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Note on the Eilenberg-Moore Spectral Sequence

By

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Abstract

In this paper, we prove two theorems on the Eilenberg-Moore spectral sequence. We give a relation between the Bockstein homomorphism in the E^2-term and the Bockstein homomorphism of the space to which the spectral sequence converges and also relate the homology suspension to the Eilenberg-Moore spectral sequence.

Introduction

The purpose of this note is to prove two theorems on the Eilenberg-Moore spectral sequence; one relates the algebraic Bockstein homomorphism between the E^2-terms to the geometric Bockstein homomorphism between the homologies of the spaces to which the spectral sequences converge (Theorem 2.2), and the other relates the homology suspension to the Eilenberg-Moore spectral sequence associated with a path fibration (Theorem 3.5).

In order to prove these theorems, we recall the definition of the Eilenberg-Moore spectral sequence. After the original work of Eilenberg and Moore ([2]), various constructions have been done by Hodgkin ([4]), Smith ([7], [8]), Rector ([6]) and Heller ([3]). In this note, we adopt a point of view of Hodgkin and Smith who construct the Eilenberg-Moore spectral sequence as the Künneth spectral sequence on the category of pointed spaces over some fixed base space.

In Section 1, we construct the Künneth spectral sequence for a generalized homology theory, dualizing the argument of Smith ([8]). We give a sufficient condition for convergence of the spectral sequence ((1.11), (1.23) (ii)) and one for identification of the E^2-term ((1.17), (1.22)). These results are also obtained in [3] under a categorical framework. Our concrete construction enables us to prove the main theorems in the following sections.

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In Section 2, we prove that the algebraic Bockstein homomorphism between the E^2-terms of the Künneth spectral sequence "converges" to the geometric Bockstein homomorphism. Our assertion (2.2) is quite similar to the main theorem on the boundary homomorphism of [11], although the proof is much easier.

In Section 3, we consider the Eilenberg-Moore spectral sequence for a generalized homology h_*, associated with the path fibration over a space B. Then, under suitable assumptions, there is a homomorphism $\tilde{h}_n(\Omega B) \to E^2_{-1,n+1}$ and a natural equivalence $E^2_{-1,n+1} \to Ph_{n+1}(B)$. We show that the composition of these coincides with the homology suspension.

The results of Sections 2, 3 are applied to determine the structure of the homology of double loop spaces of complex Stiefel manifolds ([10]).

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§ 1. Recollections on the Eilenberg-Moore Spectral Sequence

First, we define a category $\mathbf{Top}_*/B$ to formulate the Eilenberg-Moore spectral sequence as the Künneth spectral sequence in this category. We will use the notations and some results of Section 1 of [8]. (See also [4], [7].)

Definition 1.1 ([8]). *Let B be a fixed topological space. We define a category of pointed spaces over B, denoted by $\mathbf{Top}_*/B$ as follows. An object of $\mathbf{Top}_*/B$ is a pair of maps (f, s) $(f: T(f) \to B,\ s: B \to T(f))$ between topological spaces $T(f)$ and B such that $f \circ s = 1_B$ and sB is a neighborhood deformation retract of $T(f)$. A morphism $\varphi: (f, s) \to (g, t)$ of $\mathbf{Top}_*/B$ consists of a continuous map $T(\varphi): T(f) \to T(g)$ such that $g \circ T(\varphi) = f$, $T(\varphi) \circ s = t$.*

We denote by $\mathbf{Top}_*$ the category of pointed topological spaces with non-degrerate basepoints. We define functors $\Gamma: \mathbf{Top}_* \to \mathbf{Top}_*/B$ and $\Phi: \mathbf{Top}_*/B \to \mathbf{Top}_*$ which play a central role in the construction of the Eilenberg-Moore spectral sequence.

Definition 1.2 ([8]). *For each pointed topological space (X, x_0), let $p: B \times X \to B$ be the projection onto B and let $s: B \to B \times X$ be the canonical inclusion $s(b) = (b, x_0)$. We define a functor Γ by $\Gamma(X, x_0) = (p, s)$. And we* *define a functor* Φ *by* $\Phi(f, s)=(T(f)/sB, sB/sB)$.

Lemma 1.3. *The functor* Γ *is a right adjoint of* Φ; *that is, there is a natural equivalence*

$$\mathrm{Mor}_{\mathbf{Top}_*/B}((f, s), \Gamma(X, x_0)) \cong \mathrm{Mor}_{\mathbf{Top}_*}(\Phi(f, s), (X, x_0)).$$

Proof. We construct a natural map $\alpha: \mathrm{Mor}_{\mathbf{Top}_*/B}((f, s), \Gamma(X, x_0)) \to \mathrm{Mor}_{\mathbf{Top}_*}(\Phi(f, s), (X, x_0))$ as follows. Let $\varphi: (f, s)\to\Gamma(X, x_0)$ be a morphism of $\mathbf{Top}_*/B$, and let $q: B\times X\to X$ be the projection onto X. Then the composition $T(f)\xrightarrow{T(\varphi)} B\times X \xrightarrow{q} X$ maps sB to the base point x_0. We define $\alpha(\varphi)$ to be the map $T(f)/sB\to X$ induced by $q\circ T(\varphi)$. The inverse α^{-1} of α is described as follows. For a morphism $\psi: \Phi(f, s)\to(X, x_0)$, we put $T(\alpha^{-1}(\psi))(x)=(f(x), \psi\circ\pi(x))$ where $\pi: T(f)\to T(f)/sB$ is the canonical projection.

In the category $\mathbf{Top}_*/B$, we can construct mapping cones, suspensions, products, smash products and other constructions which we usually do in the category $\mathbf{Top}_*$. And we can define a cofiber sequence in $\mathbf{Top}_*/B$ as in $\mathbf{Top}_*$. Here we give the constructions of mapping cones, products, smash products, and suspensions. (See Section 1 of [8] for details.)

Constructions 1.4. (i) Let $\varphi: (f, s)\to(g, t)$ be a morphism of $\mathbf{Top}_*/B$. We define the mapping cone of φ, denoted by $(C_B(\varphi), S_{C_B(\varphi)})$ as follows:

$$TC_B(\varphi)=(T(f)\times I)\amalg T(g)\Big/ \begin{array}{l}(x, 1)\sim T\varphi(x):\ x\in T(f)\\ (s(b), r)\sim t(b):\ b\in B,\ r\in I\\ (x', 0)\sim(x'', 0)\quad \text{if}\quad f(x')=f(x'')\\ \qquad\qquad\qquad\qquad \text{for}\quad x', x''\in T(f)\end{array}$$

$C_B(\varphi): TC_B(\varphi)\to B$ and $S_{C_B(\varphi)}: B\to TC_B(\varphi)$ are defined by $C_B(\varphi)([x, r])=f(x)$ for $x\in T(f)$, $r\in I$, $C_B(\varphi)([y])=g(y)$ for $y\in T(g)$ and $S_{C_B(\varphi)}(b)=[t(b)]$ for $b\in B$ where $[x, r]$ and $[y]$ are the elements of $TC_B(\varphi)$ represented by $(x, r)\in T(f)\times I$ and $y\in T(g)$ respectively. Note that there is a natural inclusion $\iota: (g, t)\to (C_B(\varphi), S_{C_B(\varphi)})$ defined by $T(\iota)(y)=[y]$.

(ii) Let (f, s) and (g, t) be pointed space over B, their product $(f, s)\xleftarrow{\pi_f} (f, s)\times_B(g, t)\xrightarrow{\pi_g}(g, t)$ is defined by the following; We put $(f, s)\times_B(g, t)=(f\times_B g, s\times_B t)$ and $T(f\times_B g)=T(f)\times_B T(g)=\{(x, y)\in T(f)\times T(g)\mid f(x)=g(y)\}$. $f\times_B g: T(f\times_B g)\to B$, $s\times_B t: B\to T(f\times_B g)$, $T(\pi_f): T(f\times_B g)\to T(f)$ and $\pi(\pi_g): T(f\times_B g)\to T(g)$ are given by $f\times_B g(x, y)=f(x)=g(y)$, $s\times_B t(b)=(s(b), t(b))$, $T(\pi_f)(x, y)=x$ and $T(\pi_g)(x, y)=y$.

(iii) Let (f, s) and (g, t) as above. We define their smash product $(f, s)\wedge_B (g, t)=(f\wedge_B g, s\wedge_B t)$ by
$T(f\wedge_B g)=T(f\times_B g)/(x, t(b))\sim(s(b), y)$ for $(x, y)\in T(f\times_B g)$, $b\in B$. $f\wedge_B g\colon T(f\wedge_B g)\to B$ and $s\wedge_B t\colon B\to T(f\wedge_B g)$ are given by $f\wedge_B g(x\wedge_B y)=f(x)=g(y)$ and $s\wedge_B t(b)=s(b)\wedge_B t(b)$.

(iv) Let (f, s) be a pointed space over B. Let us define a suspension functor $\mathbf{Top}_*/B\to\mathbf{Top}_*/B$ by $\sum_B(f, s)=\Gamma(S^1, *)\wedge_B(f, s)$, where $(S^1, *)$ is a circle with a base point $*$.

We give some propositions we need without proofs.

Propositions 1.5 ([8]). (i) *The functor* $\Phi\colon \mathbf{Top}_*/B\to\mathbf{Top}_*$ *preserves cofibrations.*

(ii) *Let* $\wedge_B\colon \mathbf{Top}_*/B\times\mathbf{Top}_*/B\to\mathbf{Top}_*/B$ *be the smash product over* B. *For any pointed space* (f, s) *over* B, *the functors* $(f, s)\wedge_B(-)$, $(-)\wedge_B(f, s)\colon \mathbf{Top}_*/B\to\mathbf{Top}_*/B$ *preserve cofibrations.*

(iii) Let $\sum=\sum_B\colon \mathbf{Top}_*/B\to\mathbf{Top}_*/B$ *be the suspension functor on* $\mathbf{Top}_*/B$, *then there is a natural map* $\Delta\colon (h'', u'')\to\sum(h', u')$ *for each cofiber sequence* $(h', u')\xrightarrow{i}(h, u)\xrightarrow{j}(h'', u'')$ *in* $\mathbf{Top}_*/B$ *such that* $(h, u)\xrightarrow{j}(h'', u'')\xrightarrow{\Delta}\sum(h', u')$ *and* $(h'', u'')\xrightarrow{\Delta}\sum(h', u')\xrightarrow{\Sigma i}\sum(h, u)$ *are cofiber sequences.*

(iv) $\sum_B$ *preserves cofibrations and commutes with* Φ: *that is,* $\Phi\circ\sum_B=\sum\circ\Phi$, *where* $\sum$ *in the right hand is the usual suspension functor on* $\mathbf{Top}_*$.

Definition 1.6. *Consider the natural transformation* $\varphi=\alpha^{-1}(1_\Phi)\colon 1_{\mathbf{Top}_*/B}\to\Gamma\circ\Phi$. For a pointed space (f, s) over B, let $C_\varphi(f, s)$ be the mapping cone of $\varphi_{(f,s)}\colon (f, s)\to\Gamma\circ\Phi(f, s)$, and let $\iota_{(f,s)}\colon \Gamma\circ\Phi(f, s)\to C_\varphi(f, s)$ be the natural inclusion. Thus we define a functor $C_\varphi\colon \mathbf{Top}_*/B\to\mathbf{Top}_*/B$ and a natural cofiber sequence $(f, s)\xrightarrow{\varphi(f,s)}\Gamma\circ\Phi(f, s)\xrightarrow{\iota(f,s)}C_\varphi(f, s)$.

Let $\tilde{h}_*$ be a reduced homology theory on $\mathbf{Top}_*$. Putting ${}_Bh_*=\tilde{h}_*\circ\Phi$, we have a homology theory ${}_Bh_*$ on $\mathbf{Top}_*/B$ ([8], Corollary 2.2).

Construction 1.7. Let $\tilde{h}_*$ and ${}_Bh_*$ as above, and let (f, s) and (g, t) be pointed spaces over B. We form a sequence of natural cofibrations.

$$(f, s)\xrightarrow{\varphi}\Gamma\circ\Phi(f, s)\xrightarrow{\iota}C_\varphi(f, s)$$

$$C_\varphi(f, s)\xrightarrow{\varphi}\Gamma\circ\Phi C_\varphi(f, s)\xrightarrow{\iota}C_\varphi^2(f, s),\quad\text{where}\quad C_\varphi^i=\overbrace{C_\varphi\circ\cdots\circ C_\varphi}^{i\text{ factors}}$$
$$(i=0, 1, 2,\dots)$$

$$C_\varphi^i(f, s)\xrightarrow{\varphi}\Gamma\circ\Phi C_\varphi^i(f, s)\xrightarrow{\iota}C_\varphi^{i+1}(f, s)$$

We apply the functor $(g, t)\wedge(-)$ to the above cofibrations to have the following sequence of cofibrations by (1.5).

$$(g, t)\underset{B}{\wedge}(f, s) \xrightarrow{1\wedge\varphi} (g, t)\underset{B}{\wedge}\Gamma\circ\Phi(f, s) \xrightarrow{1\wedge\iota} (g, t)\wedge C_\varphi(g, t)$$
$$(g, t)\underset{B}{\wedge}C_\varphi(f, s) \xrightarrow{1\wedge\varphi} (g, t)\underset{B}{\wedge}\Gamma\circ\Phi C_\varphi(f, s) \xrightarrow{1\wedge\iota} (g, t)\underset{B}{\wedge}C^2_\varphi(g, t)$$
$$(g, t)\underset{B}{\wedge}C^i_\varphi(f, s) \xrightarrow{1\wedge\varphi} (g, t)\underset{B}{\wedge}\Gamma\circ\Phi C^i_\varphi(f, s) \xrightarrow{1\wedge\iota} (g, t)\underset{B}{\wedge}C^{i+1}_\varphi(g, t)$$

Thus we obtain the following long exact sequence for each $i=0, 1, 2,\dots$.

$$\longrightarrow {}_Bh_{q+1}((g, t)\underset{B}{\wedge}C^{i+1}_\varphi(f, s)) \xrightarrow{\Delta_*} {}_Bh_q((g, t)\underset{B}{\wedge}C^i_\varphi(f, s)) \xrightarrow{(1\wedge\varphi)_*}$$
$${}_Bh_q((g, t)\underset{B}{\wedge}\Gamma\circ\Phi C^i_\varphi(f, s)) \xrightarrow{(1\wedge\iota)_*} {}_Bh_q((g, t)\underset{B}{\wedge}C^{i+1}_\varphi(f, s)) \xrightarrow{\Delta_*} \cdots$$

We set $D^1_{p,q}={}_Bh_q((g, t)\underset{B}{\wedge}C^{-p}_\varphi(f, s))$, $E^1_{p,q}={}_Bh_q((g, t)\underset{B}{\wedge}\Gamma\circ\Phi C^{-p}_\varphi(f, s))$ $(p=0,-1, -2,\dots, q\in\mathbb{Z})$. *The Künneth spectral sequence in the category* $\mathbb{T}\mathrm{op}_*/B$ *is defined to be the spectral sequence associated with the exact couple* $\langle D^1_{p,q}, E^1_{p,q}, (1\wedge\varphi)_*, (1\wedge\iota)_*, \Delta_*\rangle$.

To discuss the convergence problem, we have to define a filtration on ${}_Bh_*((g, t)\underset{B}{\wedge}(f, s))$. Considering the suspension category associated with the category $\mathbb{T}\mathrm{op}_*/B$ ([3]), we "desuspend" the map $(g, t)\underset{B}{\wedge}C^{i+1}_\varphi(f, s)\xrightarrow{1\wedge\Delta}(g, t)\underset{B}{\wedge}\sum C^i_\varphi(f, s)\cong\sum(g, t)\underset{B}{\wedge}C^i_\varphi(f, s)$ and obtain a map $\sum^{-1}(g, t)\underset{B}{\wedge}C^{i+1}_\varphi(f, s)\to(g, t)\underset{B}{\wedge}C^i_\varphi(f, s)$. Thus we have the following sequence of maps of the suspension category.

$$(g, t)\underset{B}{\wedge}(f, s)\longleftarrow \textstyle\sum^{-1}(g, t)\underset{B}{\wedge}C_\varphi(f, s)\longleftarrow \sum^{-2}(g, t)\underset{B}{\wedge}C^2_\varphi(f, s)\longleftarrow\cdots$$
$$\longleftarrow \textstyle\sum^{-i}(g, t)\underset{B}{\wedge}C^i_\varphi(f, s)\longleftarrow \sum^{-i-1}(g, t)\underset{B}{\wedge}C^{i+1}_\varphi(f, s)\longleftarrow\cdots$$

We put $F_{p,q}=\mathrm{Im}\,\{{}_Bh_q((g, t)\underset{B}{\wedge}C^{-p}_\varphi(f, s))\cong{}_Bh_{p+q}(\sum^p(g, t)\underset{B}{\wedge}C^{-p}_\varphi(f, s))\longrightarrow {}_Bh_{p+q}((g, t)\underset{B}{\wedge}(f, s))\}$ $(p\leqq 0,\ q\in\mathbb{Z})$

$$A_{p,q}=\mathrm{Im}\,\{{}_Bh_q((g, t)\underset{B}{\wedge}\Gamma\circ\Phi C^{-p}_\varphi(f, s))\to{}_Bh_q((g, t)\underset{B}{\wedge}C^{-p+1}_\varphi(f, s))\}$$
$$\cap\,[\textstyle\bigcap_{r\geqq 1}\mathrm{Im}\,\{{}_Bh_q(\sum^{-r}(g, t)\underset{B}{\wedge}C^{-p+r+1}_\varphi(f, s))\to{}_Bh_q((g, t)\underset{B}{\wedge}C^{-p+1}_\varphi(f, s))\}].$$

By the construction of the spectral sequence, we have $E^{1-p}_{p,q}\supset E^{2-p}_{p,q}\supset\cdots\supset E^r_{p,q}\supset E^{r+1}_{q,p}\supset\cdots$. We set $E^\infty_{p,q}=\bigcap_{r\geqq 1-p}E^r_{p,q}$, and note that ${}_Bh_n((g, t)\underset{B}{\wedge}(f, s))=F_{0,n}\supset F_{-1,n+1}\supset\cdots\supset F_{m,n-m}\supset F_{m-1,n-m+1}\supset\cdots$.

Proposition 1.8. *There is a short exact sequence*

$$0 \longrightarrow F_{p,q}/F_{p-1,q+1} \longrightarrow E^{\infty}_{p,q} \longrightarrow A_{p,q} \longrightarrow 0 \quad (p \leqq 0,\ q \in \mathbb{Z}).$$

See [9] p. 464 ~ p. 470 for a proof.

Remark 1.9. There is an edge homomorphism ${}_Bh_n((g, t) \wedge_B (f, s)) = F_{0,n} \to F_{0,n}/F_{-1,n+1} \to E^{\infty}_{0,n} \subset E^2_{0,n}$.

Propositions 1.10. *Let (f, s), (g, t) be pointed spaces over B, then the following facts hold.*

(i) *$\Phi \circ \Gamma \circ \Phi(f, s) = (B_+) \wedge T(f)/sB$, where $B_+ = B \amalg \{*\}$ (disjoint union). Hence ${}_Bh_*(\Gamma \circ \Phi(f, s)) = \tilde{h}_*((B_+) \wedge T(f)/sB)$.*

(ii) *Let $T(g) \times_B T(f) = \{(y, x) \in T(g) \times T(f) | g(y) = f(x)\}$, then $\Phi((g, t) \wedge_B (f, s)) = T(g) \times_B T(f)/\bar{t}T(g) \cup \bar{s}T(f)$ where $\bar{t}\colon T(g) \to T(g) \times_B T(f)$, $\bar{s}\colon T(f) \to T(g) \times_B T(f)$ are maps defined by $\bar{t}(y) = (y, s \circ g(y))$, $\bar{s}(x) = (t \circ f(x), x)$.*

(iii) *$\Phi((g, t) \wedge_B \Gamma \circ \Phi(f, s))$ is naturally homeomorphic to $T(g)/tB \wedge T(f)/sB = T(g) \times T(f)/T(g) \times sB \cup tB \times T(f)$.*

(iv) *The map $(1 \wedge \varphi)_*\colon {}_Bh_q((g, t) \wedge_B (f, s)) \to {}_Bh_q((g, t) \wedge_B \Gamma \circ \Phi(f, s))$ coincides with the map $\tilde{h}_q(T(g) \times_B T(f)/\bar{t}T(g) \cup \bar{s}T(f)) \to \tilde{h}_q(T(g) \times T(f)/T(g) \times sB \cup tB \times T(f))$ induced by the inclusion $T(g) \times_B T(f) \subset T(f) \times T(g)$.*

Proofs are immediate from the definitions.

Lemma 1.11. *Let B be a simply connected space and let (f, s) be a pointed space over B such that ${}_BH_i(f, s) = \tilde{H}_i(T(f)/sB) = 0$ for $i < k$. Then ${}_BH_i(C_\varphi(f, s)) = 0$ for $i < k+2$, where $\tilde{H}_i$ is the ordinary homology theory.*

Proof. By the Künneth theorem of the ordinary homology, we have ${}_BH_i(\Gamma \circ \Phi(f, s)) = \tilde{H}_i((B_+) \wedge T(f)/sB) = 0$ for $i < k$ and the smash product $\tilde{H}_0(B_+) \otimes \tilde{H}_i(T(f)/sB) \to \tilde{H}_i((B_+) \wedge T(f)/sB)$ is an isomorphism for $i = k$, $k+1$. It follows that $(\varepsilon \wedge 1)_*\colon {}_BH_i(\Gamma \circ \Phi(f, s)) = \tilde{H}_i((B_+) \wedge T(f)/sB) \to \tilde{H}_i(S^0 \wedge T(f)/sB) = {}_BH_i(f, s)$ is an isomorphism for $i < k+2$, where $\varepsilon\colon B_+ \to S^0$ is the collapsing map. Since the composition ${}_BH_i(f, s) \xrightarrow{\varphi_*} {}_BH_i(\Gamma \circ \Phi(f, s)) \xrightarrow{(\varepsilon \wedge 1)_*} {}_BH_i(f, s)$ is the identity map, φ_* is an isomorphism for $i < k+2$. Consider the long exact sequence associated with the cofibration $(f, s) \xrightarrow{\varphi} \Gamma \circ \Phi(f, s) \xrightarrow{\iota} C_\varphi(f, s)$, then the result follows.

Corollary 1.12. *Let $\tilde{h}_*$ be a connective homology theory and let B be a simply connected space. For any pointed space (f, s) over B, ${}_B\tilde{h}_i(C^p_\varphi(f, s)) = 0$ for $i < 2p$.*

Proof. Applying 1.11, we see that ${}_BH_i(C^p_\varphi(f, s)) = 0$ for $i < 2p$ by induction on p. Then we have $\tilde{H}_i(\Phi C^p_\varphi(f, s); \tilde{h}_*(S^0))=0$ for $i<2p$ by the universal coefficient theorem of the ordinary homology. Consider the Atiyah-Hirzebruch spectral sequence $\tilde{H}_i(\Phi C^p_\varphi(f, s); \tilde{h}_j(S^0))\Rightarrow\tilde{h}_{i+j}(\Phi C^p_\varphi(f, s))$, then we have the result.

Lemma 1.13. *Let (f, s), (g, t) be pointed spaces over B. If f: $T(f)\to B$ is a Serre fibering whose fiber F is path connected, and if ${}_BH_i(f, s)=0$ for $i<k$, then ${}_BH_i((g, t)\underset{B}{\wedge}(f, s))=0$ for $i<k$.*

Proof. Since sB is a retract of $T(f)$, the long exact sequence associated with the cofibering $sB\to T(f)\to T(f)/sB$ splits into short exact sequences $0\to H_i(sB)\to H_i(T(f))\to\tilde{H}_i(T(f)/sB)\to 0$ $(i=0, 1, 2,\dots)$. The assumption implies that $H_i(sB)\to H_i(T(f))$ is an isomorphism for $i<k$. Hence f_*: $H_i(T(f))\to H_i(B)$ is an isomorphism for $i<k$. Since the composition $H_i(F)\to H_i(T(f))\to H_i(B)$ is zero unless $i=0$, we have $H_i(F)=0$ for $0<i<k$. Consider the Serre spectral sequence associated with the induced fibering $F\to T(g)\underset{B}{\times} T(f)\to T(g)$ by the map g. Then $E^2_{p,q}=H_p(T(g); H_q(F))=0$ for $0<q<k$ implies $E^\infty_{p,q}=0$ for $0<q<k$. Hence we have $F_{i-1,1}=0$ for $i<k$ which yields that the edge homomorphism $H_i(T(g)\underset{B}{\times} T(f))=F_{i,0}\to E^\infty_{i,0}\subset E^2_{i,0}=H_i(T(g))$ is injective for $i<k$. Since $T(g)$ is a retract of $T(g)\underset{B}{\times} T(f)$, it follows that $\bar{\imath}_*$: $H_i(T(g))\to H_i(T(g)\underset{B}{\times} T(f))$ is an isomorphism for $i<k$. Noting that $\bar{\imath}T(g)\cap\bar{s}T(f)=\{*\}\times sB$, consider the long exact sequence associated with the cofibering $T(g)\xrightarrow{\bar{\imath}}\bar{\imath}T(g)\cup\bar{s}T(f)\to T(f)/sB$. Since $T(g)$ is a retract of $\bar{\imath}T(g)\cup\bar{s}T(f)$ and $\tilde{H}_i(T(f)/sB)=0$ for $i<k$, $\bar{\imath}$: $H_i(T(g))\to H_i(\bar{\imath}T(g)\cup\bar{s}T(f))$ is an isomorphism for $i<k$. By the commutativity of the diagram

$$\begin{array}{ccc} H_i(\bar{\imath}\,T(g)\cup\bar{s}\,T(f)) & \longrightarrow & H_i(T(g)\underset{B}{\times} T(f)) \\ \nwarrow{\scriptstyle \bar{\imath}_*} & & \nearrow{\scriptstyle \bar{\imath}_*} \\ & H_i(T(g)) & \end{array}$$

$H_i(\bar{\imath}T(g)\cup\bar{s}T(f))\to H_i(T(g)\underset{B}{\times} T(f))$ is an isomorphism for $i<k$. Since $\Phi((g, t)\underset{B}{\wedge}(f, s))$ is the cofiber of the inclusion $\bar{\imath}T(g)\cup\bar{s}T(f)\subset T(g)\underset{B}{\times} T(f)$ by (1.10), we have ${}_BH_i((g, t)\underset{B}{\wedge}(f, s))=0$ for $i<k$.

Lemma 1.14. *Let (f, s) be a pointed space over B.*

(i) *If f: $T(f)\to B$ is a Serre fibration, so is $C_\varphi(f)$: $T(C_\varphi(f, s))\to B$.*

(ii) *If the total space $T(f)$ is path connected, each fiber of $C_\varphi(f)$:*

$T(C_\varphi(f, s))\to B$ is also path connected.

(iii) *If B is path connected, so is the total space $T(C_\varphi(f, s))$.*

Proofs are straightforward from the construction of C_φ.

Corollary 1.15. *Let $\tilde{h}_*$ be a connective homology theory and let B be a simply connected space. Then, for pointed spaces (g, t), (f, s) over B such that $f\colon T(f)\to B$ is a Serre fibration, we have ${}_Bh_q((g, t)\wedge_B C^p_\varphi(f, s))=0$ for $q<2p$ and $p\geqq 2$.*

Proof. By the above lemmas, $C^p_\varphi(f)\colon T(C^p_\varphi(f, s))\to B$ is a Serre fibration whose fiber is path connected if $p\geqq 2$ and ${}_BH_q(C^p_\varphi(f, s))=0$ for $q<2p$. Hence (1.12) implies ${}_BH_q((g, t)\wedge_B C^p_\varphi(f, s))=0$ for $q<2p$ and $p\geqq 2$. Applying the Atiyah-Hirzebruch spectral sequence, the result follows.

It follows from (1.15) that $A_{p,q}=0$ for $p\leqq 0$, $q\in\mathbb{Z}$, and $\cap_{p\leqq 0}F_{p,n-p}=0$ (in fact, $F_{p,n-p}=0$ for $p<\min\{-n, -1\}$). Now we have a sufficient condition for the convergence.

Theorem 1.16. *Assume that $\tilde{h}_*$ is a connective homology theory and (f, s) is a pointed space over a simply connected space B such that $f\colon T(f)\to B$ is a Serre fibration. Then, for any pointed space (g, t) over B, the Künneth spectral sequence constructed in* (1.7) *converges to ${}_Bh_*((g, t)\wedge_B (f, s))$.*

In order to make an identification of the E^2-term in terms of homological algebra, we have to assume some conditions. Let (f, s), (g, t) be pointed spaces over B.

Assumptions 1.17. (i) $\tilde{h}_*$ is a multiplicative homology theory on $\mathbf{Top}_*$ (not necessarily connective).

(ii) $\tilde{h}_*(B_+)$ is a flat $\tilde{h}_*(S^0)$-module.

(iii) Either $\tilde{h}_*(T(f)/sB)$ or $\tilde{h}_*(T(g)/tB)$ is flat over $\tilde{h}_*(S^0)$.

Under the above assumptions, the smash products $\tilde{h}_*(B_+)\otimes\tilde{h}_*(X)\to\tilde{h}_*((B_+)\wedge X)$, $\tilde{h}_*(X)\otimes\tilde{h}_*(B_+)\to\tilde{h}_*(X\wedge(B_+))$ are isomorphisms for any pointed space (X, x_0), where the tensor products are taken over $\tilde{h}_*(S^0)$. Define $\psi^L\colon T(f)/sB\to(B_+)\wedge T(f)/sB$ and $\psi^R\colon T(g)/tB\to T(g)/tB\wedge(B_+)$ by $\psi^L\circ\pi(x)=f(x)\wedge\pi(x)$ and $\psi^R\circ\rho(y)=\rho(y)\wedge g(x)$ respectively, where $\pi\colon T(f)\to T(f)/sB$ and $\rho\colon T(g)\to T(g)/tB$ are collapsing maps. Note that $\psi^L=\psi^R=$(the diagonal map of B) if $(f, s)=(g, t)=\Gamma(S^0, *)$. We put $C=\tilde{h}_*(B_+)$ $(={}_B\tilde{h}_*(\Gamma(S^0, *)))$ and define a coproduct $C\to C\otimes C$ to be the composite of the map induced by the diagonal map and the inverse of the smash product. The counit $C\to\tilde{h}_*(S^0)$ is the map induced by the collapsing map $\varepsilon: B_+\to S^0$. Let us define a left coaction of C on ${}_Bh_*(f, s)$ and a right coaction on ${}_Bh_*(g, t)$ as follows.

$$\begin{aligned}{}_Bh_*(f, s)=\tilde{h}_*(T(f)/sB) &\xrightarrow{\psi_*^L} \tilde{h}_*((B_+)\wedge T(f)/sB) \underset{\cong}{\xrightarrow{\wedge^{-1}}} \\ &\tilde{h}_*(B_+)\otimes\tilde{h}_*(T(f)/sB)=C\otimes{}_Bh_*(f, s)\\ {}_Bh_*(g, t)=\tilde{h}_*(T(g)/tB) &\xrightarrow{\psi_*^R} \tilde{h}_*(T(g)/tB\wedge(B_+)) \underset{\cong}{\xrightarrow{\wedge^{-1}}} \\ &\tilde{h}_*(T(g)/tB)\otimes\tilde{h}_*(B_+)={}_Bh_*(g, t)\otimes C\end{aligned}$$

Note that the following diagrams are commutative, where T is the swiching map.

$$\begin{array}{ccc} {}_Bh_*(f, s) & \xrightarrow{\varphi_*} & {}_Bh_*(\Gamma\circ\Phi(f, s)) \\ \| & & \| \\ \tilde{h}_*(T(f)/sB) & \xrightarrow{\psi_*^L} & \tilde{h}_*((B_+)\wedge T(f)/sB), \end{array}$$

$$\begin{array}{ccccc} {}_Bh_*(g, t) & & \xrightarrow{\varphi_*} & & {}_Bh_*(\Gamma\circ\Phi(f, s)) \\ \| & & & & \| \\ \tilde{h}_*(T(g)/tB) & \xrightarrow{\psi_*^R} & \tilde{h}_*(T(g)/tB\wedge(B_+)) & \xrightarrow{T_*} & \tilde{h}_*((B_+)\wedge T(g)/tB) \end{array}$$

We construct a natural map $\theta: {}_Bh_*((g, t)\underset{B}{\wedge}(f, s))\to{}_Bh_*(g, t)\square{}_Bh_*(f, s)$ under the assumption 1.17, where the cotensor products are taken over the coalgebra C. Let $\bar{s}$, $\bar{t}$ be maps defined in (1.10), then the composition of maps $T(g)\underset{B}{\times}T(f)/\bar{t}T(g)\cup\bar{s}T(f)\xrightarrow{\mu}T(g)\times T(f)/T(g)\times sB\cup tB\times T(f)=T(g)/tB\wedge T(f)/sB\xrightarrow{1\wedge\psi^L}T(s)/tB\wedge(B_+)\wedge T(f)/sB$ coincides with the composition $T(g)\underset{B}{\times}T(f)/\bar{t}T(g)\cup\bar{s}T(f)\xrightarrow{\mu}T(g)\times T(f)/T(g)\times sB\cup tB\times T(f)=T(g)/tB\wedge T(f)/sB\xrightarrow{\psi^R\wedge 1}T(g)/tB\wedge(B_+)\wedge T(f)/sB$, where μ is induced by the inclusion $T(g)\underset{B}{\times}T(f)\subset T(g)\times T(f)$. Noting that the smash products $\tilde{h}_*(T(g)/tB)\otimes\tilde{h}_*(T(f)/sB)\xrightarrow{\wedge}\tilde{h}_*(T(g)/tB\wedge T(f)/sB)$ and $\tilde{h}_*(T(g)/tB)\otimes\tilde{h}_*(B_+)\otimes\tilde{h}_*(T(f)/sB)\xrightarrow{\wedge}\tilde{h}_*(T(g)/tB\wedge(B_+)\wedge T(f)/sB)$ are isomorphisms, we see that the map ${}_Bh_*((g, t)\underset{B}{\wedge}(f, s))=\tilde{h}_*(T(g)\underset{B}{\times}T(f)/\bar{t}T(g)\cup\bar{s}T(f))\xrightarrow{\mu_*}\tilde{h}_*(T(g)/tB\wedge T(f)/sB)\xrightarrow{\wedge^{-1}}\tilde{h}_*(T(g)/tB)\otimes\tilde{h}_*\cdot(T(f)/sB)={}_Bh_*(g, t)\otimes{}_Bh_*(f, s)$ is lifted to the map θ.

Lemma 1.18. *${}_B\tilde{h}_*(\Gamma\circ\Phi(f, s))$ is an injective C-comodule. In fact the smash product gives an isomorphism as comodules*

$$C\otimes{}_Bh_*(f, s)=\tilde{h}_*(B_+)\otimes\tilde{h}(T(f)/sB)\xrightarrow{\wedge}\tilde{h}_*((B_+)\wedge T(f)/sB)={}_Bh_*(\Gamma\circ\Phi(f, s)).$$

Proof is straightforward.

Lemma 1.19. *The natural map* $\theta\colon {}_Bh_*((g, t)\underset{B}{\wedge}\Gamma\circ\Phi(f, s))\to {}_Bh_*(g, t)\square_Bh_*(\Gamma\circ\Phi(f, s))$ *is an isomorphism.*

Proof. Just note that the geometric fact $T(g)\underset{B}{\times}(B\times T(f)/sB)\cong T(g)\times T(f)/sB$ corresponds to the algebraic fact

$${}_Bh_*(g, t)\square(C\otimes{}_Bh_*(f, s))\cong{}_Bh_*(g, t)\otimes{}_Bh_*(f, s).$$

Lemma 1.20. $0\to{}_Bh_*(f, s)\xrightarrow{\varphi_*}{}_Bh_*(\Gamma\circ\Phi(f, s))\xrightarrow{\varphi_*\circ\iota_*}{}_Bh_*(\Gamma\circ\Phi\circ C_\varphi(f, s))\to\cdots\to{}_Bh_*(\Gamma\circ\Phi\circ C^i_\varphi(f, s))\xrightarrow{\varphi_*\circ\iota_*}{}_Bh_*(\Gamma\circ\Phi\circ C^{i+1}_\varphi(f, s))\to\cdots$ *is an injective resolution of* ${}_Bh_*(f, s)$.

Proof. The composition ${}_Bh_*(C^i_\varphi(f, s))\xrightarrow{\varphi_*}{}_Bh_*(\Gamma\circ\Phi\circ C^i_\varphi(f, s))\cong C\otimes{}_Bh_*(C^i_\varphi(f, s))\xrightarrow{\varepsilon_*\otimes 1}{}_Bh_*(C^i_\varphi(f, s))$ is the identity. Thus φ_* is a monomorphism, and the long exact sequence associated with the cofibration $C^i_\varphi(f, s)\to\Gamma\circ\Phi\circ C^i_\varphi(f, s)\to C^{i+1}_\varphi(f, s)$ splits into a short exact sequence $0\to{}_Bh_*(C^i_\varphi(f, s))\xrightarrow{\varphi_*}{}_Bh_*(\Gamma\circ\Phi\circ C^i_\varphi(f, s))\xrightarrow{\iota_*}{}_Bh_*(C^{i+1}_\varphi(f, s))\to 0$ of C-comodules. Splice these exact sequences for $i=0, 1, 2,\ldots$, and use (1.18), we have the result.

Remark 1.21. Under (i), (ii) of (1.17), the category of C-comodules becomes a relative abelian category ([1], [5]) and ${}_Bh_*$ is a homology theory $\mathbf{Top}_*/B\to$(the category of C-comodules). By the proof of (1.20), the C-comodule homomorphism $\varphi_*\colon{}_Bh_*(f, s)\to{}_Bh_*(\Gamma\circ\Phi(f, s))$ is a split monomorphism as a $h_*(S^0)$-module homomorphism.

Theorem 1.22. *Under Assumptions* 1.17, *the* E^2*-term of the Künneth spectral sequence is naturally isomorphic to* $\mathrm{Cotor}^C_{*,*}({}_Bh_*(g, t), {}_Bh_*(f, s))$ $(E^2_{p,q}\cong\mathrm{Cotor}^C_{p,q}({}_Bh_*(g, t), {}_Bh_*(f, s)))$.

Proof. This follows from (1.19) and (1.20).

Remarks 1.23. (i) The edge homomorphism ${}_Bh_*((g, t)\underset{B}{\wedge}(f, s))\to E^2_{0,n}\cong\mathrm{Cotor}^C_{0,n}({}_Bh_*(g, t), {}_Bh_*(f, s))={}_Bh_*(g, t)\square{}_Bh_*(f, s)$ coincides with the natural map θ.

(ii) (1.17) is always satisfied if $\tilde{h}_*$ is ordinary homology theory over a field or Morava K-theory.

(iii) We consider the category of spaces over B, denoted by $\mathbf{Top}/B$. An object of $\mathbf{Top}/B$ is a continuous map $f\colon T(f)\to B$, and a morphism $\varphi\colon f\to g$ in $\mathbf{Top}/B$ is a continuous map $T(\varphi)\colon T(f)\to T(g)$ such that $g\circ T(\varphi)=f$. Define a functor $G\colon\mathbf{Top}/B\to\mathbf{Top}_*/B$ as follows. Put $G(f)=(f_+, s_f)$ and $T(G(f))=T(f_+)$

$=T(f)\amalg B$ (disjoint union), f_+ and s_f are given by $f_+(x)=f(x)$ for $x\in T(f)$ $f_+(b)=b$ for $b\in B$ and $s_f(b)=b$. Then it is easy to verify that $\Phi(G(f))=T(f)_+$ and $\Phi(G(g)\underset{B}{\wedge}G(f))=(T(g)\underset{B}{\times}T(f))_+$ where f and g are spaces over B. Therefore, if a fiber product of f and g is given, the Künneth spectral sequence associated with $G(f)$ and $G(g)$ is the Eilenberg-Moore spectral sequence ([7], [8]).

§ 2. A Relation between the Algebraic Bockstein Homomorphism and the Geometric Bockstein Homomorphism

Throughout this section, we assume that B is a simply connected space such that $H_*(B: \mathbb{Z}_{(l)})$ is flat and that (f, s), (g, t) are pointed spaces over B such that both $H_*(T(f)/sB: \mathbb{Z}_{(l)})$ and $H_*(T(g)/tB: \mathbb{Z}_{(l)})$ are flat and $f: T(f)\to B$ is a Serre fibration, where l is a fixed prime number. Note that a $\mathbb{Z}_{(l)}$-module is flat if and only if it is torsion free.

Lemma 2.1. *Under the above assumptions,* ${}_BH_*(C^i_\varphi(f, s): \mathbb{Z}_{(l)})$ $(i=0, 1, 2,\dots)$ *is flat.*

Proof. Inductively, assume that ${}_BH_*(C^i_\varphi(f, s): \mathbb{Z}_{(l)})$ is flat. The cofibration $C^i_\varphi(f, s)\xrightarrow{\varphi}\Gamma\circ\Phi\circ C^i_\varphi(f, s)\xrightarrow{\iota}C^{i+1}_\varphi(f, s)$ gives a short exact sequence $0\to{}_BH_*(C^i_\varphi(f, s): \mathbb{Z}_{(l)})\to{}_BH_*(\Gamma\circ\Phi\circ C^i_\varphi(f, s): \mathbb{Z}_{(l)})\to{}_BH_*(C^{i+1}_\varphi(f, s): \mathbb{Z}_{(l)})\to 0$ which splits as $\mathbb{Z}_{(l)}$-modules. By (1.18), ${}_BH_*(\Gamma\circ\Phi\circ C^i_\varphi(f, s): \mathbb{Z}_{(l)})$ is isomorphic to $H_*(B: \mathbb{Z}_{(l)})\otimes{}_BH_*(C^i_\varphi(f, s): \mathbb{Z}_{(l)})$ which is also flat. Hence ${}_BH_*(C^{i+1}_\varphi(f, s): \mathbb{Z}_{(l)})$ is flat.

Consider two Künneth spectral sequences converging to ${}_BH_*((g, t)\underset{B}{\wedge}(f, s): F_l)$ and ${}_BH_*((g, t)\underset{B}{\wedge}(f, s): \mathbb{Z}_{(l)})$. We put

$$D^1_{p,q}={}_BH_q((g, t)\underset{B}{\wedge}C^{-p}_\varphi(f, s): F_l),\quad E^1_{p,q}={}_BH_q((g, t)\underset{B}{\wedge}\Gamma\circ\Phi\circ C^{-p}_\varphi(f, s): F_l)$$

$$\bar{D}^1_{p,q}={}_BH_q((g, t)\underset{B}{\wedge}C^{-p}_\varphi(f, s): \mathbb{Z}_{(l)}),\quad \bar{E}^1_{p,q}={}_BH_q((g, t)\underset{B}{\wedge}\Gamma\circ\Phi\circ C^{-p}_\varphi(f, s): \mathbb{Z}_{(l)}).$$

By (1.19) and (2.1), $\bar{E}^1_{p,q}$ is torsion free. Hence the Bockstein exact sequence associated splits into short exact sequences $0\to\bar{E}^1_{p,q}\xrightarrow{l\times}\bar{E}^1_{p,q}\xrightarrow{\rho}E^1_{p,q}\to 0$. Note that the multiplication by l and the mod l reduction ρ induce maps of exact couples. Taking the homologies of complexes $\{\bar{E}^1_{*,*}, \bar{d}^1\}$ and $\{E^1_{*,*}, d^1\}$, we have the algebraic Bockstein homomorphism $\tilde{\delta}: E^2_{p,q}\to\bar{E}^2_{p-1,q}$ as the boundary homomorphism.

Theorem 2.2. *If* $x\in E^2_{p,q}$ *is a permanent cycle,* $\tilde{\delta}x\in\bar{E}^2_{p-1,q}$ *is also a* *permanent cycle. Let $\bar{x}\in F_{p,q}$ be the element of ${}_BH_*((g, t)\wedge_B(f, s); F_l)$ corresponding to x, then $\delta\bar{x}\in\bar{F}_{p-1,q}=\mathrm{Im}\{\bar{D}^1_{p-1,q}\to\bar{D}^1_{0,p+q-1}\}$ and $\delta\bar{x}$ corresponds to the permanent cycle $-\tilde{\delta}x$, where δ: ${}_BH_n((g, t)\wedge_B(f, s); F_l)\to{}_BH_{n-1}((g, t)\wedge_B(f, s); Z_{(l)})$ is the geometric Bockstein homomorphism.*

Remark 2.3. In the case $x=0$ in the E^∞-term, the above statement means that $\tilde{\delta}x=0$ in the E^∞-term. If $x\neq 0$ and $\tilde{\delta}x=0$ in each E^∞-terms, we assert that $\delta\bar{x}\in\bar{F}_{p-2,q+1}$.

The following lemma implies the above theorem.

Lemma 2.4. *Let $X\xrightarrow{i}Y\xrightarrow{j}Z$ be a cofibration such that $\tilde{H}_*(Y; Z_{(l)})$ is torsion free. Suppose that a space W and a map k; $Z\to W$ such that $\tilde{H}_*(W; Z_{(l)})$ is torsion free are given. Let ∂, ∂'; $\tilde{H}_q(X; F_l)\to\tilde{H}_q(W; Z_{(l)})/\mathrm{Im}\,k_*\circ j_*$ be the maps defined as follows, for each $x\in\tilde{H}_q(X; F_l)$, take $y\in\tilde{H}_q(Y; Z_{(l)})$ such that $\rho y=i_*x$. We can take $z\in\tilde{H}_q(Z; Z_{(l)})$ such that $lz=j_*y$. Then ∂x is defined to be the image of k_*z by the projection π: $\tilde{H}_q(W: Z_{(l)})\to\tilde{H}_q(W: Z_{(l)})/\mathrm{Im}\,k_*\circ j_*$. On the other hand, there exists $z'\in\tilde{H}_q(Z; Z_{(l)})$ such that $\Delta z'=\delta x$, where δ: $\tilde{H}_q(X; F_l)\to\tilde{H}_{q-1}(X: Z_{(l)})$ is the Bockstein homomorphism and Δ: $\tilde{H}_q(Z; Z_{(l)})\to\tilde{H}_{q-1}(X; Z_{(l)})$ is the boundary homomorphism. $\partial'x$ is defined to be $\pi\circ k_*z'$. Then $\partial=-\partial'$ holds.*

Proof. It is easy to check that ∂ and ∂' are well-defined. We may assume $X\subset Y$ and replace $\tilde{H}_q(Z; Z_{(l)})$ by $H_q(Y, X; Z_{(l)})$. Let $S_*(X)$ and $S_*(Y)$ be the singular chain complexes of X and Y with $Z_{(l)}$-coefficients. For $x\in\tilde{H}_q(X; F_l)$, we take a chain $\sigma\in S_q(X)$ such that x is represented by the cycle $\rho_\sharp\sigma$ ($\rho_\sharp$ is the mod l reduction map). We put $d\sigma=l\alpha$ ($\alpha\in S_{q-1}(X)$), where d is the differential of $S_*(X)$. Since $\tilde{H}_q(Y; Z_{(l)})$ is torsion free, we can take a cycle $\bar{\sigma}\in S_q(Y)$ such that $\rho_\sharp\bar{\sigma}$ is homologous to $i_\sharp\circ\rho_\sharp\sigma$. Therefore $\sigma-\bar{\sigma}\in lS_q(Y)+d(S_{q+1}(Y))$, and we put $\sigma-\bar{\sigma}=l\beta+d\tau$ ($\beta\in S_q(Y)$, $\tau\in S_{q+1}(Y)$). Since $\bar{\sigma}$ is a cycle, $\bar{\sigma}+d\tau$ is also a cycle homologous to $\bar{\sigma}$. So we may replace $\bar{\sigma}+d\tau$ by $\bar{\sigma}$ and we have $\sigma=\bar{\sigma}+l\beta$. It follows from $d\sigma=l\alpha$ that $d\beta=\alpha$. Since $j_\sharp\bar{\sigma}=-l\beta$, $-d\beta=-\alpha$ represents $\Delta z'\in\tilde{H}_{q-1}(X; Z_{(l)})$. On the other hand, $\delta x\in\tilde{H}_{q-1}(X: Z_{(l)})$ is represented by α. This completes the proof.

Proof of (2.2). We put $X=\Phi((g, t)\wedge_B C_\varphi^{-p}(f, s))$, $Y=\Phi((g, t)\wedge_B\Gamma\circ\Phi\circ C_\varphi^{-p}(f, s))$, $Z=\Phi((g, t)\wedge_B C_\varphi^{-p+1}(f, s))$, $W=\Phi((g, t)\wedge_B\Gamma\circ\Phi\circ C_\varphi^{-p+1}(f, s))$. Suppose $x\in E^2_{p,q}$ is a permanent cycle. Let $x'\in E^1_{p,q}=\tilde{H}_q(Y; F_l)$ be a cycle which represents x. Since x' is also a permanent cycle, there exist $y\in D^1_{p,q}=\tilde{H}_q(X; F_l)$ such that

$(1 \wedge \varphi)_* y = x'$. Then, it is easy to see that $\partial y = \tilde{\delta} x$ in $E^2_{p-1,q}$. ∂y is an element coming from $\bar{D}^1_{p-1,q} = H_q(Z: \mathbb{Z}_{(l)})$ by the definition of ∂. Thus $\tilde{\delta} x$ is a permanent cycle. By the preceding lemma, $\delta \bar{x}$ belongs to $\bar{F}_{p-1,q}$ and corresponds to $-\tilde{\delta} x$.

§ 3. On the Homology Suspensions

Let B be a path connected topological space with a base point $*$ and let PB be the space of paths in B starting from $*$. And let us denote by $p: PB \to B$ the evaluation map at 1, and also denote by $i: * \to B$ the inclusion. We fix these notations throughout this section. The following lemma is easily verified.

Lemma 3.1. (i) *The total space of $\Gamma \circ \Phi \circ G(p)$ is given by $(B \times PB) \amalg B$ and the projection $(B \times PB) \amalg B \to B$ maps both $(b, l) \in B \times PB$ and $b \in B$ to b. The section $B \to (B \times PB) \amalg B$ maps b to b. Moreover, $\varphi: G(p) \to \Gamma \circ \Phi \circ G(p)$ is given by $T(\varphi)(l) = (l(1), l)$ for $l \in PB$, $T(\varphi)(b) = b$ for $b \in B$.*

(ii) *The total space of $C_\varphi \circ G(p)$ is the quotient space of $(PB \times I) \amalg (B \times PB)$ by the equivalence relation generated by $(l, 1) \sim (l(1), l)$ and $(l', 0) \sim (l'', 0)$ if $l'(1) = l''(1)$. The projection is given by $[l, r] \to l(1)$, $[b, l] \to b$ for $l \in PB$, $r \in I$, $b \in B$, and the section is given by $b \to [l_b, 0]$, where l_b is any element of PB such that $l_b(1) = b$. And the inclusion $\iota: \Gamma \circ \Phi \circ G(p) \to C_\varphi \circ G(p)$ is given by $T(\iota) \cdot (b, l) = [b, l]$, $T(\iota)(b) = [l_b, 0]$, where $b \in B$, $l \in PB$ and l_b is as above.*

We denote the total space of $C_\varphi \circ G(p)$ by T_B and denote the projection and the section by $\tilde{p}: T_B \to B$ and $\tilde{s}: B \to T_B$.

Lemma 3.2. (i) *Let (f, s) be a pointed space over B, then we have $\Phi(G(i) \wedge_B (f, s)) = f^{-1}(*)$. In particular, we have $\Phi(G(i) \wedge_B G(p)) = \Omega B_+$, $\Phi(G(i) \wedge_B \Gamma \circ \Phi \circ G(p)) = PB_+$, $\Phi(G(i) \wedge_B C_\varphi \circ G(p)) = PB \cup C\Omega B$, where $C\Omega B$ is the unreduced cone $\Omega B \times I / \Omega B \times \{0\}$ with a base point $\Omega B \times \{0\} / \Omega B \times \{0\}$ and we identify $\omega \in \Omega B (\subset PB)$ with $[\omega, 1] \in C\Omega B$. Moreover, $\Phi(1 \wedge \varphi)$ and $\Phi(1 \wedge \iota)$ are natural inclusions $\Omega B_+ \to PB_+$, $PB_+ \to PB \cup C\Omega B$.*

(ii) $\Phi(G(i) \wedge_B \Gamma \circ \Phi \circ C_\varphi \circ G(p)) = \Phi \circ C_\varphi \circ G(p) = T_B / \tilde{s} B = ((PB \times I) \amalg$ $(B \times PB) \Big/ \begin{smallmatrix} (l, 1) \sim (l(1), l) \\ (l', 0) \sim (l'', 0) \end{smallmatrix}$, *and $\Phi(1 \wedge \varphi): PB \cup C\Omega B \to T_B / \tilde{s} B$ is given by $\Phi(1 \wedge \varphi) \cdot ([l]) = [*, l]$, $\Phi(1 \wedge \varphi)([\omega, t]) = [\omega, t]$*

Proof. (i) is straightforward. (ii) is verified by applying (1.10), (iii).

Lemma 3.3. *Define $\pi: T_B / \tilde{s} B \to B$ by $\pi[b, l] = b$, $\pi[l, t] = l(t)$, then π is a* *natural (stable) homotopy equivalence and the diagram*

$$\begin{array}{ccc} \Phi\circ\Gamma\circ\Phi\circ G(p)=(B\times PB)_+ & \xrightarrow{\Phi(\iota)} & \Phi\circ C_\varphi\circ G(p)=T_B/\tilde{s}B \\ \downarrow{\scriptstyle pr_+} & & \downarrow{\scriptstyle \pi} \\ B_+ & \xrightarrow{1_B\amalg *} & B \end{array}$$

is commutative, where pr: $B\times PB\to B$ is the projection $pr(b, l)=b$.

Proof. Commutativity of the above diagram is obvious. Consider the following homotopy commutative diagram.

$$\begin{array}{ccccc} \Phi\circ G(p)=PB_+ & \xrightarrow{\Phi(\varphi)} & \Phi\circ\Gamma\circ\Phi\circ G(p) & \xrightarrow{\Phi(\iota)} & \Phi\circ C_\varphi\circ G(p) \\ \downarrow{\scriptstyle \varepsilon_+} & & \downarrow{\scriptstyle pr_+} & & \downarrow{\scriptstyle \pi} \\ S^0=\{*, +\} & \xrightarrow{i_+} & B_+ & \xrightarrow{1_B\amalg *} & B \end{array}$$

where ε: $PB\to *$. The both horizontal rows are cofiber sequences and ε_+ and pr_+ are homotopy equivalences. Hence we have the result.

Lemma 3.4. *Let c: $PB\cup C\Omega B\to\sum\Omega B$ be the map which collapses $PB\cup\{[\omega_0, r]|\omega_0$ is the constant loop at $*\}$ to the base point and let σ: $\sum\Omega B\to B$ be the adjoint of the identity map of ΩB; that is, σ is defined by $\sigma([\omega, t])=\omega(t)$. Then, c is a homotopy equivalence and the following diagram is commutative.*

$$\begin{array}{ccc} \Phi(G(i)\underset{B}{\wedge} C_\varphi\circ G(p))=PB\cup C\Omega B & \xrightarrow{c} & \sum\Omega B \\ \downarrow{\scriptstyle \Phi(1\wedge\varphi)} & & \downarrow{\scriptstyle \sigma} \\ \Phi(G(i)\underset{B}{\wedge}\Gamma\circ\Phi\circ C_\varphi\circ G(p))=T_B/\tilde{s}B & \xrightarrow{\pi} & B \end{array}$$

Proof. It is obvious that c is a homotopy equivalence, and we can verify that the diagram commutes, applying (3.2) and (3.3).

Let $\tilde{h}_*$ be a multiplicative homology theory on $\mathbf{Top}_*$, and let us consider the Künneth spectral sequence associated with $\tilde{h}_*$ and the pointed spaces $G(i)$, $G(p)$. In other words we consider the Eilenberg-Moore spectral sequence associated with the path fibering $\Omega B\to PB\to B$. The preceding lemma implies the following.

Theorem 3.5. *Assume that $h_*(B)$ is a flat $h_*(pt)$-module. Then $F_{-1,n+1}=\tilde{h}_n(\Omega B)=\ker\{h_n(\Omega B)\to h_n(pt)\}$ and the composition $\tilde{h}_n(\Omega B)=F_{-1,n+1}\to F_{-1,n+1}/F_{-2,n+2}\subset E^\infty_{-1,n+1}\subset E^2_{-1,n+1}\cong Ph_{n+1}(B)\subset h_{n+1}(B)$ coincides with the homology suspension $\tilde{h}_n(\Omega B)\cong\tilde{h}_{n+1}(\sum\Omega B)\xrightarrow{\sigma_*}h_{n+1}(B)$, where $Ph_*(B)$ is the submodule of $h_*(B)$ consists of primitive elements which is naturally identified with* $\mathrm{Cotor}^{h_*(B)}_{-1,*}$.

$(h_*(pt), h_*(pt))$ *and we identify* $E^2_{-1,n+1}$ *with* $\mathrm{Cotor}^{h_*(B)}_{-1,n+1}(h_*(pt), h_*(pt))$ *as in Section* 1.

Corollary 3.6. *Under the same assumption as above, if the homology suspension* $\sigma_*: \tilde{h}_n(\Omega B) \to h_{n+1}(B)$ *maps surjectively onto* $Ph_{n+1}(B)$, *every element of* $E^2_{-1,n+1}$ *is a permanent cycle. Assume further that the Künneth spectral sequence converges. If every element of* $E^1_{-1,n+1}$ *is a permanent cycle,* σ_* *maps* $\tilde{h}_n(\Omega B)$ *surjectively onto* $Ph_{n+1}(B)$.

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