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On the Group of Equivariant Self Equivalences of Free Actions

Dedicated to Professor Masahiro Sugawara on his 60th Birthday

By

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§ 1. Introduction

Let (P, q, B, G) be a principal fibre bundle with structure group G and with projection q. In [21] we have considered the group of G-equivariant homotopy classes of unbased (resp. based) G-equivariant self homotopy equivalences of the total space P under the free G-action on P. The group structure is given by the composition of maps. This group is denoted by $\mathscr{F}_G(P)$ (resp. $\mathscr{E}_G(P)$). In this note we shall continue to study this group and obtain a generalization of Theorem 2.1 in [21] (Theorem 2.2 in §2), which will enable us to compute the group $\mathscr{F}_G(P)$, even if P is not simply-connected. It is shown that if any finite group of order greater than 2 acts freely on the sphere S^{2n+1} $(n \geqq 0)$, then $\mathscr{F}_G(S^{2n+1}) = 1$. We also show that $\mathscr{F}_G(P)$ and $\mathscr{E}_G(P)$ are finitely presented groups under suitable conditions. In §3 we shall study the Samelson products of the classical groups $U(n)$ and $SO(n)$ to compute the group $\mathscr{F}_G(P)$. Examples are worked out in §4.

Notations are used as in [21]. For example, we denote the homotopy set $[X, \{x_0\}; Y, \{y_0\}]$ by $[X, Y]$ for spaces X, Y with base points x_0, y_0, and we do not distinguish a map and its homotopy class. We take, if necessary, the unit of a topological group as the base point.

§ 2. Statement of Theorem

We consider a numerable principal G-bundle (see [6, p. 248])

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$$(P_k,\ q,\ B,\ G)$$

with total space $P=P_k$, base space B and projection $q\colon P_k \to B$, structure group G and classifying map $k\colon B\to B_G$. We always assume that G, P, B are compactly-generated Hausdorff spaces. We also assume one of the following:

(i) B is a CW-complex,

(ii) G is compact,

(iii) B is locally compact.

Any self bundle map f on P induces naturally a self map $\bar{f}$ on the base space B such that $qf=\bar{f}q$ and this construction determines a continuous map $\Phi\colon map_G(P,\ P)\to map_k(B,\ B)$, where $map_k(B,\ B)$ is the space of maps $g\colon B\to B$ such that kg is freely homotopic to k. By the covering homotopy theorem for bundle maps (cf. [6, (7.8)]), it follows that Φ is a Serre fibration with fibre the space $I_G(P)$ of unbased bundle equivalences over B. It is easy to see $\Phi^{-1}(aut_k(B))=aut_G(P)$, where $aut_k(B)=aut\ B\cap map_k(B,\ B)$. Hence we have a Serre fibration:

$$I_G(P)\xrightarrow{i} aut_G(P)\xrightarrow{\Phi} aut_k(B).$$

By using the group isomorphism $\pi_0(I_G(P))\overset{d}{\cong}\pi_1(map(B,\ B_G),\ k)$ (see [21, p. 88]) and $\pi_1(aut_k(B),\ 1)\cong\pi_1(map(B,\ B),\ 1)$, we have the following theorem:

Theorem 2.1 ([21, *Theorem* 1.5]). *Let $(P_k,\ q,\ B,\ G)$ be a numerable principal G-bundle. Then we have the exact sequence of groups*:

$$\pi_1(map(B,\ B),\ 1)\xrightarrow{(k^{\#})_*}\pi_1(map(B,\ B_G),\ k)\xrightarrow{\nu}\mathscr{F}_G(P_k)\xrightarrow{\rho}\mathscr{F}_k(B)\longrightarrow 1,$$

where $k^{\#}\colon map(B,\ B)\to map(B,\ B_G)$ is given by $k^{\#}(f)=kf$, $\mathscr{F}_G(P_k)=\pi_0(aut_G(P))$, $\mathscr{F}_k(B)=\pi_0(aut_k(B))$, $\rho=\Phi_*$ on π_0, and $\nu=i_*d^{-1}$.

Especially if B is a suspended complex of a connected complex, then we have the following, which is a generalization of Theorem 2.1 in [21].

Theorem 2.2. *Let $(P_k,\ q,\ SZ,\ G)$ be a numerable principal G-bundle over suspended complex SZ, where Z is a connected CW-complex ($k\in[SZ,\ B_G]=[Z,\ G]$). Then we have the following commutative diagram with exact rows of groups except at θ*:

$$\begin{array}{ccccccccc}
[S^2Z,\ SZ]/[1,\ \pi_2(SZ)] & \longrightarrow & \pi_1(map(SZ,\ B_G),\ k) & \xrightarrow{\nu} & \mathscr{F}_G(P_k) & \xrightarrow{\rho} & \mathscr{F}_k(SZ) & \longrightarrow & 1\\
\downarrow{\scriptstyle \bar{\chi}} & & \| & & & & & & \\
1\longrightarrow [SZ,\ G]/\langle k,\ \pi_1(G)\rangle & \longrightarrow & \pi_1(map(SZ,\ B_G),\ k) & \xrightarrow{\omega_*} & \pi_0(G) & \xrightarrow{\theta} & [Z,\ G], & &
\end{array}$$

where $1=1_{SZ}\in[SZ, SZ]$, $[1,\ ](\langle k,\ \rangle)$ *is a generalized Whitehead (Samelson) product* (cf. [1]), *and* $\bar{\chi}$ *is induced by the characteristic homomorphism* χ: $[S^2Z, SZ]\to[SZ, G]$.

Especially if G *is* 1-*connected, then we have the following exact sequence of groups*:

$$[S^2Z, P_k]\xrightarrow{q_*}[S^2Z, SZ]\xrightarrow{\chi}[SZ, G]\xrightarrow{\nu}\mathscr{F}_G(P_k)\xrightarrow{\rho}\mathscr{F}_k(SZ)\longrightarrow 1\,.$$

Proof. By considering the evaluation fibration

$$\omega\colon map(SZ, D)\longrightarrow D \quad \text{with fibre} \quad map_*(SZ, D)$$

for $D=SZ$ or B_G, we have the following commutative diagram with exact rows of groups except at θ:

$$\text{(2.3)}\qquad \begin{array}{ccccc} \pi_2(SZ) & \xrightarrow{\Delta} & \pi_1(map_*(SZ, SZ), 1) & \longrightarrow & \pi_1(map(SZ, SZ), 1) \\ \big\downarrow k_* & & \big\downarrow (k^{\#})_* & & \big\downarrow (k^{\#})_* \\ \pi_2(B_G) & \xrightarrow{\Delta'} & \pi_1(map_*(SZ, B_G), k) & \longrightarrow & \pi_1(map(SZ, B_G), k) \end{array}$$

$$\begin{array}{l} \longrightarrow \pi_1(SZ)=1 \\ \qquad \big\downarrow k_* \\ \longrightarrow \pi_1(B_G)=\pi_0(G)\xrightarrow{\theta}\pi_0(map_*(SZ, B_G), k) \end{array}$$

Since $map_*(SZ, D)$ is an invertible H-space, it follows that for any element β of $map_*(SZ, D)$ the multiplication by β induces a self equivalence $\tilde{\beta}$ of $map_*(SZ, D)$ so that

$$\pi_i(map_*(SZ, D), \beta)\overset{\tilde{\beta}_*}{\cong}\pi_i(map_*(SZ, D), *)=[S^{i+1}Z, D]\,,$$

where * denotes the constant map. The following diagram is also commutative, since $k^{\#}$ is an H-map.

$$\begin{array}{ccccc} \pi_2(SZ) & & \xrightarrow{\quad k_* \quad} & & \pi_2(B_G) \\ & & & \cong & \\ \big\downarrow [1,\ ] & & \pi_1(G) & & \big\downarrow [k,\ ] \\ & & \big\downarrow \langle k,\ \rangle & & \\ & \overset{\chi}{\nearrow} & [SZ, G] & \cong & \\ [S^2Z, SZ] & & \xrightarrow{\quad k_* \quad} & & [S^2Z, B_G] \\ \cong\big\downarrow & & & & \cong\big\downarrow \\ \pi_1(map_*(SZ, SZ), *) & & \xrightarrow{(k^{\#})_*} & & \pi_1(map_*(SZ, B_G), *) \\ \cong\big\downarrow \tilde{1}_* & & & & \cong\big\downarrow \tilde{k}_* \\ \pi_1(map_*(SZ, SZ), 1) & & \xrightarrow{(k^{\#})_*} & & \pi_1(map_*(SZ, B_G), k) \end{array}$$

The composition of the vertical maps are Δ and Δ' by [12, Theorem 2.6]. Hence by (2.3) we have the commutative diagram with exact rows of groups except at θ.

$$\begin{array}{ccccccccc} 1 \longrightarrow & [S^2Z, SZ]/[1, \pi_2(SZ)] & \longrightarrow & \pi_1(map(SZ, SZ), 1) & \longrightarrow 1 & & & & \\ & \downarrow \bar{\chi} & & \downarrow (k^{\#})_* & & & & & \\ 1 \longrightarrow & [SZ, G]/\langle k, \pi_1(G)\rangle & \longrightarrow & \pi_1(map(SZ, B_G), k) & \xrightarrow{\omega_*} & \pi_0(G) & \xrightarrow{\theta} & [Z, G]. \end{array}$$

Therefore by Theorem 2.1 we obtain the first diagram in Theorem 2.2.

The last sequence in Theorem 2.2 follows by the first diagram and the well-known exact sequence:

$$[SY, P] \xrightarrow{q_*} [SY, B] \xrightarrow{\chi} [Y, G],$$

where (P, q, B, G) is a principal G-bundle and Y is a CW-complex. q. e. d.

Corollary 2.4. *Let (P_k, q, S^n, G) $(n \geqq 2)$ be a numerable principal G-bundle with classifying map $k \in \pi_n(B_G) = \pi_{n-1}(G)$. Then we have the following commutative diagram with exact rows of groups except at θ:*

$$\begin{array}{ccccccccc} & \pi_{n+1}(S^n) & \longrightarrow & \pi_1(map(S^n, B_G), k) & \xrightarrow{\nu} & \mathscr{F}_G(P_k) & \xrightarrow{\rho} & \mathscr{F}_k(S^n) & \longrightarrow 1 \\ & \downarrow \bar{\chi} & & \| & & & & & \\ 1 \longrightarrow & \pi_n(G)/\langle k, \pi_1(G)\rangle & \longrightarrow & \pi_1(map(S^n, B_G), k) & \xrightarrow{\omega_*} & \pi_0(G) & \xrightarrow{\theta} & \pi_{n-1}(G), & \end{array}$$

where $\mathscr{F}_k(S^n)$ is Z_2 or 1 according as $2k$ is zero or not.

In particular, if G is path-connected, then we have the exact sequence:

$$\pi_{n+1}(S^n) \xrightarrow{\bar{\chi}} \pi_n(G)/\langle k, \pi_1(G)\rangle \xrightarrow{\nu} \mathscr{F}_G(P_k) \xrightarrow{\rho} \mathscr{F}_k(S^n) \longrightarrow 1.$$

Theorem 2.5. *Let G be a finite group of order greater than 2, and let G act on the odd dimensional sphere S^{2n+1} $(n \geqq 0)$ freely. Then*

$$\mathscr{F}_G(S^{2n+1}) = 1.$$

Proof. It is well-known that $(S^{2n+1}, q, S^{2n+1}/G = M, G)$ is a principal G-bundle. By the result of [15, Theorem 1.4],

$$\mathscr{E}(M) \subset Aut\, \pi_1(M),$$

where the injection is given by the induced homomorphism on the fundamental group $\pi_1(M)$. We have the following diagram:

$$(2.6) \qquad \begin{array}{ccccc} \pi_1(map(M, M), 1) & \xrightarrow{(k^{\#})_*} & \pi_1(map(M, B_G), k) & & \\ \downarrow (\omega_1)_* & & \downarrow (\omega_2)_* & & \\ \pi_1(M) & \xrightarrow{k_*} & \pi_1(B_G) & \xrightarrow{\partial} & \pi_0(S^{2n+1}) = 0. \end{array}$$

In this diagram one can see that the evaluation homomorphism $(\omega_2)_*$ is a monomorphism, since $\pi_1(map_*(M, B_G), k)=0$ by the similar way of the proof in [8, Lemma 3]. By the evaluation fibration:

$$aut_* M \longrightarrow aut\, M \xrightarrow{\omega_1} M,$$

we have the following exact sequence:

$$\pi_1(map(M, M), 1) \xrightarrow{(\omega_1)_*} \pi_1(M) \xrightarrow{\partial} \mathscr{E}(M) \longrightarrow \mathscr{F}(M) \longrightarrow 1,$$

where $\partial(\alpha)$ $(\alpha \in \pi_1(M))$ is given by the inner automorphism of $\pi_1(M)$ by α by definition. Hence

$$Im\,(\omega_1)_* \supset Z(\pi_1(M)) \quad (\text{center of } \pi_1(M)).$$

On the contrary, by [8, p. 847]

$$Im\,(\omega_1)_* \subset Z(\pi_1(M)).$$

Therefore we have $Im\,(\omega_1)_* = Z(\pi_1(M))$. Also by [9, Lemma 2]

$$Im\,(\omega_2)_* = Z(k_*(\pi_1(M))).$$

Let $n>0$. Since $\pi_1(M)=\pi_1(B_G)=G$ is a finite group, k_* is an isomorphism in (2.6). Therefore k_* maps $Im\,(\omega_1)_*$ onto $Im\,(\omega_2)_*$. Hence $(k^\sharp)_*$ is a surjection. And by Theorem 2.1,

$$\mathscr{F}_G(S^{2n+1}) = \mathscr{F}_k(M).$$

The last equality holds even if $n=0$, since in this case M is homeomorphic to S^1 and G is a cyclic group.

By the evaluation fibrations:

$$aut_G(S^{2n+1}) \xrightarrow{\omega} S^{2n+1} \quad \text{and} \quad aut\, M \xrightarrow{\omega} M,$$

we have the following commutative diagram:

$$\begin{array}{ccccc} \mathscr{E}_G(S^{2n+1}) & = & \mathscr{F}_G(S^{2n+1}) & & \\ \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \\ \mathscr{E}(M) & \longrightarrow & \mathscr{F}(M) & \longrightarrow & 1, \end{array}$$

where the homomorphism ρ' is defined naturally. Every element f in $\mathscr{F}_G(S^{2n+1})$ is represented by an element f' in $\mathscr{E}_G(S^{2n+1})$. And we have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{1} & G \\ \downarrow & & \downarrow \\ S^{2n+1} & \xrightarrow{f'} & S^{2n+1} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\bar{f}'} & M. \end{array}$$

Hence $\bar{f}'$ induces the identity automorphism on the fundamental group $\pi_1(M)$. Therefore $\bar{f}'$ is based homotopic to the identity map so that $\rho(f)=1$. Therefore $\mathscr{F}_k(M)=1$. This completes the proof. q.e.d.

In the above proof we have shown the following:

Proposition 2.7. *Let G be a finite group of order greater than 2, and let G act on the odd dimensional sphere S^{2n+1} $(n\geqq 0)$ freely. Then the evaluation subgroup $G_1(S^{2n+1}/G)=Z(G)$* (see [8]).

By the similar way (see [8]) we have the following, which is also obtained by [3, p. 123] and [21, (1.4)].

Example 2.8. *Let $(S^n, q, P^n(R), Z_2)$ be a principal Z_2-bundle over the projective space. Then*

$$\mathscr{F}_{Z_2}(S^n)=Z_2.$$

Theorem 2.9. *Let G be a Lie group with finitely many path-components and let (P_k, q, S^n, G) $(n\geqq 2)$ be a numerable G-bundle. Then $\mathscr{F}_G(P_k)$ is an infinite group (finitely presented group) if and only if $\pi_n(G)$ is an infinite group.*

Proof. Let G_0 be the path-component of G containing the unit. By a result of Cartan-Marcev-Iwasawa, G_0 contains a maximal compact subgroup K, which is a strong deformation retract of G_0 so that $\pi_*(G_0)=\pi_*(K)$. Note that $\pi_{n+1}(S^n)$ is infinite if and only if $n=2$, and that $\pi_2(G)=\pi_2(G_0)=\pi_2(K)=0$. It follows that $\mathscr{F}_G(P_k)$ is infinite if and only if $\pi_n(G)/\langle k, \pi_1(G)\rangle$ is infinite by Corollary 2.4. By a result of Serre, $\pi_*(K)\otimes Q=\pi_*(S^{m(1)}\times S^{m(2)}\times\cdots\times S^{m(r)})\otimes Q$ for some odd integers $m(1), m(2),\dots, m(r)$. Since $\pi_i(K)$ is finitely generated and $\pi_{2i}(S^{2m+1})$ is finite, it follows that $\pi_{2i}(K)$ and hence $\pi_{2i}(G_0)$ are finite so that the Samelson product $\langle\pi_i(G_0), \pi_j(G_0)\rangle$ $(=\langle\pi_i(G), \pi_j(G)\rangle)$ is a finite subgroup of $\pi_{i+j}(G_0)$ $(=\pi_{i+j}(G))$ for every $i, j\geqq 1$. In particular $\langle k, \pi_1(G)\rangle$ is finite. Therefore $\pi_n(G)/\langle k, \pi_1(G)\rangle$ is infinite if and only if $\pi_n(G)$ is infinite. This completes the proof. q.e.d.

We have proved the following in the proof of Theorem 2.9.

Proposition 2.10. *If G is a Lie group, then the subgroup $\langle\pi_l(G), \pi_r(G)\rangle$ of $\pi_{l+r}(G)$ is finite for all $l, r\geqq 1$.*

Example 2.11. *If $G=SO(m)$ and $n=m+r$ with $m\geqq 11$, $0\leqq r\leqq 4$ and $m+r\equiv 3(4)$, or with $m\equiv 2(8)$, $m\geqq 10$ and $r=-1$, then $\pi_{n-1}(G)$ is non-trivial and $\pi_n(G)$ is infinite by* [11]. *In these cases we have non-trivial numerable principal G-bundles (P, q, S^n, G) with $\mathscr{F}_G(P)$ infinite.*

D. Sullivan [18] and C. Wilkerson [22] have shown independently $\mathscr{E}(X)$ is a finitely presented group, when X is a simply-connected finite CW-complex. We shall show that $\mathscr{F}_G(P)$ and $\mathscr{E}_G(P)$ are finitely presented groups under suitable conditions. $\mathscr{E}(B)$ acts on $[B, B_G]$ by

$$(2.12)\qquad \mathscr{E}(B)\times[B, B_G]\longrightarrow[B, B_G];\quad \bar{f}\cdot k=k\bar{f}.$$

Let $\mathscr{E}_k(B)$ be the isotropy group of $\mathscr{E}(B)$ at k of this action.

Theorem 2.13. *Let (P, q, B, G) be a numerable principal G-bundle. Assume that the base space B which is simply-connected and the structure group G are path-connected finite CW-complexes and that $\mathscr{E}(B)\cdot k$ is a finite set in* (2.12). *Then $\mathscr{F}_G(P)$ and $\mathscr{E}_G(P)$ are finitely presented groups.*

Proof. We shall show that in Theorem 2.1 $\pi_1(map(B, B_G), k)$ and $\mathscr{F}_k(B)$ are finitely presented under the above conditions. We shall make use of the Federer's spectral sequence ([7, p. 351]) converging to $\pi_*(map(B, B_G), k)$. It is easy to see that $E^2_{1,q}=H^q(B, \pi_{q+1}(B_G))$ is finitely generated abelian so that $E^\infty_{1,q}$ is finitely generated abelian. Since the extension of finitely presented groups is a finitely presented group, $\pi_1(map(B, B_G), k)$ is finitely presented by the filtration of subgroups in the spectral sequence. $\mathscr{E}(B)$ is finitely presented by [18, Theorem 10.3] or [22, Theorem 9.9]. Since B is simply-connected, $\mathscr{E}(B)=\mathscr{F}(B)$ and $\mathscr{E}_k(B)=\mathscr{F}_k(B)$. In (2.12) it is easy to see $\mathscr{E}(B)/\mathscr{E}_k(B)=\mathscr{E}(B)\cdot k$ (cf. [5, p. 40]). Since a subgroup of finite index in a finitely presented group is finitely presented ([13, p. 93]), $\mathscr{E}_k(B)$ is finitely presented by the assumption. Hence $\mathscr{F}_G(P)$ is finitely presented by Theorem 2.1.

Consider the evaluation fibration:

$$\omega:\ aut_G(P)\longrightarrow P\quad\text{with fibre}\quad aut^*_G(P).$$

Then we have the exact sequence:

$$\pi_1(P)\longrightarrow\mathscr{E}_G(P)\longrightarrow\mathscr{F}_G(P)\longrightarrow 0.$$

Now $\pi_1(P)$ is finitely presented, since $\pi_1(G)$ is finitely generated abelian. Therefore $\mathscr{E}_G(P)$ is finitely presented. q. e. d.

In the rest of this section, we assume that the base space B is a simply-connected finite CW-complex and the structure group G is a path-connected finite CW-complex.

Corollary 2.14. *If B is a double suspension of a finite CW-complex B' such that the dimension of B' is at most $2(1+\text{conn } B')$, where conn B' denotes the connectivity of B', and if k is of finite order in the group $[B, B_G]$, then $\mathscr{E}_G(P)$ and $\mathscr{F}_G(P)$ are finitely presented.*

Proof. We show that each element of $\mathscr{E}(B)\cdot k$ has the finite order. Then the conclusion follows from Theorem 2.13, since $[B, B_G]=[SB', G]$ is finitely generated abelian by [2, Lemma 1]. It follows from the Freudenthal suspension theorem that the suspension homomorphism $S_*\colon [SB', SB']\to[B, B]$ is sujective and hence for each $f\in\mathscr{E}(B)$ there exists $f'\in[SB', SB']$ with $S_*(f')=f$. Let m be the order of k. Then $m(f\cdot k)=m(kf)=m(kS_*(f'))=(mk)S_*(f')=0$. q. e. d.

We immediately have the following by Theorem 2.13.

Corollary 2.15. *If $\mathscr{E}(B)$ or $[B, B_G]$ is finite, then $\mathscr{E}_G(P)$ and $\mathscr{F}_G(P)$ are finitely presented.*

Corollary 2.16. *$\mathscr{E}_G(G\times B)$ and $\mathscr{F}_G(G\times B)$ are finitely presented.*

§3. Samelson Products

By Corollary 2.4, we must compute the Samelson product $\langle k, \pi_1(G)\rangle$ $(k\in\pi_*(G))$ to calculate the group $\mathscr{F}_G(P_k)$. In this section we shall calculate $\langle\pi_l(U(n)), \pi_1(U(n))\rangle$ for $l\leqq 2n$ and $\langle\pi_l(SO(n)), \pi_1(SO(n))\rangle$ for $l\leqq n-1$.

Let $(G(n), d)$ be one of the pairs $(SO(n), 1)$, $(U(n), 2)$ and $(Sp(n), 4)$.

Proposition 3.1. *If $l+r\leqq d(n+1)-3$, then $\langle\pi_l(G(n)), \pi_r(G(n))\rangle=0$.*

Proof. Let $i=i^m\colon G(n)\to G(n+m)$ be the inclusion defined by $i^m(A)=A\oplus I_m$, where I_m is the m-dimensional unit matrix and $1\leqq m\leqq\infty$. We define a homotopy $\phi_t\colon G(n)\to G(2n)$ by $\phi_t(A)=D_t i^n(A)D_t$, where

$$D_t=\begin{pmatrix}\left(\sin\frac{\pi t}{2}\right)I_n & \left(\cos\frac{\pi t}{2}\right)I_n\\ \left(\cos\frac{\pi t}{2}\right)I_n & \left(-\sin\frac{\pi t}{2}\right)I_n\end{pmatrix}.$$

Then $\phi_t(I_n)=I_{2n}$, $\phi_0(A)=I_n\oplus A$ and $\phi_1=i^n$. Therefore ϕ_0 is homotopic to i^n relative to I_n and $(\phi_0)_*=(i^n)_*\colon \pi_*(G(n))\to\pi_*(G(2n))$. Since $\phi_0(A)\phi_1(B)=\phi_1(B)\phi_0(A)$ for A, $B\in G(n)$, it follows that if a, $b\in\pi_*(G(n))$, then $\langle(\phi_0)_*(a), (\phi_1)_*(b)\rangle=0$ and hence

$$(3.2)\qquad (i^n)_*(\langle a, b\rangle)=\langle (i^n)_*(a), (i^n)_*(b)\rangle=\langle(\phi_0)_*(a), (\phi_1)_*(b)\rangle=0.$$

Using the homotopy exact sequence of $G(m)\xrightarrow{i^1}G(m+1)\to S^{d(m+1)-1}$, it can be seen that if $l\leqq d(n+1)-3$, then the inclusion i^1 induces isomorphisms $\pi_l(G(n))\cong\pi_l(G(n+1))\cong\cdots$, in particular $(i^n)_*\colon \pi_l(G(n))=\pi_l(G(2n))$. The conclusion then follows from (3.2) and naturality properties of the Samelson products. q. e. d.

Corollary 3.3. *$\langle\pi_l(U(n)), \pi_1(U(n))\rangle=0$ for $l\leqq 2n-2$ and $\langle\pi_l(SO(n)), \pi_1(SO(n))\rangle=0$ for $l\leqq n-3$.*

We shall use the well-known additive structure of $\pi_l(G(n))$ for $l\leqq d(n+1)-3$ without any reference.

Proposition 3.4. *$\langle\pi_{2n-1}(U(n)), \pi_1(U(n))\rangle=Z_n\subset\pi_{2n}(U(n))=Z_{n!}$ and $\langle\pi_{2n}(U(n)), \pi_1(U(n))\rangle=\pi_{2n+1}(U(n))$ which is Z_2 or 0 according as n is even or odd.*

Proof. Let $\partial\colon \pi_l(S^{2n+1})\to\pi_{l-1}(U(n))$ be the boundary homomorphism of the fibration $U(n)\xrightarrow{i}U(n+1)\to S^{2n+1}$, and $\iota=\iota_l\in\pi_l(S^l)$ be the identity map. Put $\theta'=(i^{n-1})_*(\iota_1)\in\pi_1(U(n))$. Then

$$\pi_1(U(n))=Z=\{\theta'\}.$$

Let $\beta\in\pi_{2n-1}(U(n))$ be a generator. Then

$$\pi_{2n-1}(U(n))=Z=\{\beta\}.$$

Recall that $\pi_{2n}(U(n))=Z_{n!}$ (cf. [19, p. 115]), which is generated by $\partial\iota_{2n+1}$. Since the order of $\langle\beta, \theta'\rangle$ is n by [4, Corollary], the first assertion follows.

By [10, (15.13) and (16.2)] we have

$$\langle\partial\iota, y\rangle=\pm\partial S_*Jj_*(y)\quad (y\in\pi_l(U(n)),$$

where $j\colon U(n)\subset SO(2n)$, J is the J-homomorphism and S_* is the suspension homomorphism. Recall that $Jj_*(\theta')=\eta_{2n}$, where $\eta_2\in\pi_3(S^2)$ is the Hopf map and $\eta_l=(S_*)^{l-2}(\eta_2)\in\pi_{l+1}(S^l)$. Therefore

$$\langle\partial\iota, \theta'\rangle=\partial\eta_{2n+1}=(\partial\iota_{2n+1})\eta_{2n}.$$

Since $\pi_{2n+1}(U(n))$ is Z_2 if n is even and 0 if n is odd by [19, Theorem 4.4], it follows from the homotopy exact sequence of $U(n)\to U(n+1)\to S^{2n+1}$ that $\partial\eta_{2n+1}$ generates $\pi_{2n+1}(U(n))$. This completes the proof. q. e. d.

Put $\alpha=\partial\iota\in\pi_{2n}(U(n))$. Then

$$\pi_{2n}(U(n))=Z_{n!}=\{\alpha\}.$$

Let $\Delta\colon\pi_l(S^n)\to\pi_{l-1}(SO(n))$ be the boundary homomorphism of $SO(n)\xrightarrow{i_1}SO(n+1)\to S^n$. Put $\theta=(i^{n-2})_*(\iota_1)\in\pi_1(SO(n))$. Then

$$\pi_1(SO(n))=Z_2=\{\theta\} \quad \text{if} \quad n\geqq 3.$$

By [11], $\pi_{n-1}(SO(n))$ is $Z+Z$ if $n\equiv 0\,(8)>0$ and Z_2+Z_2 if $n\equiv 1\,(8)>1$. We can see easily that $\Delta\iota_n$ is not zero and generates a direct summand of $\pi_{n-1}(SO(n))$ if $n\equiv 0,\ 1\,(8)>1$. Let $x=x_n$ denote any other generator of $\pi_{n-1}(SO(n))$. Then

$$\pi_{n-1}(SO(n))=\{\Delta\iota_n\}+\{x\} \quad \text{if} \quad n\equiv 0,\quad 1\,(8)>1.$$

Proposition 3.5. *(i) $\langle\pi_{n-2}(SO(n)),\theta\rangle$ is Z_2 if $n=5, 9$ and 0 if $n\neq 5, 9$.*

(ii) *When $n\equiv 0\,(8)>0$, we have $\pi_{n-1}(SO(n))=Z+Z=\{\Delta\iota_n\}+\{x\}$, $\pi_n(SO(n))=Z_2+Z_2+Z_2=\{\Delta\eta_n\}+\{x\eta_{n-1}\}+\{j_*(\alpha)\}$, $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n$* and *$\langle x,\theta\rangle\equiv 0$ mod $\Delta\eta_n$ if $n>8$ and $\equiv j_*(\alpha)$ mod $\Delta\eta_n$ if $n=8$. We can choose x in the image of $(i^3)_*\colon\pi_{n-1}(SO(n-3))\to\pi_{n-1}(SO(n))$ if $n>8$, and then $\langle x,\theta\rangle=0$.*

(iii) *When $n\equiv 1(8)>1$, we have $\pi_{n-1}(SO(n))=Z_2+Z_2=\{\Delta\iota_n\}+\{x\}$, $\pi_n(SO(n))=Z_2+Z_2=\{\Delta\eta_n\}+\{x\eta_{n-1}\}$, $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n$* and *$\langle x,\theta\rangle\equiv 0$ mod $\Delta\eta_n$. We can choose x with $x\in Im\,(i^3)_*$ and then $\langle x,\theta\rangle=0$.*

(iv) *When $n\equiv 2(8)>2$, we have $\pi_{n-1}(SO(n))=Z+Z_2=\{\Delta\iota_n\}+\{(i^1)_*(x_{n-1}\eta_{n-2})\}$, $\pi_n(SO(n))=Z_4=\{j_*(\alpha)\}$, $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n=2j_*(\alpha)$, and $\langle(i^1)_*(x_{n-1}\eta_{n-2}),\theta\rangle=0$.*

(v) *When $n\equiv 3(4)$, $\pi_{n-1}(SO(n))$ is Z_2 or 0 and $\pi_n(SO(n))=Z$. Thus $\langle\pi_{n-1}(SO(n)),\theta\rangle=0$.*

(vi) *When $n\equiv 4(8)$, we have $\pi_{n-1}(SO(n))=Z+Z=\{\Delta\iota_n\}+\{j_*(\beta)\}$, $\pi_n(SO(n))=Z_2+Z_2=\{\Delta\eta_n\}+\{j_*(\alpha)\}$, $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n$, and $\langle j_*(\beta),\theta\rangle=(n/2-1)!j_*(\alpha)$.*

(vii) *When $n\equiv 5(8)$, we have $\pi_{n-1}(SO(n))=Z_2=\{\Delta\iota_n\}$, $\pi_n(SO(n))=Z_2=\{\Delta\eta_n\}$, and $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n$.*

(viii) *When $n\equiv 6(8)$, we have $\pi_{n-1}(SO(n))=Z=\{\Delta\iota_n\}$, $\pi_n(SO(n))=Z_4=\{j_*(\alpha)\}$ if $n>6$, $\pi_6(SO(6))=0$, and $\langle\Delta\iota_n,\theta\rangle=\Delta\eta_n=2j_*(\alpha)$.*

Proof. We only give the proof of (ii). Others can be proved by the similar and easier methods. In the rest of this section we always assume $n\equiv 0(8)>0$.

It follows from [11] that

$$\begin{aligned}
&\pi_{n-1}(SO(n))=Z+Z=\{\Delta\iota_n\}+\{x\},\\
&\pi_{n-1}(SO(n+1))=Z=\{(i^1)_*(x)\},\\
&\pi_{n-1}(SO(\infty))=Z=\{(i^\infty)_*(x)\},\\
&\pi_n(SO(\infty))=Z_2=\{(i^\infty)_*(x\eta_{n-1})\},\\
&\pi_n(SO(n+2))=Z_2=\{(i^2)_*(x\eta_{n-1})\},\\
&\pi_n(SO(n))=Z_2+Z_2+Z_2,\\
&\pi_n(SO(n+1))=Z_2+Z_2.
\end{aligned}$$

We prove

$$\pi_n(SO(n))=Z_2+Z_2+Z_2=\{\Delta\eta_n\}+\{x\eta_{n-1}\}+\{j_*(\alpha)\}. \tag{3.6}$$

We can easily show that $\Delta\eta_n$ and $x\eta_{n-1}$ are linearly independent by the homotopy exact sequence of $SO(n)\to SO(n+1)\to S^n$. Consider the commutative diagram:

$$\begin{array}{ccccc}
 & \overset{j_*}{\nearrow} & \pi_n(SO(n)) & & \\
 & & \downarrow{\scriptstyle (i^1)_*} & & \\
\{\alpha\}=Z_{(n/2)!}=\pi_n(U(n/2)) & \longrightarrow & \pi_n(SO(n+1)) & \longrightarrow & \pi_n(SO(n+1)/U(n/2)) \\
\downarrow{\scriptstyle (i^1)_*} & & \downarrow{\scriptstyle (i^1)_*} & & \downarrow{\scriptstyle (\bar{i})_*} \\
0=\pi_n(U(n/2+1)) & \xrightarrow{j_*} & \pi_n(SO(n+2))=Z_2 & \xrightarrow{p_*} & \pi_n(\Gamma_{n/2+1}),
\end{array}$$

where $\Gamma_m=SO(2m)/U(m)$. As is easily seen, i^1 induces an embedding of differentiable closed manifolds $\bar{i}$: $SO(2m-1)/U(m-1)\to SO(2m)/U(m)$. Since both manifolds have the same dimension, $\bar{i}$ is a homeomorphism and hence $(\bar{i})_*$ is an isomorphism. We will use $\bar{i}$ to identify both manifolds. The fibration $SO(2m)/U(m)\to SO(2m+1)/U(m)\to S^{2m}$ can then be written as $\Gamma_m\to\Gamma_{m+1}\to S^{2m}$. It follows that the natural maps induce isomorphisms $\pi_l(\Gamma_m)=\pi_l(\Gamma_{m+1})=\cdots=\pi_l(\Gamma_\infty)$ if $l\leqq 2m-2$. Recall that $\pi_l(\Gamma_\infty)=\pi_l(O(\infty)/U(\infty))=\pi_l(\Omega SO(\infty))=\pi_{l+1}(SO(\infty))$. Thus $\pi_n(\Gamma_{n/2+1})=\pi_{n+1}(SO(\infty))=Z_2$ and hence p_* is an isomorphism and $\pi_n(SO(n+1)/U(n/2))=Z_2$. It follows that $(i^1)_*j_*(\alpha)\neq 0$. If $j_*(\alpha)=a\Delta\eta_n+bx\eta_{n-1}$, then $0=(i^2)_*j_*(\alpha)=b(i^2)_*(x\eta_{n-1})$ so that $b=0$ and $(i^1)_*j_*(\alpha)=0$. This is a contradiction. Thus $j_*(\alpha)$ is not a linear combination of $\Delta\eta_n$ and $x\eta_{n-1}$ and hence (3.6) follows.

By [10, (15.13) and (16.2)],

$$\langle\Delta\iota_m, y\rangle=\pm\Delta J(y)\quad (y\in\pi_l(SO(m)).$$

It follows that

$$\langle \Delta\iota_n, \theta\rangle = \Delta\eta_n, \tag{3.7}$$

since $J(\theta)=\eta_n$, which is of order 2.

We shall use the formula on $SO(2m)$ $(m\geqq 2)$ to study $\langle x, \theta\rangle$

$$D = F - \zeta_* F \zeta_*$$

of [10, 17.4], where F is the Bott suspension, ζ is the outer automorphism of $SO(2m)$ and $D(y)=\langle y, \theta\rangle$.

We shall give the proof of the following at the end of this section.

Lemma 3.8. *The following triangle is commutative:*

$$\begin{array}{ccc} \pi_*(SO(2m)) & \xrightarrow{\zeta_*} & \pi_*(SO(2m)) \\ & {\scriptstyle (i^1)_*}\searrow \quad \swarrow{\scriptstyle (i^1)_*} & \\ & \pi_*(SO(2m+1)) & \end{array}$$

Assume $n>8$ and consider the exact sequence of $SO(n-3)\to SO(n+1)\to SO(n+1)/SO(n-3)=V_{n+1,4}$:

$$\pi_{n-1}(SO(n-3)) \xrightarrow{(i^4)_*} \pi_{n-1}(SO(n+1)) \longrightarrow \pi_{n-1}(V_{n+1,4}) \longrightarrow \pi_{n-2}(SO(n-3)) \longrightarrow \pi_{n-2}(SO(n+1))=0.$$

Recall that $\pi_{n-1}(SO(n-3))=Z+Z_2$ and $\pi_{n-2}(SO(n-3))=Z_8$ from [11] and that $\pi_{n-1}(V_{n+1,4})=Z_8$ from [14]. Hence $(i^4)_*$ is an epimorphism. Let x' be any generator of a free part of $\pi_{n-1}(SO(n-3))$. Then $(i^1)_*(x)=\pm(i^4)_*(x')$ and hence

$$x \equiv \pm(i^3)_*(x') \ mod\ \Delta\iota_n \tag{3.9}$$

and $(i^1)_*(x')$ generates $\pi_{n-1}(SO(n-2))=Z$ (cf. [11]). Since $(i^2)_*(x\eta_{n-1})=(i^5)_*(x'\eta_{n-1})$ by (3.9), and since $\pi_n(SO(n-2))=Z_{12}+Z_2$ by [11], it follows that $(i^1)_*(x'\eta_{n-1})$ generates a Z_2-direct summand:

$$\pi_n(SO(n-2))=Z_{12}+Z_2=Z_{12}+\{(i^1)_*(x'\eta_{n-1})\}. \tag{3.10}$$

Consider the commutative diagram:

$$\begin{array}{ccc} \pi_{n-1}(SO(n-2)) & \xrightarrow{F} & \pi_n(SO(n-2)) \\ \downarrow{\scriptstyle (i^2)_*} & & \downarrow{\scriptstyle (i^2)_*} \\ \pi_{n-1}(SO(n)) & \xrightarrow{F} & \pi_n(SO(n)) \\ \downarrow{\scriptstyle (i^2)_*} & & \downarrow{\scriptstyle (i^2)_*} \\ \pi_{n-1}(SO(n+2)) & \xrightarrow{F} & \pi_n(SO(n+2)). \end{array}$$

Since $\pi_{n-1}(SO(n+2))$ is stable, $F(y)=y\eta_{n-1}$ on it by [10, 17.2]. Therefore

$$(i^4)_* F(i^1)_*(x')=F(i^5)_*(x')=((i^5)_*(x'))\eta_{n-1}=(i_4)_*(i^1)_*(x'\eta_{n-1})$$

and hence

$$F(i^1)_*(x')-(i^1)_*(x'\eta_{n-1})\in Ker\,(i^4)_* . \tag{3.11}$$

On the other hand, by [11] and [14], we have the exact sequence of the fibration $SO(n-2)\to SO(n)\to SO(n)/SO(n-2)=V_{n,2}$: $\pi_{n+1}(SO(n-2)=Z_2\to\pi_{n+1}(SO(n))=Z_2+Z_2+Z_2\to\pi_{n+1}(V_{n,2})=Z_{24}+Z_2\to\pi_n(SO(n-2))=Z_{12}+Z_2\xrightarrow{(i^2)_*}\pi_n(SO(n))$. Hence We have $Ker\,(i^2)_*=Z_{12}$ and $Im\,(i^2)_*=Z_2=\{x\eta_{n-1}+b\Delta\eta_n\}$ for some $b\in Z_2$ by (3.9) and (3.10). Let $(i^2)_*(F(i^1)_*(x')-(i^1)_*(x'\eta_{n-1}))=a(x\eta_{n-1}+b\Delta\eta_n)$. Then $0=(i^4)_*(F(i^1)_*(x')-(i^1)_*(x'\eta_{n-1}))=a(i^2)_*(x\eta_{n-1})$ by (3.11) and hence $a=0$. Thus $F(i^1)_*(x')-(i^1)_*(x'\eta_{n-1})\in Ker\,(i^2)_*$ and

$$F(i^3)_*(x')=(i^2)_*F(i^1)_*(x')=(i^3)_*(x'\eta_{n-1}).$$

By the naturality of ζ_* and Lemma 3.8, it follows that $\zeta_*(i^3)_*(x')=(i^2)_*\zeta_*(i^1)_*(x')=(i^3)_*(x')$ and $\zeta_* F\zeta_*(i^3)_*(x')=\zeta_*F(i^3)_*(x')=\zeta_*(i^3)_*(x'\eta_{n-1})=(i^2)_*\zeta_*(i^1)_*(x'\eta_{n-1})=(i^3)_*(x'\eta_{n-1})$. Therefore

$$\langle (i^3)_*(x'),\,\theta\rangle=D(i^3)_*(x')=F(i^3)_*(x')-\zeta_*F\zeta_*(i^3)_*(x')=0$$

and the last assertion of (ii) of Proposition 2.5 follows. It follows from (3.7) and (3.9) that

$$\langle x,\,\theta\rangle\equiv 0\ mod\ \Delta\eta_n \quad \text{if} \quad n\equiv 0\,(8)>8 .$$

We prove

Lemma 3.12. *If $n=8$, then $\langle x,\,\theta\rangle\equiv j_*(\alpha)\ mod\ \Delta\eta_8$.*

Note that

$$\pi_6(SO(6))=0,\quad \pi_7(SO(6))=Z,\ \pi_8(SO(6))=Z_{24}=\{\gamma\},\quad \pi_6(SO(7))=0,$$
$$\pi_7(SO(7))=Z=\{h\},\quad \pi_8(SO(7))=Z_2+Z_2,\quad \pi_7(V_{9,2})=Z_2$$

by [16], [17] and [14]. From the exact sequence

$$\pi_8(SO(6))\xrightarrow{(i^1)_*}\pi_8(SO(7))\xrightarrow{p_*}\pi_8(S^6)\longrightarrow\pi_7(SO(6))\longrightarrow\pi_7(SO(7))\xrightarrow{p_*}$$
$$\pi_7(S^6)\longrightarrow\pi_6(SO(6)),$$

it follows that $p_*(h)=\eta_6$, $p_*(h\eta_7)=\eta_6\eta_7$ and

$$\pi_8(SO(7))=Z_2+Z_2=\{h\eta_7\}+\{(i^1)_*(\gamma)\} .$$

Applying $\pi_*(\ )$ to the commutative diagram:

$$\begin{array}{ccccc}
SO(7) & = & SO(7) & & \\
\downarrow & & \downarrow & & \\
SO(8) & \longrightarrow & SO(9) & \longrightarrow & S^8 \\
\downarrow{\scriptstyle q} & & \downarrow & & \| \\
S^7 & \longrightarrow & V_{9,2} & \longrightarrow & S^8,
\end{array}$$

we have the commutative diagram:

$$\begin{array}{ccccccc}
 & & \pi_7(SO(7)) & = & \pi_7(SO(7)) & & \\
 & & \downarrow{\scriptstyle (i^1)_*} & & \downarrow{\scriptstyle (i^2)_*} & & \\
\pi_8(S^8) & \xrightarrow{\Delta} & \pi_7(SO(8)) & \xrightarrow{(i^1)_*} & \pi_7(SO(9)) & \longrightarrow & \pi_7(S^8) \\
\| & & \downarrow & & \downarrow & & \| \\
\pi_8(S^8) & \xrightarrow{\Delta'} & \pi_7(S^7) & \longrightarrow & \pi_7(V_{9,2}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & \pi_6(SO(7))=0. & &
\end{array}$$

By the exactness of horizontal and vertical sequences, we have

$$q_*(\Delta\iota_8)=\Delta'\iota_8=\pm 2\iota_7, \quad q_*(x)\equiv\iota_7 \bmod 2\iota_7, \quad (i^2)_*(h)=\pm 2(i^1)_*(x).$$

We can write $(i^1)_*(h)=l\Delta\iota_8+mx$. Then $\pm 2(i^1)_*(x)=(i^2)_*(h)=(i^1)_*(l\Delta\iota_8+mx)=m(i^1)_*(x)$ so that $m=\pm 2$ and $(i^1)_*(h)\equiv l\Delta\iota_8+2x \bmod 4x$. Therefore we have $0=q_*(i^1)_*(h)\equiv(2l+2)\iota_7 \bmod 4\iota_7$ and l is odd. Thus

$$(i^1)_*(h)\equiv\Delta\iota_8+2x \; mod \; \{2\Delta\iota_8\}+\{4x\},$$

and hence

(3.13) $$\langle(i^1)_*(h),\, \theta\rangle=\langle\Delta\iota_8,\, \theta\rangle=\Delta\eta_8.$$

By using the multiplication of Cayley numbers, we can define a cross-section $s\colon S^7\to SO(8)$ by

$$s(y)z=yz(y,\, z\in S^7).$$

Then $\pi_7(SO(8))=\{s\}+\{(i^1)_*(h)\}$. Thus we can write $x=a(i^1)_*(h)+bs$ for some integers a, b. Then $q_*(x)=q_*(a(i^1)_*(h)+bs)=b\iota_7$ and b is odd. Therefore

(3.14) $$x\equiv s \; mod \; \{(i^1)_*(h)\}+\{2s\}.$$

Consider the commutative diagram:

$$\begin{array}{ccccc}
 & & \pi_8(U(4)) & \longrightarrow & \pi_8(S^7) \longrightarrow \pi_7(U(3))=0 \\
 & & \downarrow j_* & & \| \\
\pi_8(SO(7)) & \xrightarrow{(i^1)_*} & \pi_8(SO(8)) & \xrightarrow{q_*} & \pi_8(S^7). \\
\| & & \| & & \| \\
\{h\eta_7\}+\{(i^1)_*(\gamma)\} & & \{\varDelta\eta_8\}+\{x\eta_7\}+\{j_*(\alpha)\} & & \{\eta_7\}
\end{array}$$

Then we have

$$q_*(x\eta_7)=(q_*(x))\eta_7=\eta_7\,,$$

$$q_*j_*(\alpha)=\eta_7\,,$$

$$(i^1)_*(h\eta_7)=(i^1)_*(h)\eta_7=(\varDelta\iota_8)\eta_7=\varDelta\eta_8\,,$$

and hence

$$(i^2)_*(\gamma)\equiv x\eta_7+j_*(\alpha)\ mod\ \varDelta\eta_8\,.$$

It follows from (3.14) that

$$\begin{aligned} x\eta_7 &\equiv s\eta_7 \text{ mod } (i^1)_*(h\eta_7) \\ &\equiv s\eta_7\ mod\ \varDelta\eta_8\,. \end{aligned}$$

and hence

$$(3.15)\qquad (i^2)_*(\gamma)\equiv s\eta_7+j_*(\alpha)\ mod\ \varDelta\eta_8\,.$$

Now we prove Lemma 3.12. We have $\langle x, \theta\rangle\equiv\langle s, \theta\rangle\ mod\ \varDelta\eta_8$ by (3.13) and (3.14), and $\langle s, \theta\rangle=(i^2)_*(\gamma)+s\eta_7\equiv j_*(\alpha)\ mod\ \varDelta\eta_8$ by [20, Lemma 4.10] and (3.15). Thus $\langle x, \theta\rangle\equiv j_*(\alpha)\ mod\ \varDelta\eta_8$ and the result follows.

Proof of Lemma 3.8. Let $\zeta'\colon SO(2m+1)\to SO(2m+1)$ be defined by $\zeta'(A)=I''AI''$, where $I''=(-I_1)\oplus I_{2m-1}\oplus(-I_1)$. Recall from [10, p. 110] that $\zeta\colon SO(2m)\to SO(2m)$ is defined by $\zeta(B)=I'BI'$, where $I'=(-I_1)\oplus I_{2m-1}$. Then $i^1\zeta=\zeta' i^1$. Since ζ' is the inner automorphism of the path-connected group, ζ' is homotopic to the identity map relative to I_{2m+1} and hence $(i^1)_*\zeta_*=(\zeta')_*(i^1)_*=(i^1)_*$. q.e.d.

This completes the proof of (ii) of Proposition 3.5.

§4. Computations

By using Corollary 2.4 and Samelson products in §3, we can compute $\mathscr{F}_G(P)$ even if P is not simply-connected (Example 4.5 and Examples 4.7–4.13).

For the trivial bundle we use Theorem 2.1 in [21].

Example 4.1 (*cf.* [21, *Example* 3.7]). *Let* (P_k, q, S^4, S^3) *be the principal* S^3*-bundle with* $k \in \pi_3(S^3)=Z$. *Then*

$$\mathscr{F}_{S^3}(P_k)=\begin{cases} Z_2 & \text{if } k\equiv 0\,(2) \quad (k\neq 0), \\ 1 & \text{if } k\equiv 1\,(2), \\ Z_2+Z_2 & \text{if } k=0. \end{cases}$$

Example 4.2 (*cf.* [21, *Example* 3.9]). *Let* (P_k, q, S^7, G_2) *be the principal* G_2*-bundle with* $k\in\pi_6(G_2)$. *Then*

$$\mathscr{F}_{G_2}(P_k)=\begin{cases} Z_2 & \text{if } k=0, \\ 1 & \text{if } k\neq 0. \end{cases}$$

Example 4.3 (*cf.* [21, *Example* 3.10]). *Let* $(P_k, q, S^6, SU(3))$ *be the principal* $SU(3)$*-bundle with* $k\in\pi_5(SU(3))$. *Then*

$$\mathscr{F}_{SU(3)}(P_k)=\begin{cases} Z_3 & \text{if } k\equiv 1\,(2), \\ Z_6 & \text{if } k\equiv 0\,(2) \quad (k\neq 0), \\ D(Z_6) & \text{if } k=0. \end{cases}$$

Example 4.4 (*cf.* [21, *Example* 3.11]). *Let* $(P_k, q, S^{2n+1}, SU(n))$ $(n\geqq 2)$ *be the principal* $SU(n)$*-bundle with* $k=m\alpha\in\pi_{2n}(SU(n))=Z_{n!}\{\alpha\}$. *Then*

$$\mathscr{F}_{SU(n)}(P_k)=\begin{cases} 1 & \text{if } 2m\not\equiv 0\,(n!),\ n\equiv 0\,(2),\ m\equiv 1\,(2) \\ & \text{or } 2m\equiv 0\,(n!),\ n\equiv 1\,(2), \\ Z_2 & \text{if } n=2,\ m\equiv 1\,(2) \\ & \text{or } 2m\not\equiv 0\,(n!),\ n=0\,(2),\ m\equiv 0\,(2) \\ & \text{or } 2m\equiv 0\,(n!),\ n\equiv 1\,(2), \\ Z_2+Z_2 & \text{if } m\equiv 0\,(n!), \end{cases}$$

and we have the following exact sequence:

$$1 \longrightarrow Z_2 \longrightarrow \mathscr{F}_{SU(n)}(P_k) \longrightarrow Z_2 \longrightarrow 1 \quad \text{if } n\equiv 0\,(2),\ m\equiv 0\,(2),\ 2m\equiv 0\,(n!).$$

Example 4.5 (*cf.* [21, *Example* 3.12]). *Let* $(P_k, q, S^{2n+1}, U(n))$ $(n\geqq 1)$ *be the principal* $U(n)$*-bundle with* $k=m\alpha\in\pi_{2n}(U(n))=Z_{n!}\{\alpha\}$. *Then if* $m\equiv 0(2)$ *or* $n\equiv 1(2)$, *then we have the same result in Example* 4.4, *and if* $m\equiv 1(2)$ *and* $n\equiv 0(2)$, *then*

$$\mathscr{F}_{U(n)}(P_k)=\begin{cases} Z_2 & \text{if } n=2, \\ 1 & \text{if } n\geqq 3. \end{cases}$$

Example 4.6 (*cf.* [21, *Example* 3.14]). *Let* $(P_k, q, S^{4n+3}, Sp(n))$ $(n \geqq 1)$ *be the principal* $Sp(n)$*-bundle with* $k = m\alpha \in \pi_{4n+2}(Sp(n)) = Z_N\{\alpha\}$, where $N = (2n+1)!$ if $n \equiv 0\,(2)$ and $N = (2n+1)!2$ if $n \equiv 1(2)$. Then

$$\mathscr{F}_{Sp(n)}(P_k) = \begin{cases} 1 & \text{if } m \equiv 1(2),\quad 2m \not\equiv 0\,(N), \\ Z_2 & \text{if } m \equiv 1(2),\quad 2m \equiv 0\,(N) \\ & \text{or } m \equiv 0(2),\quad 2m \not\equiv 0\,(N), \\ Z_2 + Z_2 & \text{if } m = 0\,(N), \end{cases}$$

and we have the following exact sequence:

$$1 \longrightarrow Z_2 \longrightarrow \mathscr{F}_{Sp(n)}(P_k) \longrightarrow Z_2 \longrightarrow 1 \quad \text{if} \quad m \equiv 0\,(2), \quad 2m \equiv 0\,(N).$$

In the following examples 4.7–4.13, we consider the principal $SO(n)$-bundles over spheres: $(P_k, q, S^n, SO(n))$. We use Proposition 3.5 to compute $\mathscr{F}_{SO(n)}(P_k)$ and we choose x in $Im\,(i^3)_*$ if $n \equiv 0(8) \geqq 16$ or $n \equiv 1(8) \geqq 9$ below.

Example 4.7. *If* $n \equiv 0(8)$ *and* $k = l\Delta\iota_n + mx \in \pi_{n-1}(SO(n)) = Z + Z = \{\Delta\iota_n\} + \{x\}$, then

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_2 + Z_2 + Z_2 & \text{if } l \equiv 0\,(2),\quad m \equiv 0\,(2) \quad ((l, m) \neq (0, 0)), \\ Z_2 + Z_2 & \text{if } l \equiv 1(2),\quad m \equiv 0(2) \\ & \text{or } n \geqq 16,\quad l \equiv 0\,(2),\quad m \equiv 1(2), \\ Z_2 & \text{if } n = 8,\quad m \equiv 1(2) \\ & \text{or } l \equiv 1(2),\quad m \equiv 1(2) \\ D(Z_2 + Z_2 + Z_2) & \text{if } l = m = 0. \end{cases}$$

Example 4.8. *If* $n \equiv 1(8) \geqq 9$ *and* $k = l\Delta\iota_n + mx \in \pi_{n-1}(SO(n)) = Z_2 + Z_2 = \{\Delta\iota_n\} + \{x\}$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_2 & \text{if } l \equiv 1\,(2),\quad m \equiv 1\,(2), \\ Z_2 + Z_2 & \text{if } l \equiv 1\,(2),\quad m \equiv 0\,(2), \\ D(Z_2 + Z_2) & \text{if } l \equiv 0\,(2),\quad m \equiv 0\,(2), \end{cases}$$

and we have the following exact sequence:

$$1 \longrightarrow Z_2 \longrightarrow \mathscr{F}_{SO(n)}(P_k) \longrightarrow Z_2 \longrightarrow 1 \quad \text{if} \quad l \equiv 0\,(2), \quad m \equiv 1\,(2).$$

Example 4.9. *If* $n \equiv 2(8)$ *and* $k = l\Delta\iota_n + mi_*(x\eta_{n-2}) \in \pi_{n-1}(SO(n)) = Z + Z_2 = \{\Delta\iota_n\} + \{i_*(x\eta_{n-2})\}$ $(n \geqq 10)$, $k = l\Delta\iota_n \in \pi_1(SO(2)) = Z = \{\Delta\iota_n\}$ $(n = 2)$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_4 & \text{if } n \geqq 10,\quad l \equiv 0\,(2) \quad (l \neq 0), \\ Z_2 & \text{if } n \geqq 10,\quad l \equiv 1(2) \\ & \text{or } n = 2,\quad k = 0, \\ 1 & \text{if } n = 2,\quad k \neq 0, \end{cases}$$

and we have the following exact sequence:

$$1 \longrightarrow Z_4 \longrightarrow \mathscr{F}_{SO(n)}(P_k) \longrightarrow Z_2 \longrightarrow 1 \quad \textit{if} \quad n \geqq 10, \quad l=0.$$

Example 4.10 (*cf.* [21, *Example* 3.3]). *If* $n \equiv 3(4)$ *and* $k \in \pi_{n-1}(SO(n)) = Z_2 = \{\Delta\iota_n\}$ $(n \geqq 11)$, $k = 0 \in \pi_{n-1}(SO(n)) = 0$ $(n = 3, 7)$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = D(Z) \quad \textit{if} \quad k=0,$$

and we have the following exact sequence which splits:

$$1 \longrightarrow Z \longrightarrow \mathscr{F}_{SO(n)}(P_k) \longrightarrow Z_2 \longrightarrow 1 \quad \textit{if} \quad k \neq 0.$$

Example 4.11. *If* $n \equiv 4(8)$ *and* $k = l\Delta\iota_n + mj_*(\beta) \in \pi_{n-1}(SO(n)) = Z + Z = \{\Delta\iota_n\} + \{j_*(\beta)\}$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_2 + Z_2 & \textit{if} \ \ n=4, \ \ l \equiv 0(2) \ \ ((l, m) \neq (0, 0)) \\ & \quad \textit{or} \ \ n>4, \ \ l \equiv 0(2) \ \ ((l, m) \neq (0, 0)), \\ Z_2 & \textit{if} \ \ n=4, \ \ l \equiv 1(2) \ \ \textit{or} \ \ m \equiv 1(2) \\ & \quad \textit{or} \ \ n>4, \ \ l \equiv 1(2), \\ D(Z_2+Z_2) & \textit{if} \ \ l=m=0. \end{cases}$$

Example 4.12. *If* $n \equiv 5(8)$ *and* $k = l\Delta\iota_n \in \pi_{n-1}(SO(n)) = Z_2 = \{\Delta\iota_n\}$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_2 + Z_2 & \textit{if} \ \ l \equiv 0(2), \\ Z_2 & \textit{if} \ \ l \equiv 1(2). \end{cases}$$

Example 4.13. *If* $n \equiv 6(8)$ *and* $k = l\Delta\iota_n \in \pi_{n-1}(SO(n)) = Z = \{\Delta\iota_n\}$, *then*

$$\mathscr{F}_{SO(n)}(P_k) = \begin{cases} Z_4 & \textit{if} \ \ n \geqq 14, \ \ l \equiv 0(2) \ \ (l \neq 0), \\ Z_2 & \textit{if} \ \ n \geqq 14, \ \ l \equiv 1(2) \\ & \quad \textit{or} \ \ n=6, \ \ l=0, \\ 1 & \textit{if} \ \ n=6, \ \ l \neq 0, \\ D(Z_4) & \textit{if} \ \ n \geqq 14, \ \ l=0. \end{cases}$$

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