'M$ on M and put $\mathscr{F}'_0:=\mathscr{L}_0(F'M)$. By i) and Lemma 4.2 $\mathscr{M}'_0\in\underline{\underline{M}}(\mathscr{E}_X^{(0)})$ and $\mathscr{E}_XM'_0=M_X$. Since these relations also hold for $\mathscr{M}_0$ we deduce: there exists $c\in N$ with $\mathscr{E}_X^{(-c)}\mathscr{M}'_0\subset\mathscr{M}_0\subset\mathscr{E}_X^{(c)}\mathscr{M}'_0$. Consequently $j^{-1}(\mathscr{E}_X^{(n-c)}\mathscr{M}'_0)\subset j^{-1}\cdot(\mathscr{E}_X^{(n)}\mathscr{M}_0)\subset j^{-1}(\mathscr{E}_X^{(n+c)}\mathscr{M}'_0)$, all $n\in Z$. Then Lemma 6.6 implies $F'_{n-c}M\subset\mathscr{G}_n(F)\subset F'_{n+c}M$, all $n\in Z$. So $\mathscr{G}(F)$ is good by Theorem 1.11. Let $\sigma(a)\in J^c$ with $v(a)=k$ and $m\in\mathscr{G}_n(F)$. Then $\sigma_k(\phi_X(a))\in J$ and $\phi_X(m)\in F_n$. So $\phi_X(am)=\phi_X(a)\phi_X(m)\in F_{n+k-1}$ (since $J\subset Ann\, gr^F(M_X)$) i.e. $am\in\mathscr{G}_{n+k-1}(F)$. Hence $J^c\subset Ann\, gr^{\mathscr{G}(F)}(M)$. Finally by (2.10) and Lemma 9.5 vi) $J(M)^e=J(M_X)$ and $J(M_X)^c=J(M)$ (Lemma 9.5 ii)) which proves the last part of Proposition 6.8.

6.9. Holonomic A-Modules

Let A, FA satisfy the conditions a) and b) of §3, and let $0\neq M\in\underline{\underline{M}}(A)$. As observed before $J(M_X)=\psi_X(gr(A)[\bar{X},\bar{X}^{-1}]J(M))$. So Corollary 9.7 implies that $\mathscr{G}(J(M_X))$ consists of the set of prime ideals $\mathfrak{p}^e$ $(=\psi_A(gr(A)[\bar{X},\bar{X}^{-1}]\mathfrak{p}))$ where $\mathfrak{p}$ runs through the set $\mathscr{G}(J(M))$. Let $d\in\bar{N}$. Since $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{p}^e$ we derive $\mathrm{ht}\,\mathfrak{p}=d$ all $\mathfrak{p}\in\mathscr{G}(J(M))$ iff $\mathrm{ht}\,\mathfrak{q}=d$, all $\mathfrak{q}\in\mathscr{G}(J(M_X))$. Applying this with $d=\mu_A$ we get: M is holonomic iff $\mathrm{ht}\,\mathfrak{q}=\mu_A$ for all $\mathfrak{q}\in\mathscr{G}(J(M_X))$. From this we derive

Corollary 6.10. *If $\mu_{\mathscr{E}_X}=\mu_A$ then M is holonomic iff M_X is holonomic.*

To investigate when the condition $\mu_A=\mu_{\mathscr{E}_X}$ is satisfied we put

$$\nu_A:=\sup_{\mathfrak{p}\in\mathscr{J}}\mathrm{ht}\,\mathfrak{p}$$

where $\mathscr{J}$ is the set of involutive prime ideals in $gr(A)$.

Proposition 6.11. *$\mu_{\mathscr{E}_X}=\mu_A$ iff $\mu_A=\nu_A$.*

Proof. Let $\mathfrak{p}$ be an involutive homogeneous prime ideal in $gr(A)$. Then by Proposition 1.19 and Proposition 9.9 $\mathfrak{p}^e$ is an involutive and homogeneous prime ideal in $gr(\mathscr{E}_X(A[X]))=gr(A)[\bar{X}, \bar{X}^{-1}]$. Since by [11], Theorem 1.9, p. 79 and Corollary 9.3 i) $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{p}^e$ we get $\mu_A \le \mu_{\mathscr{E}_X}$.

i) Suppose $\mu_A=\nu_A$. It remains to prove $\mu_{\mathscr{E}_X} \le \mu_A$. By Proposition 1.19 every involutive homogeneous prime ideal in $gr(\mathscr{E}_X(A[X]))=gr(A)[\bar{X}, \bar{X}^{-1}]$ is of the form $\mathfrak{q}^e=gr(A)[\bar{X}, \bar{X}^{-1}]_{\mathfrak{q}}$ where $\mathfrak{q}$ is a homogeneous involutive prime ideal in $gr(A)[\bar{X}]$ with $\bar{X} \notin \mathfrak{q}$. By Corollary 9.3 i) $\mathrm{ht}\,\mathfrak{q}^e=\mathrm{ht}\,\mathfrak{q}$. By Proposition 9.12 and Proposition 9.18 $\mathfrak{q}_*$ is an involutive prime ideal in $gr(A)$. So $\mathrm{ht}\,\mathfrak{q}_* \le \nu_A$. Hence Corollary 9.15 implies that $\mathrm{ht}\,\mathfrak{q}=\mathrm{ht}\,\mathfrak{q}_* \le \nu_A=\mu_A$, whence $\mathrm{ht}\,\mathfrak{q}^e \le \mu_A$. So $\mu_{\mathscr{E}_X} \le \mu_A$.

ii) Suppose $\mu_{\mathscr{E}_X}=\mu_A$. We must show $\nu_A \le \mu_A$ since obviously $\mu_A \le \nu_A$. So let $\mathfrak{p}$ be an involutive prime ideal in $gr(A)$, say $\mathrm{ht}\,\mathfrak{p}=n$. By Proposition 9.12 and Proposition 9.18 $\mathfrak{p}^*$ is an involutive prime ideal in $gr(A)[\bar{X}]$ and $\bar{X} \notin \mathfrak{p}^*$. Furthermore $\mathrm{ht}\,\mathfrak{p}^*=\mathrm{ht}\,\mathfrak{p}=n$ by Corollary 9.15. The hypothesis $\mu_{\mathscr{E}_X} \le \mu_A$ implies that there exists an involutive homogeneous prime ideal $\mathfrak{q}$ in $gr(A)$ with $n=\mathrm{ht}\,\mathfrak{p}^* \le \mathrm{ht}\,\mathfrak{q}$. So $\mathrm{ht}\,\mathfrak{p} \le \mathrm{ht}\,\mathfrak{q}$. Consequently $\nu_A \le \mu_A$ as desired.

Corollary 6.12. *Let $\mu_A=\nu_A$. Then M is holonomic iff M_X is holonomic.*

Proof. Apply Corollary 6.10 and Proposition 6.11.

6.14. A Special Result

To conclude this section we give a result which will be used in §7 to prove the main result of this paper. By Theorem 2.6 $gr(\mathscr{E}_S(A))$ and $\sigma(S)^{-1}gr(A)$ are isomorphic graded rings. We identify these rings. So we write $gr(\mathscr{E}_S(A)) =\sigma(S)^{-1}gr(A)$. Let $\mathfrak{q} \in Spec\,(gr(A))$. Put $\mathfrak{q}^e:=gr(A)[\bar{X}, \bar{X}^{-1}]_{\mathfrak{q}}$ in $gr(A)[\bar{X}, \bar{X}^{-1}]$ $(=gr(\mathscr{E}_X(A[X]))$ and $\tilde{\mathfrak{q}}:=\sigma(S_{\mathfrak{q}})^{-1}gr(A)[\bar{X}, \bar{X}^{-1}]_{\mathfrak{q}}$ in $\sigma(S_{\mathfrak{q}})^{-1}gr(A)\cdot[\bar{X}, \bar{X}^{-1}]$ $(=gr(\mathscr{E}_X(\mathscr{E}_{\mathfrak{q}}(A)[X])))$.

Lemma 6.15. *i) There exists an isomorphism of filtered rings γ from $\mathscr{E}_{\mathfrak{q}^e}(\mathscr{E}_X(A[X]))$ onto $\mathscr{E}_{\tilde{\mathfrak{q}}}(\mathscr{E}_X(\mathscr{E}_{\mathfrak{q}}(A)[X]))$.*

ii) Let $M \in \underline{\underline{M}}(A)$. There exists an isomorphism of filtered $\mathscr{E}_{\mathfrak{q}^e}(\mathscr{E}_X(A[X]))$-modules γ from $\mathscr{E}_{\mathfrak{q}^e}(\mathscr{E}_X(M[X]))$ onto $\mathscr{E}_{\tilde{\mathfrak{q}}}(\mathscr{E}_X(\mathscr{E}_{\mathfrak{q}}(M)[X]))$.

Proof. Let $\phi_{\mathfrak{q}}: A \to \mathscr{E}_{\mathfrak{q}}(A)$ be the canonical map and $\tilde{\phi}_{\mathfrak{q}}$ its obvious extension $A[X] \to \mathscr{E}_{\mathfrak{q}}(A)[X]$ with $\tilde{\phi}_{\mathfrak{q}}(X)=X$. Let $\phi_{\mathfrak{q},X}: \mathscr{E}_{\mathfrak{q}}(A)[X] \to \mathscr{E}_X (\mathscr{E}_{\mathfrak{q}}(A)[X])$ be the canonical map. Applying Theorem 2.1 iii) to the morphism $h: A[X] \to \mathscr{E}_X(\mathscr{E}_{\mathfrak{q}}(A)[X])$ defined by $h=\phi_{\mathfrak{q},X} \circ \tilde{\phi}_{\mathfrak{q}}$ we obtain a morphism of filtered rings $\rho: \mathscr{E}_X(A[X]) \to \mathscr{E}_X(\mathscr{E}_{\mathscr{g}}(A)[X])$. Take $\phi=\rho$ in Corollary 2.18 and $\mathscr{p}:=\mathscr{g}^e$ in $gr(\mathscr{E}_X(A[X]))=gr(A)[\bar{X}, \bar{X}^{-1}]$. Then $\mathscr{p}^e=\tilde{\mathscr{g}}$ and $(\mathscr{p}^e)^c=\mathscr{p}$. So by Corollary 2.18 we obtain a morphism $\tilde{\rho}: \mathscr{E}_{\mathscr{g}^e}(\mathscr{E}_X(A[X])) \to \mathscr{E}_{\tilde{\mathscr{g}}}(\mathscr{E}_X(\mathscr{E}_{\mathscr{g}}(A)\cdot [X]))$. It is left to the reader to verify that $gr(\tilde{\rho})$ is a bijection between the associated graded rings. Consequently $\tilde{\rho}$ is an isomorphism of filtered rings by Corollary 1.14, which proves i). ii) By $\tilde{\rho}$ constructed in i) $\mathscr{E}_{\tilde{\mathscr{g}}}\cdot (\mathscr{E}_X(\mathscr{E}_{\mathscr{g}}(M)[X]))$ becomes a left $\mathscr{E}_{\mathscr{g}^e}(\mathscr{E}_X(A[X]))$-module. Then arguing as in i) Corollary 2.19 gives a morphism $\gamma: \mathscr{E}_{\mathscr{g}^e}(\mathscr{E}_X(M[X])) \to \mathscr{E}_{\tilde{\mathscr{g}}}(\mathscr{E}_X(\mathscr{E}_{\mathscr{g}}(M)[X]))$ which is in fact an isomorphism, using Corollary 1.14 again.

§ 7. Modules with Regular Singularities over Filtered Rings

In this section A denotes a filtered ring with filtration FA satisfying

a) *$gr(A)$ is a commutative $\mathbb{Q}$-algebra.*

b) *FA is noetherian.*

c) $\mu_A = \nu_A$.

Furtheremore I (resp. J) is an involutive homogeneous radical ideal in $gr(A)$ (in $gr(\mathscr{E}_X)$) and $M \in \underline{\underline{M}}(A)$.

Definition 7.1. *We say that M has regular singularities along I (M has R.S. along I) if M possesses an I-good filtration. We say that M has regular singularities (M has R.S.) if M possesses a very good filtration.*

Proposition 7.2. *If M has R.S. along I then M_X has R.S. along I^e. If M_X has R.S. along J, then M has R.S. along J^c and M has R.S. iff M_X has R.S. (as an $\mathscr{E}_X$-module).*

Proof. Apply Proposition 6.8.

The main result of this paper is

Theorem 7.3. *Let M be a holonomic A-module. There is equivalence between*

i) *M has R.S.*

ii) *$\mathscr{E}_{\mathscr{p}}(M)$ is an $\mathscr{E}_{\mathscr{p}}(A)$-module with R.S. for all $\mathscr{p} \in \mathscr{G}(J(M))$.*

iii) *$\mathscr{E}_{\mathscr{p}}(M)$ is an $\mathscr{E}_{\mathscr{p}}(A)$-module with R.S. for all $\mathscr{p} \in Spec(gr(A))$.*

Proof. i)→iii) follows from Proposition 2.9 with $S:=S_{\mathscr{p}}$. iii)→ii) is obvious. So it remains to prove ii)→i). Let $\mathscr{p} \in \mathscr{G}(J(M))$. Then the hypothesis and Proposition 7.2 give $\mathscr{E}_X(\mathscr{E}_{\mathfrak{p}}(M)[X])$ is an $\mathscr{E}_X(\mathscr{E}_{\mathfrak{p}}(A)[X])$-module with R.S. Then Theorem 5.7 i)→iii) implies that $\mathscr{E}_{\tilde{\mathfrak{p}}}(\mathscr{E}_X(\mathscr{E}_{\mathfrak{p}}(M)[X]))$ is an $\mathscr{E}_{\tilde{\mathfrak{p}}}(\mathscr{E}_X(\mathscr{E}_{\mathfrak{p}}(A)[X]))$-module with R.S., where $\tilde{\mathfrak{p}}$ is as in Lemma 6.15. So by Lemma 6.15 we find $\mathscr{E}_{\mathfrak{p}^e}(\mathscr{E}_X(M[X]))$ is an $\mathscr{E}_{\mathfrak{p}^e}(\mathscr{E}_X(A[X]))$-module with R.S. for all $\mathfrak{p} \in \mathscr{G}(J(M))$, with $\mathfrak{p}^e$ as above. As observed in 6.9 the minimal components of $J(M_X)$ are all of the form $\mathfrak{p}^e$ with $\mathfrak{p} \in \mathscr{G}(J(M))$. Finally by Corollary 6.12 M_X is a holonomic $\mathscr{E}_X$-module. So we can apply Proposition 5.7 ii)→i) to the E-ring $R := \mathscr{E}_X$. Hence M_X is an $\mathscr{E}_X$-module with R.S. which implies i) using Proposition 7.2.

Obviously Definition 7.1 implies

(7.4) *If M has R.S. along I then $I \subset J(M)$.*

(7.5) *If $I \subset J(M)$ and M has R.S. then M has R.S. along I.*

Proposition 7.6. *Let* $\mathrm{ht} I = \mu_A$. *There is equivalence between*

i) M has R.S. along I.

ii) M is holonomic with R.S. and $I \subset J(M)$.

Proof. ii)→i) follows from (7.5). Conversely assume i). Then $I \subset J(M)$ by (7.4). Let $\mathfrak{p} \in \mathscr{G}(J(M))$, then $\mathfrak{p} \supset \mathfrak{q}$ for some $\mathfrak{q} \in \mathscr{G}(I)$. Hence $\mathrm{ht}\,\mathfrak{p} \geq ht\,\mathfrak{q} \geq \mu_A$ (since $\mathrm{ht} I = \mu_A$). Since $\mathfrak{p} \in \mathscr{G}(J(M))$ $\mathrm{ht}\,\mathfrak{p} \leq \mu_A$ (see §3). So $\mu_A = \mathrm{ht}\,\mathfrak{p} = \mathrm{ht}\,\mathfrak{q}$ hence $\mathfrak{p} = \mathfrak{q} \in \mathscr{G}(I)$. Consequently M is holonomic and $\mathscr{G}(J(M)) \subset \mathscr{G}(I)$. By Proposition 2.9 $\mathscr{E}_{\mathfrak{p}}(M)$ has R.S. along $\psi_A(\sigma(S_{\mathfrak{p}})^{-1}I) = \psi_A(\sigma(S_{\mathfrak{p}})^{-1}\mathfrak{p}) = \psi_A(\sigma(S_{\mathfrak{p}})^{-1}J(M)) = J(\mathscr{E}_{\mathfrak{p}}(M))$, all $\mathfrak{p} \in \mathscr{G}(J(M))$. Then apply Theorem 7.3.

Proposition 7.7. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of holonomic A-modules. Then M has R.S. iff M' and M'' have R.S.*

Proof. Put $S = \{X^n | n \in \mathbf{N}\} \subset A[X]$. As observed in (2.7) the functor $N \to \mathscr{E}_S(N)$ from $\underline{\underline{M}}(A[X])$ to $\underline{\underline{M}}(\mathscr{E}_X)$ is exact. Since the functor $M \to M[X]$ is also exact we obtain an exact sequence $0 \to M'_X \to M_X \to M''_X \to 0$ of holonomic $\mathscr{E}_X$-modules (by Corollary 6.12). Then apply Proposition 7.2 and Corollary 5.15.

§8. Rings of Differential Operators

In this section we will consider special filtered rings, the so-called rings of differential operators, often denoted by D instead of A. We show that for these rings D the condition $\nu_D=\mu_D$ is satisfied. More precisely we show that $\nu_D=\mu_D$ $=\text{gl. dim}\, D$. This enables us to prove that the notion of a holonomic D-module as introduced in §3 coincides with the usual concept of holonomicity studied in literature. We begin with some preliminaries.

Let R be a commutative k-algebra, where k is a field with char $k=0$ and P: $R\times R\to R$ a k-bi-derivation i.e. P is a bi-derivation, cf. Definition 9.1 and $P(\lambda, r)$ $=P(r, \lambda)=0$ for all $\lambda\in k$, $r\in R$. Let $\Omega_R=\Omega_{R/k}$ be the R-module of differentials over k.

Lemma 8.1. *There exists an R-bilinear form ω on Ω_R such that $\omega(da, db)$ $=P(a, b)$, all a, $b\in R$.*

Proof. Suppose $\sum g_i da_i=\sum g'_i da'_i$ and $\sum h_j db_j=\sum h'_j\ db'_j$. We must show $\sum g_i h_j P(a_i, b_j)=\sum g'_i h'_j P(a'_i, b'_j)$. It suffices to prove $\sum g_i P(a_i, b)$ $=\sum g'_i P(a'_i, b)$, all $b\in R$ and $\sum h_j P(a, b_j)=\sum h'_j P(a, b'_j)$, all $a\in R$. We only show the first equality. Put $D(b):=\sum g_i P(a_i, b)$, all $b\in R$. Then D is a k-derivation of R. So by the universal property of Ω_R there exists $\phi\in\text{Hom}\,(\Omega_R, R)$ with $\phi(db)=Db$, which implies the first equality.

Proposition 8.2. *Let $\mathfrak{p}$ be a P-stable ideal in R. Suppose that*

i) R is a regular ring

ii) Ω_R is a free R-module of rank n $(n\in\mathbb{N})$ with an R-basis $(e_1,\dots, e_n)$ such that $\det\omega(e_i, e_j)^n_{i,j=1}$ *is a unit in R.*

Then $\text{ht}\,\mathfrak{p}\geq(1/2)n$.

Before we prove this proposition we make two observations.

1. Let ϕ: $R\to R'$ be a ringhomomorphism, F a free R-module of rank n with R-basis $(f):=(f_1,\dots,f_n)$ and ω an R-bilinear form on F such that $d(\omega(f))$ $:=\det(\omega(f_i, f_j)^n_{i,j=1})$ is a unit in R. Then $F':=R'\otimes_R F$ is a free R'-module with R'-basis $(f'):=(f'_1,\dots,f'_n)$, where $f'_i=1\otimes f_i$ and we can extend ω to an R'-bilinear form ω' on F' by putting $\omega'(f'_i, f'_j)=\phi(\omega(f_i, f_j))$. Since $\phi(d(\omega(f)))$ is a unit in R' $d(\omega'(f'))$ is a unit in R'.

2. Let S be a multiplicatively closed subset of R. Then the k-bi-derivation P on R can be extended to a k-bi-derivation P' on $S^{-1}R$ (in the obvious way). Identifying $S^{-1}R\otimes_R\Omega_R$ with $\Omega_{S^{-1}R}$ it is easy to verify that the form ω' on $S^{-1}R$ $\otimes\Omega_R$ as defined in 1. equals the form on $\Omega_{S^{-1}R}$ induced by P' (according Lemma 8.1). Furthermore, if I is an R-stable ideal in R, then $S^{-1}I$ is a P'-stable ideal in $S^{-1}R$.

Proof of Proposition 8.2. Put $A:=R_{\mathfrak{p}}$, $\mathfrak{m}:=\mathfrak{p}R_{\mathfrak{p}}$, $K:=A/\mathfrak{m}$, Since char $k=0$ we have an exact sequence of K-vectorspace (cf. [8], Ex. 8.1, p. 187)

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\alpha} \Omega_A \otimes_A K$$

where $\alpha(a+\mathfrak{m}^2)=da\otimes 1$. Apply 1. to the ringhomomorphism $R\to A$. This gives a form ω' on Ω_A $(=\Omega_R\otimes_R A)$ and a basis $(e'_1,\dots, e'_n)$ with $d(\omega(e'))$ is a unit in A. By 2. ω' is the form on Ω_A induced by P' and since $\mathfrak{p}$ is P-stable, $\mathfrak{m}$ is P'-stable. By 1. applied to the ringhomomorphism $A\to K$ we get a form ω'' on $V:=\Omega_A\otimes_A K$ and a K-basis $(e''):=(e''_1,\dots, e''_n)$ of V with $d(\omega''(e''))\neq 0$. So ω'' is non-degenerated. Since $\mathfrak{m}$ is P'-stable and $\alpha(a+\mathfrak{m}^2)=da\otimes 1$, all $a\in A$, it follows that $E:=\alpha(\mathfrak{m}/\mathfrak{m}^2)$ is an isotropic K-subspace of V i.e. $\omega''(e, e')=0$ for all $e, e'\in E$. So $\dim_K E\leq 1/2 \dim V=1/2\, n$. Since α is injective $\dim_K \mathfrak{m}/\mathfrak{m}^2\leq 1/2n$. The regularity of R implies that A is a regular local ring, so $\dim_K \mathfrak{m}/\mathfrak{m}^2=\dim A$. Since $\mathrm{ht}\,\mathfrak{p}=\dim A$, we derive $\mathrm{ht}\,\mathfrak{p}\leq 1/2n$, as desired.

8.3. Applications to Rings of Differential Operators

Let B be a commutative noetherian ring which contains a field k of characteristic zero. Put $\mathfrak{g}=\mathrm{Der}_k(B, B)$ and $D(B):=U(B, \mathfrak{g})$ the *ring of universal differential operators generated by B and* $\mathfrak{g}$ (we refer to [R] for more details). Let $h\colon B\to D(B)$ and $j\colon \mathfrak{g}\to D(B)$ be the canonical maps. Then h is a monomorphism. Furthermore, $D(B)$ is a Z-filtered ring by putting

> $D(B)(v)=0$ if $v<0$, $D(B)(0)=B$ and $D(B)(v)$ is the B-submodule of $D(B)$ generated by the v-fold products of elements in $h(B)\cup j(\mathfrak{g})$.

Let $\mathfrak{m}\in \mathrm{Max}\,(B)=$the set of maximal ideals in B. Then every $\tau\in\mathfrak{g}$ induces a $B/\mathfrak{m}$-linear map $\bar{\tau}\colon \mathfrak{m}/\mathfrak{m}^2\to B/\mathfrak{m}$, since $\tau\mathfrak{m}^2\subset\mathfrak{m}$; $\bar{\tau}$ is called the *tangent map at* $\mathfrak{m}$. We say that $\mathfrak{g}$ has maximal rank at $\mathfrak{m}$ if every $B/\mathfrak{m}$-linear map from $\mathfrak{m}/\mathfrak{m}^2$ to $B/\mathfrak{m}$ is of the form $\bar{\tau}$, for some $\tau\in\mathfrak{g}$. From now on we assume that B satisfies the following conditions:

1) $\mathfrak{g}$ *has maximal rank at every* $\mathfrak{m}\in\mathrm{Max}\,(B)$.
2) *B is a regular ring of dimension n (for some* $n\in N$*).*
3) *The residue fields* $B/\mathfrak{m}$ *are algebraic over k.*
4) *For every B-module M we have*: $M\in\underline{\underline{M}}(B)$ *iff* $M_{\mathfrak{m}}\in\underline{\underline{M}}(B_{\mathfrak{m}})$, *all* $\mathfrak{m}\in\mathrm{Max}\,(B)$.
5) $\Omega_{B/k}\in\underline{\underline{M}}(B)$.

Let $\mathfrak{m}\in\mathrm{Max}\,(B)$. By 2) $\mathrm{d}:=\dim_{B/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2=\dim B_{\mathfrak{m}}\leq n$. Choose $y_1,\dots, y_d$

$\in \mathfrak{m}$ such that their images $(\bar{y}_1,\ldots,\bar{y}_d)$ form a $B/\mathfrak{m}$-basis of $\mathfrak{m}/\mathfrak{m}^2$. Then Nakayama's lemma implies that $y_1,\ldots,y_d$ generate the maximal ideal $\mathfrak{m}B_{\mathfrak{m}}$. By 1) there exist $\tau_1,\ldots,\tau_d \in \mathfrak{g}$ such that $\tau_j(y_v) \in \mathfrak{m}$ if $j \neq v$ and $\tau_j(y_j)^{-1} \in \mathfrak{m}$, all $1 \leq j \leq d$. It follows (cf. [1], p. 89) that $\mathfrak{g}_{\mathfrak{m}}$ is a free $B_{\mathfrak{m}}$-module with basis $\tau_1,\ldots,\tau_d$. Consequently, observing that $D(B)_{\mathfrak{m}} \simeq U(B_{\mathfrak{m}}, \mathfrak{g}_{\mathfrak{m}})$ as filtered rings, we get an isomorphism of graded rings

(8.4) $$gr(D(B)_{\mathfrak{m}}) \xrightarrow{\sim} B_{\mathfrak{m}}[X_1,\ldots,X_d],$$

where the polynomial ring is graded in the usual way. Furthermore $gr(D(B)_{\mathfrak{m}}) \simeq gr(D(B))_{\mathfrak{m}}$ (for example by Theorem 2.6). Identify these two rings. Obviously $gr(D(B))$ is a commutative ring so we have a Poisson product, denoted $\{\ ,\ \}$ on it, which extends to a Poisson product on $gr(D(B))_{\mathfrak{m}}$. This extension equals the Poisson product induced by $D(B)_{\mathfrak{m}} \simeq U(B_{\mathfrak{m}}, \mathfrak{g}_{\mathfrak{m}})$ of $gr(D(B)_{\mathfrak{m}})$. Identify $gr(D(B)_{\mathfrak{m}})$ with $B_{\mathfrak{m}}[X_1,\ldots,X_d]$. So X_i corresponds to the class $\tau_i + B_{\mathfrak{m}}$. Put $R := gr(D(B)_{\mathfrak{m}})$. Let ω denote the R-bilinear form on Ω_R induced by the Poisson product on R. We have the following obvious relations

(8.5) $\omega(dX_i, dy_j) = \tau_i(y_j)$ in $B_{\mathfrak{m}}$ and $\omega(dy_i, dy_j) = 0$, all i, j.

Lemma 8.6. *Ω_R is a* free *R-module with basis $(e_1,\ldots,e_{2d}) := (dy_1,\ldots,dy_d, dX_1,\ldots,dX_d)$ which satisfies*: $\det \omega(e_i, e_j)_{i,j=1}^{2d}$ *is a unit in R.*

Proof. Ω_R is generated as an R-module by the elements $dX_1,\ldots,dX_d$ and the elements da, where a runs through $A := B_{\mathfrak{m}}$. More precisely $\Omega_R \simeq (\Omega_A \otimes_A R) \oplus RdX_1 \oplus \cdots \oplus RdX_d$ (see [11], p. 189). Put $\mathfrak{n} := \mathfrak{m}B_{\mathfrak{m}}$ and $K(A) := A/\mathfrak{n}$. Let $a \in A$. Then $\bar{a} := a + \mathfrak{n} \in K(A)$ is algebraic over k (by 3)). Let $P(X) \in k[X]$ be the monic minimal polynomial of $\bar{a}$ over k. In particular $P(\bar{a}) = 0$ i.e. $P(A) \in \mathfrak{n} = \sum Ay_i$. Consequently $\left(\frac{\partial P}{\partial X}\right)(a)da \in \sum Ady_i + \mathfrak{n}\Omega_A$. Since char $k = 0$, g.c.d. $\left(P(X), \frac{\partial P}{\partial X}\right) = 1$, so there exist $r(X), s(X) \in k[X]$ with $r(X)P(X) + s(X)\frac{\partial P}{\partial X} = 1$. Hence $r(a)P(a) + s(a)\left(\frac{\partial P}{\partial X}\right)(a) + 1$, implying that $da \in \sum Ady_i + \mathfrak{n}\Omega_A$. Consequently $\Omega_A \subset \sum Ady_i + \mathfrak{n}\Omega_A$. Since $\Omega_A \in \underline{\underline{M}}(A)$ (for $\Omega_B \in \underline{\underline{M}}(B)$ by 5)) Nakayama's lemma gives $\Omega_A = \sum Ady_i$. Hence $\Omega_R = \sum Rdy_i + \sum RdX_i$. Using the relations of (8.5), it is left to the reader to verify that $\det(\omega(e_i, e_j))_{i,j=1}^{2d}$ is a unit in $B_{\mathfrak{m}}$ and hence in R $(= B_{\mathfrak{m}}[X_1,\ldots,X_d])$. Finally it follows readily that Ω_R is a free R-module with $(e_1,\ldots,e_{2d})$ as an R-basis, which proves Lemma 8.6.

Theorem 8.7. $\nu_{D(B)} = \mu_{D(B)} = \text{gl. dim } D(B) = n$.

Proof. i) Let $\mathfrak{p}$ be an involutive prime ideal in $gr(D(B))$. Put $\mathfrak{p}_0 := \mathfrak{p}$

$\cap B$ (where we identified B with the subring $h(B)$ of $gr(D(B))$. Choose $\mathfrak{m} \in \operatorname{Max}(B)$ with $\mathfrak{p}_0 \subset \mathfrak{m}$. Put $S := B \backslash \mathfrak{m}$. Then $S \cap \mathfrak{p} = \emptyset$. So $\operatorname{ht} \mathfrak{p} = \operatorname{ht} S^{-1} \mathfrak{p}$ (Corollary 9.3 i)). Put $R = S^{-1} gr(D(B)) = gr(D(B))_{\mathfrak{m}}$. Since $\mathfrak{q} := S^{-1} \mathfrak{p}$ is an involutive prime ideal in R Lemma 8.6 and Proposition 8.2 imply that $\operatorname{ht} \mathfrak{q} \leq 1/2$. $2d = d \leq n$. So $\operatorname{ht} \mathfrak{p} \leq n$. Consequently $\nu_{D(B)} \leq n$.

ii) Since $\dim B = n$ there exists $\mathfrak{m} \in \operatorname{Max}(B)$ with $\operatorname{ht} \mathfrak{m} B_{\mathfrak{m}} = n$ in $B_{\mathfrak{m}}$. Hence $\mathfrak{q} := (\mathfrak{m} B_{\mathfrak{m}})^e$ in $B_{\mathfrak{m}}[X_1, \ldots, X_n]$ is an involutive homogeneous prime ideal in $B_{\mathfrak{m}}[X_1, \ldots, X_n] = gr(D(B))_{\mathfrak{m}}$ with $\operatorname{ht} \mathfrak{q} = n$. Let $\phi: gr(D(B)) \to gr(D(B))_{\mathfrak{m}}$ be the canonical map. Put $\mathfrak{p} := \mathfrak{q}^c$. Then $\mathfrak{q} = (\mathfrak{q}^c)^e = \mathfrak{p}^e$ gives that $\mathfrak{p}$ is an involutive homogeneous prime ideal in $gr(D(B))$ with $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{q} = n$. So $\mu_{D(B)} \geq n$. Together with i) this gives: $\mu_{D(B)} = \nu_{D(B)} = n$.

iii) Finally $n = \operatorname{gl.dim} D(B)$ by [1], Chap. 3, Theorem 1.2, which completes the proof.

8.8. Final Comment

Notations and assumptions as above. Put $D := D(B)$. It is shown in [1], Chap. 3 that $gr(D)$ is a commutative noetherian ring. Furthermore $gr(D)_{\mathfrak{m}} \simeq B_{\mathfrak{m}}[X_1, \ldots, X_d]$ is a regular ring of dimension $2d$ $(d \leq n)$ for every $\mathfrak{m} \in \operatorname{Max}(B)$. It follows that $gr(D)_{\mathfrak{p}}$ is a regular local ring of dimension $\leq 2n$ for every $\mathfrak{p} \in \operatorname{Max}(gr(D))$ (since $gr(D)_{\mathfrak{p}} \simeq (gr(D)_{\mathfrak{m}})_{\mathfrak{p}_{\mathfrak{m}}}$, where $\mathfrak{m} = \mathfrak{p} \cap B$ is a maximal ideal of B because $gr(D) = B \oplus \bigoplus_{n=1}^{\infty} gr(D)(n)$). It follows that $\operatorname{gl.dim} gr(D) \leq 2n$. So we can apply the material of [2], p. 103–149.

Let $0 \neq M \in \underline{\underline{M}}(D)$. Since $n = \operatorname{gl.dim} D$ we obtain the following results

Proposition 8.9. *M is equipped with a filtration $\mathscr{B}_0(M) \subset \mathscr{B}_1(M) \subset \cdots \subset \mathscr{B}_n(M) = M$ of D-submodules $(\mathscr{B}_{-1}(M) = 0)$ and $\mathscr{B}_v(M)/\mathscr{B}_{v-1}(M)$ is isomorphic to a subquotient of* $\operatorname{Ext}_D^{n-v}(\operatorname{Ext}_D^{n-v}(M, D), D)$.

Since $\mathscr{B}_n(M) = M$ we can define:

(8.10) $\quad$ *$\delta(M)$ is the smallest positive integer with $\mathscr{B}_{\delta(M)}(M) = M$.*

Furthermore we put

(8.11) $\quad$ *$j(M)$ is the smallest positive integer with* $\operatorname{Ext}_D^{j(M)}(M, D) \neq 0$.

Obviously $0 \leq j(M) \leq n$. More precisely it can be proved that

(8.12) $$j(M) + \delta(M) = n.$$

So we get: $\delta(M) = 0$ iff $j(M) = n$ i.e. $\operatorname{Ext}_D^v(M, D) \neq 0$ iff $v = n$. A consequence of the decomposition theorem ([2], Theorem 7.3 p. 143) is:

Proposition 8.14. $\delta(M)=0$ iff ht $\mathfrak{p}=n$, all $\mathfrak{p}\in\mathscr{G}(J(M))$.

By Theorem 8.7 we therefore have:

Corollary 8.15. *M is a holonomic D-module iff* $\delta(M)=0$ *iff* $\mathrm{Ext}_D^v(M, D)\neq 0$ *only when* $v=n$.

This shows that the notion of a holonomic D-module, introduced in §3 coincides with the usual definitions given in the literature.

Examples. i) Let $\mathcal{O}_n$ be the ring of formal or convergent power series in $x_1,\dots, x_n$ over a field k of characteristic zero. Then $B:=\mathcal{O}_n$ satisfies the conditions 1)–5) and $\mathscr{D}_n=D(B)$.

ii) Let $V\subset \mathbb{C}^N$ $(N\in\mathbb{N})$ be a non-singular n-dimensional irreducible variety. Let $A(V)$ be the coordinate ring of V. Then $B:=A(V)$ satisfies 1)–5) and $\mathscr{D}(V)=D(A(V))$ (see [1], Chap. 3, §2).

§9. Some Results of Commutative Algebra

In this section *all rings are commutative.* Let $\phi\colon A\to B$ be a ring homomorphism. If I is an ideal in A we put $I^e:=B\phi(I)$, the extended ideal of A and if J is an ideal of B we put $J^c:=\phi^{-1}(J)$, the contracted ideal of J.

Definition 9.1. *A bi-derivation on a ring A is a Z-bilinear map* $D\colon A\times A\to A$ *satisfying*

$$D(a_1a_2, b)=a_1D(a_2, b)+D(a_1, b)a_2, \quad all \quad a_1, a_2, b\in A.$$
$$D(a, b_1b_2)=b_1D(a, b_2)+D(a, b_1)b_2, \quad all \quad a, b_1, b_2\in A.$$

An ideal I in A is D-stable if $D(a, b)\in I$ *for all* $a, b\in I$.

Proposition 9.2. *Let* $\phi\colon A\to B$ *be a ring homomorphism and let* D_A, D_B *be bi-derivations of A resp. B. If for some unit* $v\in B$

$$\phi(D_A(a, a'))=vD_B(\phi(a), \phi(a')), \quad all \quad a, a'\in A$$

then the following holds

i) *If I is* D_A*-stable, then* I^e *is* D_B*-stable.*

ii) *If J is* D_B*-stable, then* J^c *is* D_A*-stable.*

Proof. Left to the reader.

Let A be a ring and I a radical ideal in A. If I can be written as $\mathfrak{p}_1\cap\cdots\cap$

$\mathfrak{p}_r$, where the $\mathfrak{p}_i$ are distinct prime ideals of A satisfying $\mathfrak{p}_i \not\subset \bigcap_{j\neq i} \mathfrak{p}_j$, we call $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ *the minimal prime decomposition of* I (it is unique up to a permutation). The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ we denote by $\mathscr{G}(I)$.
If A is a noetherian ring every radical ideal admits such a decomposition.

Let S be a multiplicatively closed subset of A. Then there is a one-to-one correspondence between the prime ideals of $S^{-1}A$ and the prime ideals of A not meeting S, given by extension and contraction (under the canonical map ϕ: $A \to S^{-1}A$). An easy consequence of this fact is

Corollary 9.3. *i) If* $\mathfrak{p} \in Spec(A)$ *and* $\mathfrak{p} \cap S = \emptyset$, *then* $\text{ht}\,\mathfrak{p} = \text{ht}\,\mathfrak{p}^e$.

ii) Let I *be a radical ideal in* A. *If* $I^e \subsetneqq S^{-1}A$, *then* I^e *is a radical ideal and* $\mathscr{G}(I^e)$ *is the set of* $\mathfrak{p}^e$ *where* $\mathfrak{p} \in \mathscr{G}(I)$ *with* $\mathfrak{p} \cap S = \emptyset$.

9.4. The Adjunction of a Variable

Let A be a ring and M and A-module. We can make the ring $A[X]$ of polynomials and similarly the module $M[X]$ which is an $A[X]$-module in the obvious way (cf. §6). Let i: $A \to A[X]$ be the inclusion map. As before put $I^e = A[X]i(I)$ for an ideal I in A and $J^c = i^{-1}(J)$ for an ideal J in $A[X]$.

Lemma 9.5. *Let* I, J, K *be ideals in* A.

i) *If* $\mathfrak{p} \in Spec(A)$, *then* $\mathfrak{p}^e \in Spec(A[X])$.

ii) $I^{ec} = I$.

iii) $r(I^e) = r(I)^e$.

iv) *If* $I = J \cap K$, *then* $I^e = J^e \cap K^e$.

v) *If* M *is an* A*-module then* $Ann\, M[X] = (Ann\, M)^e$.

vi) $r(Ann\, M[X]) = (r(Ann\, M))^e$.

Proof. Left to the reader. Use the fact that $\sum a_i X^i \in I^e$ iff $a_i \in I$ for all $i \in \mathbb{N}$.

Proposition 9.6. *Assume* A *noetherian. Let* $0 \neq M$ *and let* $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ *be the minimal prime decomposition of* $r(Ann\, M)$. *Then* $\mathfrak{p}_1^e \cap \cdots \cap \mathfrak{p}_r^e$ *is the minimal prime decomposition of* $r(Ann\, M[X])$. *Furthermore* $\text{ht}\,\mathfrak{p}_i = \text{ht}\,\mathfrak{p}_i^e$, *all* i.

Proof. By Lemma 9.5 vi) and iv) $r(Ann\, M[X]) = \mathfrak{p}_1^e \cap \cdots \cap \mathfrak{p}_r^e$. The $\mathfrak{p}_i^e$ are all distinct prime ideals of $A[X]$ by Lemma 9.5 i) and ii). If $\mathfrak{p}_i^e \subset \bigcap_{j\neq i} \mathfrak{p}_j^e$ then $\mathfrak{p}_i^e \subset (\bigcap_{j\neq i} \mathfrak{p}_j)^e$ by Lemma 9.5 iv), so by Lemma 9.5 ii) $\mathfrak{p}_i \subset \bigcap_{j\neq i} \mathfrak{p}_j$ a contradiction. Finally $\text{ht}\,\mathfrak{p}_i = \text{ht}\,\mathfrak{p}_i^e$ follows from [11], Theorem 19, p. 79.

Now consider $A[X, X^{-1}]$ and $M[X, X^{-1}]$ i.e. the localization of $A[X]$ resp. $M[X]$ with respect to $S=\{X^n|n \in \mathbb{N}\}$. So $M[X, X^{-1}]$ is an $A[X, X^{-1}]$-module. Let j: $A \to A[X, X^{-1}]$ be the inclusion map. Combining 9.3 and Proposition 9.6 we obtain

Corollary 9.7. *Notations as in Proposition* 9.6. *Then* $\tilde{\mathfrak{p}}_1 \cap \cdots \cap \tilde{\mathfrak{p}}_r$ *is the minimal prime decomposition of* $r(Ann\, M[X, X^{-1}])$ *where* $\tilde{\mathfrak{p}}_i = A[X, X^{-1}]\mathfrak{p}_i$. *Furthermore* $\mathrm{ht}\, \mathfrak{p}_i = \mathrm{ht}\, \tilde{\mathfrak{p}}_i$ *all* i *and* $r(Ann\, M) = j^{-1}\, (r(Ann\, M[X, X^{-1}]))$.

9.8. Graded Rings and Modules

A ring R is called a *graded ring* (of type Z) if there is a family of additive subgroups $\{R_n|n \in Z\}$ of R such that $R = \oplus R_n$ and $R_n R_m \subset R_{n+m}$, all n, $m \in Z$. It follows that $1 \in R_0$ and R_0 is a subring of R. An R-module M is called a *graded R-module* if there exists a family $\{M_n|n \in Z\}$ of additive subgroups of M with the properties $M = \oplus M_n$ and $R_n M_n \subset M_{n+m}$, all n, $m \in Z$. If $0 \neq m \in M_n$, then m is called *a homogeneous element of degree n* and if V is a subset of M, $h(V)$ denotes the set of homogeneous elements in V. An ideal I in a graded ring R is called *homogeneous* if it is generated by homogeneous elements (equivalently: $r = \sum r_n \in I$ implies $r_n \in I$ for all $n \in Z$).

If A is an arbitrary ring, the ring $R := A[X, X^{-1}]$ is a graded ring by putting $R_n := AX^n$, all $n \in Z$. Let j: $A \to A[X, X^{-1}]$ be the inclusion map. It is left to the reader to prove

Proposition 9.9. *There is a one-to-one correspondence between Spec*(A) *and the homogeneous prime ideals of R given by extension and contraction (with respect to j).*

Proposition 9.10. *If* $R = \oplus R_n$ *is a noetherian graded ring, then* R_0 *is noetherian.*

Proof. Let I be an ideal in R_0 and $r \in R_0 \backslash I$. Then $r \notin RI$. Consequently if there exists a strictly increasing chain of ideals in R_0, say $(I_n)_{n \in N}$ then the chain $(RI_n)_{n \in N}$ of ideals in R is also strickly increasing, a contradiction.

9.11. External Homogenization, Dehomogenization

We recall some well-known facts of graded rings (cf. [14], Chap. VII, §5 and [12] part A, II.8. Let R be a graded ring. The ring $R[X]$ of polynomials can be made into a graded ring by putting $\deg X = 1$ i.e. $R[X]_n$ is the set of elements $\sum r_i X^j$ with $r_i \in R_i$ and $i+j=n$. In the same way, starting from a graded R-module M we make $M[X]$ into a graded $R[X]$-module, called *the external homogenization of* M. Let $r=r_{-m}+\cdots+r_0+\cdots+r_n \in R$. Put $r^*:=X^{n+m}r_{-m}+\cdots+X^n r_0+\cdots+r_n \in R[X]$, *the homogenized of* r. If $u=u_{-k}X^{k+j}+\cdots+u_0+\cdots+u_j \in R[X]$ is in $h(R[X])$ put $u_*:=u_{-k}+\cdots+u_0+\cdots+u_j \in R$, *the dehomogenized of* u. Then $(r^*)_*=r$ and $X^p(u_*)^*=u$ for some $p \in \bar{N}$. Let I be an ideal in R and J a homogeneous ideal in $R[X]$. We put $I^*:=$the ideal in $R[X]$ generated by the f^*, with $f \in I$. $J_*:=\{u_* | u \in h(J)\}$ this is an ideal in R.

Proposition 9.12 (cf. [14]). *There is a one-to-one correspondence between the prime ideals of* R *and the homogeneous prime ideals of* $R[X]$ *which do not contain* X. *The correspondence is described by the maps* $\mathfrak{p} \to \mathfrak{p}^*$ *and* $\mathfrak{q}_* \leftarrow \mathfrak{q}$ *which are each others inverse.*

Lemma 9.14. *If* R *is noetherian and* $\mathfrak{p}$ *a homogeneous prime ideal in* R *with* $\mathrm{ht}\,\mathfrak{p}=n$, *then there exists a chain of homogeneous prime ideals* $\mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{p}$.

Proof. See [12], Corollary I. 1.10, p. 227.

Corollary 9.15. *Let* R *be noetherian. If* $\mathfrak{q}$ *is a homogeneous prime ideal in* $R[X]$ *with* $X \notin \mathfrak{q}_*$, *then* $\mathrm{ht}\,\mathfrak{q}=\mathrm{ht}\,\mathfrak{q}_*$. *If* $\mathfrak{p}$ *is a prime ideal in* R, *then* $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{p}^*$.

Proof. Let $\mathrm{ht}\,\mathfrak{q}=n$. By Lemma 9.14 there exists homogeneous prime ideals $\mathfrak{q}_0 \subsetneqq \cdots \subsetneqq \mathfrak{q}_n=\mathfrak{q}$. Hence Proposition 9.12 gives a chain of distinct prime ideals $\mathfrak{q}_{0*} \subsetneqq \cdots \subsetneqq \mathfrak{q}_{n*}=\mathfrak{q}_*$ in R. So $\mathrm{ht}\,\mathfrak{q}_* \geq \mathrm{ht}\,\mathfrak{q}$. Conversely, since by Proposition 9.12 a chain $\mathfrak{p}_0 \subsetneqq \cdots \subsetneqq \mathfrak{p}_m=\mathfrak{q}_*$ of distinct prime ideals in R gives rise to a chain $\mathfrak{p}_0^* \subsetneqq \cdots \subsetneqq \mathfrak{p}_m^*=(\mathfrak{q}_*)^*=\mathfrak{q}$ of distinct prime ideals in $R[X]$ we get $\mathrm{ht}\,\mathfrak{q} \geq \mathrm{ht}\,\mathfrak{q}_*$. So $\mathrm{ht}\,\mathfrak{q}=\mathrm{ht}\,\mathfrak{q}_*$. Finally, by Proposition 9.12 $\mathfrak{p}=(\mathfrak{p}^*)_*$ and $\mathfrak{q}:=\mathfrak{p}^*$ is a homogeneous prime ideal of $R[X]$ with $X \notin \mathfrak{q}$. So $\mathrm{ht}\,\mathfrak{p}=\mathrm{ht}\,\mathfrak{q}_*=\mathrm{ht}\,\mathfrak{q}=\mathrm{ht}\,\mathfrak{p}^*$, as desired.

Let D be a bi-derivation on R. We extend it to a bi-derivation on $R[X]$ by the formula

$$D(\sum f_i X^i, \sum g_j X^j)=\sum_{i,j} D(f_i, g_j)X^{i+j}, \quad \text{all} \quad f_i, g_j \in R, \ i, j \in \bar{N}.$$

It readily follows that

$$D(F, G)_* = D(F_*, G_*), \quad \textit{all} \quad F, G \in R[X]. \tag{9.16}$$

Let $f, g \in R$. Then $D(f, g) = D((f^*)_*, (g^*)_*) = D(f^*, g^*)$ by (9.16). So $D(f, g)^* = (D(f^*, g^*)_*)^*$. Consequently

(9.17) $$D(f^*, g^*) = X^p D(f, g)^*, \quad \text{for some} \quad p \in \bar{N}.$$

Proposition 9.18. *If I is a D-stable ideal in R, then I^* is a D-stable ideal in $R[X]$. If J is a homogeneous D-stable ideal in $R[X]$, then J_* is a D-stable ideal in R.*

Proof. The first part follows from 9.17 and the fact that each $F \in h(I^*)$ is of the form $X^p f^*$ for some $f \in I$ and $p \in \bar{N}$. The second part follows from 9.16.

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