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The Radon-Nikodym theorem for L^p-spaces of W^*-algebras

By

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Abstract

We prove an analogue of Sakai's [15] Radon-Nikodym theorem for L^p-spaces of W^*-algebras. Following Takesaki [20] we characterize the Radon-Nikodym derivatives by analytic extensions of modular automorphism groups. Finally we relate the facial structure in L^p-spaces of W^*-algebras to Connes' [2] unitary Radon-Nikodym cocycles.

§1. Main Results

Sakai's Radon-Nikodym theorem [15] describes the facial structure in the positive cone of the predual $L^1(\mathcal{M})$ of a W^*-algebra $\mathcal{M}$. Haagerup [9] succeeded to construct the spaces $L^p(\mathcal{M})$, $1 \leq p \leq \infty$. They are obtained as certain subspaces of measurable operators affiliated with the crossed product of $\mathcal{M}$ and modular automorphisms. In what follows we shall obtain an analogue of Sakai's result for $L^p(\mathcal{M})$, $1 \leq p \leq \infty$, by use of a Radon-Nikodym lemma for measurable operators affiliated with a semifinite W^*-algebra. This result is in fact a direct consequence of an underlying facial principle for measurable operators. Takesaki [20] obtained the uniqueness of Sakai's positive Radon-Nikodym derivative and gave a characterization of non-necessarily positive derivatives. Connes [3] obtained a Radon-Nikodym theorem for weights in terms of analytic extensions of the associated unitary cocycle. We shall prove the corresponding versions for $L^p(\mathcal{M})$.

1.1. Haagerup's definition of L^p-spaces of W^*-algebras [9], [22]

Consider a W^*-algebra $\mathcal{M}$. Let φ be a normal, faithful, semifinite weight

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on $\mathcal{M}$ and let σ_t^φ, $t\in\boldsymbol{R}$, be the corresponding group of modular automorphisms of $\mathcal{M}$. We denote the crossed product [5, Part 1, Def. 2.10] of $\mathcal{M}$ and σ^φ by $\mathcal{N}=\mathcal{M}\otimes_{\sigma^\varphi}\boldsymbol{R}$. $\mathcal{M}$ has a normal, faithful representation π in $\mathcal{N}$ [5, Part 1, Prop. 2.5] and $\pi(\mathcal{M}):=L^\infty(\mathcal{M})$ is characterized as the set of fixed points under the dual action $\hat{\sigma}$ [5, Part 1, Prop. 4.12]. $\mathcal{N}$ is semifinite and has a n.f.s. trace τ, which is canonically associated to φ—see [8, Lemma 5.2]. If $\tilde{\mathcal{N}}$ denotes the space of τ-measurable operators (compare §2), then

$$L^p(\mathcal{M}):=\{T\in\tilde{\mathcal{N}}\,|\,\hat{\sigma}_t(T)=e^{-t/p}T\}\ .$$

We shall write J for the natural involution $T\mapsto T^*$ in $L^p(\mathcal{M})$. If $\psi\in\mathcal{M}_*^+$, then h_ψ is defined as the Radon-Nikodym derivative [14, Thm. 5.12] of the dual weight $\tilde{\psi}$ [6, Def. 3.1] of ψ with respect to τ, i.e. $\tilde{\psi}=\tau(h_\psi\cdot)$. It can be shown [22, Chapter II, Cor. 6], that $h_\psi\in L^1(\mathcal{M})$ and that the map $\psi\to h_\psi$ extends to an order isomorphism of $\mathcal{M}_*$ onto $L^1(\mathcal{M})$ [22, Chapter II, Thm. 7]. This gives rise to a linear functional tr on $L^1(\mathcal{M})$ defined by

$$tr(h_\psi):=\psi(1),\quad \psi\in\mathcal{M}_*\ .$$

The duality between $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$, $1\leq p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, is defined as [22, Chapter II, Thm. 32]:

$$\langle S,T\rangle_{p,q}=tr(ST)=tr(TS),\ S\in L^p(\mathcal{M}),\ T\in L^q(\mathcal{M})\ .$$

1.2. Radon-Nikodym theorem for L^p-spaces

Suppose T, $S\in L^p(\mathcal{M})$, $1\leq p\leq\infty$, *and* $0\leq T\leq S$. *Then there exists a unique* $h\in\mathcal{M}$, $0\leq\pi(h)\leq\mathrm{supp}(S)$ *such that*

$$T=\pi(h)J\pi(h)J\,S\ .$$

1.3. Separating vectors in L^p-spaces:

Suppose $1\leq p<\infty$ and $S\in L^p(\mathcal{M})^+$, i.e. $S\in L^p(\mathcal{M})$ and $S\geq 0$. We call S separating, if $x\in\mathcal{M}$, $\pi(x)S=0\Rightarrow x=0$. We call S cyclic, if $\pi(\mathcal{M})S$ is dense in $L^p(\mathcal{M})$. As one might expect, we have $\mathrm{supp}(S)=1\Leftrightarrow S$ is separating $\Leftrightarrow S$ is cyclic. Hence there exist separating vectors in $L^p(\mathcal{M})^+$ if and only if $\mathcal{M}$ is σ-finite.

Let $\mathcal{M}$ be a σ-finite W^*-algebra and consider two separating vectors S, $T\in L^1(\mathcal{M})^+$ as well as the corresponding linear functionals $\psi:=tr(\pi(\cdot)S)$, $\omega=tr(\pi(\cdot)T)\in\mathcal{M}_*^+$. Haagerup [10, §3] has shown that

1.3.1. $$\pi(\sigma_t^\psi(x))=S^{it}\pi(x)S^{-it},\ x\in\mathcal{M}\ ,$$

1.3.2. $$(D\omega\colon D\psi)_t = \pi^{-1}(T^{it}S^{-it})\,.$$

In fact (1.3.2) holds also, if $(D\omega\colon D\psi)_t$ is defined as in [4, p. 479] or [1, p. 394] and T is not necessarily a separating vector.

1.4. Analytic extensions of modular automorphism groups

Suppose ψ is a n.f.s. weight on $\mathcal{M}$. Then we shall write $\mathcal{A}^{\psi}(s)$, $s\in\mathbb{R}$, for the set of $x\in\mathcal{M}$ such that the map $t\mapsto\sigma_t^{\psi}(x)$, $t\in\mathbb{R}$, has a $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous extension $z\mapsto\sigma_z^{\psi}(x)\in\mathcal{M}$ to the strip

$$\Gamma_s = \{z \mid |\operatorname{Im} z| \leq |s|\,,\ s\cdot\operatorname{Im} z\geq 0\}\,,$$

which is $\sigma(\mathcal{M}, \mathcal{M}_*)$-analytic in Γ_s^0.

The following result has been obtained by Takesaki [20, Thm. 15.3] in the case $p=1$.

Theorem. *Suppose $\mathcal{M}$ is a σ-finite W*-algebra and $1\leq p<\infty$. Let $S\in L^p(\mathcal{M})^+$ be a separating vector, $\psi(\cdot)=tr(\pi(\cdot)S^p)\in\mathcal{M}_*^+$ be the corresponding linear functional and $x\in\mathcal{M}$. Then the following conditions are equivalent:*

(a) $$\pi(x)J\pi(x)J\,S\leq S$$

(b) $$x\in\mathcal{A}^{\psi}\left(\frac{1}{2p}\right) \text{ and } \|\sigma^{\psi}_{i/2p}(x)\|\leq 1\,.$$

1.5. Analytic extensions of Radon-Nikodym cocycles

Our final result has been obtained by Connes [3] not only in the case $p=1$ but for n.f.s. weights.

Theorem. *Suppose $\mathcal{M}$ is a σ-finite W*-algebra and $1\leq p<\infty$. Let $S, T\in L^p(\mathcal{M})^+$, $\operatorname{supp}(S)=1$ and consider the corresponding linear functionals $\psi:=\operatorname{tr}(\pi(\cdot)S^p)$, $\omega:=\operatorname{tr}(\pi(\cdot)T^p)\in\mathcal{M}_*^+$. Then the following two conditions are equivalent:*

(a) $T\leq S$

(b) *The function $t\mapsto(D\omega\colon D\psi)_t$, $t\in\mathbb{R}$, has a $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous extension to $\Gamma_{-1/2p}$, which is analytic in $\Gamma^0_{-1/2p}$ and*

$$\|(D\omega\colon D\psi)_{-i/2p}\|\leq 1\,.$$

Furthermore if one of the above conditions holds, then

$$y := (D\omega\colon D\psi)_{-i/2p}$$

satisfies

$$T = \pi(y)\, J\pi(y)\, J\, S\,.$$

Remark. The theorem shows that the (operator) order relation in $L^p(\mathcal{M})$ $T \leq S$ is exactly the order relation $\psi \leq \omega\left(\frac{1}{2p}\right)$ for the corresponding functionals in the sense of Connes-Takesaki [4, Ch. II, Def. 4.1].

§2. Measurable Operators with Respect to a Trace

2.1. Results of Haagerup-Terp [22] — compare also [11] —

Suppose $\mathcal{N}$ is a semifinite W^*-algebra acting on a Hilbert space H. Let τ be a n.f.s. trace on $\mathcal{N}$. A subspace $\mathcal{X}$ of H is called τ-dense, if for $\varepsilon \in \mathbb{R}_*^+$, there exists a projection $e \in \mathcal{N}$ such that

$$eH \subseteq \mathcal{X} \text{ and } \tau(1-e) < \varepsilon\,.$$

A closed, densely defined operator T affiliated with $\mathcal{N}$ is called τ-measurable, if its domain of definition $\mathcal{D}(T)$ is τ-dense. We denote the set of τ-measurable operators by $\tilde{\mathcal{N}}$. If T is a selfadjoint operator affiliated with $\mathcal{N}$ and $T = \int_{-\infty}^{\infty} t\, dE(t)$ is its spectral decomposition, then $T \in \tilde{\mathcal{N}}$ if and only if for every $\varepsilon \in \mathbb{R}_*^+$, there exists $t > 0$ such that $\tau(1 - E([-t, t])) < \varepsilon$ [22, Chapter I, Prop. 21]. If $T \in \tilde{\mathcal{N}}$, then $(T|_{\mathcal{X}})^- = T$ for every τ-dense subspace $\mathcal{X} \subseteq \mathcal{D}(T)$ [22, Chapter I, Prop. 12]. If $S, T \in \tilde{\mathcal{N}}$, then $S^* \in \tilde{\mathcal{N}}$ and $S+T$, $S \circ T$ are densely defined, preclosed and $\overline{S+T}$, $\overline{S \circ T} \in \tilde{\mathcal{N}}$. In what follows, we shall always consider the strong sum and product in $\tilde{\mathcal{N}}$ and omit the closure sign. $\tilde{\mathcal{N}}$ is a *-algebra with respect to strong sum, strong product and adjoint operation [22, Chapter I, Prop. 24].

2.2. Lemma. *Suppose* $S, T \in \tilde{\mathcal{N}}$.

(a) *If* $T \circ S = 0$, *then* rightsupp $(T) \cdot$ leftsupp $(S) = 0$.

(b) *If* leftsupp $(T) \vee$ rightsupp $(T) \leq$ leftsupp (S) *and* $S^*TS = 0$, *then* $T = 0$.

(c) *Suppose* $0 \leq T$, S *and* $\mathrm{supp}(S) = 1$. *Then the following two statements are equivalent*:

i) $T \leq S^2$

ii) *The operator* $T^{1/2}S^{-1}$ *has domain of definition* $\mathcal{D}(S^{-1})$ *and norm less than* 1.

(d) **Facial principle**: *If* $0 \leq T \leq S^*S$, *then there exists a unique* $x \in \mathcal{N}$,

$0 \leq x \leq$ leftsupp (S) *such that*

$$T = S^* x S .$$

(e) *If* leftsupp $(S)=1$ *and* $S^*TS \geq 0$, *then* $T \geq 0$.

Proof. (a) Using polar decomposition and spectral calculus one reduces (a) to the bounded case, which is known.

In the remainder of the proof we shall assume w.l.o.g. that $S \geq 0$ and $\text{supp}(S)=1$.

(b) Let $S=\int_0^\infty t \, dE(t)$ be the spectral decomposition of S. Then we have $E\left(\left(\frac{1}{n}, \infty\right)\right)TS=0 \Rightarrow T=0$.

(c) and (d) $0 \leq T \leq S^2$ implies $\mathcal{D}(S) \subseteq \mathcal{D}(T^{1/2})$.

For $\xi \in \mathcal{D}(S)=S^{-1}\mathcal{D}(S^{-1})$ we set

$$y(S\xi) = T^{1/2}\xi .$$

It follows from the hypothesis that $y=T^{1/2}S^{-1}$ has norm less than 1.

Conversely, if $T^{1/2}S^{-1}$ has domain of definitions $\mathcal{D}(S^{-1})$ and norm less than 1, then let $y \in \mathcal{N}$ be its closure. We have

$$yS = T^{1/2} \text{ on } \mathcal{D}(S)$$

and $||y|| \leq 1$.

Since $\mathcal{D}(S)$ is τ-dense we obtain

$$yS = T^{1/2} .$$

Hence $T=Sy^*yS=SxS \leq S^2$ where $0 \leq x := y^*y \leq 1$

(e) We obtain from (b) that $T=T^*$ since

$$STS = \frac{1}{2} S(T+T^*)S .$$

Now $0 \leq STS \leq S|T|S$ and (d) imply the existence of $x \in \mathcal{N}_+$ such that

$$STS = S|T|^{1/2}x|T|^{1/2}S$$

So
$$T = |T|^{1/2}x|T|^{1/2} \geq 0 \quad \text{by (b)} .$$

□

The following lemma generalizes a well known result for bounded operators. The main idea in the proof is due to Pedersen [12].

2.3. Lemma. *Suppose* $S, T \in \tilde{\mathcal{N}}$. *If* $0 \leq T \leq S$ *and* $0 \leq \alpha \leq 1$, *then* $T^\alpha \leq S^\alpha$.

Proof. Assume first that T is bounded and S has a bounded inverse. Then the proof of [20, Chapter 1, Prop. 6.3] applies and yields the desired result. In general let

$$S = \int_0^\infty t\, dE(t) \quad \text{and} \quad T = \int_0^\infty t\, dF(t)$$

be the spectral decomposition of S and T respectively. If $e_n = E([0, n]) \wedge F([0, n])$, then $\mathfrak{X} = \bigcup_{n \in N} e_n H$ is a τ-dense subspace of H. Hence it is a core for $S^\alpha - T^\alpha$. If $\xi \in e_n H$, then

$$\begin{aligned} \langle T^\alpha \xi, \xi \rangle &= \langle T^\alpha F([0, n])\xi, \xi \rangle \\ &= \langle (T F([0, n]))^\alpha \xi, \xi \rangle \\ &\leq \inf_{\varepsilon > 0} \langle (S + \varepsilon I)^\alpha \xi, \xi \rangle \\ &= \inf_{\varepsilon > 0} \langle (S + \varepsilon I)^\alpha E([0, n])\xi, \xi \rangle \\ &= \langle S^\alpha \xi, \xi \rangle . \end{aligned}$$

This completes the proof. □

The following result has been obtained by Pedersen and Takesaki [13] in the bounded case

2.4. Lemma. *Suppose* $S, T \in \tilde{\mathcal{M}}$, $S, T \geq 0$. *If*

(a) $(S^{1/2} T S^{1/2})^{1/2} = |T^{1/2} S^{1/2}| \leq S$ *and* $\operatorname{supp}(T) \leq \operatorname{supp}(S)$

then there exists a unique $h \in \mathcal{M}$ *such that*

(b) $0 \leq h \leq \operatorname{supp}(S)$

and $T = h S h$.

Conversely (b) *implies* (a). *Furthermore, if* $T \leq S$, *then* (a) *is satisfied thanks to Lemma* 2.3.

Proof. If (a) holds, then Lemma 2.2 (d) implies that

$$(S^{1/2} T S^{1/2})^{1/2} = S^{1/2} h S^{1/2}$$

for an $h \in \mathcal{M}$ with $0 \leq h \leq \operatorname{supp}(S)$. Hence

$$S^{1/2} T S^{1/2} = S^{1/2} h S h S^{1/2} .$$

Using Lemma 2.2 (b) this implies (b). If (b) is satisfied, then

$$S^{1/2} T S^{1/2} = S^{1/2} h S h S^{1/2} .$$

The uniqueness of the square root shows that

$$(S^{1/2}T\,S^{1/2})^{1/2} = S^{1/2}h\,S^{1/2} \leq S\,.$$

This and Lemma 2.2 (d) prove that h is uniquely determined. □

§3. Proof of theorems

Proof of 1.2. This is an immediate consequence of Lemma 2.4 if one observes, that the uniqueness-statement forces the delivered h to be a fixed point under the dual action $\hat{\sigma}$.

□

Proof of 1.3. Using Lemma 2.2 (a) we show that

$$\{\pi(\mathcal{M})S\}^{\perp} = \mathrm{supp}(S)^{\perp}L^q(\mathcal{M})\,.$$

With this in mind the first statement in (1.3) is a direct consequence of the bipolar theorem and Lemma 2.2 (a). (1.3.1) and (1.3.2) for separating T follow from [6, Thm. 3.2], [14, Thm. 4.6] and [2, Lemma 1.2.3]:

$$\pi(\sigma_t^{\psi}(x)) = \sigma_t^{\tilde{\psi}}(\pi(x)) = h_{\psi}^{it}\pi(x)h_{\tilde{\psi}}^{-it} = S^{it}\pi(x)S^{-it}\,,$$

$$\pi((D\omega\colon D\psi)_t) = (D\tilde{\omega}\colon D\tilde{\psi})_t = (D\tilde{\omega}\colon D\tau)_t(D\tau\colon D\tilde{\psi})_t = T^{it}S^{-it}\,.$$

If T is not separating, fix $R\in L^1(\mathcal{M})^+$ with $\mathrm{supp}(R)=\mathrm{supp}(T)^{\perp}$, let $\omega_1\in\mathcal{M}_*^+$ denote the functional corresponding to $T+R$ and $p=\pi^{-1}(\mathrm{supp}\,T)=\mathrm{supp}(\omega)$. Now we obtain

$$\begin{aligned}\pi(D\omega\colon D\psi)_t) &= \pi((D\omega\colon D\omega_1)_t\,(D\omega_1\colon D\psi)_t) = \\ &= \pi(p(D\omega_1\colon D\psi)_t = \pi(p)\,(T+R)^{it}\,S^{-it} = T^{it}\,S^{-it}\,.\end{aligned}$$

□

Proof of 1.4 *Theorem.* (a)⇒(b). If (a) holds, then

$$\sigma_t^{\psi}(x)\,S\sigma_t^{\psi}(x^*) \leq S, \quad t\in\mathbb{R}\,.$$

Reducing to a τ-dense subspace of $\mathcal{D}(S^{1/2})\cap\mathcal{D}(|\,S^{1/2}\sigma_t^{\psi}(x^*)\,|)$ as in the proof of Lemma 2.3 one shows

$$|\langle\xi,\, S^{1/2}\,\sigma_t^{\psi}(x^*)\eta\rangle| \leq \|\xi\|\cdot\|S^{1/2}\eta\|,\ \xi,\,\eta\in\mathcal{D}(S^{1/2})\,. \tag{3.1}$$

Now for $\xi,\,\eta\in\mathcal{D}(S^{1/2})$ we define

$$f_{\xi,\eta}(z) = \langle\pi(x)S^{-ip^z}\xi,\, S^{1/2-ip^{\bar{z}}}\eta\rangle$$

$f_{\xi,\eta}$ is continuous and bounded in $\Gamma_{1/2p}$ and is analytic in $\Gamma^0_{1/2p}$.

(3.1) and the three lines principle show that this defines a norm bounded, ope-

rator valued function $z \mapsto \sigma_z(x)$, $z \in \Gamma_{1/2p}$, which satisfies

$$\langle \sigma_z(x)\xi, S^{1/2}\eta \rangle = f_{\xi,\eta}(z),\ \xi, \eta \in \mathcal{D}(S^{1/2}) .$$

$$||\sigma_{i/2p}(x)|| \leq 1 .$$

Since this construction is invariant under unitary operators in $\mathcal{N}'$ and under the dual action $\hat{\sigma}$ we conclude $\sigma_z(x) \in \pi(\mathcal{M})$. If ξ, η are arbitrary vectors in $\mathcal{H}$, then there exist sequences $\xi_n, \eta_n \in \mathcal{D}(S^{1/2})$ with $\xi = \lim_{n\to\infty} \xi_n$, $\eta = \lim_{n\to\infty} S^{1/2}\eta_n$. Then

$$\langle \sigma_z(x)\xi, \eta \rangle = \lim_{n\to\infty} f_{\xi_n,\eta_n}(z)$$

uniformly on $\Gamma_{1/2p}$. Hence $z \mapsto \sigma_z(x)$ is continuous on $\Gamma_{1/2p}$ and analytic in $\Gamma^0_{1/2p}$ for the weak operator topology. Using the iemma, in [19, Sec. 9.24] as well as the equivalence between $\sigma(\mathcal{M}, \mathcal{M}_*)$ and the weak operator topology on bounded sets we obtain (b).

(b) $\Rightarrow$ (a). First we claim

$$\pi(x)S^{1/2} = S^{1/2}(\sigma^{\psi}_{i/2p}(x)) \qquad \text{for } x \in \mathcal{A}^{\psi}\left(\frac{1}{2p}\right). \tag{3.2}$$

In order to obtain this we consider for ξ, $\eta \in \mathcal{D}(S^{1/2})$ the following two functions on $\Gamma_{1/2p}$:

$$z \mapsto \langle \pi(x)S^{-ipz}\xi, S^{1/2-ip\bar{z}}\eta \rangle$$

$$z \mapsto \langle \pi(\sigma^{\psi}_z(x))\xi, S^{1/2}\eta \rangle .$$

These functions are continuous $\Gamma_{1/2p}$ and analytic in $\Gamma^0_{1/2p}$. So they are equal, since they are equal on the real line by (1.3.1). This yields (3.2). (a) follows now from (b) and (3.2). □

Proof of 1.6 *Theorem.* If (a) holds, then by Lemma 2.2 (c) we know that the operator

$$(T + \mathrm{supp}(T)^{\perp})^{1/2}\ \mathrm{supp}(T)S^{-1/2}$$

has domain of definition $\mathcal{D}(S^{-1/2})$ and norm less than 1. Hence by (i) $\Rightarrow$ (ii) of the proposition in [19, Sec. 9.24] and 1.3.2 we obtain that the map

$$f(z) = (T^{iz}S^{-iz})^{-} = ((T + \mathrm{supp}(T)^{\perp})^{iz}\ \mathrm{supp}(T)S^{-iz})^{-}$$

is a well defined $\pi(\mathcal{M})$-valued extension of

$$t \mapsto \pi((D\omega : D\psi)_t)$$

which is continuous with respect to the strong operator topology on $\Gamma_{-1/2p}$ and analytic in $\Gamma^0_{-1/2p}$. The three lines principle then shows that f is uniformly bounded by 1. So it is even $s(\mathcal{M}, \mathcal{M}_*)$-continuous and we obtain (b).

If (b) holds, then by 1.3.2 and (iii)$\Rightarrow$(ii) of the proposition in [19, Sec. 9.24] we know that $\mathcal{D}(T^{1/2}S^{-1/2})=\mathcal{D}(S^{-1/2})$. The uniqueness of analytic extensions shows that

$$\pi((D\omega\colon D\psi)_{-i/2p}) = (T^{1/2}S^{-1/2})^- .$$

Hence we can apply Lemma 2.2 (c) again to conclude (a). The remainder of Theorem 1.6 is clear by the proof of Lemma 2.2 (c).

□

3.3. Remark. By the way we have obtained the following:
Suppose $\mathcal{M}$ is a σ-finite W^*-algebra, $x\in\mathcal{M}^+$ and $\psi\in\mathcal{M}_*^+$ is faithful. Then for $s\geq 0$ there exists a unique $y\in\mathcal{A}^\psi(s)$ such that

$$y\geq 0$$

and
$$x = \sigma^\psi_{is}(y)\sigma^\psi_{is}(y)^* = \sigma^\psi_{is}(y)\sigma^\psi_{-is}(y) .$$

3.4. Remark. The facial principle has further applications:
For σ-finite W^*-algebras it allows to define completely positive, dense embeddings of $\mathcal{M}$ into $L^p(\mathcal{M})$ and of $L^q(\mathcal{M})$ into $L^1(\mathcal{M})$. As in [17, Thm. 3.1], [18] this allows to transport semi-discreteness and injectivity from $\mathcal{M}$ to $L^p(\mathcal{M})$ and vice versa.

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