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A Study of Variation of Mixed Hodge Structure

By

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§ 0. Introduction

In [S–Z], Steenbrink and Zucker proposed and studied "admissible variation of Mixed Hodge structure" on a curve as an "appropriate" notion. Here, we give a definition of admissible variation of mixed Hodge structure on a complex variety by "curve test" and afford the evidence that this is a good notion.

0.1. We shall recall what is an admissible VMHS on a curve, proposed by Steenbrink-Zucker. Let X be a curve and X^* a Zariski open subset of X. A graded polarizable variation of mixed Hodge structure on X^* is called admissible, if it satisfies

(a) There exists the relative monodromy filtration of the logarithm of the unipotent part of the monodromy around each point in $X\backslash X^*$ with respect to the weight filtration.

(b) The Hodge filtration F extends to a subsheaf of the canonical extension such that $\mathrm{Gr}^W\mathrm{Gr}_F$ is locally free. Furthermore for any morphism $f\colon D\to X$ from unit disc D to X the Hodge filtration of f^*H satisfies the same conditions.

0.2. Let X be a complex analytic space, and X^* a non singular Zariski open subset of X. A graded polarizable VMHS on X is called admissible (on X) if its restriction to any curve in X (not totally contained in $X\backslash X^*$) is admissible. Such a definition by "curve test" is seldom a good notion unless it is a property of codimension 1 (or 0, 2). This means that if this property is true outside a closed analytic subset of codimension 2 (or 1, 3) then this holds on the whole

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space. As an example we recall the regularity of integrable connections or the quasi-unipotency of constructible sheaves.

In this article, we prove that "admissible" VMHS has a property of codimension 1 when X is non-singular and X^* is the complement of normally crossing hypersurface.

The proof makes an essential use of the purity theorem for a variation of Hodge structure.

This paper includes also a study of general notions which will be useful for further studies of mixed Hodge structures.

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§ 1. Admissible Variation of Mixed Hodge Structure

1.1. We shall recall the definition of Hodge structure. For an integer w, a *Hodge structure* H of weight w is a triplet $(\mathrm{H}_{\boldsymbol{C}}, F(\mathrm{H}), \bar{F}(\mathrm{H}))$, where $\mathrm{H}_{\boldsymbol{C}}$ is a finite-dimensional $\boldsymbol{C}$-vector space and $F(\mathrm{H})$, $\bar{F}(\mathrm{H})$ are two finite filtrations of $\mathrm{H}_{\boldsymbol{C}}$ such that $\mathrm{H}_{\boldsymbol{C}} \simeq \bigoplus_{p+q=w} \mathrm{H}^{pq}(\mathrm{H})$. Here $\mathrm{H}^{pq}(\mathrm{H}) = F^p(\mathrm{H}) \cap \bar{F}^q(\mathrm{H})$. Let $\mathrm{C}(\mathrm{H})$ be the Weil operator defined by $\mathrm{C}(\mathrm{H})|_{H^{pq}(\mathrm{H})} = i^{p-q}$.

1.2. For a $\boldsymbol{C}$-vector space V, let us denote by $\overline{V}$ the complex conjugate and let $-: V \to \overline{V}$ be the $\boldsymbol{R}$-linear isomorphism such that $\overline{av} = \bar{a}\bar{v}$ for $a \in \boldsymbol{C}$ and $v \in V$. For a homomorphism $f: V \to V'$ of $\boldsymbol{C}$-vector space let $\bar{f}: \overline{V} \to \overline{V}'$ be the homomorphism given by $\bar{f}(\bar{v}) = \overline{f(v)}$.

For a Hodge structure H of weight w, the complex conjugate $\overline{\mathrm{H}}$ of H is the Hodge structure of weight w defined by $(\overline{\mathrm{H}})_{\boldsymbol{C}} = (\mathrm{H}_{\boldsymbol{C}})^-$, $F^p(\overline{\mathrm{H}}) = \bar{F}^p(\mathrm{H})^-$, $\bar{F}^q(\overline{\mathrm{H}}) = F^q(\mathrm{H})^-$. We have $\mathrm{H}^{pq}(\overline{\mathrm{H}}) = \mathrm{H}^{qp}(\mathrm{H})^-$ and $\mathrm{C}(\overline{\mathrm{H}}) = \mathrm{C}(\mathrm{H})^-$

1.3. A *polarization* of a Hodge structure H of weight w is a bilinear homomorphism $S: \mathrm{H}_{\boldsymbol{C}} \otimes \overline{\mathrm{H}}_{\boldsymbol{C}} \to \boldsymbol{C}$ such that

(1.3.1) $S(F^p(\mathrm{H}), F^q(\overline{\mathrm{H}})) = S(\bar{F}^p(\mathrm{H}), \bar{F}^q(\overline{\mathrm{H}})) = 0$ for $p+q > w$.

(1.3.2) $S(\mathrm{C}(\mathrm{H})u, \bar{v})$ is a positive definite Hermitian form on $\mathrm{H}_{\boldsymbol{C}}$.

1.4. A *mixed Hodge structure* H of weight w is data $(\mathrm{H}_{\boldsymbol{C}}, W(\mathrm{H}), F(\mathrm{H}), \bar{F}(\mathrm{H}))$. Here, $\mathrm{H}_{\boldsymbol{C}}$ is a finite-dimensional $\boldsymbol{C}$-vector space, and $W(\mathrm{H})$, $F(\mathrm{H})$ and $\bar{F}(\mathrm{H})$ are filtrations of $\mathrm{H}_{\boldsymbol{C}}$ such that $\mathrm{Gr}_k^W(\mathrm{H})$ is a Hodge structure of weight

$w+k$ together with the induced filtrations of $F(\mathrm{H})$ and $\bar{F}(\mathrm{H})$. If $w=0$, we call it simply a mixed Hodge structure.

1.5. Let X be a complex manifold and $\bar{X}$ its complex conjugate. A *variation of Hodge structure* H is data $(\mathrm{H}_C, F(\mathrm{H}), \bar{F}(\mathrm{H}))$. Here H_C is a locally free $\mathbb{C}_X$-module with finite rank and $F(\mathrm{H})$ (resp. $\bar{F}(\mathrm{H})$) is a filtration of $\mathcal{O}_X\otimes\mathrm{H}_C$ (resp. $\mathcal{O}_{\bar{X}}\otimes\mathrm{H}_C$) by vector subbundles such that for any $x\in X$, $\mathrm{H}(x)=(\mathrm{H}_{C,x}, F(\mathrm{H})(x), \bar{F}(\mathrm{H})(x))$ is a Hodge structure of weight w and $vF^p(\mathrm{H})\subset F^{p-1}(\mathrm{H})$ (resp. $v\bar{F}^p(\mathrm{H})\subset\bar{F}^{p-1}(\mathrm{H})$) for any vector field v on X (resp. $\bar{X}$). Here $\bar{X}$ is the complex conjugate of X.

A polarization S of a variation of Hodge structure H is a homomorphism $S\colon \mathrm{H}_C\otimes\bar{\mathrm{H}}_C\to C$ such that at each point $x\in X$, S_x gives a polarization of $\mathrm{H}(x)$. A variation of Hodge structure is called *polarizable* if it admits a polarization.

1.6. A *variation of mixed Hodge structure* H on X consists of data $(\mathrm{H}_C, W(\mathrm{H}), F(\mathrm{H}), \bar{F}(\mathrm{H}))$; H_C is a locally free $\mathbb{C}_X$-module of finite rank, $W(\mathrm{H})$ is a filtration of H_C by locally free $\mathbb{C}_X$-modules and $F(\mathrm{H})$ (resp. $\bar{F}(\mathrm{H})$) is a filtration of $\mathcal{O}_X\otimes\mathrm{H}_C$ (resp. $\mathcal{O}_{\bar{X}}\otimes\mathrm{H}_C$) by vector subbundles, such that $vF^p(\mathrm{H})\subset F^{p-1}(\mathrm{H})$ (resp. $v\bar{F}^p(\mathrm{H})\subset\bar{F}^{p-1}(\mathrm{H})$) for any vector field v on X (resp. $\bar{X}$) and that $\mathrm{Gr}_k^{W(\mathrm{H})}$ is a variation of Hodge structure of weight k.

A variation of mixed Hodge structure H is called *graded polarizable* if $\mathrm{Gr}_k^{W(\mathrm{H})}$ is polarizable for any k.

1.7. Let $f\colon X\to Y$ be a holomorphic map. Then for a variation of mixed Hodge structure H on Y, we can naturally define $f^*(\mathrm{H})$ a variation of mixed Hodge structure on X. If H is graded polarizable, then so is $f^*(\mathrm{H})$.

1.8. Let D be the unit disc and D^* the punctured unit disc; $D=\{z\in\mathbb{C};\ |z|<1\}$, $D^*=D\backslash\{0\}$. Following Steenbrink-Zucker [S–Z], we say that a VMHS (variation of mixed Hodge structure) H on D^* is *pre-admissible* if it satisfies (1.8.1)–(1.8.4):

(1.8.1) H is graded polarizable.

(1.8.2) The monodromy M of H_C around the origin is quasi-unipotent.

(1.8.3) The logarithm N of the unipotent part of M admits a weight filtration relative to $W(\mathrm{H})$ (see §3).

(1.8.4) Let $\mathcal{O}_D(\mathrm{H})$, $W_k(\mathcal{O}_D(\mathrm{H}))$ (resp. $\mathcal{O}_{\bar{D}}(\mathrm{H})$ and $W_k(\mathcal{O}_{\bar{D}}(\mathrm{H}))$) be the canonical

extension of $\mathcal{O}_{D^*}\otimes \mathrm{H}_C$ and $\mathcal{O}_{D^*}\otimes W_k(\mathrm{H})$ (resp. $\mathcal{O}_{\bar{D}^*}\otimes \mathrm{H}_C$ and $\mathcal{O}_{\bar{D}^*}\otimes W_k(\mathrm{H})$). Then $F(\mathrm{H})$ (resp. $\bar{F}(\mathrm{H})$) extends to a subbundle of $\mathcal{O}_D(\mathrm{H})$ (resp. $\mathcal{O}_{\bar{D}}(\mathrm{H})$) such that $\mathrm{Gr}^p_{F(\mathrm{H})}\mathrm{Gr}_k^{W(\mathcal{O}_D(\mathrm{H}))}$ (resp. $\mathrm{Gr}^p_{\bar{F}(\mathrm{H})}\mathrm{Gr}_k^{W(\mathcal{O}_{\bar{D}}(\mathrm{H}))}$) is locally free.

1.9. Let X be an analytic space, and X^* a non-singular Zariski open subset of X. Let H be a VMHS on X^*. We say that H is *admissible* if we have

(1.9.1) H is graded polarizable.

(1.9.2) For any $f\colon D\to X$ such that $f(D^*)\subset X^*$, $f_0^*(H)$ is pre-admissible. Here $f_0\colon D^*\to X^*$ is the restriction of f.

When $X=D$ and $X^*=D^*$ and the monodromy is unipotent, any pre-admissible VMHS on D^* is admissible by the following lemma.

Lemma 1.9.1. *Let $f\colon D\to D$ be $z\mapsto z^n$ $(n\geqq 1)$ and let* H *be a* VMHS *on D^* and $\mathrm{H}'=f^*\mathrm{H}$. Then we have*

(1.9.3) *Assume that the monodromy of* H′ *is unipotent. Then the condition* (1.8.4) *for* H′ *implies* (1.8.4) *for* H.

(1.9.4) *Assume that the monodromy of* H *is unipotent. Then the condition* (1.8.4) *for* H *implies* (1.8.4) *for* H′.

Proof. Let $G=\mathbb{Z}_n$ be the Galois group of $f\colon D\to D$. Then we have

$$(f_*\mathcal{O}(W_k(\mathrm{H}')))^G=\mathcal{O}(W_k(\mathrm{H}))\quad\text{or}\quad(f_*z^{-(n-1)}\mathcal{O}(W_k(\mathrm{H}')))^G=\mathcal{O}(W_k(\mathrm{H}))\,,$$
$$(f_*F^p(\mathrm{H}'))^G=F^p(\mathrm{H})\quad\text{or}\quad(f_*(z^{-(n-1)}F^p(\mathrm{H})))^G=F^p(\mathrm{H})$$

according to the meaning of canonical extension (i.e. left or right). In either case, (1.8.4) for H′ implies (1.8.4) for H because the G-invariant of a vector bundle is also a vector bundle. (1.9.4) follows from the fact

$$\mathcal{O}(W_k(\mathrm{H}'))=f^*\mathcal{O}(W_k(\mathrm{H}'))\quad\text{and}\quad F^p(\mathrm{H}')=f^*F^p(\mathrm{H})\,.$$ Q.E.D.

If H is admissible, then $F^p(\mathrm{H})$ (resp. $\bar{F}^p(\mathrm{H})$) extends to a coherent submodule of the canonical extension of $\mathcal{O}_{X^*}(\mathrm{H})$ (resp. $\mathcal{O}_{\bar{X}^*}(\mathrm{H})$).

1.10. We conjecture the following

Conjecture: *Assume that X is equi-dimensional and let Z be a closed analytic subset of X with codimension $\geqq 2$. If $\mathrm{H}|_{X^*\setminus Z}$ is admissible in $X\setminus Z$, then* H *is admissible in X.*

Note that the quasi-unipotency (1.8.2) is a consequence of [K].

Later, we shall prove this conjecture when X is non-singular and $X\backslash X^*$ is a normally crossing hypersurface (Theorem 4.5.2). As a corollary of this theorem, we have

Proposition 1.10.1. *Let X^* be a non-singular Zariski open subset of a complex analytic variety X and Y a closed analytic subset of X such that $X^*\backslash Y$ is dense in X^*. Then, for any* VMHS H *on X^* if $\mathrm{H}|_{X^*\backslash Y}$ is admissible in X then H is admissible in X.*

Proof. Let us take $f\colon X'\to X$ satisfying (1.10.1)–(1.10.4).

(1.10.1) X' is non-singular.

(1.10.2) f is proper.

(1.10.3) $f^{-1}(X^*)\to X^*$ is an isomorphism.

(1.10.4) $f^{-1}(X\backslash X^*)$ is a normally crossing hypersurface.

Then H is admissible if and only if $f^*\mathrm{H}|_{f^{-1}(X^*)}$ is admissible. In fact, any curve $\varphi\colon D\to X$ such that $\varphi(D^*)\subset X^*$ comes from a curve in X'. Similarly $\mathrm{H}|_{X^*\backslash Y}$ is admissible if and only if $f^*\mathrm{H}|_{f^{-1}(X^*\backslash Y)}$ is admissible in X'. Therefore, by replacing H with $f^*\mathrm{H}$, we may assume from the beginning that X is non-singular and $X\backslash X^*$ is a normally crossing hypersurface. Set $Z=\overline{Y\cap X^*}\cap(X\backslash X^*)$. Then $\mathrm{H}|_{X\backslash Z}$ is admissible in $X\backslash Z$. Since $\operatorname{codim} Z\geqq 2$, we can apply Theorem 4.5.2.

1.11. Remark also the following.

Proposition 1.11.1. *Let us assume X non-singular. If* $\operatorname{codim}(X\backslash X^*)\geqq 2$, *then a graded polarizable* VMHS *on X^* extends to a graded polarizable* VMHS *on X.*

This is a consequence of the following lemma and W. Schmid's theorem [S]: a polarizable variation of Hodge structure outside a closed hypersurface extends to a globally defined polarizable VHS, if the associated local system extends.

Lemma 1.11.2. *Let X be a complex manifold, Z a closed analytic subspace of codimension $\geqq 2$. Let $\mathscr{L}$ be a vector bundle of X and W a filtration by subbundles of $\mathscr{L}$. Let $\mathscr{F}$ be a coherent $\mathcal{O}_{X\backslash Z}$ submodule of $\mathscr{L}|_{X\backslash Z}$ such that $\mathrm{Gr}_k^W/\mathscr{F}(\mathrm{Gr}_k^W)$ is a locally free $\mathcal{O}_{X\backslash Z}$-module. Assume further that, for any k, $\mathscr{F}(\mathrm{Gr}_k^W)$ extends to a subbundle of Gr_k^W defined on X. Then $\mathscr{F}$ extends to a*

subbundle $\tilde{\mathscr{F}}$ *of* $\mathscr{L}$ *defined on* X *such that* $\tilde{\mathscr{F}}\mathrm{Gr}_k^W$ *is a subbundle of* Gr_k^W.

Proof. Let $j\colon X\backslash Z\hookrightarrow X$ be the inclusion map. Set $\tilde{\mathscr{F}}=j_*\mathscr{F}\subset j_*j^{-1}\mathscr{L}=\mathscr{L}$. We shall show that $\tilde{\mathscr{F}}\cap W_k$ is a subbundle of W_k by the induction on k. Consider the following exact sequence

$$0\longrightarrow \frac{W_{k-1}}{\tilde{\mathscr{F}}\cap W_{k-1}}\longrightarrow \frac{W_k}{\tilde{\mathscr{F}}\cap W_k}\longrightarrow \frac{Gr_k^W}{\tilde{\mathscr{F}}(Gr_k^W)}\longrightarrow 0.$$

By the hypothesis of the induction, $W_{k-1}/\tilde{\mathscr{F}}\cap W_{k-1}$ is locally free and hence $\mathscr{H}^1_Z(W_{k-1}/\tilde{\mathscr{F}}\cap W_{k-1})=0$. By the construction of $\tilde{\mathscr{F}}$ we have $\mathscr{H}^0_Z(W_k/\tilde{\mathscr{F}}\cap W_k)=0$. Therefore we have $\mathscr{H}^0_Z(\mathrm{Gr}_k^W/\tilde{\mathscr{F}}(\mathrm{Gr}_k^W))=0$. This means that $\tilde{\mathscr{F}}(\mathrm{Gr}_k^W)$ is the extension as a subbundle of Gr_k^W. Therefore $\mathrm{Gr}_k^W/\tilde{\mathscr{F}}\mathrm{Gr}_k^W$ is locally free and hence so is $W_k/\tilde{\mathscr{F}}\cap W_k$. Q.E.D.

Similarly, the preceding lemma, together with Schmid's nilpotent orbit theorem [S], implies the following.

Proposition 1.11.3. *Let* X *be a complex manifold,* X^* *the complement of a normally crossing hypersurface,* Z *a closed analytic subset of* X *of codimension* $\geqq 2$ *and* H *a graded polarizable* VMHS *on* X^*. *Then if* $\mathrm{H}|_{X^*\backslash Z}$ *is admissible on* $X\backslash Z$, *then the Hodge filtration* $F(\mathrm{H})$ *extends to coherent subsheave of the canonical extension* $\mathscr{O}_X(\mathrm{H})$ *of* $\mathscr{O}_{X^*}\otimes\mathrm{H}_{\mathbb{C}}$ *such that* $\mathrm{Gr}_{F(\mathrm{H})}\mathrm{Gr}^{W(\sigma_X(\mathrm{H}))}$ *is a locally free* $\mathscr{O}_X$*-module.*

§ 2. Perverse Sheaves in a Normally Crossing Case

2.1. Let us consider $X=\boldsymbol{C}^l$ and for $J\subset I=\{1,\dots,l\}$, we set $X_J=\{z\in X;\ z_j=0 \text{ for } j\in J\}$ and $X_J^*=\{z\in X_J;\ z_j\neq 0 \text{ for } j\notin J\}$. Let $\mathscr{P}$ be the category of perverse sheaves $\mathscr{F}^\cdot$ on X such that $\mathscr{F}^\cdot|X_J^*$ has locally constant cohomology groups. A. Galligo, M. Granger and Ph. Maisonobe [G-G-M] shows that $\mathscr{P}$ is equivalent to the category $\{M_\alpha; f_{\alpha\beta}, g_{\beta\alpha}\}$. Here M_α $(\alpha\subset I)$ is a vector space and $f_{\alpha\beta}\colon M_\beta\to M_\alpha$, $g_{\beta\alpha}\colon M_\alpha\to M_\beta$ are given for $\beta\subset\alpha\subset I$, such that they satisfy

$$f_{\alpha\beta}\circ f_{\beta\gamma}=f_{\alpha\gamma},\quad g_{\gamma\beta}g_{\beta\alpha}=g_{\gamma\alpha}\qquad\text{for}\quad \alpha\supset\beta\supset\gamma, \tag{2.1.1}$$

$$f_{\alpha\alpha}=g_{\alpha\alpha}=id, \tag{2.1.2}$$

$$g_{\alpha,\alpha\cup\beta}f_{\alpha\cup\beta,\beta}=f_{\alpha,\alpha\cap\beta}g_{\alpha\cap\beta,\beta}\qquad\text{for any}\quad \alpha,\ \beta, \tag{2.1.3}$$

(2.1.4) If $\alpha\supset\beta$ and $\sharp\alpha=\sharp\beta+1$ then $1-g_{\beta\alpha}f_{\alpha\beta}$ is invertible.

Now consider the condition that $\mathscr{F}^\cdot$ is the minimal extension of a locally

constant sheaf on X_γ^*. In terms of $\{M_\alpha\}$ this is to saying that $M_\alpha=0$ for $\gamma\not\subset\alpha$, and $f_{\alpha\gamma}$ is surjective and $g_{\gamma\alpha}$ is injective for $\gamma\subset\alpha$. If we consider the condition that $\mathscr{F}^\cdot$ is a direct sum of such minimal extensions, we arrive to the arguments that we are going to describe in this section.

2.2. Let $f: A\to B$ and $g: B\to A$ be morphisms in an abelian category. We write $A\underset{g}{\overset{f}{\rightleftarrows}}B$ if we have the following equivalent conditions:

(2.2.1) $\quad B\xleftarrow{\sim}\operatorname{Im}f\oplus\operatorname{Ker}g$.

(2.2.2) $\quad \operatorname{Im}(g\circ f)\xrightarrow{\sim}\operatorname{Im}g$ and $\operatorname{Ker}f\xrightarrow{\sim}\operatorname{Ker}(g\circ f)$.

This is a self-dual notion, i.e., if T is an exact contravarient functor, $A\underset{g}{\overset{f}{\rightleftarrows}}B$ implies $T(A)\underset{T(f)}{\overset{T(g)}{\rightleftarrows}}T(B)$.

2.3. More generally, let I be a finite set and let $\mathfrak{S}$ be the set of subsets of I. For an abelian category $\mathscr{A}$ let $\mathrm{P}(I,\mathscr{A})$ be the category defined as follows. $\mathrm{Ob}(\mathrm{P}(I,\mathscr{A}))$ consists of $\{M_\alpha\}_{\alpha\in\mathfrak{S}}$ and $f_{\alpha\beta}: M_\beta\to M_\alpha$ and $g_{\beta\alpha}: M_\alpha\to M_\beta$ for $\beta\subset\alpha$ satisfying (2.1.1), (2.1.2) and (2.1.3). The morphisms are defined in an obvious way. For $\gamma\in\mathfrak{S}$ let $M_\gamma(I,\mathscr{A})$ be the set of objects $((M_\alpha),(f_{\alpha\beta}),(g_{\beta\alpha}))$ in $\mathrm{P}(I,\mathscr{A})$ satisfying

(2.3.1) $\quad M_\alpha=0$ if $\alpha\not\supset\gamma$.

(2.3.2) $\quad f_{\alpha\gamma}$ is surjective and $g_{\gamma\alpha}$ is injective if $\alpha\supset\gamma$.

Let $\mathrm{M}(I,\mathscr{A})$ be the set of objects which is isomorphic to the direct sum of objects in $\bigcup_\alpha \mathrm{M}_\alpha(I,\mathscr{A})$.

Proposition 2.3.1. *Let $M=((M_\alpha),(f_{\alpha\beta}),(g_{\beta\alpha}))$ be an object of $\mathrm{P}(I,\mathscr{A})$. Then the following conditions are equivalent.*

(2.3.3) $M\in M(I,\mathscr{A})$.

(2.3.4) $M_\beta\underset{g_{\beta\alpha}}{\overset{f_{\alpha\beta}}{\rightleftarrows}}M_\alpha$ *for any* α,β *with* $\alpha\supset\beta$.

(2.3.5) (2.3.4) *holds for any* α,β *with* $\alpha\supset\beta$ *and* $\sharp\alpha=\sharp\beta+1$.

(2.3.6) *If we set* $\mathrm{P}_\alpha(M)=\bigcap_{\beta\subsetneq\alpha}\operatorname{Ker}g_{\beta\alpha}$ *then* $M_\alpha\xleftarrow{\sim}\bigoplus_{\beta\subset\alpha}f_{\alpha\beta}(\mathrm{P}_\beta(M))$ *and* $g_{\beta\alpha}$: $f_{\alpha\beta}\mathrm{P}_\beta(M)\to\mathrm{P}_\beta(M)$ *is injective for* $\alpha\supset\beta$.

Proof. Since (2.3.6)⇔(2.3.3)⇒(2.3.4)⇒(2.3.5) is evident, we shall show (2.3.5)⇒(2.3.3). When $|I| \leqq 1$, this is evident. We shall assume $|I| > 1$. Taking an element i_0 of I, let $I_0 = I \setminus \{i_0\}$ and $\mathfrak{S}_0$ the set of subsets of I_0. Set $\mathscr{B} = \mathrm{P}(I_0, \mathscr{A})$. Then $\mathrm{P}(I, \mathscr{A})$ is equivalent to $\mathrm{P}(\{i_0\}, \mathscr{B})$, by the following functor. For $M = ((M_\alpha), (f_{\alpha\beta}), (g_{\beta\alpha})) \in \mathrm{Ob}(\mathrm{P}(I, \mathscr{A}))$ set

$$i(M) = ((M_\alpha), (f_{\alpha\beta}), (g_{\beta\alpha}); \alpha, \beta \in \mathfrak{S}_0) \in \mathrm{Ob}(\mathscr{B})$$

$$j(M) = (M_{\alpha \cup \{i_0\}}, (f_{\alpha \cup i_0, \beta \cup i_0}), (g_{\beta \cup \{i_0\}, \alpha \cup \{i_0\}}); \alpha, \beta \in \mathfrak{S}_0) \in \mathrm{Ob}(\mathscr{B}).$$

Then $(f_{\alpha \cup \{i_0\}, \alpha})_\alpha$ and $(g_{\alpha, \alpha \cup \{i_0\}})_\alpha$ define $f(M)\colon i(M) \to j(M)$ and $g(M)\colon j(M) \to i(M)$, respectively. Thus $t(M) = (i(M), j(M), f(M), g(M))$ gives an object of $\mathrm{P}(\{i_0\}, \mathscr{B})$. Now assume that $M \in \mathrm{Ob}(P(I, \mathscr{A}))$ satisfies (2.3.5).

Since $t(M)$ satisfies (2.3.5), $t(M)$ belongs to $\mathrm{M}((i_0), \mathscr{B})$. Hence $M \cong M_1 \oplus M_2$ such that $i(M_1) = 0$ and $i(M_2) \to j(M_2)$ is surjective and $j(M_2) \to i(M_2)$ is injective. As (2.3.5) and (2.3.3) are stable by direct summands, we may assume either $i(M) = 0$ or $i(M) \twoheadrightarrow j(M) \rightarrowtail i(M)$. In the first case $i(M) = 0$, the hypothesis of the induction implies $j(M) \in \mathrm{M}(I_0, \mathscr{A})$. Hence M belongs to $\mathrm{M}(I, \mathscr{A})$. Therefore, we may assume $i(M) \twoheadrightarrow j(M) \rightarrowtail i(M)$, or equivalently, $f_{\alpha\beta}$ is surjective and $g_{\beta\alpha}$ is injective if $\alpha \supset \beta$ and $\alpha \ni i_0$, $\beta \not\ni i_0$. Now, we may assume this for all $i_0 \in I$. Hence M belongs to $\mathrm{M}(I, \mathscr{A})$. Q. E. D.

§ 3. Relative Weight Filtration (cf. [S–Z])

3.1. Let W be a filtration of an object H in an abelian category and let N be a nilpotent endomorphism of H such that $NW_k \subset W_k$. A relative weight filtration M is a filtration of H such that

(3.1.1) $NM_k \subset M_{k-2}$

(3.1.2) $N^l\colon \mathrm{Gr}^M_{k+l}\mathrm{Gr}^W_k \longrightarrow \mathrm{Gr}^M_{k-l}\mathrm{Gr}^W_k$ is an isomorphism for any $l \geqq 1$ and k.

Remark that it is unique if it exists. We shall denote it by $M(N, W)$. As shown in [S–Z] we have

(3.1.3) $M_l(N, W) \cap W_k = W_k \cap (N^j)^{-1} M_{l-2j}(N, W)$ for $j \geqq 0$ and $l \geqq k+j-1$.

(3.1.4) $M_l(N, W) \cap W_k = N^j(M_{l+2j}(N, W) \cap W_k) + M_l(N, W) \cap W_{k-1}$
for $j \geqq 0$, $l \leqq k-j$.

The last one implies

(3.1.5) $M(N, W)_l \subset N^r M(N, W)_{l+2r} + M(N, W)_l \cap W_{l+r-1}$ for $r \geqq 0$.

Note also the following result in [S–Z].

Lemma 3.1.1 ([S–Z]). *Assume that $H=W_k$ and that the weight filtration of $N|W_{k-1}$ relative to $W\cap W_{k-1}$ exists. Then the relative weight filtration of N exists if and only if we have*

$$N^jW_k\cap W_{k-1}\subset N^jW_{k-1}+M_{k-j-1}(N|W_{k-1},\ W\cap W_{k-1}) \qquad \text{for any} \quad j\geqq 1.$$

3.2. Now assuming the existence of relative weight filtration, we shall deduce their properties. We denote by M the weight filtration of a nilpotent endomorphism N of H relative to a filtration W of H. After [Z], we define

$$K_k=K_k(W)=\bigcap_{j=0}^{\infty}(N^j)^{-1}W_{k-j}, \tag{3.2.1}$$

$$I_k=I_k(W)=\sum_{j=0}^{\infty}N^jW_{k+j}.$$

They are the dual notions of each other. We have

$$K_k=W_k\cap N^{-1}K_{k-1}, \tag{3.2.2}$$

$$I_k=W_k+NI_{k+1}.$$

Lemma 3.2.1. $M_k=\sum_{j=o}^{\infty}N^j(W_{k+j}\cap M_{k+2j})=\bigcap_{j=0}^{\infty}(N^j)^{-1}(W_{k-j}+M_{k-2j})$.

Proof. By the duality, it is enough to show

$$M_k\cap W_{k+r}\subset\sum_{j=0}^{r}N^j(W_{k+j}\cap M_{k+2j}) \qquad \text{for} \quad r\geqq 0. \tag{3.2.3}$$

We shall prove this by the induction of r. This is true for $r=0$ and for $r>0$, we have

$$M_k\cap W_{k+r}\subset N^r(W_{k+r}\cap M_{k+2r})+M_k\cap W_{k+r-1}$$

$$\subset N^r(W_{k+r}\cap M_{k+2r})+\sum_{j=0}^{r-1}N^j(W_{k+j}\cap M_{k+2j})$$

by (3.1.4) and the hypothesis of the induction. Q.E.D.

Corollary 3.2.2. $K_k\subset M_k\subset I_k$.

Lemma 3.2.3. *For $p\geqq q$, $K(W(W_p/W_q))$ and $I(W(W_p/W_q))$ are the filtration induced from $K(W)$ and $I(W)$, respectively. Here $W(W_p/W_q)$ is the filtration of W_p/W_q induced from W.*

Proof. It is enough to show

$$I_k \cap W_p \subset \sum_{j=0}^{\infty} N^j(W_{k+j} \cap W_p). \tag{3.2.4}$$

This is trivial for $p \leqq k$. When $p \geqq k$, we shall prove

$$I_k \cap W_p \subset \sum_{j=0}^{p-k} N^j(W_{k+j}) \tag{3.2.5}$$

by the descending induction on p. We have

$$I_k \cap W_p \subset I_k \cap W_{p+1} \subset \sum_{j=0}^{p-k+1} N^j(W_{k+j}).$$

Therefore, we have

$$I_k \cap W_p \subset \sum_{j=0}^{p-k} N^j W_{k+j} + N^{p-k+1} W_{p+1} \cap W_p.$$

The last term is contained in $N^{p-k+1}W_p + M_{k-1} \cap W_p$ by Lemma 2.2.1, and the preceding lemma implies $M_{k-1} \cap W_p \subset \sum N^j(W_{k-1+j} \cap W_p)$. Thus the induction proceeds.

Lemma 3.2.4. *$M_k \cap W_{k+r} = N^r(M_{k+2r} \cap W_{k+r}) + K_{k+r-1} \cap M_k$ for $r \geqq 0$ and $M_k \cap W_{k+r} = N^r(M_{k+2r} \cap W_{k+r}) + N^{r-1}(M_{k+2r-2} \cap W_{k+r-1}) + K_{k+r-2} \cap M_k$ for $r \geqq 1$.*

Proof. We have, for $r \geqq 1$

$$M_k \cap W_{k+r} \subset N^r(M_{k+2r} \cap W_{k+r}) + M_k \cap W_{k+r-1}. \tag{3.2.6}$$

Therefore, the second assertion follows from the first. We shall show the first one by the induction on k. By the hypothesis of the induction, we have

$$\begin{aligned} N(M_k \cap W_{k+r-1}) &\subset M_{k-2} \cap W_{k+r-1} \\ &\subset N^{r+1}(M_{k+2r} \cap W_{k+r-1}) + K_{k+r-2} \cap M_{k-2}. \end{aligned} \tag{3.2.7}$$

Therefore, we have

$$M_k \cap W_{k+r-1} \subset N^r(M_{k+2r} \cap W_{k+r-1}) + N^{-1}K_{k+r-2} \cap W_{k+r-1} \cap M_k.$$

Since the last term is $K_{k+r-1} \cap M_k$ by (3.2.2), we are done. Q. E. D.

Proposition 3.2.5. $M_k = \sum_{j=0}^{\infty} N^j K_{k+2j} = \bigcap_{j=0}^{\infty} (N^j)^{-1} I_{k-2j}$.

Proof. By the duality, it is enough to show $M_k \subset \sum_{j=0}^{\infty} N^j K_{k+2j}$. We shall prove this by the descending induction on k. By the preceding lemma, we have

$$M_k \cap W_{k+1} \subset NM_{k+2} + K_k.$$

By (3.1.5) we have

$$M_k \subset NM_{k+2} + M_k \cap W_k .$$

Thus, we have

$$M_k \subset NM_{k+2} + K_k \subset \sum_{j=1}^{\infty} N^j K_{k+2j} + K_k . \qquad \text{Q. E. D.}$$

Lemma 3.2.5.1. *For $j \geqq 0$, $W_{k+1} \cap (N^j)^{-1} K_k \subset K_{k+j}$.*

Proof. For $0 \leqq v < j$, we have

$$N^v (W_{k+1} \cap (N^j)^{-1} K_k) \subset W_{k+1} \subset W_{k+j-v} .$$

For $v \geqq j$, we have

$$N^v (W_{k+1} \cap (N^j)^{-1} K_k) \subset N^{v-j} K_k \subset W_{k+j-v} .$$

Lemma 3.2.6. $M_k \cap W_{k-r} = \sum_{j=0}^{\infty} N^j W_{k-r} \cap N^{-(j+r+1)} K_{k-r-2} + M_k \cap W_{k-r-1}$ *for $r \geqq 0$.*

Proof. We have, by the preceding lemma and Corollary 3.2.2,

$$N^j W_{k-r} \cap N^{-(j+r+1)} K_{k-r-2} \subset N^j (W_{k-r} \cap N^{-(2j+r+1)} K_{k-r-1}) \subset N^j K_{k+2j} \subset M_k .$$

Therefore, it is enough to show

$$M_k \cap W_{k-r} \subset \sum_{j=0}^{\infty} N^j W_{k-r} \cap N^{-(j+r+1)} K_{k-r-2} + W_{k-r-1} . \tag{3.2.8}$$

We shall show this by the descending induction on r. Note that (3.2.8) is true if r is so big that $N^{r+1} = 0$. Now we have, for any $r \geqq 0$,

$$\begin{aligned} N^{1+r}(M_k \cap W_{k-r}) &\subset M_{k-2r-2} \cap W_{k-r} \\ &\subset N^{r+2}(M_{k+2} \cap W_{k-r}) + N^{r+1}(M_k \cap W_{k-r-1}) + K_{k-r-2} . \end{aligned}$$

by Lemma 3.2.4. This implies

$$M_k \cap W_{k-r} \subset N(M_{k+2} \cap W_{k-r}) + N^{-(r+1)} K_{k-r-2} \cap W_{k-r} + M_k \cap W_{k-r-1} .$$

The first term of the right hand side is contained, by the hypothesis of the induction, in $N \sum_{j=0}^{\infty} N^j W_{k-r} \cap N^{-(r+3+j)} K_{k-r-2} \subset \sum_{j=0}^{\infty} N^{j+1} W_{k-r} \cap N^{-(r+1+j+1)}$. K_{k-r-2} modulo W_{k-r-1}. Thus the induction proceeds. Q. E. D.

Lemma 3.2.7. *For any r and k we have*

$$M_k \cap W_{k+r} = \sum_{j, j+r \geqq 0} N^{j+r} W_{k+r} \cap N^{-(j+1)} K_{k+r-2} + M_k \cap W_{k+r-1} .$$

Proof. For $r \leqq 0$, this is nothing but the preceding lemma. For $r>0$, we have

$$\begin{aligned} M_k \cap W_{k+r} &= N^r(M_{k+2r} \cap W_{k+r}) + M_k \cap W_{k+r-1} \\ &= N^r\Big(\sum_{j=0}^{\infty} N^j W_{k+r} \cap N^{-(j+r+1)} K_{k+r-2} + M_{k+2r} \cap W_{k+r-1}\Big) + M_k \cap W_{k+r-1} \\ &= \sum_{j=0}^{\infty} N^{j+r} W_{k+r} \cap N^{-(j+1)} K_{k+r-2} + M_k \cap W_{k+r-1}. \end{aligned}$$

Q. E. D.

Lemma 3.2.8. *Let φ be the projection $M_k \to \mathrm{Gr}_k^M$. Then we have the direct sum decomposition*

$$\varphi(M_k \cap W_{k+r}) = \bigoplus_{j,\, j+r \geqq 0} \varphi(N^{j+r} W_{k+r} \cap N^{-(j+1)} K_{k+r-2}) \oplus \varphi(M_k \cap W_{k+r-1}).$$

Proof. By the preceding lemma it is enough to show that for $m+r$, $m \geqq 0$

$$\begin{aligned} (3.2.9)\quad & X = N^{m+r} W_{k+r} \cap N^{-(m+1)} K_{k+r-2} \\ & \cap \Big(\sum_{0 \leqq j < m,\, j+r \geqq 0} N^{j+r} W_{k+r} \cap N^{-(j+1)} K_{k+r-2} + M_k \cap W_{k+r-1} + M_{k-1}\Big) \subset M_{k-1}. \end{aligned}$$

In order to see this, it is enough to show

$$N^{-(m+r)} X \cap W_{k+r} \subset M_{k+2m+2r-1}.$$

By Lemma 3.2.1, this follows from

$$(3.2.10)\qquad N^v(N^{-(m+r)} X \cap W_{k+r}) \subset W_{k+2m+2r-1-v} + M_{k+2m+2r-1-2v},$$

which we are going to prove.

For $v \geqq 2m+r-1$, this follows from

$$W_{k+r} \subset W_{k+2m+2r-1-v}.$$

If $v \geqq 2m+r+1 > m+r$, we have

$$\begin{aligned} N^{v-m-r} X &\subset N^{v-m-r-m-1} K_{k+r-2} \\ &\subset K_{k+r-2-(v-2m-r-1)} = K_{k+2m+2r-v-1} \subset W_{k+2m+2r-1-v}. \end{aligned}$$

The remaining case is $v = 2m+r$. Then, we have

$$\begin{aligned} N^{v-m-r} X &\subset N^m\Big(\sum_{0 \leqq j < m} N^{-(j+1)} K_{k+r-2} + W_{k+r-1} + M_{k-1}\Big) \\ &\subset K_{k+r-2} + W_{k+r-1} + M_{k-2m-1} \subset W_{k+r-1} + M_{k-2m-1} \\ &= W_{k+2m+2r-1-v} + M_{k+2m+2r-1-2v}. \end{aligned}$$

Q. E. D.

Theorem 3.2.9. *We have a canonical decomposition*

$$\mathrm{Gr}_l^M = \bigoplus_k \mathrm{Gr}_k^W \mathrm{Gr}_l^M.$$

This theorem is one of the key points of this article. (See Remark 5.3.2.)

3.3. Let W and W' be a filtration of objects H and H' in an abelian category, respectively. Let f: H$\to$H' and g: H'$\to$H be morphisms preserving filtrations.

Proposition 3.3.1. *Assume* (3.3.1)–(3.3.2):

(3.3.1) $f\circ g$ is nilpotent.

(3.3.2) *The weight filtration of $f\circ g$ relative to W' exists.*

(3.3.3) *For any k we have* $\mathrm{Gr}_k^W \underset{\triangleright}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Gr}_k^{W'}$.

Then the weight filtration of $g\circ f$ relative to W exists and we have

$$f M_k(g\circ f, W)\subset M_{k-1}(f\circ g, W'). \tag{3.3.4}$$

$$g M_k(f\circ g, W')\subset M_{k-1}(g\circ f, W). \tag{3.3.5}$$

We shall denote $f\circ g$ and $g\circ f$ by the same letter N. We shall first show

Lemma 3.3.2. *If $M=M(N, W)$ and $M'=M(N, W')$ exist, then we have* (3.3.4) *and* (3.3.5).

Proof. By the duality, it is enough to show one of them, say (3.3.4). We set

$$K_k=\bigcap_{j=0}^{\infty}(N^j)^{-1}W_{k-j} \quad\text{and}\quad K'_k=\bigcap_{j=0}^{\infty}(N^j)^{-1}W'_{k-j}.$$

By Proposition 3.2.5, it is enough to show

$$fK_k\subset K'_{k-1}. \tag{3.3.6}$$

We have, for $j\geqq 0$

$$(N^jK_k)\subset W_{k-j} \quad\text{and}\quad (g\circ f)(N^jK_k)\subset W_{k-j-1}$$

Hence (3.3.3) implies $f(N^jK_k)\subset W'_{k-j-1}$. This shows (3.3.6). Q. E. D.

Now, we shall show the existence of the weight filtration of N relative to W. We may assume, by the induction, that $\mathrm{H}=W_0$, $\mathrm{H}'=W'_0$ and that $\tilde{M}=M(N|_{W_{-1}}, W|_{W_{-1}})$ exists. Therefore, the preceding lemma implies $f\tilde{M}_k\subset M'_{k-1}$ and $g(M'_k\cap W'_{-1})\subset\tilde{M}_{k-1}$. Here M' denotes $M(N, W')$. By Lemma 3.1.1, it is enough to show

$$\mathrm{Im}\,N^j\cap W_{-1}\subset N^jW_{-1}+\tilde{M}_{-j-1} \qquad \text{for} \quad j\geqq 1. \tag{3.3.7}$$

We have, by (3.3.3),

$$\begin{aligned}\operatorname{Im} N^j \cap W_{-1} &\subset g(N^{j-1}W'_0 \cap W'_{-1}) \\ &\subset g(N^{j-1}W'_{-1} + M'_{-j} \cap W'_{-1}) \subset gN^{j-1}W'_{-1} + \tilde{M}_{-j-1}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}W'_k &\subset fW_k + W'_k \cap g^{-1}W_{k-1} \subset fW_k + (M'_k \cap W'_k + W'_{k-1}) \cap g^{-1}W_{k-1} \\ &\subset fW_k + M'_k \cap W'_k + W'_{k-1}.\end{aligned}$$

The induction on k deduced from this

$$W'_k = fW_k + M'_k \cap W'_k.$$

Therefore, we obtain

$$\begin{aligned}gN^{j-1}W'_{-1} &\subset gN^{j-1}(fW_{-1} + M'_{-1} \cap W'_{-1}) \\ &\subset N^jW_{-1} + N^{j-1}\tilde{M}_{-2} \subset N^jW_{-1} + \tilde{M}_{-2j} \subset N^jW_{-1} + \tilde{M}_{-j-1}.\end{aligned}$$

This shows (3.3.7) and we have completed the proof of Proposition 3.3.1.

Remark that Proposition 3.3.1 means that the existence of relative weight filtration is a micro-local property.

3.4. For a filtration W and a nilpotent endomorphism N such that $NW_k \subset W_k$, if the relative weight filtration $M(N, W)$ exists, we define the filtration N_*W by

$$\begin{aligned}(N_*W)_k &= NW_{k+1} + M_k(N, W) \cap W_k \\ &= NW_{k+1} + M_k(N, W) \cap W_{k+1} \\ &= NW_{k+1} + \bigcap_{j \geq 0} (N^j)^{-1}W_{k-j}.\end{aligned} \tag{3.4.1}$$

The first equality follows from (3.1.4) and the second follows from Proposition 3.2.5.

Note that its dual notion is

$$\begin{aligned}(N_!W)_k &= W_{k-1} + M_k(N, W) \cap N^{-1}W_{k-1} \\ &= W_{k-1} + M_k(N, W) \cap N^{-1}W_{k-2}.\end{aligned} \tag{3.4.2}$$

In the sequel, we only treat N_*W, as we can treat $N_!W$ similarly.

Lemma 3.4.1. *We have*

$$(NW_{k+1} + M(N, W)_k \cap W_k) \cap W_l = N(W_{k+1} \cap W_l) + M(N, W)_k \cap W_k \cap W_l \tag{3.4.3}$$

and

$$\begin{aligned}&N(W_{k+1} + W_l) + (M(N, W)_k + W_l) \cap (W_k + W_l) \\ &\quad = NW_{k+1} + M(N, W)_k \cap W_k + W_l.\end{aligned} \tag{3.4.4}$$

Proof. (3.4.3) is trivial when $l \geqq k+1$. When $l \leqq k$, we shall prove (3.4.3) by the induction on l. Writing M for $M(N, W)$, we have

$$\begin{aligned} W_l \cap (N_* W)_k &\subset W_{l+1} \cap (N_* W)_k \cap W_l \\ &\subset (NW_{l+1} + M_k) \cap W_l \subset N(M_{k+2} \cap W_{l+1} + W_l) + M_k \subset NW_l + M_k . \end{aligned}$$

Hence we obtain $W_l \cap (N_* W)_k \subset NW_l + M_k \cap W_l$. The equality (3.4.4) follows from $(M(N, W)_k + W_l) \cap (W_k + W_l) = M(N, W)_k \cap W_k + W_l$.

Lemma 3.4.2. *We have*

(0) $NW_{k+1} \cap W_k \subset (N_* W)_{k-1}$.

(i) $\mathrm{Gr}^W_{k+1} \underset{\triangleright}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Gr}^{N*W}_k$.

(ii) *The relative weight filtration of* $N_* W$ *is* $M(N, W)$.

(iii) $$\mathrm{Gr}^W_k \mathrm{Gr}^{N*W}_l = \begin{cases} 0 & \text{for } l < k-1 \\ N\mathrm{Gr}^W_k & \text{for } l = k-1 \\ \mathrm{Coker}\,(N;\ \mathrm{Gr}^{M(N,W)}_{l+2}\mathrm{Gr}^W_k \to \mathrm{Gr}^{M(N,W)}_l \mathrm{Gr}^W_k) & \text{for } l \geqq k. \end{cases}$$

Proof. (0) follows from Lemma 3.1.1.

We write M for $M(N, W)$. Then we have

$$(N_* W)_k \cap W_k = NW_k + M_k \cap W_k \subset M_k \cap W_k + (N_* W)_{k-1} . \tag{3.4.5}$$

This shows $\mathrm{Ker}\,(\mathrm{Gr}^{N*W}_k \to \mathrm{Gr}^W_{k+1}) = M_k \cap W_k / M_k \cap W_k \cap (N_* W)_{k-1}$. Now we have

$$NW_{k+1} \cap (M_k \cap W_k + (N_* W)_{k-1}) \subset (N_* W)_{k-1} , \tag{3.4.6}$$

In fact, the left hand side of (3.4.6) is contained in $NW_{k+1} \cap W_k$.

The property (i) then follows from (3.4.6).

Now we shall prove (ii). It is enough to show the following properties.

$$M_{k-l} \cap (N_* W)_k \subset N^l (M_{k+l} \cap (N_* W)_k) + (N_* W)_{k-1} \quad \text{for } l \geqq 1 , \tag{3.4.7}$$

and

$$(N_* W)_k \cap (N^l)^{-1} (M_{k-l-1} + (N_* W)_{k-1}) \subset M_{k+l-1} + (N_* W)_{k-1} \quad \text{for } l \geqq 1. \tag{3.4.8}$$

We have, by (3.1.4)

$$\begin{aligned} M_{k-l} \cap (N_* W)_k &\subset M_{k-l} \cap W_{k+1} \subset N^{l+1}(M_{k+l+2} \cap W_{k+1}) + W_k \cap M_{k-l} \\ &\subset N^l (M_{k+l} \cap (N_* W)_k) + (N_* W)_{k-1} . \end{aligned}$$

We have also

$$\begin{aligned} &(N_* W)_k \cap (N^l)^{-1} (M_{k-l-1} + (N_* W)_{k-1}) \\ &\quad \subset N(W_{k+1} \cap (N^{l+1})^{-1} (M_{k-l-1} + W_k)) + W_k \cap M_k \\ &\quad \subset N(W_{k+1} \cap M_{k+l+1}) + NW_k + W_k \cap M_k \subset M_{k+l-1} + (N_* W)_{k-1} . \end{aligned}$$

Thus we have (3.4.7) and (3.4.8).

Finally we shall show (iii). By Lemma 3.4.1, we may assume that W is the trivial filtration. Then we have

$$(N_*W)_k=\begin{cases} 0 & k<-1 \\ \operatorname{Im} N+M_k & \text{for} \quad k\geqq -1. \end{cases}$$

Therefore $\operatorname{Gr}_1^{N_*W}=\operatorname{Im} N$ and, for $k\geqq 0$,

$$\begin{aligned}\operatorname{Gr}_k^{N_*W}&=(\operatorname{Im} N+M_k)/(\operatorname{Im} N+M_{k-1})=M_k/(\operatorname{Im} N+M_{k-1})\cap M_k\\ &=M_k/(M_{k-1}+\operatorname{Im} N\cap M_k)=M_k/(M_{k-1}+NM_{k+2}).\end{aligned}$$

Thus we obtain (iii).

Corollary 3.4.3.

$$\operatorname{Gr}_k^{N_*W}=\operatorname{Im}(N\colon \operatorname{Gr}_{k+1}^W\to\operatorname{Gr}_{k+1}^W)\oplus\operatorname{Coker}(N\colon W_k\operatorname{Gr}_{k+2}^M\to W_k\operatorname{Gr}_k^M).$$

Proof. By the preceding lemma we have

$$\operatorname{Gr}_k^{N_*W}=\operatorname{Im}(N\colon \operatorname{Gr}_{k+1}^W\to\operatorname{Gr}_{k+1}^W)\oplus\operatorname{Ker}(\operatorname{Gr}_k^{N_*W}\to\operatorname{Gr}_{k+1}^W). \tag{3.4.9}$$

The last term is isomorphic to

$$(N_*W)_k\cap W_k/(N_*W)_{k-1}\cap W_k.$$

Since $(N_*W)_k\cap W_k\subset NW_{k+1}\cap W_k+W_k\cap M_k\subset(N_*W)_{k-1}+W_k\cap M_k$, we have

$$\begin{aligned}&\operatorname{Ker}(\operatorname{Gr}_k^{N_*W}\to\operatorname{Gr}_{k+1}^W)\cong W_k\cap M_k/M_k\cap(N_*W)_{k-1}\\ &\cong W_k\cap M_k/(M_k\cap NW_k+M_{k-1}\cap W_k)=W_k\cap M_k/(N(M_{k+2}\cap W_k)+M_{k-1}\cap W_k).\end{aligned}$$

This shows the desired result.

§4. Infinitesimal Mixed Hodge Modules

4.1. Let w be an integer, let F and $\bar{F}$ be two filtrations of a finite-dimensional $\boldsymbol{C}$-vector space H, and let $S\colon \mathrm{H}\otimes\overline{\mathrm{H}}\to\boldsymbol{C}$ be a non-degenerate bilinear form satisfying

$$\overline{S(x,\bar{y})}=(-1)^wS(y,\bar{x}) \qquad \text{for} \quad x,\ y\in\mathrm{H}, \tag{4.1.1}$$

$$S(F^p,(\bar{F}^q)^-)=0 \qquad \text{for} \quad p+q>w. \tag{4.1.2}$$

Let $\{N_1,\dots,N_l\}$ be a mutually commuting set of nilpotent endomorphisms of H such that $S(N_jx,\bar{y})+S(x,\overline{N_jy})=0$ and $N_jF^p\subset F^{p-1}$, $N_j\bar{F}^p\subset\bar{F}^{p-1}$. Then the following conditions are equivalent (essentially due to [C–K]. See [K–K]).

(4.1.3) There exists c such that $(\mathrm{H}, e^{i\Sigma t_j N_j}F, e^{-i\Sigma t_j N_j}\bar{F})$ is a Hodge structure of weight w and S polarizes it when $t_j > c$.

(4.1.4) The weight filtration W of $N=\sum_j t_j N_j$ does not depend on $t_1,\ldots, t_l>0$ and $(W, F, \bar{F})$ is a mixed Hodge structure of weight w and the bilinear form $S_k: P_k\otimes\bar{P}_k\to\mathbb{C}$ given by $S_k(x, y)=S(x, \overline{N^k y})$ polarizes the primitive part P_k $\mathrm{Ker}\,(N^{k+1}: \mathrm{Gr}_k^W\to\mathrm{Gr}_{-k-2}^W)$.

If these equivalent conditions are satisfied, we say that $(\mathrm{H}; F, \bar{F}; N_1,\ldots, N_l; S)$ is a *nilpotent orbit* of weight w. If $(\mathrm{H}; F, \bar{F}; N_1,\ldots, N_l; S)$ is a nilpotent orbit for some S, then we say also that $(\mathrm{H}; F, \bar{F}; N_1,\ldots, N_l)$ is a nilpotent orbit.

4.2. Let W, F and $\bar{F}$ be three filtrations of a finite-dimensional $\mathbb{C}$-vector space H and let $N_1,\ldots, N_l$ be nilpotent endomorphisms of H such that

(4.2.1) $\qquad N_j F^p\subset F^{p-1}, \quad N_j\bar{F}^p\subset\bar{F}^{p-1}$ and $N_j W_k\subset W_k$.

If $(\mathrm{Gr}_k^W, F(\mathrm{Gr}_k^W), \bar{F}(\mathrm{Gr}_k^W); N_1|_{\mathrm{Gr}_k^W},\ldots, N_l|_{\mathrm{Gr}_k^W})$ is a nilpotent orbit of weight k for any k, then we say that $(\mathrm{H}; W; F, \bar{F}; N_1,\ldots, N_l)$ is a *pre-infinitesimal mixed Hodge module* (that we shall abbreviate to *pre-IMHM*, in the sequel).

4.3. Let $(\mathrm{H}; W; F, \bar{F}; N_1,\ldots, N_l)$ be a pre-IMHM. Set $I=\{1,\ldots, l\}$. If, for any $J\subset I$, there exists a filtration $M(J)$ of H such that

(4.3.1) $N_j M_k(J)\subset M_{k-2}(J)$ for $j\in J$

(4.3.2) $M(J)$ is the weight filtration of $\sum_{j\in J} N_j$ relative to W.

Then we say that $(\mathrm{H}; W; F, \bar{F}; N_1,\ldots, N_l)$ is an *infinitesimal mixed Hodge module* (abbreviated to IMHM in the sequel). Then by the result of [C-K],

(4.3.3) $M(J)$ is the weight filtration of N relative to W for any $N\in C(J)=\{\sum_{j\in J} t_j N_j;\ t_j>0\}$.

If $(\mathrm{H}; W; F, \bar{F}; N_1,\ldots, N_l)$ and $(\mathrm{H}'; W'; F', \bar{F}'; N_1',\ldots, N_l')$ are pre-IMHM's, then their tensor product $(\mathrm{H}\otimes\mathrm{H}'; W\otimes W'; F\otimes F', F\otimes\bar{F}'; N_1\otimes N_1',\ldots, N_l\otimes N_l')$ and their inner-Hom $(\mathrm{Hom}(\mathrm{H}, \mathrm{H}'); \mathrm{Hom}(W, W'); \mathrm{Hom}(F, F'), \mathrm{Hom}(\bar{F}, \bar{F}'); -\mathrm{Hom}(N_1, N_1'),\ldots, -\mathrm{Hom}(N_l, N_l'))$ are also pre-IMHM's. The tensor product and the inner-Hom of IMHM's are also IMHM's.

4.4. We shall prove in §6 the following theorem.

Theorem 4.4.1. *If* $(\mathrm{H}; W; F, \bar{F}; N_1,\ldots, N_l)$ *is a pre-IMHM and if there*

exists the weight filtration of N_j relative to W for any $j=1,\ldots, l$, then it is an IMHM.

4.5. Let X be the n-dimensional polydisc D^n and let X^* be D^{*n}. Here, D^* is the punctured disc $D\backslash\{0\}$. Let $\mathrm{H}=(\mathrm{H}_C, W(\mathrm{H}), F(\mathrm{H}), \bar{F}(\mathrm{H}))$ be a graded polarizable variation of mixed Hodge structure on X^*. Letting $\mathcal{O}_X(\mathrm{H})$ be a canonical extension of $\mathcal{O}_{X^*}\otimes\mathrm{H}_C$, we extend $\mathcal{O}_{X^*}\otimes W(\mathrm{H})$ as a filtration of subbundles $W(\mathcal{O}_X(\mathrm{H}))$ of $\mathcal{O}_X(\mathrm{H})$. Similarly we define $\mathcal{O}_{\bar{X}}(\mathrm{H})$ and $W(\mathcal{O}_{\bar{X}}(\mathrm{H}))$. Let us assume that

(4.5.1) $F(\mathrm{H})$ (resp. $\bar{F}(\mathrm{H})$) extends to a filtration of $\mathcal{O}_X(\mathrm{H})$ (resp. $\mathcal{O}_{\bar{X}}(\mathrm{H})$) such that $\mathrm{Gr}^{W(\mathcal{O}_X(\mathrm{H}))}\mathrm{Gr}_{F(\mathrm{H})}$ (resp. $\mathrm{Gr}^{W(\mathcal{O}_{\bar{X}}(\mathrm{H}))}\mathrm{Gr}_{F(\mathrm{H})}$) is locally free.

For the sake of simplicity, we assume further

(4.5.2) The monodromy M_j of H_C around $z_j=0$ is unipotent.

Let N_j be the logarithm of M_j.

Set $U=\{z\in X^*;\ \mathrm{Re}\, z_j>0\}$ and we define $\Psi(\mathrm{H}_C)=\Gamma(U;\mathrm{H}_C)$. We identify $\Psi(\mathrm{H}_C)$ and $\boldsymbol{C}_{\{0\}}\otimes_{\mathcal{O}_X}\mathcal{O}_X(\mathrm{H})$ by $s\mapsto e^{\frac{1}{2\pi i}\sum N_j\log z_j}s\in\mathcal{O}_X(\mathrm{H})$. Then $F(\mathrm{H})$ and $\bar{F}(\mathrm{H})$ give the filtration $F(\Psi(\mathrm{H}))$ and $F(\Psi(\mathrm{H}_C))$ of $\Psi(\mathrm{H}_C)$. By the nilpotent orbit theorem ([S]) due to W. Schmid, we have

Proposition 4.5.1. *$\Psi(\mathrm{H})=\{\Psi(\mathrm{H}_C), \Psi(W(\mathrm{H}_C)), F(\Psi(\mathrm{H})), \bar{F}(\Psi(\mathrm{H}));\ N_1,\ldots, N_n\}$ is a pre-IMHM.*

Note that H is admissible in X if $\Psi(\mathrm{H})$ is an infinitesimal mixed Hodge module. Thus, together with Proposition 1.11.3, Theorem 4.4.1 implies

Theorem 4.5.2. *Let X be a complex manifold, X^* the complements of a normally crossing hypersurface of X and Z a closed analytic subset of X of codimension $\geqq 2$. If* H *is a variation of mixed Hodge structure on X^* and if $H|_{X^*\backslash Z}$ is admissible in $X\backslash Z$, then* H *is admissible in X.*

§ 5. Properties of Infinitesimal Mixed Hodge Modules

5.1. In this section, we shall study the properties of infinitesimal mixed Hodge modules. Hereafter we write IMHM for infinitesimal mixed Hodge module. In the course of arguments, we shall make a frequent use of the following

Theorem 5.1.1. *Let* $\mathrm{H}=(W(\mathrm{H}), F(\mathrm{H}), \bar{F}(\mathrm{H}))$ *be a mixed Hodge structure. Let* $\{G_1,\ldots, G_l\}$ *be a distributive family* (see [K]) *of filtrations of* H *in the category of mixed Hodge structure. Then* $\{G_1,\ldots, G_l, W(\mathrm{H}), F(\mathrm{H})\}$ *is a distributive family.*

Proof. We shall first show $\{G_1,\ldots, G_l, W(H)\}$ is distributive. Since $\{G_1,\ldots, G_l\}$ is distributive, it is enough to show

$$\begin{aligned} &I(G_1,\ldots, G_l)\cap W_k(\mathrm{H})\\ &\quad=I(G_1\cap W_k(\mathrm{H}),\ldots, G_l\cap W_k(\mathrm{H}))\\ &\quad=S(G_1\cap W_k(\mathrm{H}),\ldots, G_l\cap W_k(\mathrm{H})) \end{aligned}$$

(see [K]).
This follows from the fact that $\mathrm{H}\mapsto W_k(\mathrm{H})$ is an exact functor on the category of mixed Hodge structure. Similarly $\{G_1,\ldots, G_l, F(\mathrm{H})\}$ is a distributive family. Since $W(\mathrm{H})$ is a filtration of H in the category of mixed Hodge structure, we can argue as :$\{G_1,\ldots, G_l\}$ is distributive$\Rightarrow\{G_1,\ldots, G_l, W(\mathrm{H})\}$ is distributive$\Rightarrow\{G_1,\ldots, G_l, W(\mathrm{H}), F(\mathrm{H})\}$ is distributive.

5.2. The following proposition is proved by Deligne (see [S–Z]). We shall give another proof.

Proposition 5.2.1. *Let* $W, F, \bar{F}$ *be filtrations of* H *and let* N *be a nilpotent endomorphism of* H *satisfying*

$$NF^p\subset F^{p-1},\quad N\bar{F}^p\subset\bar{F}^{p-1}\quad\text{and}\quad NW_k\subset W_k.$$

Assume further

(5.2.1) *There exists the weight filtration* M *of* N *relative to* W.

(5.2.2) Gr_k^W *is a mixed Hodge structure with the induced filtration of* M *as a weight filtration and with the induced filtrations from* F *and* $\bar{F}$ *as Hodge filtration.*

Then, $(M; F, \bar{F})$ *is a mixed Hodge structure.*

Proof. By the induction, we may assume that

(5.2.3) $$H=W_0$$

(5.2.4) $$(M\cap W_k; F\cap W_k, \bar{F}\cap W_k)$$

is a mixed Hodge structure if $k<0$.

Lemma 5.2.2. (M, F, W) *is a distributive family of filtrations.*

By Theorem 5.1.1, $(M \cap W_{-1}, F \cap W_{-1}, W \cap W_{-1})$ is distributive. Hence it is enough to show, for $k \geqq 1$

$$(A_k) \qquad M_{k-1} \cap (F^p + W_{-1}) \subset M_{k-1} \cap F^p + M_{k-1} \cap W_{-1}.$$

$$(B_k) \qquad M_{-k} \cap (F^p + W_{-1}) \subset M_{-k} \cap F^p + M_{-k} \cap W_{-1}.$$

We shall show $(A_{k+1}) \Rightarrow (B_k)$ for $k \geqq 1$ and $(A_{k+2}) \Rightarrow (A_k)$ for $k \geqq 1$. They establish (A_k) and (B_k) for all k.

$(A_{k+1}) \Rightarrow (B_k)$: Since $(F^p \cap M_{-k})(\mathrm{Gr}_0^W) = N^k(F^{p+k} \cap M_k)(\mathrm{Gr}_0^W)$, we have

$$\begin{aligned} M_{-k} \cap (F^p + W_{-1}) &\subset N^k(M_k \cap (F^{p+k} + W_{-1})) + W_{-1} \\ &\subset N^k(M_k \cap F^{p+k} + M_k \cap W_{-1}) + W_{-1} \subset M_{-k} \cap F^p + W_{-1} \end{aligned}$$

$(A_{k+2}) \Rightarrow (A_k)$: we have

$$\begin{aligned} (5.2.5) \quad N^k(F^p \cap (M_{k-1} + W_{-1})) &\subset F^{p-k} \cap (M_{-k-1} + W_{-1}) \\ &\subset N^{k+1}(F^{p+1} \cap (M_{k+1} + W_{-1})) + W_{-1} \subset N^{k+1}(F^{p+1} \cap M_{k+1}) + W_{-1} \\ &\subset N^k(F^p \cap M_{k-1}) + W_{-1}. \end{aligned}$$

Furthermore we have

$$(5.2.6) \quad N^k(F^p \cap (M_{k-1} + W_{-1})) \cap W_{-1} \subset F^{p-k} \cap (N^k W_{-1} + M_{-k-1} \cap W_{-1}).$$

Since $N^k W_{-1}$ is also a mixed Hodge structure, $\{N^k W_{-1}, F^{p-k} \cap W_{-1}, M_{-k-1} \cap W_{-1}\}$ is distributive by Theorem 5.1.1. Thus the right hand-side of (5.2.6) is contained in $F^{p-k} \cap N^k W_{-1} + F^{p-k} \cap M_{-k-1} \cap W_{-1} = N^k(F^p \cap W_{-1}) + F^{p-k} \cap M_{-k-1} \cap W_{-1}$.

Finally, (5.2.5) shows

$$\begin{aligned} F^p \cap (M_{k-1} + W_{-1}) \subset F^p \cap M_{k-1} + F^p \cap W_{-1} \\ + (N^k)^{-1} M_{-k-1} \subset M_{k-1} + F^p \cap W_{-1}. \end{aligned}$$

Q. E. D.

Now, we shall prove the proposition 5.2.1. By the assumption $(\mathrm{Gr}_l^M \mathrm{Gr}_k^W, F(\mathrm{Gr}_l^M \mathrm{Gr}_k^W), \bar{F}(\mathrm{Gr}_l^M \mathrm{Gr}_k^W))$ is a Hodge structure of weight l for any l. By the preceding proposition, we have $F(\mathrm{Gr}_l^M \mathrm{Gr}_k^W) = F(\mathrm{Gr}_k^W \mathrm{Gr}_l^M)$, which implies $(\mathrm{Gr}_l^M, F(\mathrm{Gr}_l^M), \bar{F}(\mathrm{Gr}_l^M))$ is a Hodge structure of weight l because an extension of Hodge structures of weight l is also a Hodge structure of the same weight. This completes the proof of Proposition 5.2.1.

As a corollary of Proposition 5.2.1, we have

Proposition 5.2.3. *If* $(W; F, \bar{F}; N_1, \ldots, N_l)$ *is an IMHM, then* $(M(\{1, \ldots, l\}),$

$F, \bar{F})$ *is a mixed Hodge structure. Moreover* N_j *is a morphism of mixed Hodge structure (up to the Tate twist).*

In particular, $M(J)$, W, etc. are filtrations in the category of mixed Hodge structures.

Proposition 5.2.4. *Let* $(W; F, \bar{F}; N_1,\ldots, N_l)$ *be an IMHM, and let* $J_1 \subset \cdots \subset J_k \subset \{1,\ldots, l\}$ *be an increasing sequence. Then* $\{W, F, M(J_1),\ldots, M(J_k)\}$ *is distributive.*

Proof. We shall prove this by the induction of k. We may assume that $J_k = \{1,\ldots, l\}$ and that $\{W, M(J_1),\ldots, M(J_{k-1})\}$ is distributive. Since W and $M(J_p)$ $(1 \leqq p < k)$ are filtrations of $(M(J_k); F, \bar{F})$ in the category of mixed Hodge structure, $\{W, F, M(J_1),\ldots, M(J_k)\}$ is distributive by Theorem 5.1.1.

The following is proved by Cattani-Kaplan [C–K] in the pure case and asked by Steenbrink-Zucker ([S–Z]) in the mixed case.

Proposition 5.2.5. *Let* $(W; F, \bar{F}; N_1,\ldots, N_l)$ *be an IMHM. Then, for* $J_1, J_2 \subset \{1,\ldots, l\}$ *and* $N \in C(J_2) = \{\sum_{j\in J_2} t_j N_j;\ t_j > 0\}$, $M(J_1 \cup J_2)$ *is the weight filtration of* N *relative to* $M(J_1)$.

Proof. We have $NM_k(J_1 \cup J_2) \subset M_{k-2}(J_1 \cup J_2)$. Hence it is enough to show that

$$N^l : \mathrm{Gr}^{M(J_1\cup J_2)}_{k+l}\mathrm{Gr}^{M(J_1)}_k \longrightarrow \mathrm{Gr}^{M(J_1\cup J_2)}_{k-l}\mathrm{Gr}^{M(J_1)}_k$$

is an isomorphism. This follows from the fact that $\{M(J_1 \cup J_2), M(J_1), W\}$ is a distributive family and the fact that

$$N^l : \mathrm{Gr}^{M(J_1\cup J_2)}_{k+l}\mathrm{Gr}^{M(J_1)}_k\mathrm{Gr}^W_p \longrightarrow \mathrm{Gr}^{M(J_1\cup J_2)}_{k-l}\mathrm{Gr}^{M(J_1)}_k\mathrm{Gr}^W_p$$

is an isomorphism, which is due to Cattani-Kaplan [C–K].

Proposition 5.2.6. (i) The category of pre-IMHM's is an abelian category.

(ii) The category of IMHM's is abelian, on which W, $M(N_1, W)$, Gr^W, $\mathrm{Gr}^{M(N_1,W)}$, etc., are exact functors

Proof. (i) is almost evident. We shall show (ii). The problem is only the existence of relative weight filtrations of the kernel, the cokernel, etc. Therefore it is enough to show that for a morphism f: $H = (W(H), F(H), \bar{F}(H); N_1,\ldots, N_l) \to H' = (W(H'), F(H), \bar{F}(H'); N_1,\ldots, N_l)$ in the category of IMHM's, the restriction of $M(N_1, W(H))$ to $\mathrm{Ker} f$ is the weight filtration of N_1 relative to $W(H) \cap \mathrm{Ker} f$. Set $H'' = \mathrm{Ker} f$. Now H and H′ have a mixed Hodge structure

with W as a weight filtration and $e^{i\Sigma t_j N_j}F$ and $e^{-i\Sigma t_j N_j}\bar{F}$ as Hodge filtrations for $t_1,\dots, t_l \gg 0$, respectively. Hence f is strict with respect to W-filtration. On the other hand, H and H′ have another mixed Hodge structure with $M(N_1, W)$ as a weight filtration, respectively. Moreover, W is a filtration of them by sub-mixed Hodge structures. Therefore if we denote by $M(\mathrm{H}'')$ and $W(\mathrm{H}'')$ the restriction of $M(N_1, W(\mathrm{H}))$ and $W(\mathrm{H})$ to H″, then the rows in the following diagram are exact.

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Gr}^{M(\mathrm{H}'')}_{k+l}\mathrm{Gr}^{W(\mathrm{H}'')}_{k} & \longrightarrow & \mathrm{Gr}^{M(\mathrm{H})}_{k+l}\mathrm{Gr}^{W(\mathrm{H})}_{k} & \longrightarrow & \mathrm{Gr}^{M(\mathrm{H}')}_{k+l}\mathrm{Gr}^{W(\mathrm{H}')}_{k} \\ & \downarrow{\scriptstyle N_1^l} & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Gr}^{M(\mathrm{H}'')}_{k-l}\mathrm{Gr}^{W(\mathrm{H}'')}_{l} & \longrightarrow & \mathrm{Gr}^{M(\mathrm{H})}_{k-l}\mathrm{Gr}^{W(\mathrm{H})}_{k} & \longrightarrow & \mathrm{Gr}^{M(\mathrm{H}')}_{k-l}\mathrm{Gr}^{W(\mathrm{H}')}_{k}. \end{array}$$

Here $M(\mathrm{H})=M(N_1, W(\mathrm{H}))$ and $M(\mathrm{H}')=M(N_1, W(\mathrm{H}'))$. Since the vertical arrows are isomorphisms except the left one, the left one is also an isomorphism. This shows that $M(\mathrm{H}'')$ is the weight filtration of N_1 relative to $W(\mathrm{H}'')$. Q. E. D.

5.3. As an application of Theorem 3.2.9, we shall prove the following

Proposition 5.3.1. *Let $(W; F, \bar{F}; N_1,\dots, N_l)$ be a pre-IMHM. Assume that the weight filtration M of N_1 relative to W exists. Let $N_{1*}W$ be the filtration as defined in* (3.4.1). *Then $(M; F, \bar{F}; N_2,\dots, N_l)$ and $(N_{1*}W; F, \bar{F}; N_1, N_2,\dots, N_l)$ are pre-IMHM's.*

Proof. In order to show the first, we have to prove the existence of a polarization on Gr^M_l, by which $(F(t), \bar{F}(t))=(e^{i\Sigma t_j N_j}F, e^{-i\Sigma t_j N_j}\bar{F})$ is polarized for $t_1,\dots, t_l \gg 0$. Now by Theorem 3.2.9, Gr^M_l is a product of $\mathrm{Gr}^M_l\mathrm{Gr}^W_k$'s canonically. This is true also as a Hodge structure with $(F(t), \bar{F}(t))$ as Hodge filtrations. On the other hand, $\mathrm{Gr}^M_l\mathrm{Gr}^W_k$ has a polarization free from t, by which $F(t)$ is polarized. Therefore Gr^M_l has a polarization by which $(F(t), \bar{F}(t))$ is polarized. Thus, $(M; F, \bar{F}; N_2,\dots, N_l)$ is a pre-IMHM. Now we shall show $(N_{1*}W; F, \bar{F}; N_1, N_2,\dots, N_l)$ is a pre-IMHM. We have by Corollary 3.4.3

$$\mathrm{Gr}^{N_{1*}W}_k = N_1\mathrm{Gr}^W_{k+1} \oplus \mathrm{Coker}\,(N_1 : W_k\mathrm{Gr}^M_{k+2} \to W_k\mathrm{Gr}^M_k).$$

Since Gr^M_k is polarized, so is $\mathrm{Coker}(N_1 : W_k\mathrm{Gr}^M_{k+2} \to W_k\mathrm{Gr}^M_k)$. Since $(\mathrm{Gr}^W_{k+1}; F, \bar{F}; N_1,\dots, N_l)$ is a nilpotent orbit, the vanishing cycle theorem ([K–K], [C–K–S]) shows that $(N_1\mathrm{Gr}^W_{k+1}; F, \bar{F}; N_1,\dots, N_l)$ is also a nilpotent orbit.

Remark 5.3.2. Similarly to Proposition 5.3.1, Theorem 3.2.9 implies the following. Let X be a complex manifold, $f: X \to \boldsymbol{C}$ be a holomorphic map

such that $f^{-1}(0)$ is non-singular. Let H be a graded polarizable VMHS on $X\backslash f^{-1}(0)$ admissible on X. Then the near-by cycle of H with respect to f is also a graded polarizable VMHS.

In the following sections 5.4 and 5.5, we shall give the weight filtration relative to $N_{1*}W$.

5.4. Let $(\mathrm{H}; F, \bar{F}; N_1, N_2)$ be a nilpotent orbit of weight w. Set $W(\mathrm{H}) = M(N_1+N_2)$, $W' = M(N_1)$. Here, $M(*)$ shows the weight filtration.

Lemma 5.4.1. $\operatorname{Ker} N_1^k \cap \operatorname{Im} N_2 \subset N_2(\operatorname{Ker} N_1^k) + W_{k-2}$

$$\operatorname{Im} N_1^k \cap \operatorname{Im} N_2 \cap W_{-k-1} \subset \operatorname{Im} N_2 N_1^k .$$

Proof. By the purity theorem ([K–K], [C–K–S]) we have

$$\operatorname{Ker} N_1^k N_2 \subset \operatorname{Ker} N_1^{k-1} N_2 + \operatorname{Ker} N_1^k + W_k .$$

Therefore the induction on k shows

$$\operatorname{Ker} N_1^k N_2 \subset \operatorname{Ker} N_1^k \operatorname{Ker}\ N_2 + W_k .$$

This shows the first statement. The second statement is the dual one.

Proposition 5.4.2.

$$M_k(N_1|\operatorname{Im} N_2) = (N_{2*}M(N_1))_{k-1} \cap \operatorname{Im} N_2 = N_2 W'_k + W_{k-1} \cap W'_{k-1} \cap \operatorname{Im} N_2$$
$$= N_2 W'_k + W_{k-1} \cap W'_k \cap \operatorname{Im} N_2 .$$

Proof. Let $\tilde{W}_k$ be $(N_{2*}M(N_1))_{k-1} \cap \operatorname{Im} N_2$. It is enough to show (5.4.1)–(5.4.3).

(5.4.1) $\qquad N_1 \tilde{W}_k \subset \tilde{W}_{k-2}$

(5.4.2) $\qquad \tilde{W}_{-k} \subset N_1^k \tilde{W}_k \qquad$ for $\quad k \geqq 1 .$

(5.4.3) $\qquad \operatorname{Ker} N_1^k \cap \operatorname{Im} N_2 \subset \tilde{W}_{k-1} \qquad$ for $\quad k \geqq 1 .$

The first one is obvious. The last one follows from the preceding lemma. We shall show the second. In the expression $\tilde{W}_{-k} = N_2 W'_{-k} + W_{-k-1} \cap W'_{-k} \cap \operatorname{Im} N_2$, the first term is contained in $N_2 N_1^k W'_k \subset N_1^k \tilde{W}_k$ and the second term is contained in

$$W_{-k-1} \cap \operatorname{Im} N_1^k \cap \operatorname{Im} N_2 \cap W'_{-k} \subset \operatorname{Im} N_1^k N_2 \cap W_{-k-1} \cap W'_{-k}$$
$$\subset N_1^k N_2 (W_{k+1}) \cap W'_{-k} \subset N_1^k (N_2 W_{k+1} \cap W'_k) \subset N_1^k \tilde{W}_k .$$

5.5. Let $(W; F, \bar{F}; N_1, N_2)$ be an IMHM.

Theorem 5.5.1.

$$N_{2*}M(N_1, W)=M(N_1, N_{2*}W).$$

We shall reduce this theorem to Proposition 5.4.2. In order to perform this smoothly, we shall first show that these filtrations commutes with Gr^W.

For simplicity, we write

$$\begin{aligned}(5.5.1)\qquad & M'=M(N_1, W),\\ & M''=M(N_2, W),\quad M=M(N_1+N_2, W)=M(N_2, M')\\ & \tilde{W}'=N_{1*}W,\quad \tilde{W}''=N_{2*}W \text{ and } \tilde{M}=N_{2*}M'.\end{aligned}$$

Hence we have

$$\begin{aligned}(5.5.2)\qquad & \tilde{W}''_k=N_2W_{k+1}+M''_k\cap W_k,\\ & \tilde{M}_k=N_2M'_{k+1}+M_k\cap M'_k.\end{aligned}$$

Lemma 5.5.2. (i) $\tilde{W}''_k\cap W_a=N_2(W_{k+1}\cap W_a)+M''_k\cap W_k\cap W_a$.

(ii) $\tilde{W}''_k+W_a=N_2(W_{k+1}+W_a)+(M''_k+W_a)\cap(W_k+W_a)$.

(iii) $\tilde{M}_l\cap W_a=N_2(M'_{l+1}\cap W_a)+M_l\cap M'_l\cap W_a$.

(iv) $\tilde{M}_l+W_a=N_2(M'_{l+1}+W_a)+(M_l+W_a)\cap(M'_l+W_a)$.

Proof. (i) follows from Lemma 3.4.1.

(ii) is trivial.

(iv) follows from the fact that $\{M, M', W\}$ is distributive (Proposition 5.2.3).

Finally, we shall show (iii). Since $M(H/W_a)=M(N_2, M'(H/W_a))$, we have, using (3.1.3) and the fact that $\{W, M, M'\}$ is distributive,

$$\begin{aligned}\tilde{M}_l\cap W_a&=(N_2M'_{l+1}+M'_l\cap M_l)\cap W_a\\ &\subset(N_2(M'_{l+1}\cap(M_{l+2}+W_a)+M'_{l+1}\cap W_a)+M'_l\cap M_l)\cap W_a\\ &\subset N_2(M'_{l+1}\cap W_a)+M_l\cap M'_{l+1}\cap W_a\\ &=N_2(M'_{l+1}\cap W_a)+M_l\cap M'_l\cap W_a.\end{aligned}$$

Q.E.D.

Lemma 5.5.3. *$\{\tilde{M}, W, \tilde{W}''\}$ is distributive.*

Proof. It is enough to show

$$(5.5.3)\qquad \tilde{M}_l\cap(W_a+\tilde{W}''_k)\subset\tilde{M}_l\cap W_a+\tilde{M}_l\cap\tilde{W}''_k.$$

When $a\geqq k+1$, this is trivial because $W_a\supset\tilde{W}''_k$. So, assume $a\leqq k$. Then we have

$$\begin{aligned}\tilde{M}_l\cap(W_a+\tilde{W}''_k)&\subset(N_2M'_{l+1}+M_l)\cap(W_a+\tilde{W}''_k)\\ &\subset M_l\cap(W_a+\tilde{W}''_k)+N_2M'_{l+1}\cap(W_a+\tilde{W}''_k)\end{aligned}$$

because $\{M, N_2M', W_a+\tilde{W}''_k\}$ is distributive by Theorem 5.1.1. Since $\mathrm{Ker}(N_2|M'_{l+1}(\mathrm{H}/W_{k+1}))\subset M_{l+1}(\mathrm{H}/W_{k+1})$, we have

$$\begin{aligned}N_2M'_{l+1}\cap(W_a+\tilde{W}''_k)&\subset N_2M'_{l+1}\cap W_{k+1}\cap(W_a+\tilde{W}''_k)\\&\subset N_2(M'_{l+1}\cap M_{l+1}+M'_{l+1}\cap W_{k+1})\cap(W_a+\tilde{W}''_k)\\&\subset N_2(M'_{l+1}\cap W_{k+1})+M_l\cap(W_a+\tilde{W}''_k)\subset\tilde{M}_l\cap\tilde{W}''_k+M_l\cap(W_a+\tilde{W}''_k).\end{aligned}$$

Hence we have

$$\tilde{M}_l\cap(W_a+\tilde{W}''_k)\subset\tilde{M}_l\cap\tilde{W}''_k+M_l\cap M'_{l+1}\cap(W_a+\tilde{W}''_k).$$

Since $M_l\cap W_a\subset N_2(M_{l+2}\cap W_a)+M_l\cap M'_l\cap W_a\subset\tilde{W}''_k+M_l\cap M'_l\cap W_a$, we have

$$\begin{aligned}M_l\cap M'_{l+1}\cap(W_a+\tilde{W}''_k)&\subset M'_{l+1}\cap(M_l\cap W_a+M_l\cap\tilde{W}''_k)\\&\subset M'_{l+1}\cap(M_l\cap M'_l\cap W_a+M_l\cap\tilde{W}''_k)\\&\subset M_l\cap M'_l\cap W_a+M_l\cap M'_{l+1}\cap\tilde{W}''_k\subset\tilde{M}_l\cap W_a+\tilde{M}_l\cap\tilde{W}''_k.\end{aligned}$$

Q. E. D.

Now, we are ready to prove Theorem 5.5.1. In order to prove this it is enough to show

(5.5.4) $N_1^l\colon \mathrm{Gr}^{\tilde{M}}_{k+l}\mathrm{Gr}^{\tilde{W}''}_k\longrightarrow\mathrm{Gr}^{\tilde{M}}_{k-l}\mathrm{Gr}^{\tilde{W}''}_k$ *is an isomorphism for* $l\geqq 1$.

This follows from

(5.5.5) $N_1^l\colon \mathrm{Gr}^W_a\mathrm{Gr}^{\tilde{M}}_{k+l}\mathrm{Gr}^{\tilde{W}''}_k\longrightarrow\mathrm{Gr}^W_a\mathrm{Gr}^{\tilde{M}}_{k-l}\mathrm{Gr}^{\tilde{W}''}_k$ *is an isomorphism.*

By virtue of Lemma 5.5.3, $\mathrm{Gr}^W\mathrm{Gr}^{\tilde{M}}\mathrm{Gr}^{\tilde{W}''}=\mathrm{Gr}^{\tilde{M}}\mathrm{Gr}^{\tilde{W}''}\mathrm{Gr}^W$. Then, by Lemma 3.4.2, when $k<a-1$, $\mathrm{Gr}^W_a\mathrm{Gr}^{\tilde{W}''}_k=0$. When $k=a-1$, $\mathrm{Gr}^W_a\mathrm{Gr}^{\tilde{W}''}_k=N_2\mathrm{Gr}^W_a$ and (5.5.5) follows from Proposition 5.4.2 along with Lemma 5.5.2. Therefore we can assume $k\geqq a$. We shall show, in this case

(5.5.6) $$\tilde{M}(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a)=M(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a).$$

Note that $\tilde{W}''_k\cap W_a=N_2W_a+M''_k\cap W_a$ and $\tilde{M}_l\cap W_a=N_2(M'_{l+1}\cap W_a)+M_l\cap M'_l\cap W_a$ by Lemma 5.5.2. Since $N_2W_a\subset\tilde{W}''_{k-1}$, we have

$$\tilde{M}_l(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a)\subset M_l(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a).$$

Conversely, (3.1.5) implies

$$M_l\cap W_a\subset N_2(M_{l+2}\cap W_a)+M_l\cap M'_l\cap W_a\subset\tilde{W}''_{k-1}+\tilde{M}_l\cap W_a$$

and hence $M(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a)\subset\tilde{M}(\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a)$. Thus (5.5.5) follows from

(5.5.7) $N_1^l\colon \mathrm{Gr}^M_{k+l}\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a\hookrightarrow\mathrm{Gr}^M_{k-l}\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a$ for $k\geqq a$, $l\geqq 1$.

Since $\mathrm{Gr}^{\tilde{W}''}_k\mathrm{Gr}^W_a$ is a direct summand of $\mathrm{Gr}^{\tilde{M}''}_k\mathrm{Gr}^W_a$ by Lemma 3.4.2 and

$M(\mathrm{Gr}_a^W)=M(N_1|\mathrm{Gr}_a^W;\ M''(\mathrm{Gr}_a^W))$, we have (5.5.7) and this completes the proof of Theorem 5.5.1.

As a corollary of Theorem 5.5.1 we have

Corollary 5.5.4. *If* $(W;\ F,\ \bar{F};\ N_1,\dots,\ N_l)$ *is an IMHM, then so are* $(N_{1*}W;\ F,\ \bar{F};\ N_1,\dots,\ N_l)$ *and* $(M(N_1,\ W);\ F,\ \bar{F};\ N_2,\dots,\ N_l)$.

Use Proposition 5.3.1

Proposition 5.5.5.

$$\begin{aligned}(N_{1*}N_{2*}W)_k&=N_1(N_{2*}W)_{k+1}+N_2(N_{1*}W)_{k+1}\\&\qquad+M_k(N_1+N_2,\ W)\cap(N_{1*}W)_k\cap(N_{2*}W)_k\\&=N_1(N_{2*}W)_{k+1}+N_2(N_{1*}W)_{k+1}\\&\qquad+M_k(N_1+N_2,\ W)\cap(N_{1*}W)_{k+1}\cap(N_{2*}W)_{k+1}.\end{aligned}$$

In particular $N_{1*}N_{2*}W=N_{2*}N_{1*}W$.

Proof. We shall use the notation in (5.5.1). Noting $(N_{1*}N_{2*}W)_k=N_1\tilde{W}''_{k+1}+\tilde{M}_k\cap\tilde{W}''_k=(N_1\tilde{W}''_{k+1}+\tilde{M}_k)\cap\tilde{W}''_{k+1}$, it is enough to prove

$$\tilde{M}_k\cap\tilde{W}''_k\subset N_2\tilde{W}'_{k+1}+M_k\cap\tilde{W}''_k\cap\tilde{W}'_k, \tag{5.5.8}$$

$$N_2\tilde{W}'_{k+1}\subset N_1\tilde{W}''_{k+1}+\tilde{M}_k\cap\tilde{W}''_k, \tag{5.5.9}$$

and

$$M_k\cap\tilde{W}'_{k+1}\subset N_1\tilde{W}''_{k+1}+\tilde{M}_k. \tag{5.5.10}$$

Note that

$$\tilde{M}_k=N_2M'_{k+1}+M_k\cap M'_k. \tag{5.5.11}$$

By Lemma 5.5.2 we have

$$\begin{aligned}\tilde{M}_k\cap\tilde{W}''_k\subset\tilde{M}_k\cap W_{k+1}&\subset N_2(M'_{k+1}\cap W_{k+1})+M_k\cap M'_k\cap W_{k+1}\\&\subset N_2\tilde{W}'_{k+1}\cap\tilde{W}''_k+M_k\cap\tilde{W}'_k.\end{aligned}$$

Therefore we have (5.5.8).

Secondly, (5.5.9) follows from

$$\begin{aligned}N_2\tilde{W}'_{k+1}=N_2(N_1W_{k+2}+M'_{k+1}\cap W_{k+1})&\subset N_1N_2W_{k+2}+N_2M'_{k+1}\cap N_2W_{k+1}\\&\subset N_1\tilde{W}''_{k+1}+\tilde{M}_k\cap\tilde{W}''_k.\end{aligned}$$

Finally, by (3.1.5), we have

$$M_k\cap\tilde{W}'_{k+1}=M_k\cap(N_1W_{k+2}+M'_{k+1}\cap W_{k+1})\subset N_1(M_{k+2}\cap W_{k+2})+M_k\cap M'_{k+1}$$

$$\subset N_1(N_2W_{k+2}+M_{k+2}\cap M'_{k+2}\cap W_{k+2})+M_k\cap M'_{k+1}$$
$$\subset N_1N_2W_{k+2}+M_k\cap M'_{k+1}\subset N_1\tilde{W}''_{k+1}+\tilde{M}_k.$$

5.6. Let I be a finite set and let $\mathfrak{S}$ be the set of subsets of I. Let us consider the following category $\mathrm{MH}(I)$ consisting of $\mathrm{H}=\{\mathrm{H}(\alpha), W^\alpha(\mathrm{H}), F_\alpha(\mathrm{H}), \bar{F}_\alpha(\mathrm{H}), f_{\alpha\beta}(\mathrm{H}), g_{\beta\alpha}(\mathrm{H})\}_{\beta\subset\alpha\subset I}$. Here, $\mathrm{H}(\alpha)$ is a finite-dimensional vector space, $F_\alpha(\mathrm{H})$, $\bar{F}_\alpha(\mathrm{H})$ and $W^\alpha(\mathrm{H})$ are filtrations of $\mathrm{H}(\alpha)$ and $f_{\alpha\beta}(\mathrm{H})$: $\mathrm{H}(\beta)\to\mathrm{H}(\alpha)$, $g_{\beta\alpha}(\mathrm{H})$: $\mathrm{H}(\alpha)\to\mathrm{H}(\beta)$ satisfy the following conditions (5.6.1)–(5.6.7):

(5.6.1) $f_{\alpha\alpha}=g_{\alpha\alpha}=id$,

(5.6.2) $f_{\alpha\beta}\circ f_{\beta\gamma}=f_{\alpha\gamma}$, $g_{\gamma\beta}\circ g_{\beta\alpha}=g_{\gamma\alpha}$ for $\alpha\supset\beta\supset\gamma$.

(5.6.3) $g_{\alpha,\alpha\cup\beta}f_{\alpha\cup\beta,\beta}=f_{\alpha,\alpha\cap\beta}g_{\alpha\cap\beta,\beta}$ for any α and β.

(5.6.4) $f_{\alpha\beta}F^p_\beta(\mathrm{H})\subset F^{p-|\alpha\setminus\beta|}_\alpha(\mathrm{H})$, $f_{\alpha\beta}\bar{F}^p_\beta(\mathrm{H})\subset\bar{F}^{p-|\alpha\setminus\beta|}_\alpha(\mathrm{H})$,
$f_{\alpha\beta}W^\beta_k(\mathrm{H})\subset W^\alpha_{k-|\alpha\setminus\beta|}(\mathrm{H})$, $g_{\beta\alpha}F^p_\alpha(\mathrm{H})\subset F^p_\beta(\mathrm{H})$,
$g_{\beta\alpha}\bar{F}^p_\alpha(\mathrm{H})\subset\bar{F}^p_\beta(\mathrm{H})$ and $g_{\beta\alpha}W^\alpha_k(\mathrm{H})\subset W^\beta_{k+|\alpha\setminus\beta|}(\mathrm{H})$ for $\beta\subset\alpha\subset I$.

(5.6.5) Letting $N_j\in\mathrm{End}(\mathrm{H}(\alpha))$ be $N_j=g_{\alpha,\alpha\cup\{j\}}f_{\alpha\cup\{j\},\alpha}$ for $a\not\ni j$ and
$N_j=f_{\alpha,\alpha\setminus\{j\}}g_{\alpha\setminus\{j\},j}$ for $a\ni j$, N_j is nilpotent.

(5.6.6) $\{\mathrm{H}(\alpha); W^\alpha(\mathrm{H}); F_\alpha(\mathrm{H}), \bar{F}_\alpha(\mathrm{H}); \{N_j\}_{j\in I}\}$ is an IMHM for any $\alpha\subset I$.

(5.6.7) $\mathrm{Gr}^{W\alpha}_k \underset{\triangleright}{\overset{\longrightarrow}{\longleftarrow}} \mathrm{Gr}^{W\beta}_{k-|\alpha\setminus\beta|}$ for $\alpha\supset\beta$.

Note that if H is in $\mathrm{MH}(I)$, then H^* defined as follows is also an object of $\mathrm{MH}(I)$

(5.6.8)
$$H^*(\alpha)=H(\alpha)^*$$
$$W^\alpha_k(\mathrm{H}^*)=(\mathrm{H}(\alpha)/W^\alpha_{-k+2|\alpha|-1}(\mathrm{H}))^*,$$
$$F^p_\alpha(\mathrm{H}^*)=(\mathrm{H}(\alpha)/F^{1-p+|\alpha|}_\alpha(\mathrm{H}))^*,$$
$$\bar{F}^p_\alpha(\mathrm{H}^*)=(\mathrm{H}(\alpha)/\bar{F}^{1-p+|\alpha|}_\alpha(\mathrm{H}))^*,$$
$$f_{\alpha\beta}(\mathrm{H}^*)=g_{\beta\alpha}(\mathrm{H})^*$$
$$g_{\beta\alpha}(\mathrm{H}^*)=(-1)^{|\beta|-|\alpha|}f_{\alpha\beta}(\mathrm{H})^*.$$

For $\mathrm{H}\in\mathrm{MH}(I)$, let us define the Tate twist $\mathrm{H}(n)$ of H by

(5.6.9)
$$\mathrm{H}(n)(\alpha)=\mathrm{H}(\alpha),\quad W^\alpha_k(\mathrm{H}(n))=W^\alpha_{k+2n}(\mathrm{H}),$$
$$F^p_\alpha(\mathrm{H}(n))=F^{p+n}(\mathrm{H}),\quad \bar{F}^p_\alpha(\mathrm{H}(n))=\bar{F}^{p+n}_\alpha(\mathrm{H}),$$
$$f_{\alpha\beta}(\mathrm{H}(n))=f_{\alpha\beta}(\mathrm{H}),\quad g_{\beta\alpha}(\mathrm{H}(n))=g_{\beta\alpha}(\mathrm{H}).$$

Lemma 5.6.1. *If $\alpha\subset\beta$, and $\gamma\supset\beta\setminus\alpha$,*

$$f_{\beta\alpha}(\mathrm{H})M_k(\gamma,\mathrm{H}(\alpha))\subset M_{k-2|\beta\setminus\alpha|}(\gamma,\mathrm{H}(\beta)),$$
$$g_{\alpha\beta}(\mathrm{H})M_k(\gamma,\mathrm{H}(\alpha))\subset M_k(\gamma,\mathrm{H}(\beta)).$$

Here, $M(\gamma, \mathrm{H}(\alpha))=M(N, W^{\alpha}(\mathrm{H}))$ for $N \in \mathrm{C}(\gamma)$.

Proof. We shall prove them by the induction of $|\beta\backslash\alpha|$. If $|\beta\backslash\alpha|=1$ and $\gamma=\beta\backslash\alpha$, they follow from Proposition 3.3.1. When $|\beta\backslash\alpha|=1$, they follow from this case and the fact that $M(\gamma, \mathrm{H}(\alpha))=M(N, M(\beta\backslash\alpha, \mathrm{H}(\alpha)))$ for $N \in \mathrm{C}(\gamma)$. Now assume $|\beta\backslash\alpha|>1$. Then taking δ such that $\alpha \subsetneqq \delta \subsetneqq \beta$, they follow from the hypothesis of the inductions for α, δ and δ, β.

Lemma 5.6.2. *If we set*

$$P_k^{\alpha}(\mathrm{H})=\bigcap_{\beta \supsetneqq \alpha} \mathrm{Ker}(g_{\beta,\alpha}: \mathrm{Gr}_k^{W^{\alpha}(\mathrm{H})} \longrightarrow \mathrm{Gr}_{k+|\alpha\backslash\beta|}^{W^{\beta}(\mathrm{H})}) \subset \mathrm{Gr}_k^{W^{\alpha}(\mathrm{H})}, \tag{5.6.10}$$

then $P_k^{\alpha}(\mathrm{H})$ *has weight* k *with respect to the weight filtration* $M(\alpha, \mathrm{H}(\alpha))$.

Proof. We may assume that $\alpha=I$. Take $N \in \mathrm{C}(I)$. Then $M(\alpha, \mathrm{H}(\alpha))$ induces the weight filtration of $N|\mathrm{Gr}_k^{W^{\alpha}(\mathrm{H})}$, shifted by k. On ther other hand, $P_k^{\alpha}(\mathrm{H})$ is a direct summand of $\mathrm{Gr}_k^{W^{\alpha}(\mathrm{H})}$ as an $\boldsymbol{C}[N_1, \ldots, N_l]$-module by Proposition 3.3.1. Therefore $M(\alpha, \mathrm{H}(\alpha))$ induces on $P_k^{\alpha}(\mathrm{H})$ the weight filtration of $N|P_k^{\alpha}(\mathrm{H})$ shifted by k. Then this Lemma follows from $N|P(\mathrm{H}_k^{\alpha})=0$. Q. E. D.

For $\gamma \in I$, we define $\psi_{\gamma}(\mathrm{H})$ and $\varphi_{\gamma}(\mathrm{H})$ in $\mathrm{MH}(I\backslash\gamma)$ as follow

$$\begin{aligned} &\psi_{\gamma}(\mathrm{H})(\alpha)=\mathrm{H}(\alpha), \quad W^{\alpha}(\psi_{\gamma}(\mathrm{H}))=M(\gamma, \mathrm{H}(\alpha)), \\ &F_{\alpha}(\psi_{\gamma}(\mathrm{H}))=F_{\alpha}(\mathrm{H}), \quad \bar{F}_{\alpha}(\psi_{\gamma}(\mathrm{H}))=\bar{F}_{\alpha}(\mathrm{H}), \\ &f_{\alpha\beta}(\psi_{\gamma}(\mathrm{H}))=f_{\alpha\beta}(\mathrm{H}) \quad \text{and} \quad g_{\beta\alpha}(\psi_{\gamma}(\mathrm{H}))=g_{\beta\alpha}(\mathrm{H}) \qquad \text{for} \quad \alpha \subset \beta \subset I\backslash\gamma \end{aligned} \tag{5.6.11}$$

$$\begin{aligned} &\varphi_{\gamma}(\mathrm{H})(\alpha)=\mathrm{H}(\alpha \cup \gamma), \quad W^{\alpha}(\psi_{\gamma}(\mathrm{H}))=M(\gamma, \mathrm{H}(\alpha \cup \gamma)) \\ &F_{\alpha}(\varphi_{\gamma}(\mathrm{H}))=F_{\alpha\cup\gamma}(\mathrm{H}), \quad \bar{F}_{\alpha}(\varphi_{\gamma}(\mathrm{H}))=\bar{F}_{\alpha\cup\gamma}(\mathrm{H}) \\ &f_{\alpha\beta}(\varphi_{\gamma}(\mathrm{H}))=f_{\alpha\cup\gamma,\beta\cup\gamma}(\mathrm{H}) \quad \text{and} \\ &g_{\beta\gamma}(\varphi_{\alpha}(\mathrm{H}))=g_{\beta\cup\gamma,\alpha\cup\gamma}(\mathrm{H}) \qquad \text{for} \quad \alpha \subset \beta \subset I\backslash\gamma. \end{aligned} \tag{5.6.12}$$

Proposition 5.6.4. $\psi_{\gamma}(\mathrm{H})$ *and* $\varphi_{\gamma}(\mathrm{H})$ *are objects in* $\mathrm{MH}(I\backslash\gamma)$.

Proof. (5.6.1)–(5.6.5) are evident and (5.6.6) follows from Proposition 5.3.1.

In order to prove (5.6.7), we prepare the following

Lemma 5.6.5. *Let* $(\mathrm{H}; F, \bar{F}; N_1, N_2)$ *and* $(\mathrm{H}'; F', \bar{F}'; N_1', N_2')$ *be a nilpotent orbit of weight* w *and* $w-1$, *respectively. Let* $f: \mathrm{H} \to \mathrm{H}'$ *and* $g: \mathrm{H}' \to \mathrm{H}$ *be homomorphisms such that*

$$g \circ f=N_1, \quad f \circ g=N_1', \tag{5.6.13}$$

$$N_2' \circ f=f \circ N_2, \quad N_2 \circ g=g \circ N_2', \tag{5.6.14}$$

(5.6.15) $$H \underset{\underset{g}{\triangleright}}{\overset{f}{\rightleftarrows}} H'.$$

Then, we have for any k

(5.5.16) $$\mathrm{Gr}_k^{M(N_2)} \underset{\underset{g}{\triangleright}}{\overset{f}{\rightleftarrows}} \mathrm{Gr}_k^{M(N_2')}.$$

Here $M(*)$ is the weight filtration.

Proof. By the duality, it is enough to show

(5.6.17) $$gM_k(N_2') \subset N_1 M_k(N_2) + M_{k-1}(N_2).$$

We have, by Proposition 5.4.2,

$$gM_k(N_2') \subset M_k(N_2|\mathrm{Im}\, g) = M_k(N_2|\mathrm{Im}\, N_1) \subset N_1 M_k(N_2) + M_{k-1}(N_2).$$

Q. E. D.

Now, we shall show (5.6.7) for $\psi_\gamma(\mathrm{H})$ and $\varphi_\gamma(\mathrm{H})$. For this, it is enough to show, by Proposition 2.3.1

(5.6.18) $$\mathrm{Gr}_l^{M(\gamma, W^\beta)} \underset{\triangleright}{\rightleftarrows} \mathrm{Gr}_{l-1}^{M(\gamma, W^\alpha)} \quad \text{for} \quad \beta \subset \alpha \quad \text{and} \quad |\alpha \backslash \beta| = 1.$$

By Theorem 3.2.9, $\mathrm{Gr}_l^{M(\gamma, W^\beta)}$ is canonically a direct sum of $\mathrm{Gr}_l^{M(\gamma, W^\beta)}\mathrm{Gr}_k^{W^\beta}$'s. Hence (5.6.19) follows from

(5.6.20) $$\mathrm{Gr}_l^{M(\gamma, W^\beta)}\mathrm{Gr}_k^{W^\beta} \underset{\triangleright}{\rightleftarrows} \mathrm{Gr}_{l-1}^{M(\gamma, W^\beta)}\mathrm{Gr}_{k-1}^{W^\beta},$$

which is an immediate consequence of Lemma 5.6.5.

5.7. By Lemma 5.6.1, $f_{\alpha\beta}(\mathrm{H})$ and $g_{\beta\alpha}(\mathrm{H})$ are considered as morphisms of mixed Hodge structures $f_{\alpha\beta}\colon \mathrm{H}(\beta) \to \mathrm{H}(\alpha)(-|\alpha\backslash\beta|)$ $g_{\beta\alpha}\colon \mathrm{H}(\alpha) \to \mathrm{H}(\beta)$, with $M(I, \mathrm{H}(\alpha))$ as weight filtrations.

Letting e_j $(j \in I)$ be the base of $\mathbb{Z}^{(I)}$, we shall define the complex $\Pi(\mathrm{H})$ by

(5.7.1) $$\Pi(\mathrm{H})^n = \bigoplus_{|\alpha|=n} \mathrm{H}(\alpha)(-n) \underset{\mathbb{Z}}{\otimes} \mathbb{Z} e_\alpha.$$

Here $e_\alpha = \underset{j \in \alpha}{\wedge} e_j$. The differential is given by $\sum_{j,\alpha} f_{\alpha \cup \{j\}, \alpha} e_j$. Let us define $W_k(\Pi(\mathrm{H}))$ by

$$W_k(\Pi(\mathrm{H})^n) = \bigoplus_{|\alpha|=n} W_{k-n}(\mathrm{H}(\alpha))(-n) \otimes e_\alpha.$$

Then $\Pi(\mathrm{H})$ is a filtrered complex in the category of mixed Hodge structures.

Theorem 5.7.1. $\mathrm{H}^n(W_k(\Pi(\mathrm{H})))$ has weight $\leqq k+n$.

Proof. It is enough to show that $\mathrm{H}^n(\mathrm{Gr}_a^W\Pi(\mathrm{H}))$ has weight $\leqq a+n$. Then replacing H with Gr^W we may assume

$$W_k^a(\mathrm{H})=\begin{cases}\mathrm{H}(\alpha) & k=a-|\alpha|\\ 0 & k\neq a-|\alpha|\,.\end{cases}$$

Then Proposition 2.3.1 implies that H is a direct sum of M (I, (vector spaces)). Thus, we may assume that, there is $\gamma\subset I$ such that $\mathrm{H}(\alpha)=0$ if $\alpha\not\supset\gamma$ and $\mathrm{H}(\alpha)=(\prod_{j\in\gamma\setminus\alpha}N_j)\mathrm{H}(\gamma)$ for $\alpha\supset\gamma$. In this case, the theorem is nothing but the purity theorem ([K–K], [C–K–S]).

5.8. Let $\mathrm{H}=(W;\ F,\ \bar{F};\ N_1,\ldots,N_l)$ be an IMHM. For $J\subset I=\{1,\ldots,l\}$, we set

$$M(J)=M(J;\ W)=M(\textstyle\sum_{j\in J}N_j,\ W)\,, \tag{5.8.1}$$

$$\Psi_J W=N_{j_1}*\cdots N_{j_r}*W \quad\text{where}\quad J=\{j_1,\ldots,j_r\}\,. \tag{5.8.2}$$

By Corollary 5.5.4 and Proposition 5.5.5 $\Psi_J W$ is well-defined.

Set

$$\mathrm{C}(J)=\{\textstyle\sum_{j\in J}t_jN_j;\ t_j>0\}\,. \tag{5.8.3}$$

Then we have already remarked in Proposition 5.2.5.

$$M(J_1\cup J_2)=M(N,\ M(J_2))\qquad\text{for}\quad N\in\mathrm{C}(J_1)\,, \tag{5.8.4}$$

or

$$M(J_1\cup J_2)=M(J_1,\ M(J_2))\,.$$

Applying Theorem 5.5.1, we have

$$M(J_1,\ \Psi_{J_2}W)=\Psi_{J_2}M(J_1,\ W)\,. \tag{5.8.5}$$

We have also

$$M(K,\ \Psi_J W)=M(K,\ W)\qquad\text{for}\quad K\supset J\,. \tag{5.8.6}$$

In fact, for $K\ni j$, we have $N_jM_k(K,\ W)\subset M_{k-2}(K,\ W)$, which implies $(N_j)_*M(K,\ W)=M(K,\ W)$.

By Proposition 2.3.1 and Lemma 3.4.2, we can associate to W the object $DI(\mathrm{H})$ of $\mathrm{MH}(I)$ given by $DI(\mathrm{H})(\alpha)=\mathrm{H}$, $W^\alpha(DI(\mathrm{H}))=\Psi_\alpha W$, $F_\alpha(DI(\mathrm{H}))=F$, $\bar{F}_\alpha(DI(\mathrm{H}))=\bar{F}$, $f_{\alpha\beta}(DI(\mathrm{H}))=\prod_{j\in\alpha\setminus\beta}N_j$, $g_{\beta\alpha}(DI(\mathrm{H}))=id$.

Proposition 5.8.1. *Let* $\mathrm{H}'=(W; F, \bar{F}; N_1,\dots, N_l)$ *be an IMHS and* $H\in \mathrm{MH}(I)$. *Then we have*

$$\mathrm{Hom}_{\mathrm{MH}(I)}(\mathrm{H}, DI(\mathrm{H}'))=\mathrm{Hom}(\mathrm{H}(\phi), \mathrm{H}').$$

Here the last Hom *is the set of homomorphisms preserving* W, F, $\bar{F}$ and *commuting with* $N_1,\dots, N_l$.

Proof. It is enough to show that $g_{\phi\beta}W^{\beta}(\mathrm{H})\subset \Psi_{\beta}W^{\phi}(\mathrm{H})$. We shall prove

$$g_{\beta\alpha}W^{\alpha}(\mathrm{H})\subset \Psi_{\alpha\setminus\beta}W^{\beta}(\mathrm{H}) \quad \text{for} \quad \beta\subset\alpha, \tag{5.8.7}$$

by the induction of $|\alpha\setminus\beta|$. If $\alpha=\beta$, there is nothing to prove. Assume $|\alpha\setminus\beta|=1$. Set $\alpha\setminus\beta=\{j\}$. Then by (5.6.7)

$$\begin{aligned} g_{\beta\alpha}W^{\alpha}_k(\mathrm{H}) &= N_jW^{\beta}_{k+1}(\mathrm{H})+g_{\beta\alpha}W^{\alpha}_k(\mathrm{H})\cap W^{\beta}_k(\mathrm{H})\\ &\subset N_jW^{\beta}_{k+1}(\mathrm{H})+g_{\beta\alpha}(M_k(N_j, W^{\alpha}(\mathrm{H}))\cap W^{\alpha}_k(\mathrm{H})+W^{\alpha}_{k-1}(\mathrm{H}))\\ &\subset N_jW^{\beta}_{k+1}(\mathrm{H})+M_k(N_j, W^{\beta}(\mathrm{H}))\cap W^{\beta}_{k+1}(\mathrm{H})+g_{\beta\alpha}W^{\alpha}_{k-1}(\mathrm{H})\\ &=(N_{j*}W^{\beta}(\mathrm{H}))_k+g_{\beta\alpha}W^{\alpha}_{k-1}(\mathrm{H}). \end{aligned}$$

Here the last inclusion follows from Proposition 3.3.1.

Now assume $|\alpha\setminus\beta|\geqq 2$. Take γ such that $\beta\subsetneqq\gamma\subsetneqq\alpha$. Then the induction on $|\alpha\setminus\beta|$ shows

$$g_{\gamma\alpha}W^{\alpha}(\mathrm{H})\subset \Psi_{\alpha\setminus\gamma}W^{\gamma}(\mathrm{H})$$

$$g_{\beta\gamma}W^{\gamma}(\mathrm{H})\subset \Psi_{\gamma\setminus\beta}W^{\beta}(\mathrm{H}).$$

Therefore we have

$$\begin{aligned} g_{\beta\alpha}W^{\alpha}(\mathrm{H}) &\subset g_{\beta\gamma}\Psi_{\alpha\setminus\gamma}W^{\gamma}(\mathrm{H})\\ &\subset \Psi_{\alpha\setminus\gamma}g_{\beta\gamma}W^{\gamma}(\mathrm{H})\subset \Psi_{\alpha\setminus\gamma}\Psi_{\gamma\setminus\beta}W^{\beta}(\mathrm{H})=\Psi_{\alpha\setminus\beta}W^{\beta}(\mathrm{H}). \end{aligned}$$

Proposition 5.8.2.

$$\begin{aligned} (\Psi_JW)_k &= \sum_{j\in J} N_j(\Psi_{J\setminus\{j\}}W)_{k+1}+M_k(J)\cap \bigcap_{j\in J}(\Psi_{J\setminus\{j\}}W)_{k+1}\\ &= \sum_{j\in J} N_j(\Psi_{J\setminus\{j\}}W)_{k+1}+M_k(J)\cap \bigcap_{j\in J}(\Psi_{J\setminus\{j\}}W)_{k}. \end{aligned}$$

Proof. We shall prove this by the induction of $|J|$. For $j\in J$, we have

$$M(J)\cap(\Psi_{J\setminus\{j\}}W)_{k+1}=M(N_j, M(J\setminus\{j\})\cap(\Psi_{J\setminus\{j\}}W)_{k+1})$$

because $M(J\setminus\{j\})=M(J\setminus\{j\}, \Psi_{J\setminus\{j\}}W)$. Therefore we have

$$\begin{aligned} &M_k(J)\cap(\Psi_{J\setminus\{j\}}W)_{k+1}\\ &\quad\subset N_j(M_{k+2}(J)\cap(\Psi_{J\setminus\{j\}}W)_{k+1})\\ &\qquad+M_k(J)\cap M_k(J\setminus\{j\})\cap(\Psi_{J\setminus\{j\}}W)_{k+1}. \end{aligned} \tag{5.8.8}$$

Note that

$$N_j\Psi_{J\setminus\{j\}}W\subset\Psi_{J\setminus\{i\}}W \quad \text{for any } i,j\in J. \tag{5.8.9}$$

In fact, if $i\neq j$, $N_j(\Psi_{J\setminus\{j\}}W)_k\subset N_j(\Psi_{J\setminus\{j,i\}}(W))_{k+1}\subset(\Psi_{J\setminus\{i\}}W)_k$. Then (5.8.8) and (5.8.9) imply

$$\begin{aligned}M_k(J)\cap\bigcap_{i\in J}(\Psi_{J\setminus\{i\}}W)_{k+1}&\\ \subset N_j(\Psi_{J\setminus\{j\}}W)_{k+1}+M_k(J)\cap M_k(J\setminus\{j\})\cap&\bigcap_{i\in J\setminus\{j\}}(\Psi_{J\setminus\{i\}}W)_{k+1}.\end{aligned}$$

We have

$$\begin{aligned}M_k(J)\cap M_k(J\setminus\{j\})&=M_k(N_j,\ M(J\setminus\{j\}))\cap M_k(J\setminus\{j\})\\ &\subset(N_{j*}M(J\setminus\{j\}))_k=M(J\setminus\{j\},\ \Psi_{\{j\}}W)_k.\end{aligned}$$

Therefore by the induction hypothesis,

$$\begin{aligned}M_k(J)\cap M_k(J\setminus\{j\})\cap&\bigcap_{i\in J\setminus\{j\}}(\Psi_{J\setminus\{i\}}W)_{k+1}\\ &\subset M_k(J\setminus\{j\},\ \Psi_{\{j\}}W)\cap\bigcap_{i\in J\setminus\{j\}}(\Psi_{J\setminus\{j\}\setminus\{i\}}\Psi_{\{j\}}W)_{k+1}\\ &\subset(\Psi_{J\setminus\{j\}}\Psi_{\{j\}}W)_k=(\Psi_JW)_k.\end{aligned}$$

This shows that

$$\sum_{j\in J}N_j(\Psi_{J\setminus\{j\}}W)_{k+1}+M_k(J)\cap\bigcap_{j\in J}(\Psi_{J\setminus\{j\}}W)_{k+1}\subset(\Psi_JW)_k.$$

On the other hand, since DI(H) belongs to MH(I), we can apply Lemma 5.6.2 and we obtain

$$\begin{aligned}\mathrm{Gr}_k^{\Psi_J(W)}=\sum\mathrm{Im}(\mathrm{Gr}_{k+1}^{\Psi_{J\setminus\{j\}}W}&\longrightarrow\mathrm{Gr}_k^{\Psi_JW})\\ &+M_k(J)\cap\bigcap_{j\in J}\mathrm{Ker}\,(\mathrm{Gr}_k^{\Psi_JW}\longrightarrow\mathrm{Gr}_{k+1}^{\Psi_{J\setminus\{j\}}W}),\end{aligned}$$

or

$$\begin{aligned}(\Psi_JW)_k\subset\sum_{j\in J}N_j(\Psi_{J\setminus\{j\}}W)_{k+1}&\\ +M_k(J)\cap\bigcap_{j\in J}(\Psi_{J\setminus\{j\}}W)_k&+(\Psi_JW)_{k-1}.\end{aligned}$$

Then the induction in k shows that $(\Psi_JW)_k$ is contained in

$$\sum_{j\in J}N_j(\Psi_{J\setminus\{j\}}W)_{k+1}+M_k(J)\cap\bigcap_{j\in J}(\Psi_{J\{\setminus j\}}W)_k.$$

§ 6. The Proof of Theorem 4.4.1.

6.1. In order to prove Theorem 4.4.1, it is enough to show the following.

Lemma 6.1.1. *Let* $\mathrm{H}=(W; F, \bar{F}; N_1, N_2)$ *be a pre-IMHM and assume that there exist the weight filtration of* N_1 *relative to* W *and that of* N_2. *Then there exists the weight filtration* M *of* N_1+N_2 *relative to* W *and we have* $N_jM_k \subset M_{k-2}$ *for* $j=1, 2$.

6.2. We shall show this by the induction. Hence, we may assume that $\mathrm{H}=W_0$ and that H induces on W_{-1} an IMHM. Let H_0 be Gr_0^W. Then we have an exact sequence

$$0 \longrightarrow W_{-1} \xrightarrow{\psi} \mathrm{H} \xrightarrow{\varphi} \mathrm{H}_0 \longrightarrow 0. \tag{6.2.1}$$

Let $M(\mathrm{H}_0)$ and $M(W_{-1})$ be the relative weight filtration of N_1+N_2 on H_0 and W_{-1}, respectively. In order to prove 6.1.1, it is enough to show that there exists a section $s\colon \mathrm{H}_0 \to \mathrm{H}$ of φ in (6.2.1) satisfying the following condition (6.2.2). Note first that $[N_j, s]\colon \mathrm{H}_0 \to \mathrm{H}$ splits as the composition of $\alpha_j(s)\colon \mathrm{H}_0 \to W_{-1}$ and ψ.

$$\alpha_j(s)M_k(\mathrm{H}_0) \subset M_{k-2}(W_{-1}). \tag{6.2.2}$$

In fact, if there exists such a section s that satisfies (6.2.2),

$$M_k=\psi(M_k(W_{-1}))+s(M_k(\mathrm{H}_0)) \quad \text{satisfies} \quad N_jM_k \subset M_{k-2},$$

and it is evident that M induces on H_0 the weight filtration of N_1+N_2, shifted by k.

6.3. Now, we shall show the existence of such a section s. Set $\tilde{\mathrm{H}}=\mathrm{Hom}(\mathrm{H}_0, W_{-1})$. Then $\tilde{\mathrm{H}}$ is an IMHM. Then (6.2.2) is equivalent to $\alpha_j(s) \in M_{-2}(\tilde{\mathrm{H}})$. Let $M'(\mathrm{H}_0)$, $M'(\mathrm{H})$, $M'(W_{-1})$ and $M'(\tilde{\mathrm{H}})$ be the relative weight filtration of N_1 on H_0, H, W_{-1} and $\tilde{\mathrm{H}}$, respectively and let $M''(\mathrm{H}_0)$, $M''(\mathrm{H})$, $M''(W_{-1})$ and $M''(\tilde{\mathrm{H}})$ be the relative weight filtration of N_2. Since the exact sequence (6.2.1) is strict with respect to the filtration M', there exists a section $s'\colon \mathrm{H}_0 \to \mathrm{H}$ such that $s'M'_k(\mathrm{H}_0) \subset M'_k(\mathrm{H})$. Therefore, we have

$$\alpha_1(s') \in M'_{-2}(\tilde{\mathrm{H}}). \tag{6.3.1}$$

Similarly if we take a section $s''\colon \mathrm{H}_0 \to \mathrm{H}$ such that $s''M''_k(\mathrm{H}_0) \subset M''_k(\mathrm{H})$ we have

$$\alpha_2(s'') \in M''_{-2}(\tilde{\mathrm{H}}). \tag{6.3.2}$$

By the assumption $t=s'-s'' \in \tilde{\mathrm{H}}$. Note that $\tilde{\mathrm{H}}=W_{-1}(\tilde{\mathrm{H}})$. Thus we have

(6.3.3)
$$\alpha_1(s') \in W_{-1}(\tilde{H}) \cap M'_{-2}(\tilde{H}) \subset (N_{1*}W(\tilde{H}))_{-2}$$

and $\alpha_2(s') = N_2 t + \alpha_2(s'') \in N_2 W_{-1}(\tilde{H}) + M''_{-2}(\tilde{H}) \cap W_{-1}(\tilde{H}) = (N_{2*}W(\tilde{H}))_{-2}$.

Now, we consider the complex

$$W_{-1}(\Pi)\colon W_{-1}(\tilde{H}) \longrightarrow (N_{1*}W(\tilde{H}))_{-2} \oplus (N_{2*}W(\tilde{H}))_{-2} \longrightarrow (N_{1*}N_{2*}W(\tilde{H}))_{-3}.$$

Then $(\alpha_1(s'), \alpha_2(s'))$ is a cochain of $W_{-1}(\Pi)$ of degree 1. By the purity theorem for IMHM (Theorem 5.7.1) and the results in §5.8, $H^1(W_{-1}(\Pi))$ has weight $\leqq 0$, and hence the cohomology class of $(\alpha_1(s'), \alpha_2(s'))$ is represented by a cochain of weight $\leqq 0$. Thus, there exists $u \in \tilde{H}$ such that

(6.3.4)
$$\alpha_1(s') - N_1 u \in M_{-2}(\tilde{H}),$$
$$\alpha_2(s') - N_2 u \in M_{-2}(\tilde{H}).$$

Then $s = s' - u$ satisfies $\alpha_j(s) \in M_{-2}(\tilde{H})$. This completes the proof of Theorem 4.4.1.

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