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BP-Hopf Module Spectrum and BP_*-Adams Spectral Sequence

Dedicated to Professor Masahiro Sugawara on his 60th birthday

By

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Let BP be the Brown-Peterson spectrum for a fixed prime p. It is an associative and commutative ring spectrum whose homotopy is $BP_*=Z_{(p)}[v_1,\dots,v_n,\dots]$. For any CW-spectrum Y, the Brown-Peterson homology BP_*Y is not only an associative BP_*-module but also an associative BP_*BP-comodule. In this note we deal with associative BP-module spectra E whose homotopies E_* are associative BP_*BP-comodules. An associative BP-module spectrum with such a structure is called a BP-Hopf module spectrum (see 1.1 for the definition). For every invariant regular sequence $J=\{q_0,\dots,q_{n-1}\}$, the associative BP-module spectrum BPJ with homotopy BP_*/J is a BP-Hopf module spectrum if $n<2(p-1)$ (Proposition 1.2).

As is well known [1], $BP\wedge Y$ has the Adams geometric resolution $W_{BP,Y}=\{W_kY=\overline{BP}^k\wedge BP\wedge Y,\ d_k\colon W_kY\to W_{k+1}Y\}_{k\geq 0}$ where $\overline{BP}$ denotes the cofiber of unit $i\colon S\to BP$ and $\overline{BP}^k=\overline{BP}\wedge\cdots\wedge\overline{BP}$ with k-factors. Applying BP_*-homology to $W_{BP,Y}$ we obtain a relative injective resolution of BP_*Y by extended BP_*BP-comodules. We will show that each BP-Hopf module spectrum E admits a BP-geometric resolution $W_E=\{W_k=\overline{BP}^k\wedge E,\ d_k\colon W_k\to W_{k+1}\}_{k\geq 0}$ inducing a relative injective resolution of E_* (Theorem 3.3).

Let K_mY denote the fiber of the map $\overline{BP}^{m+1}\wedge Y\to\sum^{m+1}Y$. Then there is a cofiber sequence $K_{m-1}Y\xrightarrow{b_{m-1}}W_mY\xrightarrow{c_m}K_mY\xrightarrow{a_m}\sum^1K_{m-1}Y$ and the differential map $d_m\colon W_mY\to W_{m+1}Y$ is factorized as $d_m=b_mc_m\colon W_mY\to K_mY\to W_{m+1}Y$. We will give a sufficient condition under which a BP-geometric resolution $W=$

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$\{W_k, d_k\}_{k\geqq 0}$ admits an ∞-factorized system $X(\infty)$ like $KY=\{K_mY, a_m, b_{m-1}, c_m\}_{m\geqq 1}$ (Theorem 4.6). Moreover we will show that the BP-geometric resolution $W_{v_n^{-1}BPJ}=\{\overline{BP}^k\wedge v_n^{-1}BPJ, d_k\}_{k\geqq 0}$ admits an ∞-factorized system under some restriction on the fixed prime p and the length n of J (Theorem 4.9).

The BP_*-Adams spectral sequence $E_2^{s,t}(S, KY)=\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*Y)\Rightarrow\pi_*(BP^\wedge Y)$ is derived from the tower $\{\sum^{-m}K_mY, a_m\}_{m\geqq 1}$ with homotopy inverse limit $BP^\wedge Y$. With an ∞-factorized system $X=\{X_m, a_m, b_{m-1}, c_m\}_{m\geqq 1}$ of a BP-geometric resolution over a BP-Hopf module spectrum E, we associate the spectral sequence $E_2^{s,t}(S, X)=\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, E_*)\Rightarrow\pi_*(X_\infty)$ where X_∞ denotes homotopy inverse limit of the tower $\{\sum^{-m}X_m, a_m\}_{m\geqq 1}$. Discussing the convergence of the spectral sequence we will prove our main result (Theorem 5.7) that there exists a unique BP-local CW-spectrum Y_J such that $BP\wedge Y_J$ is isomorphic to $v_n^{-1}BPJ$ as BP-Hopf module spectra under some restriction on p and n.

In this note we work in the *homotopy* category of CW-spectra, and we do not necessarily assume that a ring spectrum or a module spectrum is associative if not stated.

§1. *BP*-Hopf Module Spectrum

1.1. The Brown-Peterson spectrum BP is an associative and commutative ring spectrum with a multiplication $m: BP\wedge BP\to BP$ and a unit $i: SBP$. We call a CW-spectrum E a *BP-Hopf module spectrum* if E is an associative (left) BP-module spectrum together with a (left) BP-module map $\eta_E: E\to BP_\wedge E$ such that $\phi_E\eta_E=1$ and $(1\wedge\eta_E)\eta_E=(1\wedge i\wedge 1)\eta_E$ where ϕ_E is the BP-module structure map of E and 1 denotes the identity map. If the coassociativity of η_E is not necessarily satisfied, we call such an E a *quasi* BP-Hopf module spectrum. As an obvious example we have

(1.1) *For any CW-spectrum* X, $BP\wedge X$ *is a BP-Hopf module spectrum whose structure maps are given by* $\phi_{BP\wedge X}=m\wedge 1$ *and* $\eta_{BP\wedge X}=1\wedge i\wedge 1$.

Given BP-Hopf module spectra E and F, a map $f: E\to F$ is said to be a BP-Hopf module map if f is a (left) BP-module map such that $\eta_F f=(1\wedge f)\eta_E$. For any CW-spectra X and Y we have easily

(1.2) *Let* $f: BP\wedge X\to BP\wedge Y$ *be a BP-Hopf module map and* Y *be a*

BP-module spectrum. Then there exists a unique map $f'\colon X \to Y$ *such that* $1 \wedge f' = f$.

In fact, f' is given by the composite map $\phi_Y f(i \wedge 1)$.

(1.3) i) *Let E be a BP-Hopf module spectrum. Then* $E_* X$ *is an associative* BP_*BP*-comodule whose coaction map is given by* $\psi_X\colon E_*X \to BP_*(E \wedge X) \cong BP_*BP \underset{BP^*}{\otimes} E_*X$ *induced by* η_E.

ii) *Let* $f\colon E \to F$ *be a BP-Hopf module map. Then it induces a homomorphism* $f_*\colon E_*X \to F_*X$ *of* BP_*BP*-comodules.*

Let E be an associative BP-module spectrum. Given an (associative) BP-module spectrum Y, E^*Y is an (associative) BP^*BP-comodule whose coaction map is given by $\psi_Y\colon E^*Y \to E^*(BP \wedge Y) \cong BP^*BP \underset{BP^*}{\hat{\otimes}} E^*Y$. A map $f\colon Y \to \sum^d E$ is a BP-module map if and only if it represents a primitive element in E^dY (see [14, 15]). We denote by $\mathrm{Pr}\, E^*Y$ the BP^*-module consisting of all primitive elements in E^*Y. If $f\colon Y \to Z$ is a BP-module map, then it induces a homomorphism $f^*\colon E^*Z \to E^*Y$ of BP^*BP-comodules, and hence $f^*\colon \mathrm{Pr}\, E^*Z \to \mathrm{Pr}\, E^*Y$.

1.2. Let $J = \{q_0, \dots, q_{n-1}\}$ be an invariant regular sequence in BP_* of length n (see [5]) and $J_m = \{q_0, \dots, q_{m-1}\}$ the subsequences for each m, $0 \leqq m \leqq n$, in which $J_n = J$. By Baas [2] there exists an associative BP-module spectrum BPJ_m with pairing $\phi_m\colon BP \wedge BPJ_m \to BPJ_m$, whose homotopy is $BPJ_{m*} \cong BP_*/(q_0, \dots, q_{m-1})$. BPJ_m and BPJ_{m+1} are related by a cofiber sequence

$$\sum^{d_m} BPJ_m \xrightarrow{\cdot q_m} BPJ_m \xrightarrow{j_m} BPJ_{m+1} \xrightarrow{k_m} \sum^{d_m+1} BPJ_m \tag{1.4}$$

of BP-module spectra, where $d_m = \dim q_m$ is the dimension of q_m in BP_* and $\cdot q_m$ acts as left multiplication by q_m, thus it is the composite map $\phi_m(q_m \wedge 1)$. Further we have a multiplication $\mu_m\colon BPJ_m \wedge BPJ_m \to BPJ_m$ which makes BPJ_m into a quasi-associative ring spectrum (see [4, Proposition 5.5]). Putting $j = j_{n-1} \cdots j_0\colon BP \to BPJ$ it is a map of ring spectra as well as BP-module spectra.

A BPJ-module spectrum F is said to be *quasi-associative* if the following two equalities hold (cf., [4, Remark 5.3]):

(i) $\mu_F(\phi \wedge 1) = \phi_F(1 \wedge \mu_F)\colon BP \wedge BPJ \wedge F \to F$,

(ii) $\phi_F(1 \wedge \mu_F)(T \wedge 1) = \mu_F(1 \wedge \phi_F)\colon BPJ \wedge BP \wedge F \to F$,

where μ_F and $\phi_F = \mu_F(j \wedge 1)$ denote the BPJ- and BP-module structure maps of F respectively, and $T\colon BPJ \wedge BP \to BP \wedge BPJ$ is the switching map.

Let E be an associative BP-module spectrum, F be a quasi-associative BPJ-

module spectrum and X be a CW-spectrum such that BPJ_*X is BPJ_*-free. For $0 \leqq m < n$ we consider the homomorphism

$$\kappa\colon [BPJ_m \wedge X, E \wedge F] \longrightarrow \mathrm{Hom}_{BPJ_*}(BPJ_*(BPJ_m \wedge X), E_*F)$$

defined to be $\kappa(f) = (1 \wedge \mu_F)_*(T \wedge 1)_*(1 \wedge f)_*$, which is an isomorphism in our case because of [1, Proposition 13.5]. Then the cofiber sequence (1.4) gives rise to a split short exact sequence $0 \to (E \wedge F)^{*-d_m-1}(BPJ_m \wedge X) \to (E \wedge F)^*(BPJ_{m+1} \wedge X) \to (E \wedge F)^*(BPJ_m \wedge X) \to 0$ of BP^*-modules. This sequence splits as BP^*BP-comodules, because $(E \wedge F)^*(BPJ_m \wedge X) \cong BP^*BP \hat{\otimes}_{BP^*} \Lambda_{(E\wedge F)^*X}(x_0, \dots, x_{m-1})$ and hence it is an extended BP^*BP-comodule (use [4, Lemmas 5.1 and 5.2]). Here $\Lambda_R(x_0, \dots, x_{m-1})$ is the exterior algebra over R in the variables x_i with dimension $d_i + 1$. Therefore we see

(1.5) i) $\mathrm{Pr}\,(E \wedge F)^*(BPJ_{m+1} \wedge X) \cong \Lambda_{(E\wedge F)^*X}(x_0, \dots, x_m)$, *and*

ii) $j_m\colon BPJ_m \to BPJ_{m+1}$ *induces an epimorphism* $j_m^*\colon \mathrm{Pr}\,(E \wedge F)^*(BPJ_{m+1} \wedge X) \to \mathrm{Pr}\,(E \wedge F)^*(BPJ_m \wedge X)$ *for each* m, $0 \leqq m < n$. (*Cf.*, [14, 15]).

Lemma 1.1. *Let* J *be an invariant regular sequence in* BP_* *of finite length. Then* BPJ *is a quasi* BP*-Hopf module spectrum such that* $j\colon BP \to BPJ$ *is a quasi* BP*-Hopf module map.*

Proof. Let $J = \{q_0, \dots, q_{n-1}\}$. For $0 \leqq m < n$ we inductively show that BPJ_{m+1} is a quasi BP-Hopf module spectrum so that the cofiber sequence (1.4) is of quasi BP-Hopf module spectra. Assume that there exists a BP-module map $\eta_m\colon BPJ_m \to BP \wedge BPJ_m$ with $\phi_m \eta_m = 1$. We observe that $(\cdot q_m \wedge 1)_* = (1 \wedge \cdot q_m)_*\colon BPJ_{m*}BPJ_m \to BPJ_{m*}BPJ_m$ since $\eta_L(q_m) \equiv \eta_R(q_m) \bmod J_m$. Using the isomorphism $\kappa\colon [BPJ_m, BP \wedge BPJ_m] \to \mathrm{Hom}_{BPJ_{m*}}(BPJ_{m*}BPJ_m, BP_*BPJ_m)$, it is shown that $\kappa(\eta_m \cdot q_m) = (1 \wedge \mu_m)_*(T \wedge 1)_*(\cdot q_m \wedge \eta_m)_* = \kappa((1 \wedge \cdot q_m)\eta_m)$, and hence $\eta_m \cdot q_m = (1 \wedge \cdot q_m)\eta_m$. So we can find a map $\eta'_{m+1}\colon BPJ_{m+1} \to BP \wedge BPJ_{m+1}$ such that $\eta'_{m+1} j_m = (1 \wedge j_m)\eta_m$ and $(1 \wedge k_m)\eta'_{m+1} = \eta_m k_m$.

We next replace this map η'_{m+1} with a BP-module one. By (1.5) we observe that $j_m\colon BPJ_m \to BPJ_{m+1}$ induces an epimorphism $j_m^*\colon \mathrm{Pr}\,(BP \wedge BPJ_{m+1})^* BPJ_{m+1} \to \mathrm{Pr}\,(BP \wedge BPJ_{m+1})^* BPJ_m$. Pick a BP-module map $\eta''_{m+1}\colon BPJ_{m+1} \to BP \wedge BPJ_{m+1}$ such that $\eta''_{m+1} j_m = (1 \wedge j_m)\eta_m$. In order to show that $(1 \wedge k_m)\eta''_{m+1} = \eta_m k_m$ we consider the commutative diagram

$$\begin{array}{ccc}
[\Sigma^{d_m+1} BPJ_m, BP\wedge BPJ_{m+1}] & \longrightarrow & [BPJ_{m+1}, BP\wedge BPJ_{m+1}] \\
\downarrow & & \downarrow \\
[BPJ_m, BP\wedge BPJ_m] & \longrightarrow & [BPJ_{m+1}, \Sigma^{d_m+1} BP\wedge BPJ_m]
\end{array}$$

$$\longrightarrow [BPJ_m, BP\wedge BPJ_{m+1}]$$

$$\downarrow$$

$$\longrightarrow [BPJ_m, \Sigma^{d_m+1} BP\wedge BPJ_m]$$

with exact rows. Since the left vertical arrow is trivial, the equality $\eta'_{m+1}j_m = \eta''_{m+1}j_m$ implies that $(1\wedge k_m)\eta'_{m+1} = (1\wedge k_m)\eta''_{m+1}$ and hence $(1\wedge k_m)\eta''_{m+1} = \eta_m k_m$ as desired.

Applying Five lemma we see that the BP-module map $\rho_{m+1} = \phi_{m+1}\eta''_{m+1}$: $BPJ_{m+1}\to BPJ_{m+1}$ is a homotopy equivalence with $\rho_{m+1}j_m = j_m$ and $k_m\rho_{m+1} = k_m$. Putting $\eta_{m+1} = \eta''_{m+1}\rho^{-1}_{m+1}$, it is a BP-module map such that $\phi_{m+1}\eta_{m+1} = 1$, $\eta_{m+1}j_m = (1\wedge j_m)\eta_m$ and $(1\wedge k_m)\eta_{m+1} = \eta_m k_m$, as desired.

Proposition 1.2. *Let J be an invariant regular sequence in BP_* of length n. If n is less than $2(p-1)$, then BPJ is a BP-Hopf module spectrum.*

Proof. By (1.5) we observe that the map j: $BP\to BPJ$ induces an epimorphism j^*: $\Pr(BP\wedge BP\wedge BPJ)^*BPJ\to \Pr(BP\wedge BP\wedge BPJ)^*BP$, and $\Pr(BP\wedge BP\wedge BPJ)^*BPJ \cong A_{(BP\wedge BP\wedge BPJ)_*}(x_0,\dots, x_{n-1})$. Since $(BP\wedge BP\wedge BPJ)^* = 0$ unless $*\equiv 0 \bmod 2(p-1)$ and $\dim x_0\cdots x_{n-1}\equiv n \bmod 2(p-1)$, j^* becomes an isomorphism at dimension 0 when $n<2(p-1)$. Hence the coassociativity of η_n is immediately shown, because $j^*((1\wedge\eta_n)\eta_n) = 1\wedge i\wedge ji = j^*((1\wedge i\wedge 1)\eta_n)$ by Lemma 1.1.

Hereafter we only treat of a fixed invariant regular sequence $J=\{q_0,\dots, q_{n-1}\}$ for which BPJ_{m+1} are BP-Hopf module spectra and the cofiber sequences (1.4) are of BP-Hopf module spectra for each m, $0\leqq m<n$. Thus BPJ is assumed to be a BP-Hopf module spectrum such that j: $BP\to BPJ$ is a BP-Hopf module map.

§2. Extended *BP*-Hopf Module Spectrum

2.1. A BP-Hopf module spectrum E is called an *extended* BP-Hopf module spectrum if there exists an associative BP-module spectrum Y and a homotopy equivalence h: $E\to BP\wedge Y$ of BP-Hopf module spectra. If E is an extended BP-Hopf module spectrum, then E_*X is an extended BP_*BP-comodule for any CW-spectrum X.

Lemma 2.1. *Let E be a BP-Hopf module spectrum with comodule structure map η_E. Then there exists a homotopy equivalence $\tau_E: E\wedge BP \to BP\wedge E$ of BP-Hopf module spectra such that $\tau_E(1\wedge i)=\eta_E$ and $T\tau_E T\tau_E=1$, where T: $BP\wedge E\to E\wedge BP$ denotes the switching map.*

Proof. Set $\tau_E=(1\wedge\phi_E)(1\wedge T)(\eta_E\wedge 1)$, which is a BP-Hopf module map. It has an inverse τ_E^{-1} given by $\tau_E^{-1}=(\phi_E\wedge 1)(1\wedge T)(1\wedge\eta_E)$.

For the BP-Hopf module spectrum BPJ such that $j: BP\to BPJ$ is a BP-Hopf module map, we have

Corollary 2.2. *There exists a homotopy equivalence $\tau: BPJ\wedge BP\to BP\wedge BPJ$ of BP-Hopf module spectra such that $\tau(1\wedge i)=\eta$, $\tau(j\wedge 1)=1\wedge j$ and $T\tau T\tau=1$, where η denotes the comodule structure map of BPJ.*

The BPJ_*-module BP_*BPJ admits the following structure maps to be considered: (i) A product map $\nabla: BP_*BPJ\underset{BPJ_*}{\otimes}BP_*BPJ\to BP_*BPJ$ defined as usual, (ii) two unit maps η_L, η_R: $BPJ_*\to BP_*BPJ$ induced by η, $i\wedge 1$ respectively, (iii) a counit map $\varepsilon: BP_*BPJ\to BPJ_*$ induced by BP-module structure map $\phi=\mu(j\wedge 1)$, (iv) a coproduct map $\Delta: BP_*BPJ\to BP_*(BP\wedge BPJ)\cong BP_*BP\underset{BP_*}{\otimes}BP_*BPJ\cong BP_*BPJ\underset{BPJ_*}{\otimes}BP_*BPJ$ induced by $1\wedge i\wedge 1$, and (v) a conjugation map $c: BP_*BPJ\to BP_*BPJ$ induced by τT.

Proposition 2.3. *(BPJ_*, BP_*BPJ) is a Hopf algebroid, and $(j_*, (1\wedge j)_*)$: $(BP_*, BP_*BP)\to(BPJ_*, BP_*BPJ)$ is a morphism of Hopf algebroids.*

Proof. As is easily checked, Δ and ε are BPJ_*-bimodule maps and $(\varepsilon\otimes 1)\Delta=1=(1\otimes\varepsilon)\Delta$, $(\Delta\otimes 1)\Delta=(1\otimes\Delta)\Delta$, $c\eta_L=\eta_R$, $c\eta_R=\eta_L$, $\eta_L\varepsilon=\nabla(1\otimes c)\Delta$ and $\eta_R\varepsilon=\nabla(c\otimes 1)\Delta$. So the former part is obtained. The latter part is immediate.

For a quasi-associative BPJ-module spectrum F, $BP_*F\cong BP_*BP\underset{BP_*}{\otimes}F_*\cong BP_*BPJ\underset{BPJ_*}{\otimes}F_*$ and it is an extended BP_*BPJ-comodule. Let E be a BP-Hopf module spectrum which is a quasi-associative BPJ-module spectrum, and X be a CW-spectrum. Then E_*X is an associative BP_*BPJ-comodule with coaction map $\psi_X: E_*X\to BP_*(E\wedge X)\cong BP_*BPJ\underset{BPJ_*}{\otimes}E_*X$ induced by η_E. As is easily seen, we have

$$\mathrm{Hom}_{BP_*BP}(E_*X, BP_*F)=\mathrm{Hom}_{BP_*BPJ}(E_*X, BP_*F). \tag{2.1}$$

Further we recall that there exists an isomorphism

$$\theta: \mathrm{Hom}_{BP_*BP}(E_*X, BP_*F)\longrightarrow \mathrm{Hom}_{BPJ_*}(E_*X, F_*) \tag{2.2}$$

given by $\theta(u)=\phi_{F*}u$ and $\theta^{-1}(v)=(1\otimes v)\psi_X$, where $\phi_F=\mu_F(j\wedge 1)$.

Given BP-module spectra M, N we denote by $[M, N]_{BP}$ the subset of $[M, N]$ consisting of all the homotopy classes of BP-module maps. For a quasi-associative BPJ-module spectrum F we define a map

$$\tilde{\kappa}\colon [X, F] \longrightarrow [BPJ\wedge X, F]_{BP}$$

to be $\tilde{\kappa}(f)=\mu_F(1\wedge f)$. Denote by κ the composite map

$$\kappa=\pi\tilde{\kappa}\colon [X, F] \longrightarrow [BPJ\wedge X, F]_{BP} \longrightarrow \mathrm{Hom}_{BPJ_*}(BPJ_*X, F_*)$$

where π assigns to a map f the induced homomorphism f_*. Notice that κ is an isomorphism when BPJ_*X is BPJ_*-free.

For BP-Hopf module spectra M, N we also denote by $[M, N]_\Gamma$ the subset of $[M, N]_{BP}$ consisting of all the homotopy classes of BP-Hopf module maps. Let E be a BP-Hopf module spectrum and F be an associative BP-module spectrum. Then we have an isomorphism

$$\Theta\colon [E\wedge X, BP\wedge F]_\Gamma \longrightarrow [E\wedge X, F]_{BP} \tag{2.3}$$

defined to be $\Theta(f)=\phi_F f$. The inverse Θ^{-1} is given by $\Theta^{-1}(g)=(1\wedge g)(\eta_E\wedge 1)$ as in (2.2). For a quasi-associative BPJ-module spectrum F we denote by λ the composite map

$$\lambda=\Theta^{-1}\tilde{\kappa}\colon [X, F] \longrightarrow [BPJ\wedge X, F]_{BP}\cong[BPJ\wedge X, BP\wedge F]_\Gamma \tag{2.4}$$

which is given as $\lambda(f)=(1\wedge\mu_F)(\eta\wedge 1)(1\wedge f)$.

Lemma 2.4. *Let F be a quasi-associative BPJ-module spectrum such that F_* is BPJ_*-free and $F_*=0$ unless $*\equiv 0 \bmod 2(p-1)$. If the length of J is less than $p-1$, then the map $\lambda\colon [X, F]\to[BPJ\wedge X, BP\wedge F]_\Gamma$ is natural with respect to F.*

Proof. Let F and G be a quasi-associative BPJ-module spectra such that F_* is BPJ_*-free and $F_*=0=G_*$ unless $*\equiv 0 \bmod 2(p-1)$. For any map $h\colon F\to G$ it is sufficient to show that $(1\wedge h)(1\wedge\mu_F)(\eta\wedge 1)=(1\wedge\mu_G)(\eta\wedge 1)(1\wedge h)\colon BPJ\wedge F\to BP\wedge G$. The map $j\colon BP\to BPJ$ induces an epimorphism $(j\wedge 1)^*\colon \mathrm{Pr}\,(BP\wedge G)^*(BPJ\wedge F)\to \mathrm{Pr}\,(BP\wedge G)^*(BP\wedge F)$ by (1.5). Note that $\mathrm{Pr}\,(BP\wedge G)^*(BPJ\wedge F)\cong\Lambda_{(BP\wedge G)^*F}(x_0,\dots, x_{n-1})$ and $(BP\wedge G)^*F\cong\mathrm{Hom}^*_{BPJ_*}(BPJ_*F, BP_*G)=0$ unless $*\equiv 0, 1,\dots, n \bmod 2(p-1)$, where n denotes the length of J. Therefore $(j\wedge 1)^*$ becomes an isomorphism at dimension 0 when $n<p-1$. Then the desired equality follows immediately, since $(j\wedge 1)^*((1\wedge h)(1\wedge\mu_F)(\eta\wedge 1))=1\wedge h=(j\wedge 1)^*((1\wedge\mu_G)(\eta\wedge 1)(1\wedge h))$.

2.2. For an invariant regular sequence $J=\{q_0,\dots, q_{n-1}\}$ in BP_* we denote by Λ_J the set of the numbers $\sum_{0\leq i\leq n-1} t_i(d_i+1)$ for all n-tuples $(t_0,\dots, t_{n-1})$ of zeros and ones, where $d_i=\dim q_i$. Let $\sum_J = \vee_{d\in\Lambda_J} \sum^d$, the wedge of the suspended sphere spectra, and $\iota: S\to\sum_J$ be the canonical inclusion.

Lemma 2.5. *For each BPJ-module spectrum F there exists a homotopy equivalence $e_F: BPJ\wedge F\to BP\wedge F\wedge\sum_J$ of BP-Hopf module spectra such that $e_F(j\wedge 1)=1\wedge 1\wedge\iota$.*

Proof. For $0\leqq m<n$ we inductively construct a homotopy equivalence $e_{m+1}: BPJ_{m+1}\wedge F\to BP\wedge F\wedge\sum_{J_{m+1}}$ of BP-Hopf module spectra, where n denotes the length of J. By (1.5) we recall that $j_m: BPJ_m\to BPJ_{m+1}$ induces an epimorphism $j_m^*: \mathrm{Pr}\,(BPJ_m\wedge BPJ)^*BPJ_{m+1}\to\mathrm{Pr}\,(BPJ_m\wedge BPJ)^*BPJ_m$ for any m, $0\leqq m<n$. Then we can choose a BP-module map $\eta_{m+1,m}: BPJ_{m+1}\to BPJ_m\wedge BPJ$ such that $\eta_{m+1,m}j_m=1\wedge ji$. Setting $r_m=(1\wedge\mu_F)(\eta_{m+1,m}\wedge 1): BPJ_{m+1}\wedge F\to BPJ_m\wedge F$, it is a BP-module map with $r_m(j_m\wedge 1)=1$. We change r_m into a BP-Hopf module map $\tilde r_m: BPJ_{m+1}\wedge F\to BPJ_m\wedge F$ defined to be the composition $\tilde r_m=e_m^{-1}(1\wedge\mu_F\wedge 1)(1\wedge j\wedge 1\wedge 1)(1\wedge e_m)(1\wedge r_m)(\eta_{m+1}\wedge 1)$. It is easily seen that $\tilde r_m(j_m\wedge 1)=1$. Thus the sequence $BPJ_m\wedge F\to BPJ_{m+1}\wedge F\to\sum^{d_m+1} BPJ_m\wedge F$ is a split cofibering of BP-Hopf module spectra. So we have a homotopy equivalence $e_{m+1}: BPJ_{m+1}\wedge F\to(BPJ_m\wedge F)\vee(\sum^{d_m+1} BPJ_m\wedge F)\to BP\wedge F\wedge\sum_{J_{m+1}}$ of BP-Hopf module spectra.

Let F and G be BPJ-module spectra. For any map $f: F\to G$ there exists a unique map

$$f_J: F\wedge\textstyle\sum_J\longrightarrow G\wedge\sum_J \tag{2.5}$$

such that $(1\wedge f_J)e_F=e_G(1\wedge f)$. This is easily shown by use of (1.2). If $f: F\to G$ is a BPJ-module map, then $(1\wedge f)r_m=r_m(1\wedge f)$ and hence $(1\wedge f\wedge 1)e_F=e_G(1\wedge f)$. So we see

$$f_J=f\wedge 1 \text{ if } f: F\longrightarrow G \text{ is a } BPJ\text{-module map.} \tag{2.6}$$

Let F, G and H be BPJ-module spectra, and X and Y be CW-spectra. For any maps $f: F\to G$, $g: G\to H$ and $h: X\to Y$ the following results are immediately obtained.

(2.7) $1_J=1: F\wedge\sum_J\longrightarrow F\wedge\sum_J$ *and* $(gf)_J=g_Jf_J: F\wedge\sum_J\longrightarrow H\wedge\sum_J$.

(2.8) $(h\wedge f)_J=h\wedge f_J: X\wedge F\wedge\sum_J\longrightarrow Y\wedge G\wedge\sum_J$.

(2.9) *The diagram below is commutative*:

$$\begin{array}{ccccc} F\wedge\sum_{J_m} & \xrightarrow{1\wedge i} & F\wedge\sum_{J_{m+1}} & \xrightarrow{1\wedge k} & \sum^{d_m+1}F\wedge\sum_{J_m} \\ \downarrow{\scriptstyle f_{J_m}} & & \downarrow{\scriptstyle f_{J_{m+1}}} & & \downarrow{\scriptstyle f_{J_m}} \\ G\wedge\sum_{J_m} & \xrightarrow[1\wedge j]{} & G\wedge\sum_{J_{m+1}} & \xrightarrow[1\wedge k]{} & \sum^{d_m+1}G\wedge\sum_{J_m} \end{array}$$

where j and k are the canonical maps.

Lemma 2.6. i) *Let F be a quasi-associative BPJ-module spectrum with structure map μ_F. Then $F\wedge\sum_J$ is an associative BP-module spectrum whose structure map is $\phi_{F,J}=(\mu_F\wedge 1)(j\wedge 1\wedge 1)$: $BP\wedge F\wedge\sum_J\to F\wedge\sum_J$.*

ii) *Let E be a BP-Hopf module spectrum with comodule structure map η_E. If E is a quasi-associative BPJ-module spectrum, then $E\wedge\sum_J$ is a BP-Hopf module spectrum whose comodule structure map is $\eta_{E,J}$: $E\wedge\sum_J\to BP\wedge E\wedge\sum_J$.*

Proof. From the quasi-associativity of μ_F it follows that the map $\phi_F=\mu_F(j\wedge 1)$ is a BPJ-module map. Then (2.6) implies that $\phi_{F,J}=\phi_F\wedge 1$. Hence i) is obtained. ii) is immediate by means of (2.7) and (2.8).

It is easy to show

Lemma 2.7. *Let F and G be quasi-associative BPJ-module spectra, and f: $F\to G$ be a BP-module map. Then,*

i) *f_J: $F\wedge\sum_J\to G\wedge\sum_J$ is a BP-module map. Moreover,*

ii) *if F and G are BP-Hopf module spectra and f is a BP-Hopf module map, then f_J is a BP-Hopf module map, too.*

§3. Geometric Resolution

3.1. Let E and M be BP-Hopf module spectra. A complex $W=\{W_k, d_k: W_k\to W_{k+1}\}_{k\geqq 0}$ consisting of CW-spectra and maps is called an *E-geometric resolution* over M if the following three conditions are satisfied:

(i) There exists a BP-Hopf module map δ: $M\to E\wedge W_0$ with $(1\wedge d_0)\delta=0$.

(ii) The long sequence

$$*\longrightarrow M\xrightarrow{\delta}E\wedge W_0\xrightarrow{1\wedge d_0}E\wedge W_1\longrightarrow\cdots\longrightarrow E\wedge W_k\xrightarrow{1\wedge d_k}E\wedge W_{k+1}\longrightarrow\cdots$$

splits as a sequence of BP-module spectra. That is, there exist BP-module maps ε: $E\wedge W_0\to M$ and s_k: $E\wedge W_{k+1}\to E\wedge W_k$, $k\geqq 0$, such that $\varepsilon s_0=0=s_k s_{k+1}$, $\varepsilon\delta=1$, $\delta\varepsilon+s_0(1\wedge d_0)=1$ and $(1\wedge d_k)s_k+s_{k+1}(1\wedge d_{k+1})=1$ for each $k\geqq 0$.

(iii) $E\wedge W_k$ is an extended BP-Hopf module spectrum for each $k\geqq 0$.

From (1.3) we verify that if $W=\{W_k, d_k: W_k\to W_{k+1}\}_{k\geqq 0}$ is an E-geometric resolution over M, then

(3.1) $E_*W=\{E_*W_k, (1\wedge d_k)_*: E_*W_k\to E_*W_{k+1}\}_{k\geq 0}$ *is a relative injective resolution of* M_* *by extended* BP_*BP*-comodules.*

Let us denote by $\overline{BP}$ the cofiber of unit $i: S\to BP$, although the fiber of unit i was denoted as $\overline{BP}$ in [1] or [3]. Let E be a BP-module spectrum with structure map $\phi_E: BP\wedge E\to E$. The cofibering $E\xrightarrow{i\wedge 1}BP\wedge E\xrightarrow{\pi\wedge 1}\overline{BP}\wedge E$ splits, and hence there exists a unique map

$$\psi_E: \overline{BP}\wedge E\longrightarrow BP\wedge E \tag{3.2}$$

such that $(\pi\wedge 1)\psi_E=1$ and $(i\wedge 1)\phi_E+\psi_E(\pi\wedge 1)=1$. When E is a BP-Hopf module spectrum whose comodule structure map is $\eta_E: E\to BP\wedge E$, the cofibering $\overline{BP}\wedge E\xrightarrow{\psi_E}BP\wedge E\xrightarrow{\phi_E}E$ admits another splitting. Thus there exists a unique map

$$\rho_E: BP\wedge E\longrightarrow \overline{BP}\wedge E \tag{3.3}$$

such that $\rho_E\psi_E=1$ and $\eta_E\phi_E+\psi_E\rho_E=1$. We define two maps $\bar{\phi}_E: BP\wedge\overline{BP}\wedge E\to\overline{BP}\wedge E$ and $\bar{\eta}_E: \overline{BP}\wedge E\to BP\wedge\overline{BP}\wedge E$ to be

$$\begin{aligned}\bar{\phi}_E=\rho_E(m\wedge 1)(1\wedge\psi_E)&: BP\wedge\overline{BP}\wedge E\longrightarrow BP\wedge BP\wedge E\\ &\longrightarrow BP\wedge E\longrightarrow\overline{BP}\wedge E,\\ \bar{\eta}_E=(1\wedge\rho_E)(1\wedge i\wedge 1)\psi_E&: \overline{BP}\wedge E\longrightarrow BP\wedge E\\ &\longrightarrow BP\wedge BP\wedge E\longrightarrow BP\wedge\overline{BP}\wedge E.\end{aligned} \tag{3.4}$$

Lemma 3.1. *Let E be a BP-Hopf module spectrum. Then $\overline{BP}\wedge E$ is a BP-Hopf module spectrum such that $\rho_E: BP\wedge E\to\overline{BP}\wedge E$ is a BP-Hopf module map.*

Proof. By routine computations we can show the equalities $\bar{\phi}(i\wedge 1\wedge 1)=1$, $\bar{\phi}(1\wedge\bar{\phi})=\bar{\phi}(m\wedge 1\wedge 1)$, $\bar{\phi}\bar{\eta}=1$, $\bar{\eta}\bar{\phi}=(m\wedge 1\wedge 1)(1\wedge\bar{\eta})$ and $\bar{\phi}(1\wedge\rho)=\rho(m\wedge 1)$ without use of the coassociativity of η_E. Here the subscript E is omitted in $\bar{\phi}_E$, $\bar{\eta}_E$ and ρ_E. Moreover we obtain the equalities $(1\wedge\bar{\eta})\bar{\eta}=(1\wedge i\wedge 1\wedge 1)\bar{\eta}$ and $\bar{\eta}\rho=(1\wedge\rho)(1\wedge i\wedge 1)$ under the assumption that η_E is coassociative.

Remark. Such BP-Hopf module structure maps $\phi_{\overline{BP}\wedge E}$ and $\eta_{\overline{BP}\wedge E}$ on $\overline{BP}\wedge E$ that $\rho_E: BP\wedge E\to\overline{BP}\wedge E$ becomes a BP-Hopf module map are uniquely determined.

3.2. Given any BP-Hopf module spectrum E two maps $d_E: E\to\overline{BP}\wedge E$ and $s_E: BP\wedge\overline{BP}\wedge E\to BP\wedge E$ are defined to be

$$(3.5)\quad \begin{aligned} d_E &= (\pi \wedge 1)\eta_E \colon E \longrightarrow BP \wedge E \longrightarrow \overline{BP} \wedge E \\ s_E &= -(m \wedge 1)(1 \wedge \psi_E) \colon BP \wedge \overline{BP} \wedge E \longrightarrow BP \wedge BP \wedge E \longrightarrow BP \wedge E . \end{aligned}$$

Note that $d_E = -\rho_E(i \wedge 1)$, $s_E = -\psi_E \bar{\phi}_E$ and s_E is a BP-module map. Similarly $d_{\overline{BP} \wedge E} \colon \overline{BP} \wedge E \to \overline{BP}^2 \wedge E$ and $s_{\overline{BP} \wedge E} \colon BP \wedge \overline{BP}^2 \wedge E \to BP \wedge \overline{BP} \wedge E$ are defined to be $d_{\overline{BP} \wedge E} = (\pi \wedge 1 \wedge 1)\bar{\eta}_E$ and $s_{\overline{BP} \wedge E} = -(m \wedge 1 \wedge 1)(1 \wedge \bar{\psi}_E)$, where $\overline{BP}^2 = \overline{BP} \wedge \overline{BP}$. Obviously $d_{\overline{BP} \wedge E} = -1 \wedge d_E$. By easy calculations we have

Lemma 3.2. *Let E be a BP-Hopf module spectrum. Then $\phi_E s_E = 0 = s_E s_{\overline{BP} \wedge E}$, $s_E(1 \wedge d_E) = \psi_E \rho_E$, $(1 \wedge d_E)s_E + s_{\overline{BP} \wedge E}(1 \wedge d_{\overline{BP} \wedge E}) = 1$ and moreover $(1 \wedge d_E)\eta_E = 0 = d_{\overline{BP} \wedge E} d_E$.*

Let E be a BP-Hopf module spectrum with structure maps ϕ_E and η_E. For each $k \geqq 1$, $\overline{BP}^k \wedge E$ becomes a BP-Hopf module spectrum whose structure maps $\phi_k \colon BP \wedge \overline{BP}^k \wedge E \to \overline{BP}^k \wedge E$ and $\eta_k \colon \overline{BP}^k \wedge E \to BP \wedge \overline{BP}^k \wedge E$ are inductively constructed by $\phi_k = \bar{\phi}_{k-1}$ and $\eta_k = \bar{\eta}_{k-1}$, where $\phi_0 = \phi_E$, $\eta_0 = \eta_E$ and $\overline{BP}^k = \overline{BP} \wedge \cdots \wedge \overline{BP}$ with k-factors.

Theorem 3.3. *Let E be a BP-Hopf module spectrum. Then there exists a BP-geometric resolution $W_E = \{W_k = \overline{BP}^k \wedge E,\ d_k \colon W_k \to W_{k+1}\}_{k \geqq 0}$ over E.*

Proof. Consider the map $d_k \colon \overline{BP}^k \wedge E \to \overline{BP}^{k+1} \wedge E$ defined to be $d_k = (\pi \wedge 1 \wedge 1)\eta_k$. Then Lemma 3.2 implies that the long sequence $* \to E \xrightarrow{\eta_E} BP \wedge E \xrightarrow{1 \wedge d_0} BP \wedge \overline{BP} \wedge E \xrightarrow{1 \wedge d_1} BP \wedge \overline{BP}^2 \wedge E \to \cdots$ splits as a sequence of BP-module spectra. Hence the complex $W_E = \{W_k = \overline{BP}^k \wedge E,\ d_k\}_{k \geqq 0}$ is a BP-geometric resolution over E.

Proposition 3.4. *Let $W = \{W_k, d_k\}_{k \geqq 0}$ be a BP-geometric resolution over M. Assume that M and W_k, $k \geqq 0$, are quasi-associative BPJ-module spectra. Then $W = \{W_k, d_k\}_{k \geqq 0}$ is a BPJ-geometric resolution over $M \wedge \sum_J$.*

Proof. $W = \{W_k, d_k\}_{k \geqq 0}$ possesses a split sequence

$$* \longrightarrow M \underset{\varepsilon}{\overset{\delta}{\rightleftarrows}} BP \wedge W_0 \underset{s_0}{\overset{1 \wedge d_0}{\rightleftarrows}} BP \wedge W_1 \underset{s_1}{\overset{1 \wedge d_1}{\rightleftarrows}} BP \wedge W_2 \rightleftarrows \cdots$$

in which δ is a BP-Hopf module map and ε and s_k, $k \geqq 0$, are BP-module maps. This gives rise to another split sequence

$$* \longrightarrow M \wedge \textstyle\sum_J \underset{\varepsilon_J}{\overset{\delta_J}{\rightleftarrows}} BP \wedge W_0 \wedge \sum_J \underset{s_{0,J}}{\overset{1 \wedge d_{0,J}}{\rightleftarrows}} BP \wedge W_1 \wedge \sum_J \underset{s_{1,J}}{\overset{1 \wedge d_{1,J}}{\rightleftarrows}} BP \wedge W_2 \wedge \sum_J \rightleftarrows \cdots$$

by means of (2.5), (2.7), (2.8) and Lemmas 2.6 and 2.7. Set $\tilde{\delta} = e_{W_0}^{-1} \delta_J$:

$M \wedge \sum_J \to BPJ \wedge W_0$, which is a BP-Hopf module map by Lemmas 2.5 and 2.7 ii). Then the long sequene

$$* \longrightarrow M \wedge \sum_J \underset{\tilde{\varepsilon}}{\overset{\tilde{\delta}}{\rightleftarrows}} BPJ \wedge W_0 \underset{\tilde{s}_0}{\overset{1\wedge d_0}{\rightleftarrows}} BPJ \wedge W_1 \underset{\tilde{s}_1}{\overset{1\wedge d_1}{\rightleftarrows}} BPJ \wedge W_2 \rightleftarrows \cdots$$

becomes a split sequence of BP-module spectra, too. Here the BP-module maps $\tilde{\varepsilon}$ and $\tilde{s}_k$, $k \geqq 0$, are defined to be $\tilde{\varepsilon} = \varepsilon_J e_{W_0}$ and $\tilde{s}_k = e_{W_k}^{-1} s_{k,J} e_{W_{k+1}}$. Since $BPJ \wedge W_k$ is an extended BP-Hopf module spectrum by Lemma 2.5, the desired result is obtained.

Combining Proposition 3.4 with Theorem 3.3 we have

Corollary 3.5. *Let E be a BP-Hopf module spectrum which is a quasi-associative BPJ-module spectrum. Then the complex $W_E = \{W_k = \overline{BP}^k \wedge E, d_k\}_{k \geqq 0}$ is a BPJ-geometric resolution over $E \wedge \sum_J$.*

§4. Factorized System

4.1. Let $W = \{W_k, d_k\}_{k \geqq 0}$ be an E-geometric resolution over M. We say W admits an *m-factorized system* $X(m) = \{X_j, a_j, b_{j-1}, c_j\}_{1 \leqq j \leqq m}$ if the following properties are satisfied:

(i) $X_{j-1} \xrightarrow{b_{j-1}} W_j \xrightarrow{c_j} X_j \xrightarrow{a_j} \sum^1 X_{j-1}$ is a cofiber sequence, and

(ii) $d_{j-1} = b_{j-1} c_{j-1}$ and $d_m b_{m-1} = 0$ for each j, $1 \leqq j \leqq m$,

where $X_0 = W_0$, $b_0 = d_0$, $c_0 = 1$ and $1 \leqq m \leqq \infty$.

Let $X(m) = \{X_j, a_j, b_{j-1}, c_j\}_{1 \leqq j \leqq m}$ be an m-factorized system of $W = \{W_k, d_k\}_{k \geqq 0}$. Pick up a map b_m: $X_m \to W_{m+1}$ with $b_m c_m = d_m$ and a split sequence

$$* \longrightarrow M \underset{\varepsilon}{\overset{\delta}{\rightleftarrows}} E \wedge W_0 \underset{s_0}{\overset{1\wedge d_0}{\rightleftarrows}} E \wedge W_1 \underset{s_1}{\overset{1\wedge d_1}{\rightleftarrows}} E \wedge W_2 \rightleftarrows \cdots$$

of BP-module spectra in which δ is a BP-Hopf module map and fix them. Choose a map u_m: $E \wedge X_{m-1} \to \sum^{-1} E \wedge X_m$ such that $(1 \wedge a_m) u_m = 1 - (1 \wedge c_{m-1}) s_{m-1}(1 \wedge b_{m-1})$, and then replace it with the map

$$t_m \colon E \wedge X_{m-1} \longrightarrow \textstyle\sum^{-1} E \wedge X_m \tag{4.1}$$

given by $t_m = u_m - (1 \wedge c_m) s_m (1 \wedge b_m) u_m$. Since $(1 \wedge a_m) u_m (1 \wedge a_m) = 1 \wedge a_m$ and $(1 \wedge a_m) u_m (1 \wedge c_{m-1}) s_{m-1} = 0$, we can easily check

(4.2)
(i) $s_m(1 \wedge b_m) t_m = 0 = t_m (1 \wedge c_{m-1}) s_{m-1}$,
(ii) $t_m (1 \wedge a_m) + (1 \wedge c_m) s_m (1 \wedge b_m) = 1$ *and*

$(1\wedge a_m)t_m+(1\wedge c_{m-1})s_{m-1}(1\wedge b_{m-1})=1$.

Notice that the map t_m is a BP-module map, because $(1\wedge a_m)t_m(\phi_E\wedge 1)=(1\wedge a_m)(\phi_E\wedge 1)(1\wedge t_m)$ by use of (4.1). Hence the long sequence

(4.3) $\cdots\longrightarrow E\wedge X_{m-1}\xrightarrow{1\wedge b_{m-1}}E\wedge W_m\xrightarrow{1\wedge c_m}E\wedge X_m\xrightarrow{1\wedge a_m}\sum^1 E\wedge X_{m-1}\longrightarrow\cdots$

splits as a sequence of BP-module spectra. Immediately (4.2) implies

(4.4) $(1\wedge a_m)t_m=t_{m-1}(1\wedge a_{m-1})$, $(1\wedge a_m)t_m(1\wedge a_m)=1\wedge a_m$ *and* $t_m(1\wedge a_m)t_m=t_m$.

Lemma 4.1. *Let $W=\{W_k, d_k\}_{k\geqq 0}$ be an E-geometric resolution over M and $X(m)=\{X_j, a_j, b_{j-1}, c_j\}_{1\leqq j\leqq m}$ be its m-factorized system. Then there exists a BP-Hopf module map ε_m: $E\wedge X_m\to\sum^m M$ such that $\varepsilon_m=\varepsilon_{m-1}(1\wedge a_m)$ and the long sequence*

$$\cdots\xrightarrow{1\wedge c_m}\sum^{-1}E\wedge X_m\xrightarrow{1\wedge a_m}E\wedge X_{m-1}\xrightarrow{1\wedge b_{m-1}}E\wedge W_m\xrightarrow{1\wedge c_m}E\wedge X_m\xrightarrow{\varepsilon_m}\sum^m M\longrightarrow *$$

splits as a sequence of BP-module spectra.

Proof. Consider the composite map $\varepsilon_m=\varepsilon(1\wedge a_1)\cdots(1\wedge a_m)$: $E\wedge X_m\to\sum^m M$. Obviously $\delta\varepsilon_m=(1\wedge a_1)\cdots(1\wedge a_m)$ and it is a BP-Hopf module map. Therefore ε_m is also a BP-Hopf module map since $(1\wedge\delta)\eta_M\varepsilon_m=(1\wedge\delta)(1\wedge\varepsilon_m)(\eta_E\wedge 1)$. Set $\delta_m=t_m\cdots t_1\delta$: $M\to\sum^{-m}E\wedge X_m$, then (4.2) and (4.4) imply that $\varepsilon_m\delta_m=1$ and $\delta_m\varepsilon_m+(1\wedge c_m)s_m(1\wedge b_m)=1$. The result is now immediate from (4.3).

Let $W=\{W_k, d_k\colon W_k\to W_{k+1}\}_{k\geqq 0}$ be a complex of CW-spectra, and $X\xrightarrow{b}W_m\xrightarrow{c}Y\xrightarrow{a}\sum^1 X$ be a cofiber sequence. Suppose that two sequences $[\sum^1 W_m, W_{m+2}]\to[\sum^1 W_m, W_{m+3}]\to 0$ and $[\sum^1 X, W_{m+1}]\to[\sum^1 X, W_{m+2}]\to[\sum^1 X, W_{m+3}]$ induced by d's are both exact. Then an easy diagram chasing shows that there exists a map $\bar b$: $Y\to W_{m+1}$ satisfying $\bar bc=d_m$ and $d_{m+1}\bar b=0$ if $d_mb=0$. Hence we obtain immediately.

Proposition 4.2. *Let $W=\{W_k, d_k\}_{k\geqq 0}$ be an E-geometric resolution over M such that $[\sum^1 W_m, W_{m+3}]=0$. Assume that W admits an m-factorized system $X(m)=\{X_j\}_{1\leqq j\leqq m}$. Then W admits an $(m+1)$-factorized system $X(m+1)=\{X_j\}_{1\leqq j\leqq m+1}$ if the sequence $[\sum^1 X_{m-1}, W_{m+1}]\to[\sum^1 X_{m-1}, W_{m+2}]\to[\sum^1 X_{m-1}, W_{m+3}]$ is exact.*

Let $W=\{W_k, d_k\}_{k\geqq 0}$ and $W'=\{W'_k, d'_k\}_{k\geqq 0}$ be two complexes of CW-

spectra, and $g=\{g_k\}_{k\geqq 0}$: $W\to W'$ be a map of complexes. Let $X\xrightarrow{b}W_m\xrightarrow{c}Y\xrightarrow{a}\sum^1 X$ and $X'\xrightarrow{b'}W'_m\xrightarrow{c'}Y'\xrightarrow{a'}\sum^1 X'$ be two cofiber sequences, and $\bar{b}$: $Y\to W_{m+1}$ and $\bar{b}'$: $Y'\to W'_{m+1}$ be maps satisfying $\bar{b}c=d_m$, $\bar{b}'c'=d'_m$ and $d_{m+1}\bar{b}=0=d'_{m+1}\bar{b}'$ respectively. Suppose that $[\sum^1 W_m, W'_{m+1}]\to[\sum^1 W_m, W'_{m+2}]\to 0$ and $[\sum^1 X, W'_m]\to[\sum^1 X, W'_{m+1}]\to[\sum^1 X, W'_{m+2}]$ are both exact. Given a map f: $X\to X'$ with $b'f=g_m b$, we can easily choose a map h: $Y\to Y'$ such that $\bar{b}'h=g_{m+1}\bar{b}$, $hc=c'g_m$ and $a'h=(\sum^1 f)a$. Hence we have

Proposition 4.3. *Let $W=\{W_k, d_k\}_{k\geqq 0}$ and $W'=\{W'_k, d'_k\}_{k\geqq 0}$ be E-geometric resolutions over M and N respectively, and $X(m)=\{X_j\}_{1\leqq j\leqq m}$ and $X'(m)=\{X'_j\}_{1\leqq j\leqq m}$ be their m-factorized systems. Given a map g: $W\to W'$ of complexes, there exists a map $f(m)$: $X(m)\to X'(m)$ of m-factorized systems if $[\sum^1 W_k, W'_{k+2}]=0$ and the sequences $[\sum^1 X_{k-1}, W'_k]\to[\sum^1 X_{k-1}, W'_{k+1}]\to[\sum^1 X_{k-1}, W'_{k+2}]$ are exact for all k, $1\leqq k\leqq m-1$.*

4.2. Let $W=\{W_k, d_k\}_{k\geqq 0}$ be a BPJ-geometric resolution over N and F be a quasi-associative BPJ-module spectrum. Suppose that F satisfies the condition:

$(4.5)_W$ κ: $[\sum^t W_k, F]\to \mathrm{Hom}^{-t}_{BPJ_*}(BPJ_*W_k, F_*)$ is an isomorphism for each $k\geqq 0$.

For example, all F satisfy the condition $(4.5)_W$ whenever BPJ_*W_k is BPJ_*-free (see **2.1**).

Let $X(m)=\{X_j, a_j, b_{j-1}, c_j\}_{1\leqq j\leqq m}$ be an m-factorized system of W. By making use of (4.3) and Five lemma we see that κ: $[\sum^t X_m, F]\to\mathrm{Hom}^{-t}_{BPJ_*}(BPJ_*X_m, F_*)$ is an isomorphism, too. From Lemma 4.1 we obtain that the sequence $BPJ_*W_m\to BPJ_*X_m\to N_{*-m}\to 0$ is split exact of BPJ_*-modules. This gives rise to a split exact sequence $0\to\mathrm{Hom}^{m-t}_{BPJ_*}(N_*, F_*)\xrightarrow{\varepsilon^\#_m}[\sum^t X_m, F]\to[\sum^t W_m, F]$. Recall that there exists an isomorphism θ: $\mathrm{Hom}_{BP_*BP}(N_*, BP_*F)\to\mathrm{Hom}_{BPJ_*}(N_*, F_*)$ by (2.2). Replacing $\varepsilon^\#_m$ with the composite map $\varepsilon^\#_m\theta$, denoted by ξ_m, we have a split exact sequence

$$(4.6)\qquad 0\longrightarrow \mathrm{Hom}^{m-t}_{BP_*BP}(N_*, BP_*F)\xrightarrow{\xi_m}[\textstyle\sum^t X_m, F]\xrightarrow{c^*_m}[\textstyle\sum^t W_m, F]$$

Lemma 4.4. *Let $W=\{W_k\}_{k\geqq 0}$ be a BPJ-geometric resolution over N and $X(m)=\{X_j\}_{1\leqq j\leqq m}$ be its m-factorized system. Let F be a quasi-associative BPJ-module spectrum satisfying the condition $(4.5)_W$ such that $F_*=0$ unless $*\equiv 0 \bmod 2(p-1)$. Suppose that the length of J is less than $p-1$. Then the* *map* ξ_m: $\mathrm{Hom}_{BP_*BP}^{-m-t}(N_*, BP_*F) \to [\sum^t X_m, F]$ *is natural with respect to* F. *Moreover it is an isomorphism if* $[\sum^t W_m, F]=0$.

Proof. The composite map $\xi_m(\delta_*)^*\theta^{-1}\kappa$: $[\sum^{m+t} W_0, F] \to [\sum^t X_m, F]$ is induced by the composition $a_1 \cdots a_m$, because $\delta\varepsilon_m=(1 \wedge a_1)\cdots(1 \wedge a_m)$. Obviously $\theta^{-1}\kappa=\pi\Theta^{-1}\kappa=\pi\lambda$, so it follows from Lemma 2.4 that $(\delta_*)^*\theta^{-1}\kappa$: $[\sum^{m+t} W_0, F] \to \mathrm{Hom}_{BP_*BP}^{-m-t}(N_*, BP_*F)$ is natural with respect to F. Since $(\delta_*)^*\theta^{-1}\kappa$ is an epimorphism it is obvious that ξ_m is also natural with respect to F. The latter part is immediate from (4.6).

As a sufficient condition under which $[\sum^t G, F]=0$ holds we have

Lemma 4.5. *Let F and G be quasi-associative BPJ-module spectra such that $F_*=0=G_*$ unless $*\equiv 0 \bmod 2(p-1)$ and G_* is BPJ_*-free. If the length n of J is less than $2p-3$, then $[\sum^t G, F]=0$ for each t, $1 \leqq t < 2(p-1)-n$.*

Proof. Note that $BPJ_*G=0$ unless $*\equiv 0, 1, \dots, n \bmod 2(p-1)$. This implies that $[\sum^t G, F] \cong \mathrm{Hom}_{BPJ_*}^{-t}(BPJ_*G, F_*)=0$ when $1 \leqq t < 2(p-1)-n$.

Theorem 4.6. *Suppose that the length of J is less than $p-1$. Let $W=\{W_k, d_k\}_{k \geqq 0}$ be a BP-geometric resolution over M such that M and W_k, $k \geqq 0$, are quasi-associative BPJ-module spectra with W_{k*} BPJ_*-free and $W_{k*}=0$ unless $*\equiv 0 \bmod 2(p-1)$. If $\mathrm{Ext}_{BP_*BP}^{m+2,-m-t}(M_*, M_*)=0$ for all $m \geqq 1$ and $t \in \Lambda_J$, then W admits an ∞-factorized system $X(\infty)$. Moreover, its ∞-factorized system is uniquely given if $\mathrm{Ext}_{BP_*BP}^{m+1,-m-t}(M_*, M_*)=0$ for all $m \geqq 1$ and $t \in \Lambda_J$. (Cf.,* [13, Lemma 3.1]).

Proof. $W=\{W_k, d_k\}_{k \geqq 0}$ is a BPJ-geometric resolution over $M \wedge \sum_J$ by Proposition 3.4. Note that $[\sum^1 W_i, W_k]=0$ for all i, $k \geqq 0$, because of Lemma 4.5. Inductively we assume that W admits an m-factorized system $X(m)=\{X_j\}_{1 \leqq j \leqq m}$, to show the existence of its ∞-factorized system $X(\infty)$. By Lemma 4.4 we have an isomorphism ξ_m: $\mathrm{Hom}_{BP_*BP}^{-m}(M_*\sum_J, BP_*W_k) \to [\sum^1 X_{m-1}, W_k]$ which is natural with respect to W_k. The sequence $0 \to M_*\sum_{J_i} \to M_*\sum_{J_{i+1}} \to M_{*-d_i-1}\sum_{J_i} \to 0$ is exact of BP_*BP-comodules and split exact of (free) BPJ_*-modules. Hence our first hypothesis implies that $\mathrm{Ext}_{BP_*BP}^{m+2,-m}(M_*\sum_J, M_*)=0$ for all $m \geqq 1$. Using the natural isomorphism ξ_m this means that the sequence $[\sum^1 X_{m-1}, W_{m+1}] \to [\sum^1 X_{m-1}, W_{m+2}] \to [\sum^1 X_{m-1}, W_{m+3}]$ is exact. Apply Proposition 4.2 to botain an $(m+1)$-factorized system $X(m+1)=\{X_j\}_{1 \leqq j \leqq m+1}$.

The uniqueness of $X(\infty)$ is easily shown by use of Proposition 4.3, because our latter hypothesis implies that the sequences $[\sum^1 X_{m-1}, W_m] \to [\sum^1 X_{m-1},$

$W_{m+1}]\to[\sum^1 X_{m-1}, W_{m+2}]$ are exact for all $m\geqq 1$.

4.3. Let $W=\{W_k, d_k\}_{k\geqq 0}$ be a BP-geometric resolution over M such that W_k is a quasi-associative BPJ-module spectrum for each $k\geqq 0$. Let L and N be BP-Hopf module spectra and $f\colon L\to N$ be a BP-Hopf module map inducing an isomorphism $f_*\colon BPJ_*\underset{BP_*}{\otimes} L_*\cong N_*$. Then the map f induces an isomorphism $(f_*)^*\colon \mathrm{Hom}_{BP_*BP}(N_*, BP_*W_k)\to\mathrm{Hom}_{BP_*BP}(L_*, BP_*W_k)$. Hence we have an isomorphism

$$\mathrm{Ext}^{s,t}_{BP_*BP}(N_*, M_*)\cong\mathrm{Ext}^{s,t}_{BP_*BP}(L_*, M_*). \tag{4.7}$$

Specially $j\colon BP\to BPJ$ induces an isomorphism

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BPJ_*, M_*)\cong\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M_*). \tag{4.8}$$

Lemma 4.7. *Let C be an associative BP_*BP-comodule which is a direct limit of finitely presented v_{n-1}-torsion comodules. If n is not divided by $p-1$, then* $\mathrm{Ext}^s_{BP_*BP}(BP_*, v_n^{-1}C)=0$ *for all $s>n^2$.*

Proof. We may assume that C itself is finitely presented and v_{n-1}-torsion. Choose a Landweber prime filtration $C=C_0\supset C_1\supset\cdots\supset C_r=\{0\}$ so that each subquotient C_k/C_{k+1} is a suspension of $BP_*/I_{n(k)}$ for some $n(k)\geqq n$. Then $v_n^{-1}C$ has a filtration $v_n^{-1}C=B_0\supset B_1\supset\cdots\supset B_q=\{0\}$ so that all subquotients are suspensions of $v_n^{-1}BP_*/I_n$. By Morava's Theorem [8, Theorem 3.16] $\mathrm{Ext}^s_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)=0$ for all $s>n^2$ whenever $p-1\nmid n$. The desired result is easily shown.

Let us denote by L_n, $n\geqq 0$, the localization functor with respect to $v_n^{-1}BP_*$-homology (see [3] or [11]). Consider the functor N_n, $n\geqq 0$, derived from the cofibering $X\to L_{n-1}X\to\sum^{-n+1}N_nX$, where $N_0=1$. We put $M_n=L_nN_n$, $n\geqq 0$. By [17, Theorem 2.3] we notice that N_nX is v_k-torsion for each k, $0\leqq k<n$, and $M_nX=v_n^{-1}N_nX$ if X is an associative BP-module spectrum.

Corollary 4.8. *Let n be a positive integer not less than the length of J. Suppose that p is odd and $n^2+n<2p$. Then* $\mathrm{Ext}^{m+k,-m-t}_{BP_*BP}(BP_*, M_nBPJ_*)=0$ *for all $m\geqq 1$, $k\geqq 1$ and $t\in\Lambda_J$.*

Proof. In the $m+k>n^2$ case the result is immediate from Lemma 4.7. In the $m+k\leqq n^2$ case it is obvious that $\mathrm{Ext}^{*,-m-t}_{BP_*BP}(BP_*, M_nBPJ_*)=0$ for all $t\in\Lambda_J$, since $1\leqq m\leqq m+n\leqq n^2+n-1<2(p-1)$.

Given an E-geometric resolution $W=\{W_k, d_k\}_{k\geqq 0}$ over M, $L_nW=\{L_nW_k, L_nd_k\}_{k\geqq 0}$ is also an E-geometric resolution over L_nM, because $E\wedge L_nX$

$=L_n(E \wedge X)=L_nE \wedge X$ by Ravenel's result [12, Theorem 1]. Recall that the radical of J is just $I_n=(p, v_1, \dots, v_{n-1})$ where n denotes the length of J. So it follows from [17, Proposition 2.2] that $L_nF=v_n^{-1}F$ whenever F is a quasi-associative BPJ-module spectrum.

Let $W_{BPJ}=\{W_k=\overline{BP}^k \wedge BPJ, d_k\}_{k\geqq 0}$ be the BP-geometric resolution over BPJ constructed in Theorem 3.3. The BP-geometric resolution L_nW_{BPJ}, obtained by applying the localization functor L_n to the BP-geometric resolution W_{BPJ}, coincides with the BP-geometric resolution $W_{v_n^{-1}BPJ}=\{\overline{BP}^k \wedge v_n^{-1}BPJ, d_k\}_{k\geqq 0}$ over $v_n^{-1}BPJ$.

Theorem 4.9. *Let J be an invariant regular sequence of length n. Suppose that p is odd and $n^2+n<2p$. Then the BP-geometric resolution $W_{v_n^{-1}BPJ}=\{L_nW_k=\overline{BP}^k \wedge v_n^{-1}BPJ, d_k\}_{k\geqq 0}$ over $v_n^{-1}BPJ$ admits a unique ∞-factorized system $Y(\infty)$.*

Proof. For any quasi-associative BPJ-module spectrum F the map κ: $[\sum^t L_nW_k, L_nF]\to \mathrm{Hom}_{BPJ_*}^{-t}(BPJ_*L_nW_k, L_nF_*)$ is an isomorphism because $[\sum^t L_nW_k, L_nF]\cong[\sum^t W_k, L_nF]\cong \mathrm{Hom}_{BPJ_*}^{-t}(BPJ_*W_k, v_n^{-1}F_*)$. Thus all L_nF satisfy the condition $(4.5)_W$ where the BPJ-geometric resolution $W_{v_n^{-1}BPJ}$ over $v_n^{-1}BPJ \wedge \sum_J$ is abbreviated as W. Moreover it follows from Lemma 4.5 that $[\sum^1 L_nW_i, L_nW_k]\cong[\sum^1 W_i, L_nW_k]=0$ for all $i, k\geqq 0$. Inductively we assume that $W_{v_n^{-1}BPJ}$ admits an m-factorized system $Y(m)=\{Y_j\}_{1\leqq j\leqq m}$. By Lemma 4.4 there exists an isomorphism ξ_m: $\mathrm{Hom}_{BP_*BP}^{-m}(BPJ_*\sum_J, BP_*L_nW_k)\to[\sum^1 Y_{m-1}, L_nW_k]$, which is natural with respect to W_k. Combining Corollary 4.8 with (4.7) it is shown that $\mathrm{Ext}_{BP_*BP}^{m+k, -m-t}(BPJ_*, v_n^{-1}BPJ_*)=0$ for all $m\geqq 1$, $k\geqq 1$ and $t\in A_J$, when $n\geqq 1$. As in the proof of Theorem 4.6 this implies that the sequence $[\sum^1 Y_{m-1}, L_nW_{m+k-1}]\to[\sum^1 Y_{m-1}, L_nW_{m+k}]\to[\sum^1 Y_{m-1}, L_nW_{m+k+1}]$ is exact. In the $n=0$ case the exactness is easily shown since $[Y, L_0W]\cong \mathrm{Hom}(\pi_*Y, \pi_*W \otimes Q)$. Applying Propositions 4.2 and 4.3 we obtain the desired result.

§ 5. Homotopy Inverse Limit

5.1. Let $W=\{W_k, d_k\}_{k\geqq 0}$ be an E-geometric resolution over M. Assume that W admits an ∞-factorized system $X=\{X_m, a_m, b_{m-1}, c_m\}_{m\geqq 1}$. For a CW-spectrum Y the tower $\{\sum^{-m} Y\wedge X_m, 1\wedge a_m\}_{m\geqq 1}$ has a homotopy inverse limit $\varprojlim_m \sum^{-m}Y\wedge X_m$ denoted by $(Y\wedge X)_\infty$. It possesses the canonical projections q_m: $(Y\wedge X)_\infty\to\sum^{-m}Y\wedge X_m$ such that $(1\wedge a_m)q_m=q_{m-1}$. The BP-Hopf module maps ε_m: $\sum^{-m}E \wedge X_m \to M$ given in Lemma 4.1 induce a BP-Hopf module map

(5.1) $$\varepsilon_\infty : E \wedge X_\infty \longrightarrow M$$

defined to be $\varepsilon_\infty = \varepsilon_m(1 \wedge q_m) = \varepsilon(1 \wedge q_0)$.

Proposition 5.1. *Let $W = \{W_k, d_k\}_{k \geqq 0}$ be an E-geometric resolution over M which admits an ∞-factorized system $X = \{X_m\}_{m \geqq 1}$. If the canonical map $q: E \wedge X_\infty \to (E \wedge X)_\infty$ is a homotopy equivalence, then the BP-Hopf module map $\varepsilon_\infty: E \wedge X_\infty \to M$ is a homotopy equivalence, too.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} E_{*+m+1}W_{m+1} & \xrightarrow{(1\wedge c_{m+1})_*} & E_{*+m+1}X_{m+1} & \xrightarrow{\varepsilon_{m+1*}} & M_* & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ E_{*+m}W_m & \xrightarrow[(1\wedge c_m)_*]{} & E_{*+m}X_m & \xrightarrow[\varepsilon_{m*}]{} & M_* & \longrightarrow & 0 \end{array}$$

in which the left vertical arrow is trivial. The two rows are exact by Lemma 4.1. Hence we observe that $\varprojlim_m E_{*+m}X_m \cong M_*$ and $\varprojlim_m^1 E_{*+m}X_m = 0$. This implies that the map ε_∞ induces an isomorphism $\varepsilon_{\infty*}: E_*X_\infty \to M_*$ under our hypothesis.

Corollary 5.2. *Assume that E is connective and of finite type and that all W_k, $k \geqq 0$, are $(N+k)$-connected for some N independent on k. Then the BP-Hopf module map $\varepsilon_\infty: E \wedge X_\infty \to M$ is a homotopy equivalence.*

Proof. From [1, Theorem 15.2] it follows that the canonical map $q: E \wedge X_\infty \to (E \wedge X)_\infty$ is a homotopy equivalence.

Given a ring spectrum E we form a cofibering $S \xrightarrow{i} E \xrightarrow{\pi} \bar{E}$ and put $\bar{E}^k = \bar{E} \wedge \cdots \wedge \bar{E}$ with k-factors. Consider the Adams geometric resolution $W_{E,Y} = \{W_kY = \bar{E}^k \wedge E \wedge Y,\ d_k: W_kY \to W_{k+1}Y\}_{k \geqq 0}$ for a CW-spectrum Y, where d_k is defined to be $d_k = (-1)^k(1 \wedge \pi \wedge 1 \wedge 1)(1 \wedge 1 \wedge i \wedge 1)$. Note that the Adams geometric resolution $W_{E,Y}$ gives an E-geometric resolution over $E \wedge Y$ when E is a ring and BP-Hopf module spectrum. Let K_mY denote the fiber of the obvious map $\bar{E}^{m+1} \wedge Y \to \sum^{m+1} Y$, thus $\sum^m Y \xrightarrow{\alpha_m} K_mY \to \bar{E}^{m+1} \wedge Y \to \sum^{m+1} Y$ be a cofiber sequence, $m \geqq 0$, where $K_0Y = E \wedge Y$ and $\alpha_0 = i \wedge 1$. Then we have a cofiber sequence $K_{m-1}Y \xrightarrow{b_{m-1}} W_mY \xrightarrow{c_m} K_mY \xrightarrow{a_m} \sum^1 K_{m-1}Y$ such that $b_mc_m = d_m$ and $a_m\alpha_m = \alpha_{m-1}$ (see [3]). Hence we see

(5.2) $KY = \{K_mY, a_m, b_{m-1}, c_m\}_{m \geqq 1}$ *is an ∞-factorized system of the E-geometric resolution $W_{E,Y}$.*

The tower $\{\sum^{-m}K_mY, a_m\}_{m \geqq 1}$ has a homotopy inverse limit $E^\wedge Y$ with a map $\alpha\colon Y\to E^\wedge Y$ inducing the maps $\alpha_m\colon Y\to\sum^{-m}K_mY$. A CW-spectrum Y is said to be *E-nilpotent complete* if the map $\alpha\colon Y\to E^\wedge Y$ is a homotopy equivalence. Any E-module spectrum is obviously E-nilpotent complete. Note that $1\wedge\alpha\colon E\wedge Y\to E\wedge(E^\wedge Y)$ has a left inverse constructed using the map $q_0\colon E^\wedge Y\to E\wedge Y$. The left inverse $(m\wedge 1)(1\wedge q_0)$ coincides with ε_∞ given in (5.1), where m denotes the multiplication of E. Since $E\wedge Y$ is E-nilpotent complete, it follows that $1\wedge\alpha\colon E\wedge Y\to E\wedge(E^\wedge Y)$ is a homotopy equivalence if and only if the canonical map $q\colon E\wedge(E^\wedge Y)\to E^\wedge(E\wedge Y)$ is so. Therefore we have

(5.3) *An E-local CW-spectrum Y is E-nilpotent complete if the canonical map* $q\colon E\wedge(E^\wedge Y)\to E^\wedge(E\wedge Y)$ *is a homotopy equivalence.*

5.2. Let $W=\{W_k, d_k\}_{k\geqq 0}$ be an E-geometric resolution over M, which admits an ∞-factorized system $X=\{X_m, a_m, b_{m-1}, c_m\}_{m\geqq 1}$. Consider the tower $\{\sum^{-m}E\wedge X_m, 1\wedge a_m\}_{m\geqq 1}$ with homotopy inverse limit $(E\wedge X)_\infty$. Since $(1\wedge a_m)\cdots(1\wedge a_k)t_k\cdots t_m\,(1\wedge a_m)=1\wedge a_m$ by means of (4.4), it is easily seen that

$$\begin{aligned}(5.4)\qquad &\operatorname{Im}\{(1\wedge a_m)_*\colon [Y, \textstyle\sum^{-1} E\wedge X_m]\longrightarrow[Y, E\wedge X_{m-1}]\}\\ &=\operatorname{Im}\{(1\wedge a_m\cdots a_k)_*\colon [Y, \textstyle\sum^{-k+m-1}E\wedge X_k]\longrightarrow [Y, E\wedge X_{m-1}]\}\end{aligned}$$

for any $k\geqq m$. Thus the inverse system $\{[Y, \sum^{-m}E\wedge X_m], (1\wedge a_m)_*\}_{m\geqq 1}$ satisfies the Mittag-Leffler condition. This result implies that

$$(5.5)\qquad q_\#\colon [Y, (E\wedge X)_\infty]\longrightarrow \varprojlim_m\, [Y, \textstyle\sum^{-m} E\wedge X_m]$$

is an isomorphism because $\varprojlim_m^1 [\sum^1 Y, \sum^{-m}E\wedge X_m]=0$.

Lemma 5.3. *Let* $W=\{W_k, d_k\}_{k\geqq 0}$ *be an E-geometric resolution over M with an* ∞*-factorized system* $X=\{X_m\}_{m\geqq 1}$. *Then* $q^\#\colon \varinjlim_m [\sum^{-m} E\wedge X_m, Y]\to[(E\wedge X)_\infty, Y]$ *is an isomorphism for any CW-spectrum Y.*

Proof. Since $(1\wedge a_{m+1})t_{m+1}\cdots t_1=t_m\cdots t_1$ for any $m\geqq 1$, the maps $t_m\cdots t_1\colon E\wedge X_0\to\sum^{-m} E\wedge X_m$ give rise to a unique map $t\colon E\wedge X_0\to(E\wedge X)_\infty$ such that $q_0t=(1\wedge a_1)t_1$ and $q_mt=t_m\cdots t_1$ for each $m\geqq 1$. By use of (4.4) it is obvious that $q_mtq_0=q_m$, since $(1\wedge a_{j+1})q_{j+1}=q_j$. Applying (5.5) in the $Y=(E\wedge X)_\infty$ case we obtain that $q_\#(tq_0)=q_\#(1)$, and hence $tq_0=1$. Therefore it is easy to show that $q^\#$ is an epimorphism. Next, choose a map $f_m\colon \sum^{-m} E\wedge X_m\to Y$ with $f_mq_m=0$. Then $f_m(1\wedge a_{m+1})=f_mt_m\cdots t_1(1\wedge a_1)\cdots(1\wedge a_{m+1})=0$, so $q^\#$ is a monomorphism.

For a ring spectrum F, a tower $\{Z_m, f_m\}_{m\geqq 1}$ of CW-spectra is said to be an

F-nilpotent resolution of a *CW*-spectrum Z if (i) Z_m is F-nilpotent for each $m\geqq 1$, and (ii) there exists a map $g: Z\to\varprojlim_m Z_m$ inducing an isomorphism $g^{\#}: \varinjlim_m [Z_m, N]\to[Z, N]$ for any F-nilpotent spectrum N (see [3, Definition 5.6]).

Theorem 5.4. *Let* $W=\{W_k, d_k\}_{k\geqq 0}$ *be an E-geometric resolution over M such that each* W_k *is E-nilpotent, and* $X=\{X_m, a_m, b_{m-1}, c_m\}_{m\geqq 1}$ *be its* ∞*-factorized system. Assume that the canonical map* $q: E\wedge X_\infty\to(E\wedge X)_\infty$ *is a homotopy equivalence. Then the tower* $\{\Sigma^{-m} X_m, a_m\}_{m\geqq 1}$ *is an E-nilpotent resolution of homotopy inverse limit* X_∞ *when E is a ring and BP-Hopf module spectrum.*

Proof. Each X_m is obviously E-nilpotent. So it is sufficient to show that $q^{\#}: \varinjlim_m [\Sigma^{-m} X_m, E\wedge Y]\to[X_\infty, E\wedge Y]$ is an isomorphism, when taking $E\wedge Y$ specially as an E-nilpotent spectrum N. Consider the commutative diagram

$$\begin{array}{ccccc} \varinjlim_m [\Sigma^{-m}E\wedge X_m, E\wedge Y] & \longrightarrow & [(E\wedge X)_\infty, E\wedge Y] & \longrightarrow & [E\wedge X_\infty, E\wedge Y] \\ \downarrow{\scriptstyle (i\wedge 1)^*} & & & & \downarrow{\scriptstyle (i\wedge 1)^*} \\ \varinjlim_m [\Sigma^{-m}X_m, E\wedge Y] & & \longrightarrow & & [X_\infty, E\wedge Y] \end{array}$$

in which $i: S\to E$ denotes the unit map of E. The upper arrows are both isomorphisms by Lemma 5.3 and the assumption on the map q. Note that the vertical arrows $(i\wedge 1)^*$ are split epimorphisms because E is a ring spectrum. Hence we see easily that the bottom arrow is an isomorphism as desired.

5.3. Let $W=\{W_k, d_k\}_{k\geqq 0}$ be a BP-geometric resolution over M and $X=\{X_m, a_m, b_{m-1}, c_m\}_{m\geqq 1}$ be its ∞-factorized system. Consider the BP_*-Adams spectral sequence $\{E_r^{s,t}(Y, X)\}_{r\geqq 1}$ constructed using the tower $\{\Sigma^{-m} X_m, a_m\}_{m\geqq 1}$ with homotopy inverse limit X_∞. Denote by $X_{m,j}$ and $X_{\infty,m}$, $-1\leqq j<m<\infty$, the fibers of the composite maps $a_{j+1}\cdots a_m: \Sigma^{-m} X_m\to\Sigma^{-j} X_j$ and $q_m: X_\infty\to\Sigma^{-m} X_m$ respectively. Obviously $X_{m,m-1}=\Sigma^{-m} W_m$, $X_{m,-1}=\Sigma^{-m} X_m$ and the sequence $X_{k,m}\to X_{k,j}\to X_{m,j}$ is a cofibering, $-1\leqq j<m<k\leqq\infty$.

Given a CW-spectrum Y we set

$$Z_r^{s,t}(Y, X)=\mathrm{Ker}\{[Y, \Sigma^{s+t}X_{s,s-1}]\longrightarrow[Y, \Sigma^{s+t+1}X_{s+r-1,s}]\}$$
$$B_r^{s,t}(Y, X)=\mathrm{Im}\{[Y, \Sigma^{s+t-1}X_{s-1,s-r}]\longrightarrow[Y, \Sigma^{s+t}X_{s,s-1}]\}$$
$$E_r^{s,t}(Y, X)=Z_r^{s,t}(Y, X)/B_r^{s,t}(Y, X)$$

for each r, $1\leqq r\leqq\infty$. Further we define a decreasing filtration of $[Y, \Sigma^d X_\infty]$ by

$F^{s,d-s}(Y, X)=F^s[Y, \sum^d X_\infty]=\mathrm{Ker}\{[Y, \sum^d X_\infty] \longrightarrow [Y, \sum^{d-s+1} X_{s-1}]\}$.

The composite map $F^{s,t}(Y, X)/F^{s+1,t-1}(Y, X) \cong E_\infty^{s,t}(Y, X) \to \varprojlim_{r>s} E_r^{s,t}(Y, X)$ is always a monomorphism, and the map $[Y, \sum^d X_\infty] \to \varprojlim_s [Y, \sum^d X_\infty]/F^{s,d-s}(Y, X)$ is always an epimorphism. We say the spectral sequence $\{E_r^{s,t}(Y, X)\}_{r\geq 1}$ converges *completely* to $[Y, X_\infty]$ if the above two maps are both isomorphisms. Use the cofiberings $X_{\infty,m} \to X_\infty \to \sum^{-m} X_m$ to show that $\varprojlim_m X_{\infty,m}=pt$ by means of Verdier's lemma. This implies that $\varprojlim_m [Y, X_{\infty,m}] =0=\varprojlim_m{}^1 [Y, X_{\infty,m}]$. Then [1, Theorem 8.2] says

(5.6) *the spectral sequence* $\{E_r^{s,t}(Y, X)\}_{r\geq 1}$ *converges completely to* $[Y, X_\infty]$ *if and only if* $\varprojlim_{r>s}{}^1 E_r^{s,t}(Y, X)=0$ *for each* s, t.

We say the spectral sequence $\{E_r^{s,t}(Y, X)\}_{r\geq 1}$ converges *finitely* to $[Y, X_\infty]$ if for each s, t there exists $r_0=r_0(s, t)<\infty$ such that $E_{r_0}^{s,t}(Y, X)=E_r^{s,t}(Y, X)$ whenever $r_0 \leq r<\infty$. From (5.4) it follows that

(5.7) *the spectral sequence* $\{E_r^{s,t}(Y, X)\}_{r\geq 1}$ *converges completely if it converges finitely.*

Under the assumption that BP_*Y is BP_*-free, $E_1^{s,t}(Y, X)\cong \mathrm{Hom}^t_{BP_*BP}(BP_*Y, BP_*W_s)$ and $E_2^{s,t}(Y, X)\cong \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*Y, M_*)$ in the BP_*-Adams spectral sequence $\{E_r^{s,t}(Y, X)\}_{r\geq 1}$.

Proposition 5.5. *Let n be a positive integer not less than the length of J and Y be a CW-spectrum. Let $W=\{W_k, d_k\}_{k\geq 0}$ be a BP-geometric resolution over $M_n(BPJ \wedge Y)$ which admits an ∞-factorized system $X=\{X_m\}_{m\geq 1}$. If n is not divided by $p-1$, then the canonical map $q: Z \wedge X_\infty \to (Z \wedge X)_\infty$ is a homotopy equivalence for any CW-spectrum Z.*

Proof. Consider the BP_*-Adams spectral sequence $\{E_r^{s,t}(Z)=E_r^{s,t}(S, Z \wedge X)\}_{r\geq 1}$ associated with the tower $\{Z \wedge \sum^{-m} X_m, 1 \wedge a_m\}_{m\geq 1}$ for each CW-spectrum Z. By Lemma 4.7 we observe that $E_2^{s,*}(Z)\cong \mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, M_nBPJ_*(Y \wedge Z))=0$ for all $s>n^2$. Therefore $E_{n^2+1}^{s,t}(Z)=E_{n^2+m}^{s,t}(Z)$ for all $m\geq 1$. Thus the spectral sequence $\{E_r^{s,t}(Z)\}_{r\geq 1}$ converges completely to $\pi_*(Z \wedge X)_\infty$ by (5.7). Hence $\pi_*(Z \wedge X)_\infty$ has a decreasing filtration $\pi_*(Z \wedge X)_\infty=F^0(Z) \supset F^1(Z) \supset \cdots \supset F^{n^2+1}(Z)=\{0\}$ such that $F^s(Z)/F^{s+1}(Z)\cong E_{n^2+1}^{s,*}(Z)$. Let $\{Z_\lambda\}$ be a set of finite subspectra of Z whose union is just Z. Since $\varinjlim_\lambda \pi_*(Z_\lambda \wedge X_{m,j})\cong \pi_*(Z \wedge X_{m,j})$, the canonical map $\varinjlim_\lambda E_r^{s,t}(Z_\lambda) \to E_r^{s,t}(Z)$ is an isomorphism for every r, $1 \leq r<\infty$. By a downward induction on s we verify that the canonical map

$\varinjlim_{\lambda} F^s(Z_\lambda) \to F^s(Z)$ is an isomorphism, and hence the map $\varinjlim_{\lambda} \pi_*(Z_\lambda \wedge X)_\infty \to \pi_*(Z \wedge X)_\infty$ becomes an isomorphism when taking $s=0$ especially. Therefore it is shown that the map $q\colon Z \wedge X_\infty \to (Z \wedge X)_\infty$ induces an isomorphism in homotopy, since the canonical map $q\colon Z_\lambda \wedge X_\infty \to (Z_\lambda \wedge X)_\infty$ is a homotopy equivalence for every finite CW-spectrum Z_λ.

Theorem 5.6. *Let n be a positive integer not divided by $p-1$, and Y be a BP-local CW-spectrum such that BP_*Y is v_k-torsion for every k, $0 \leqq k < n$, and it is uniquely v_n-divisible. Then Y is BP-nilpotent complete, and the BP_*-Adams spectral sequence $\{E_r^{s,t}(S, KY)\}_{r \geqq 1}$ converges completely to $\pi_*(Y)$.* (Cf., [12, Theorem 9]).

Proof. The hypothesis on BP_*Y implies that $BP \wedge Y = \sum^{-n} M_n(BP \wedge Y)$ by [17, Proposition 2.2]. Apply Proposition 5.5 to the Adams BP-geometric resolution $W_{BP,Y}$ with the ∞-factorized system $KY = \{K_m Y\}_{m \geqq 1}$. Then we observe that the canonical map $q\colon BP \wedge (BP^\wedge Y) \to BP^\wedge(BP \wedge Y)$ is a homotopy equivalence. From (5.3) the result follows immediately, since the BP_*-Adams spectral sequence derived from KY converges completely to $\pi_*(BP^\wedge Y)$ as shown in the proof of Proposition 5.5.

Let $f\colon BP \wedge Y \to BP \wedge Y'$ be a BP-Hopf module map. By Proposition 4.3 the map f induces a map $f_\infty\colon BP^\wedge Y \to BP^\wedge Y'$, whenever $[\sum^1 W_m Y, W_{m+2} Y'] = 0$ and the sequences $[\sum^1 K_{m-1} Y, W_m Y'] \to [\sum^1 K_{m-1} Y, W_{m+1} Y'] \to [\sum^1 K_{m-1} Y, W_{m+2} Y']$ are exact for all $m \geqq 1$. Note that

(5.8) *$f_\infty\colon BP^\wedge Y \to BP^\wedge Y'$ is a homotopy equivalence if a BP-Hopf module map $f\colon BP \wedge Y \to BP \wedge Y'$ is so.*

Theorem 5.7. *Let J be an invariant regular sequence of length n. Suppose that p is odd and $n^2 + n < 2p$. Then there exists a unique BP-local CW-spectrum Y_J such that $BP \wedge Y_J$ is isomorphic to $v_n^{-1}BPJ$ as BP-Hopf module spectra.*

Proof. Putting Theorem 4.9 and Propositions 5.1 and 5.5 together we can show the existence of a $v_n^{-1}BP$-local CW-spectrum Y_J with the desired property. The uniqueness of Y_J is immediate by use of (5.8) and Theorem 5.6 because the assumption on (5.8) is satisfied as shown in the proof of Theorem 4.9.

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