Steady Solutions of the Boltzmann Equation for a Gas Flow past an Obstacle, II. Stability

Dedicated to Professor Mizohata on his 60th anniversary

By

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§1. Introduction

This is a continuation of our previous work [9] on the exterior problem for the Boltzmann equation describing a gas flow past an obstacle. In [9], we established the existence of stationary solutions, and the subject of this paper is their stability.

The paper [9] will be referred to as Part I and any formulae number of Part I will be preceded by the symbol I, when quoted in this paper. Thus, (I.1.1a) means Equation (1.1a) of Part I, and so on.

Our problem is the initial boundary value problem (I.1.1):

$$
\begin{aligned}
f_t &= -\xi \cdot \nabla_x f + Q[f, f], & (t, x, \xi) &\in \mathbb{R}_+ \times \Omega \times \mathbb{R}^n, \\
r^- f &= M r^+ f, & (t, x, \xi) &\in \mathbb{R}_+ \times S^-, \\
f &\to g_c(\xi) \ (|x| \to \infty), & (t, \xi) &\in \mathbb{R}_+ \times \mathbb{R}^n, \\
f|_{t=0} &= f^0, & (x, \xi) &\in \Omega \times \mathbb{R}^n.
\end{aligned}
\tag{1.1}
$$

For the notations, we refer the readers to Part I. The stationary problem induced from (1.1) is (I.1.10). Let $u = u(c)$ be the solution obtained in Theorem I.9.3 to (I.1.12). Then,

$$
f_c = g_0 + g_c^{1/2} u(c)
$$

is a (unique) solution to (I.1.10). The aim of the present paper is to show that f_c is asymptotically stable in t. In other words, we shall show that whenever the initial f^0 in (1.1) is close to f_c, a unique solution $f = f(t)$ exists to (1.1) in the large in t and approaches f_c as $t \to \infty$.

In order to prove this, we seek a solution f of the form

$$f = f(t) = f_c + g_c^{1/2} w(t) = g_0 + g_c^{1/2}(u(c) + w(t)) \, .$$

Substract (I.1.10) from (1.1) and recall the definition (I.1.11) of L_c, Γ_0 and $\tilde{M}_0$. Then, $w = w(t)$ should solve,

(1.2)
$$\begin{aligned}
w_t &= -\xi \cdot \nabla_x w + L_c w + 2\Gamma_0[u(c), w] + \Gamma_0[w, w] \, , \\
\tilde{M}_0 w &= 0 \, , \\
w &\to 0 \; (|x| \to \infty) \, , \\
w|_{t=0} &= w_0 \, .
\end{aligned}$$

Obviously, the desired stability of f_c follows if for each small w_0, (1.2) has a unique solution in the large in t tending to 0 as $t \to \infty$.

This has been shown in [1] for the special case $c=0$ in which $u(0)=0$ (see Theorem I.9.3 (i)). The method employed there is based on the contraction mapping principle supplemented by nice decay estimates of the semigroup for the linearized equation of (1.2). This method is the only possible method, at least up to the present, which makes accessible the nonlinear Boltzmann equation in the large in t, see [2, 4–8].

To apply it to (1.2) for the case $c \neq 0$, recall the operator B_c of (I.1.13). It is the linear operator

(1.3) $B_c = \xi \cdot \nabla_x + L_c \, , \qquad (x, \xi) \in \Omega \times \mathbb{R}^n \, ,$

associated with the boundary condition $\tilde{M}_0 w = 0$ on S^- and $w \to 0$ $(|x| \to \infty)$. It was shown in § I.7 that B_c generates a semigroup in $L^2(\Omega \times \mathbb{R}^n)$. Denote the semigroup by $E_c(t)$;

$$E_c(t) = \exp(tB_c) \, .$$

Then, (1.2) is reduced to the integral equation,

(1.4) $w(t) = E_c(t) w_0 + \displaystyle\int_0^t E_c(t-s) \{2\Gamma_0[u(c), w(s)] + \Gamma_0[w(s), w(s)]\} ds \, .$

In § 5, we will solve (1.4) in the large in t by the help of the contraction mapping principle. Then, a decay estimate of $E_c(t)$ is needed, and will be discussed in § 4 by a perturbation argument based on the explicit expression of the resolvent of B_c found in § I.7. The unperturbed operator is B_c^∞ of § I.6, i.e., (1.3) defined for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. We shall discuss the semigroup $E_c^\infty(t)$ generated by B_c^∞ in § 3. § 2 is a preliminary and § 6 gives the proof of Proposition 4.5 which is a key to § 4. As announced in Part I, the proof of

Proposition I.7.4 which plays an essential role in § 6 as well as § I.7 will be given in § 7.

Observe that the linear operator associated to (1.2) is not B_c but $B_c + 2\Gamma_0$ $[u(c), \cdot]$, and that if the semigroup for the latter is used, then the linear term on the right hand side of (1.4) is absent. However, our perturbation technique no longer works to get any decay estimates for the latter semigroup, so, only $E_c(t)$ is available. On the other hand, the extra linear term on the right hand side of (1.4) can be handled only by the decay estimate of $E_c(t)$ sharper than required when the linear term is absent; we need the estimate $E_c(t) = 0$ $(t^{-\alpha})$, $\alpha > 1$ for (1.4) while $\alpha > 1/2$ suffices if the linear term is absent. Also, to handle the linear term, we are required that $u(c)$ is small, namely, c is small.

§2. Preliminaries

Given a Banach space X, we define the Banach space X_α, $\alpha \in \mathbb{R}$, of functions $f(t)$ of $t \geq 0$ with values in X by

$$(2.1) \qquad f(t) \in x_\alpha \Leftrightarrow (1+t)^\alpha f(t) \in L^\infty([0, \infty); X).$$

When X is any one of the spaces $L_\beta^{p,q}(Q)$, $Y_\beta^{p,q,\pm}$, etc. of Part I, we write $L_{\beta;\alpha}^{p,q}(Q)$, $Y_{\beta;\alpha}^{p,q,\pm}$, etc. for X_α.

Let Y be another Banach space and $\mathbb{B}(X, Y)$ denote the space of all bounded linear operators from X to Y. Then $\mathbb{B}_\alpha(X, Y)$ will be defined by (2.1) with $\mathbb{B}(X, Y)$ substituted for X. If $A(t) \in \mathbb{B}_\alpha(X, Y)$, it induces the multiplication operator A,

$$(Au)(t) = A(t)u, \qquad u \in X,$$

and the convolution operator $A*$,

$$(A*f)(t) = \int_0^t A(t-s)f(s)ds.$$

The following two lemmas will be useful in the sequel.

Lemma 2.1. Let $A(t) \in \mathbb{B}_\alpha(X, Y)$.

(i) $A \in \mathbb{B}(X, Y_\alpha)$.

(ii) Suppose $\alpha > 0$. For each $\gamma > 0$, put $\mu = \min(\alpha, \gamma, \alpha + \gamma - 1)$. If $\alpha \neq 1$ or $\gamma \neq 1$, then

$$A* \in \mathbb{B}(X_\gamma, Y_\mu).$$

If $\alpha > 1$ and $\gamma \in [0, \alpha]$, then $\mu = \gamma$.

   Proof.   (i) is obvious and (ii) is a consequence of

$$\int_0^t (1+t-s)^{-\alpha}(1+s)^{-\gamma}ds \leq C(1+t)^{-\mu}, \qquad t \geq 0,$$

which can be shown splitting the integral over $(0, t/2)$ and $(t/2, t)$.

   Lemma 2.2.   *Suppose $A(t) \in L^\infty([\varepsilon, \infty); B(X, Y))$ for all $\varepsilon > 0$ and satisfy*

$$\|A(t)\|_{B(X,Y)} \leq Ct^{-\delta}e^{-\nu t}, \qquad t > 0,$$

with some constants $C \geq 0$, $\delta < 1$ and $\nu > 0$. Then for all $\alpha \geq 0$,

$$A* \in B(X_\alpha, Y_\alpha).$$

   Proof.   This is immediate from

(2.2) $$\int_0^t (t-s)^{-\delta}e^{-\nu(t-s)}(1+s)^{-\alpha}ds \leq C\nu^{\delta-1}(1+\nu^{-\alpha})(1+t)^{-\alpha}, \qquad t \geq 0.$$

   For a function $f(t) \in X$ of $t \geq 0$, the Laplace transform $\hat{f} = \mathcal{L}_t f$ is defined by

$$\hat{f}(\lambda) = (\mathcal{L}_t f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt.$$

If $e^{-\sigma t}f(t) \in X_\alpha$ with some α, $\sigma \in \mathbf{R}$, then $\hat{f}(\lambda) \in X$ exists for all $\lambda \in \mathbf{C}_+(\sigma)$ and is analytic in λ. If $f_1(t), f_2(t)$ are two such functions, then

(2.3) $$(\mathcal{L}_t f_1 * f_2)(\lambda) = \hat{f}_1(\lambda)\hat{f}_2(\lambda)$$

holds for all $\lambda \in \mathbf{C}_+(\sigma)$.

   For a function $g(\lambda) \in X$ of $\lambda \in \mathbf{C}_+(\sigma)$, the inverse Laplace transform $\check{g} = \mathcal{L}_t^{-1}g$ is defined by

(2.4) $$\check{g}(t) = s-\lim_{a \to \infty} \frac{1}{2\pi i} \int_{\eta+ia}^{\eta+ia} e^{\lambda t}g(\lambda)d\lambda, \qquad \eta > \sigma,$$

the topology of the convergence being that of X. The following simple sufficient condition for $g(\lambda)$ to be a Laplace transform will be used later.

   Lemma 2.3.   *Let $g(\lambda) \in X$ be defined and analytic in $\lambda \in \bar{C}_+(\sigma)$. Suppose*

$$\sup_{\tau \in \mathbf{R}} \|g(\eta+i\tau)\|_X \to 0 \ (\eta \to \infty),$$

$$\sup_{\eta \geq \sigma} \|g(\eta+i\tau)\|_X \leq C(1+|\tau|)^{-\delta}, \qquad \tau \in \mathbf{R}$$

with some constants $C \geq 0$ *and* $\delta > 1$. *Then there exists a unique* $\check{g}(t)$ *such that* $e^{-\sigma t}\check{g}(t) \in \mathscr{B}^0([0, \infty); X)$, *and for all* $\lambda \in \mathbb{C}_+(\sigma)$, *there holds*

$$g(\lambda) = (\mathscr{L}_t \check{g})(\lambda).$$

Obviously, the above result remains valid if X is replaced by $\mathcal{B}(X, Y)$.

§3. Asymptotic Behaviors of $E_c^\infty(t)$

Recall the Definition (I.6.1) of the operator B_c^∞. As shown there, it generates a strongly continuous semigroup in the space $L^2(Q^\infty)$, $Q^\infty = \mathbb{R}_x^n \times \mathbb{R}_\xi^n$. Denote this semigroup as $E_c^\infty(t)$. According to the convention stated in §I.2, $E_c^\infty(t)$ will also express its realizations in other spaces.

Let $\tilde{L}_\beta^{p,q}(Q^\infty)$, etc., be the spaces defined in §I.2. In this section we write

$$\tilde{L}_\beta^{p,q} = \tilde{L}_\beta^{p,q}(Q^\infty), \quad \tilde{L}_{\beta;\alpha}^{p,q} = \tilde{L}_{\beta;\alpha}^{p,q}(Q^\infty),$$
$$Z^q = \tilde{L}_0^{2,2} \cap \tilde{L}_0^{q,2}, \quad Z_\alpha^q = \tilde{L}_{0;\alpha}^{2,2} \cap \tilde{L}_{0;\alpha}^{q,2}.$$

The constants $C \geq 0$ and operator norms appearing in the followings are all locally bounded in $c \in \mathbb{R}^n$ and possibly independent of t.

Let P_c be the projection of Lemma I.3.8 and let $\Lambda_c^\delta = \nu_c(\xi)^\delta \times$. The main result of this section is as follows.

Theorem 3.1. *Let* $1 \leq q \leq 2 \leq p \leq \infty$, $\beta \in \mathbb{R}$, $m = 0, 1$ *and put* $\alpha = \dfrac{n}{2}\left(\dfrac{1}{q} - \dfrac{1}{p}\right) + \dfrac{m}{2}$.

(i)   *There is a constant* $C \geq 0$ *such that*

$$\|E_c^\infty * (I - P_c)^m u\|_{\tilde{L}_{\beta;\alpha}^{p,\infty}} \leq C(\|u\|_{\tilde{L}_\beta^{p,\infty}} + \|u\|_{Z^q}).$$

(ii)   *Let* $\gamma \geq 0$ *and* $\delta \leq 1$, *and put* $\mu = \min(\alpha, \gamma, \alpha + \gamma - 1)$. *Then there is a constant* $C \geq 0$ *such that if* $\gamma \neq 1$,

$$\|E_c^\infty * (I - P_c)^m \Lambda_c^\delta f\|_{\tilde{L}_{\beta;\mu}^{p,\infty}} \leq C(\|f\|_{\tilde{L}_{\beta;\gamma}^{p,\infty}} + \|\Lambda_c^\delta f\|_{Z_\gamma^q}).$$

This theorem for the case $c = 0$ has been established essentially in [4] and [6]. Since the case $c \neq 0$ can be dealt with in the same way, only the outline of the proof will be presented.

Let us begin by recalling from the theory of semigroups [3] that a semigroup e^{tA} is an inverse Laplace transform of the resolvent of the generator A, or more precisely,

(3.1) $$e^{tA}u = s - \lim_{a\to\infty} \frac{1}{2\pi i} \int_{\sigma-ia}^{\sigma+ia} e^{\lambda t}(\lambda-A)^{-1} u\, d\lambda$$

holds for $u\in D(A)$ and $t\geq 0$ if $\sigma\in\mathbb{R}$ is large enough.

Write (3.1) for B_c^∞ in $L^2(Q^\infty)$ and substitute in it the decomposition (I.6.11) of $(\lambda-B_c^\infty)^{-1}$ to deduce

(3.2) $$E_c^\infty(t) = \sum_{j=0}^{\bar{m}} \check{U}_j(t, c),$$

where $\check{U}_j = \mathcal{L}_t^{-1} U_j$ are given as follows. Let $\hat{B}_c(k)$ be that of (I.6.4), which also generates a semigroup in $L^2(\mathbb{R}_\xi^n)$, denoted as $\check{E}_c(t, k)$. Write (3.1) at this time for $\hat{B}_c(k)$. Then by (I.6.11),

(3.3) $$\check{U}_0(t, c) = \mathcal{F}_x^{-1} \check{E}_c(t, k)\{\chi(|k|>\kappa_0) + \chi(|k|<\kappa_0)Q(k, c)\}\mathcal{F}_x,$$

where $Q(k, c) = I - \Sigma P_j(k, c)$, while since $\mathcal{L}_t^{-1}(\lambda-\mu)^{-1} = e^{\mu t}$,

(3.4) $$\check{U}_j(t, c) = \mathcal{F}_x^{-1} e^{\mu_j(k, c)t}\chi(|k|<\kappa_0)P_j(k, c)\mathcal{F}_x, \qquad 1\leq j\leq m.$$

Proposition 3.2. *Let $\sigma_0>0$ be the constant of (I.6.7). There exists a constant $C\geq 0$ and*

$$\|\check{U}_0(t, c)\|_{\mathcal{B}(L^2(Q^\infty))} \leq Ce^{-\sigma_0 t}, \qquad t\geq 0.$$

Proof. Let $\hat{A}_c(k)$ be defined by (I.6.2). It generates a semigroup in $L^2(\mathbb{R}_\xi^n)$, denoted as $\check{F}_c(t, k)$. By (3.1) for $\hat{B}_c(k)$ and $\hat{A}_c(k)$, and by the second resolvent equation of the form (I.6.14) for them,

(3.5) $$\check{E}_c(t, k) = \sum_{h=0}^{l-1} H_h(t, k, c) + D_l(t, k, c)$$

$$H_h = \{(\check{F}_c(\circ, k)K_c*)^h \check{F}_c(\circ, k)\}(t),$$
$$D_l = \mathcal{L}_t^{-1}\hat{D}_l, \quad \hat{D}_l(\lambda, k, c) = \hat{G}^l(\lambda-\hat{B}_c(k))^{-1},$$
$$\hat{G} = (\lambda-\hat{A}_c(k))^{-1}K_c.$$

Write $B = B(L^2(\mathbb{R}_\xi^n))$. Obviously $\check{F}_c(t, k) = \{\exp(-i\xi\cdot k - \nu_c(\xi))t\}\times$, so in view of (I.3.4),

$$\|H_h(t, k, c)\|_{\mathcal{B}} \leq Ct^h e^{-\nu_0 t},$$

uniformly for $k\in\mathbb{R}^n$. Next, combine Lemma I.6.12 (i) with (ii) restricted for $|k|>\kappa$, and choose $\kappa = (1+|\tau|)^{\gamma/(\gamma+1)}$ to obtain

(3.6) $$\|\hat{G}(\sigma+i\tau, k, c)\|_{\mathcal{B}} \leq C(1+|\tau|)^{-\gamma^2/(\gamma+1)},$$

uniformly for $k\in\mathbb{R}^n$ and for $\sigma\geq -\nu_0+\delta$, $\delta>0$ being fixed. Then, by Proposition I.6.3 and (I.6.19) (i), $\hat{D}_l(\lambda, k, c)$ with the choice $l>\gamma^{-2}(\gamma+1)$ satisfies

the condition of Lemma 2.3 of $g(\lambda)$ uniformly for $|k| > \kappa_0$ while so does $\hat{D}_l(\lambda, k, c)Q(k, c)$ uniformly for $|k| < \kappa_0$, both with $\sigma = -\sigma_0$. Hence

$$
\left.\begin{array}{c}
\sup_{|k|>\kappa_0} \|D_l(t, k, c)\|_B , \\[2mm]
\sup_{|k|<\kappa_0} \|D_l(t, k, c)Q(k, c)\|_B
\end{array}\right\} \leq Ce^{-\sigma_0 t}.
$$

Substitution of these estimates into (3.5) and then into (3.3) yields the proposition by the aid of Parseval.

Proposition 3.3. *Under the same situation of Theorem 3.1*

$$
\check{U}_j(t, c)(I-P_c)^m \in \mathscr{B}_\alpha(\tilde{L}_0^{q,2}, \tilde{L}_\beta^{p,\infty}) , \qquad 1 \leq j \leq \bar{m} \leq n+2 .
$$

Proof. Using (3.4) and proceeding exactly in the same way as in (I.6.12), we get

$$
\|\check{U}_j(t, c)(I-P_c)^m u\|_{\tilde{L}_\beta^{p,\infty}} \leq CI(t)\|u\|_{\tilde{L}_0^{q,2}},
$$

with

$$
I(t) = \left(\int_{|k|<\kappa_0} |e^{\mu_j(k,c)t}|k|^m|^r dk\right)^{1/r}, \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{p} .
$$

Since $\mathrm{Re}\,\mu_j(k, c) = \mathrm{Re}\,\lambda_j(|k|)$ by Theorem I.6.2 (i), and by (I.6.6), $I(t) \leq C(1+t)^{-\alpha}$ for all $t \geq 0$, which proves the proposition.

To establish Theorem 3.1, we still need the following lemma on the semi-group $F_c^\infty(t)$ defined by (I.5.5). Put for $h \in \bar{N}_+$,

$$
\text{(3.7)} \qquad
\begin{aligned}
G_h* &= (F_c^\infty K_c*)^h , \\
H_h(t) &= (G_h*F_c^\infty)(t).
\end{aligned}
$$

Lemma 3.4. (i) $\forall p \in [1,\infty], \forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \forall h \in \bar{N}_+,$

$$
G_h* \in \mathscr{B}(\tilde{L}_{\beta;\alpha}^{p,\infty}) , \qquad H_h \in \mathscr{B}(\tilde{L}_\beta^{p,\infty}, \tilde{L}_{\beta,\alpha}^{p,\infty}) .
$$

(ii) $\forall p \in [2, \infty], \forall \beta \in \mathbb{R}, \exists l \in \bar{N}_+, \forall \alpha \in \mathbb{R},$

$$
G_l* \in \mathscr{B}(\tilde{L}_{0;\alpha}^{2;2}, \tilde{L}_{\beta,\alpha}^{p,\infty}) .
$$

(iii) $\forall p \in [1, \infty], \forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \forall \delta \leq 1,$

$$
F_c^\infty \Lambda_c^\delta* \in \mathscr{B}(\tilde{L}_{\beta,\alpha}^{p,\infty}) .
$$

Proof. Lemma I.5.1 implies that $F_c^\infty(t)$ satisfies the condition for $A(t)$ of Lemma 2.2 with $\delta = 0$ and $\nu = \nu_0$, $X = \tilde{L}_\beta^{p,\infty}$. Then (i) follows readily by

the aid of Proposition I.3.5. Similarly (ii) follows by a repeated use of Lemma I.5.1 (ii) (cf. the proof of Lemma I.5.4). And (iii) is a direct consequence of (2.2) for $\delta=0$ and $\nu=\nu_c(\xi)$. Note that the constant $C\geq0$ of (2.2) is independent of ν.

Proof of Theorem 3.1. Similarly to (3.5), we get

$$E_c^\infty(t) = \sum_{h=0}^{l-1} H_h(t)+(G_l*E_c^\infty)(t),$$

where G_l and H_h are of course those of (3.7). In view of Lemmas 3.4 (i) and (iii), it suffices to show that $G_l*E_c^\infty(t)$ enjoys Theorem 3.1. Choose $l\in N_+$ of Lemma 3.4 (ii). Then Proposition 3.2 and Lemma 2.1 show that $G_l*\overset{\vee}{U}_0(\cdot, c)\in \boldsymbol{B}(\tilde{L}_0^{2,2}, \tilde{L}_{\beta,\alpha}^{p,\infty})$ for all $\alpha\geq0$, while due to Proposition 3.3 and Lemma 3.4 (i), $G_l*\overset{\vee}{U}_j(\cdot, c)$, $1\leq j\leq l$, $\in \boldsymbol{B}(\tilde{L}_0^{q,2}, \tilde{L}_{\beta,\alpha}^{p,\infty})$ with the α specified in Theorem 3.1. Now the proof is completed by (3.2).

Let $r^\pm$ be the trace operators given by Theorem I.4.1. According to the theory of semigroups [3], $E_c^\infty(t)u\in D(B_c^\infty)$ whenever $u\in D(B_c^\infty)$. In virtue of Proposition I.4.4 and (I.5.1), therefore, $r^\pm E_c^\infty(t)\in \boldsymbol{B}(W_{\nu_c}^{2,-}(Q^\infty), Y^{2,\pm})$. We shall use this operator in L^∞-space. Note that

(3.8)
$$||r^\pm v||_{\tilde{Y}_{\beta,\alpha}^{p,\infty,\pm}}\leq||v||_{L_{\beta+1/p,\alpha}^\infty}.$$

Combine this and Theorem 3.1 and put $p=\infty$ in both. Then we obtain

Proposition 3.5. *Let* $q\in[1.2]$ *and* $\beta\in\boldsymbol{R}$.

(i) *Put* $\alpha=\dfrac{n}{2q}$. *There is a constant* $C\geq0$ *and*

$$||r^\pm E_c^\infty u||_{\tilde{Y}_{\beta,\alpha}^{\infty,\pm}}\leq C(||u||_{L_\beta^\infty}+||u||_{Z^q}).$$

(ii) *Let* $m=0, 1$, $\alpha\geq0$, $\delta\leq1$. *Put* $\alpha=\dfrac{1}{2}\left(\dfrac{n}{q}+m\right)$ *and* $\mu=\min(\alpha, r, \alpha+r-1)$. *There is a constant* $C\geq0$ *and if* $r\neq1$,

$$||r^\pm E_c^\infty*(I-P_0)^m \Lambda_c^\delta f||_{Y_{\beta,\mu}^{\infty,\pm}}\leq C(||f||_{L_{\beta,\gamma}^\infty}+||\Lambda_c^\delta f||_{Z_\gamma^q}).$$

§4. Asymptotic Behaviors of $E_c(t)$

$E_c(t)$ is the semigroup on the space $L^2(Q)$, $Q=\Omega_x\times R_\xi^n$, generated by the operator B_c defined by (1.3) or more precisely by (I.7.10), and will also denote its realizations in other spaces.

In this and following sections we use the spaces

$$\tilde{L}_\beta^{p,q} = \tilde{L}_\beta^{p,q}(Q), \qquad\qquad \tilde{L}_{\beta,\alpha}^{p,q} = \tilde{L}_{\beta,\alpha}^{p,q}(Q),$$

$$X_\beta^p = \tilde{L}_\beta^{\infty,\infty} \cap \tilde{L}_{\beta-1/p}^{p,\infty}, \qquad X_{\beta,\alpha}^p = \tilde{L}_{\beta,\alpha}^{\infty,\infty} \cap \tilde{L}_{\beta-1/p,\alpha}^{p,\infty},$$

$$Z^q = \tilde{L}_0^{2,2} \cap \tilde{L}_0^{q,2}, \qquad\qquad Z_\alpha^q = \tilde{L}_{0,\alpha}^{2,2} \cap \tilde{L}_{0,\alpha}^{q,2}.$$

Let $\bar{c}_0 > 0$ be the constant of Proposition I.7.5 and write

$$B[\bar{c}_0] = \{c \in \mathbb{R}^n \mid |c| \le \bar{c}_0\}.$$

In contrast to $E_c^\infty(t)$, asymptotic behaviors of $E_c(t)$ can be established only for $c \in B[\bar{c}_0]$.

Theorem 4.1. *Let* $n \ge 3$, $1 \le q \le 2 \le p \le \infty$, $\beta > (n+1)/2$ *and* $\theta \in [0, 1)$. *Put*

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{q} - \frac{1}{p}\right), \quad \alpha_2 = \begin{cases} \dfrac{1}{2}(n-1+\theta), & (n, \text{ odd}), \\[2mm] \dfrac{1}{2}(n-1), & (n, \text{ even}). \end{cases}$$

(i) *Put* $\alpha = \min(\alpha_1, \alpha_2)$. *There is a constant* $C \ge 0$ *and for all* $c \in B[\bar{c}_0]$,

$$\|E_c u\|_{X_{\beta,\alpha}^p} \le C \, |c|^{-\theta} (\|u\|_{X_\beta^p} + \|u\|_{Z^q}).$$

(ii) *Put* $\alpha = \min\left(\alpha_1 + \dfrac{1}{2}, \alpha_2, \dfrac{1}{2}\left(1 - \dfrac{1}{p}\right)\right)$. *Let* $\gamma \ge 0$ *and put* $\mu = \min(\alpha, \gamma, \alpha + \gamma - 1)$. *There is a constant* $C \ge 0$, *and for all,* $c \in B[\bar{c}_0]$ *and if* $\gamma \ne 1$,

$$\|E_c*(I-P_0)A_c f\|_{X_{\beta,\mu}^p} \le C \, |c|^{-\theta} (\|f\|_{X_{\beta,\gamma}^p} + \|A_c f\|_{Z_\gamma^q}).$$

The proof is based on the expression of the resolvent of B_c obtained in Theorem I.7.7, namely,

(4.1) $$(\lambda - B_c)^{-1} = r(\lambda - B_c^\infty)^{-1}e + S_c(\lambda)D_c(\lambda)M_0(\lambda - B_c^\infty)^{-1}e,$$

where

$$S_c(\lambda) = R_c(\lambda) + r(\lambda - B_c^\infty)^{-1}eK_cR_c(\lambda),$$
$$D_c(\lambda) = (I - T_c(\lambda))^{-1},$$
$$T_c(\lambda) = M_0(\lambda - B_c^\infty)^{-1}eK_cR_c(\lambda).$$

Suppose $\check{S}_c = \mathcal{L}_t^{-1}S_c$ and $\check{D}_c = \mathcal{L}_t^{-1}D_c$ exist. Then (4.1) implies, with the aid of (2.3), that

(4.2) $$E_c(t) = rE_c^\infty(t)e + (\check{S}_c * \check{D}_c * M_0 E_c^\infty e)(t).$$

First of all, we shall establish the existence of $\check{R}_c = \mathscr{L}_t^{-1} R_c$.

Proposition 4.2. $\check{R}_c*$ *exists for all* $c \in \boldsymbol{R}^n$ *and has the following properties. Let* $p, r \in [1, \infty]$, $r \geq p$ *and* $\alpha, \beta \in \boldsymbol{R}$.

(i) $\check{R}_c* \in \boldsymbol{B}(Y_{\beta,\alpha}^{\infty,-}, \tilde{L}_{\beta+\delta,\alpha}^{p,r})$ *if* $\delta > \dfrac{1}{p} + \dfrac{n}{r}$.

(ii) $K_c \check{R}_c* \in \boldsymbol{B}(Y_{\beta,\alpha}^{\infty,-}, \tilde{L}_{\beta+\delta,\alpha}^{p,r})$ *if* $\delta > \dfrac{1}{p} + \dfrac{1}{r} - 1$.

All the operator norms are locally bounded in c.

Proof. Let $h(t, x, \xi) \in \tilde{Y}_{\beta,\alpha}^{p,r,-}$ and define $v = v(t, x, \xi)$ by

(4.3) $v = e^{-\nu_c(\xi) t^-(x,\xi)} h(t - t^-(x, \xi),\, x - t^-(x, \xi)\xi,\, \xi)$

for $(x, \xi) \in Q^-$ and $t > t^-(x, \xi)$, and $v = 0$ otherwise, where Q^- was defined in the course of the proof of Theorem I.4.1 and $t^-(X, \xi)$ in that of Proposition I.7.1. Put $w = |v|^p$ in $(I.4.2)_-$ and use (I.7.3) to obtain

$$\||v(t, \cdot, \xi)\|_{L^p(\Omega_x)}^p = \int_0^t \int_{\partial\Omega^-(\xi)} |e^{-\nu_c(\xi)(t-s)} h(s, X, \xi)|^p \times \rho(X, \xi) d\sigma_x ds\,,$$

which, together with (I.3.4) and (2.2), gives

(4.4) $\|e^{-\sigma t} v\|_{\tilde{L}_{\beta,\alpha}^{p,r}} \leq C \|h\|_{\tilde{Y}_{\beta,\alpha}^{p,r,-}}\,,$

for all $\sigma > -\nu_0$ and $\alpha \in \boldsymbol{R}$. Hence $\hat{v}(\lambda) = (\mathscr{L}_t v)(\lambda)$ exists for $\lambda \in \boldsymbol{C}_+(-\nu_0)$. Compare (I.7.4) with (4.3) to see that $\hat{v}(\lambda) = R_c(\lambda)\hat{h}(\lambda)$, and recall (2.3). Thus (4.3) gives an explicit expression of $\check{R}_c*$ and (4.4) implies

$$\check{R}_c* \in \boldsymbol{B}(\tilde{Y}_{\beta,\alpha}^{p,r,-}, \tilde{L}_{\beta,\alpha}^{p,r})\,.$$

Now (i) follows since $Y_{\beta,\alpha}^{\infty,-} \subset \tilde{Y}_{\beta-\delta,\alpha}^{p,r}$ is a continuous embedding, and then (ii) by Proposition I.3.5.

Remark 4.3. (4.3) indicates that $\check{R}_c(t)$ itself exists only in the distribution sense in t.

By virtue of the above proposition, $\check{S}_c*$ exists and is given as

(4.5) $\check{S}_c* = \check{R}_c* + r E_c^\infty e * K_c \check{R}_c*\,.$

Lemma 4.4. *Let* $p \in [2, \infty]$, $\beta > n/2$, $r \geq 0$ *and put*

$$\alpha_3 = \frac{n}{2}\left(1 - \frac{1}{p}\right)\,, \quad \mu = \min(\alpha_3,\, r,\, \alpha_3 + r - 1)\,.$$

Then for all $c \in \boldsymbol{R}^n$,

$$\overset{\lor}{S_c}* \in \mathbb{B}(T^{\infty,-}_{\beta-1/p,\gamma}, \tilde{L}^{p,\infty}_{\beta,\mu}) \,,$$

the norm being locally bounded in c.

Proof. In Theorem 3.1 (ii), put $q=1$, $m=\delta=0$ and substitute $eK_c R_c*f$ for f. Use Proposition 4.2 (ii) thrice with $r=\infty$, with $p=r=2$, $\beta=0$ and with $p=1$, $r=2$, $\beta=0$. Then

$$rE^{\infty}_c e * K_c \overset{\lor}{R_c} * \in \mathbb{B}(Y^{\infty,-}_{\beta,\gamma}, \tilde{L}^{p,\infty}_{\beta,\mu}) \,.$$

Combine this and Proposition 4.2 (i) in (4.5) to conclude the lemma.

As for $\overset{\lor}{D_c}(t)$, we have the

Proposition 4.5. $\overset{\lor}{D_c}*$ *exists for all* $c \in B[\bar{c}_0]$ *and the following holds. Let* $n \geq 3$, $\beta > n/2$, $r \geq 0$ *and* $\theta \in [0, 1)$. *Let* $\alpha_2 > 0$ *be as in Theorem* 4.1 *and put* $\mu = \min(\alpha_2, r, \alpha_2 + r - 1)$. *There is a constant* $c \geq 0$ *and for all* $c \in B[\bar{c}_0]$ *and if* $r \neq 1$,

$$\|\overset{\lor}{D_c}*\|_{\mathbb{B}(Y^{\infty,-}_{\beta,\gamma}, Y^{\infty,-}_{\beta,\mu})} \leq C |c|^{-\theta}.$$

We relegate the lengthy proof of § 6 and proceed to the

Proof of Theorem 4.1. In view of the assumption $[M]_2$ (iii) and Remark 1.1 of § I.1, $\tilde{M}_0 E^{\infty}_c(t)$ also enjoys Proposition 3.5. Combine this with Lemma 4.4 and Proposition 4.5 and choose $r=\alpha_2$ in the last two. Add the results for $p=\infty$ and $p=p$, and note that $\alpha_2 \geq 1$ for $n \geq 3$. It then follows that $\overset{\lor}{S_c}*\overset{\lor}{D_c}*\tilde{M}_0 E^{\infty}_c(t)$ enjoys Theorem 4.1 (i) with $\alpha = \min\left(\alpha_2, \alpha_3, \dfrac{n}{2q}\right)$ and Theorem 4.1 (ii) with $\alpha = \min\left(\alpha_2, \alpha_3, \dfrac{1}{2}\left(\dfrac{n}{q}+1\right)\right)$. This and Theorem 3.1, substituted in (4.2), complete the proof.

Finally we shall discuss the continuity in t of $E_c(t)$.

Proposition 4.6. $E_c(t)$ *is a strongly continuous semigroup on* $L^p(Q)$, $2 \leq p < \infty$.

Proof. The case $p=2$ was already proven. Therefore, the proposition follows if

(4.6) $$\|E_c(t)\|_{\mathbb{B}(L^p(Q))} \leq C e^{\alpha t}$$

holds for all $p \in [2, \infty]$. The constant $C \geq 0$ and $\alpha \in \mathbb{R}$ may depend on p. Indeed, let $u \in L^{\infty} \cap L^2$ and put $v = (E_c(t+h) - E_c(t))u$, where $L^p = L^p(Q)$. Then by (4.6) for $p=\infty$, we get for each fixed t,

$$||v||_{L^p} \leq ||v||_{L^\infty}^{1-2/p} ||v||_{L^2}^{2/p} \leq C||v||_{L^2}^{2/p},$$

so that $v \to 0$ $(h \to 0)$ in L^p if $p \in [2, \infty)$. Again by (4.6) and since $L^\infty \cap L^2$ is dense in L^p, this results in the proposition. It remains to prove (4.6). Clearly this is the case for $p=2$, because $E_c(t)$ is strongly continuous, [3]. Let A_c be the operator of § I.7. According to Theorem 7.6 (i), it generates a contraction semigroup in L^2, denoted as $F_c(t)$, and according to Theorem I.7.6 (iii),

$$F_c(t) = rF_c^\infty(t)e + (\check{R}_c * \tilde{M}_0 F_c^\infty e)(t)$$

should hold. Evaluate this right side in $B(L^\infty)$ using Lemma I.5.1 (i), (3.8), (4.3) and $[M]_2$ (iii). Then $F_c(t)$ enjoys (4.6) for $p = \infty$ and hence so does $E_c(t)$ as seen from the equation $E_c = F_c + F_c * K_c E_c$ and by Gronwall's inequality. Now the interpolation yields (4.6) for $p \in (2, \infty)$.

As a property of the convolution operator, we have

Corollary 4.7. E_c* *maps* $L^\infty([0, \infty); L^p(Q))$ *into* $\mathcal{E}^0 ([0, \infty); L^p(Q))$, $2 \leq p < \infty$.

§5. Stability of Stationary Solutions

Define the nonlinear map H by writing the right side of (1.4) as $H[w](t)$, that is,

$$H[w] = E_c w_0 + 2E_c * \Gamma_0[u(c), w(\cdot)] + E_c * \Gamma_0[w(\cdot), w(\cdot)].$$

We shall find fixed points of H. The spaces used in this section are all the same of the previous section.

In Theorem I.9.3, we change the notations of the parameters p, θ and α to p_0, θ_0 and α_0 respectively, and fix them as well as β in the region

$$(5.1) \qquad \theta_0 \in [0, 2/7), \, p_0 \in [2.4] \cap \left(\left(1 - \frac{2}{n+\theta_0}\right)^{-1}, n\right),$$

$$\alpha_0 \in \left(\theta_0\left(1 + \frac{2}{p_0}\right), \, 1 - \theta_0\left(2 - \frac{1}{p_0}\right)\right), \, \beta > n+1 \,.$$

Here the restriction on p_0 is somewhat stringent compared with (I.9.1) for a technical reason which will be revealed below.

In what follows, $\bar{c} = \bar{c}(\theta_0, p_0, \alpha_0, \beta)$ will denote the constant $\bar{c}$ of Theorem I.9.3 with values of the parameters specified by (5.1). Then the stationary solution $u = u(c)$ exists for each $c \in B[\bar{c}]$ with

(5.2) $$\|u(c)\|_{X^p_{\beta^0}} \leq C\,|c|^{\alpha_0}.$$

The space in which the fixed point of H is sought is $X^p_{\beta,\gamma}$ and the space from which the initials w_0 should be taken is $X^p_\beta \cap Z^q$. For $n \geq 4$, we can choose $p = p_0$ but not for $n = 3$. More specifically, we should make the following choice.

(5.3) $$p \in [2,4] \cap \left(\left(1 - \frac{2}{n}\right)^{-1}, \left(\frac{1}{2} - \frac{1}{p_0}\right)^{-1} \right),$$

$$q \in [1,2] \cap \left[1, \left(\frac{1}{p} + \frac{1}{n}\right)^{-1}\right),$$

$$r \in \left(\frac{1}{2}, r_0\right], \quad r_0 = \min\left(\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right), \frac{1}{2}\left(\frac{n}{p_0}+1\right), \frac{1}{2}\left(\frac{n}{p}+1\right)\right).$$

$$\beta > n+1.$$

Under this choice of parameters, we write

$$\|\|\ \|\| = \|\ \|_{X^p_{\beta,\gamma}}, \quad \|\ \| = \|\ \|_{X^p_\beta} + \|\ \|_{Z^q}.$$

The complicated choice of parameters in the above is necessary in order for H to be a contraction, which follows from the

Lemma 5.1. *Let $n \geq 3$ and suppose (5.1) and (5.3) be fulfilled. Then for each $\alpha \in [0, \alpha_0)$, there are constants C_0, $C_1 \geq 0$ such that for all. $c \in B[\bar{c}]$,*

$$\|\|H[w]\|\| \leq C_0\|w_0\| + C_1(|c|^\alpha + \|\|w\|\|)\|\|w\|\|,$$

$$\|\|H[w] - H[w']\|\| \leq C_1(|c|^\alpha + \|'w\|\| + \|\|w'\|\|)\|\|w - w'\|\|.$$

Proof. By Theorem 4.1 (i) with $\theta = 0$,

(5.4) $$\|\|E_c w_0\|\| \leq C_0\|w_0\|$$

holds if $1 \leq q \leq 2 \leq p \leq \infty$, $0 \leq r \leq \min\left(\alpha_1, \frac{1}{2}(n-1)\right)$. Next, put $f(t) = \Lambda_c^{-1}\Gamma_0[u(c), w(t)]$. (I.9.3) readily gives

(5.5) $$\|f\|_{X^p_{\beta,\gamma}} + \|\Lambda_c f\|_{Z^q_\gamma} \leq C\|u(c)\|_{X^p_{\beta^0}}\|w\|_{X^p_{\beta,\gamma}},$$

if $\beta > n+1$ and $\dfrac{1}{p_0} + \dfrac{1}{p} = \dfrac{1}{q}$. With this p, q and f and with $\theta = \alpha_0 - \alpha$, Theorem 4.1 (ii) yields

(5.6) $$\|\|E_c * \Gamma_0[u(c), w]\|\| \leq C_1|c|^{\alpha - \alpha_0}\|u(c)\|_{X^p_{\beta^0}}\|\|w\|\|.$$

Indeed, $\alpha_1 + \dfrac{1}{2} = \dfrac{1}{2}\left(\dfrac{n}{p_0}+1\right) > 1$ since $p_0 < n$ due to (5.1), $\alpha_2 > 1$ since $\theta > 0$

and $\dfrac{n}{2}\left(1-\dfrac{1}{p}\right) > 1$ by (5.3), so that Theorem 4.1 (ii) applies with $\mu = \min(\alpha, r) = r$ for any r subject to (5.3). And it is at this point that p_0 should satisfy the condition $p_0 < n$ besides (I.9.1). Thus (5.1) is required. Finally, if $f(t) = A_c^{-1} \Gamma_0[v(t), w(t)]$, then (I.9.3) and (I.9.4) give

(5.7) $\|f\|_{X_{\beta,2\gamma}^p} + \|A_c f\|_{Z_{2\gamma}^q} \le C \|\|v\|\| \, \|\|w\|\|$.

Use again Theorem 4.1 (ii) but at this time with $q=p/2$, $\theta=0$ and with r replaced by $2r$. Since $p \in [2, 4]$, then $q \in [1, 2]$, and by (5.3),

$$\alpha_2 = \frac{1}{2}(n-1) > \alpha_1 = \frac{1}{2}\left(\frac{n}{p}+1\right) \ge r_0 \ge r > \frac{1}{2} \, ,$$

so that $\mu = \min(\alpha, 2r) \ge r$. In view of Lemma I.3.9 (i), therefore,

(5.8) $\|\|E_c * \Gamma_0[v, w]\|\| \le C_1 \|\|v\|\| \, \|\|w\|\|$.

Combine (5.2), (5.4), (5.6) and (5.8), and recall that Γ_0 is bilinear symmetric. This finishes the proof of the lemma.

Now we can prove the desired stability of $u(c)$ as a simple corollary to the

Theorem 5.2. *Let $n \ge 3$. Under the assumptions $[\mathcal{O}]$, $[q]$ and $[M]$ of §I.1, and for each p, q, β, r satisfying (5.3) and for $\alpha \in (0, \alpha_0)$, there exist positive constants a_0 and $\bar{\bar{c}}$ such that for any initial, $w_0 \in X_\beta^p \cap Z^q$ with $\|w_c\| \le a_0$, (1.4) possesses a unique solution*

$$w = w(t) \in X_{\beta,\gamma}^p \cap \mathcal{E}^0([0, \infty); L^p(Q))$$

in the large in time, if $c \in B[\bar{c}]$.

Proof. Let C_0, C_1 be the constants of Lemma 5.1 and choose constants a_0 and $\bar{\bar{c}}$ such that

$$0 < \bar{\bar{c}} \le \bar{c} \, , \quad \bar{\bar{c}} < C_1^{-1/\alpha} \, ,$$
$$0 < a_0 (1 - C_1 \bar{\bar{c}}^\alpha)^2/(4C_0 C_1) \, ,$$

and put

$$\mu = 1 \{(1 - C_1 \bar{\bar{c}}^\alpha)^2 - 4C_0 C_1 a_0\}^{1/2} \, , \quad a_1 = (\mu_1 - C_1 \bar{\bar{c}}^\alpha)/(2C_1) \, .$$

Let V be a ball

$$V = \{w(t) \in X^p_{\beta,\gamma} | \; |||w||| \leq a_1\} \, .$$

Endowed with the metric $d(v, w) = |||v-w|||$, V is a complete metric space. Suppose $||w_0|| \leq a_0$ and $v, w \in V$. Then by the previous lemma,

$$|||H[w]||| \leq C_0 a_0 + (C_1(\bar{c}^\alpha + a_1) a_1 = a_1 \, ,$$
$$|||H[v]-H[w]||| \leq C_1(\bar{c}^\alpha + 2a_1))|||v-w||| = \mu|||v-w||| \, .$$

Since $\mu \in (0, 1)$, this proves that $H: V \to V$ is a contraction. Hence a unique fixed point $w = H[w]$ exists in V and gives a solution to (1.4) in the large in time. It is easy to check that this w is a unique fixed point also in $X^p_{\beta,\gamma}$. It remains to show that

$$w(t) \in \mathcal{E}^0([0, \infty); L^p(Q)) \, .$$

Note that $X^p_\beta \subset L^p(Q)$ is a continuous embedding if $\beta > n/p$. Hence $w_0 \in L^p(Q)$, and in view of (5.5) and (5.7),

$$\Gamma_0[u(c), w(t)] \, , \quad \Gamma_0[w(t), w(t)] \in L^\infty([0, \infty); L^p(Q)) \, .$$

Then $H[w](t) \in \mathcal{E}^0([0, \infty); L^p(Q))$ by Proposition 4.6 and its corollary. This completes the proof of the theorem.

Since $w \in V$, it follows that

$$|w(t)|_{X^p_\beta} \leq a_1(1+t)^{-\gamma}$$

whenever $||w_0|| \leq a_0$. Thus $w(t) \to 0$ $(t \to \infty)$ in X^p_β, which shows that the stationary solution $u = u(c)$ is asymptotically stable as $t \to \infty$.

§6. Proof of Proposition 4.5

Write $D_{c,l}(\lambda) = (I - T_c(\lambda))^{-1} T_c(\lambda)^l$, $l \in \mathbb{N}_+$, and note the expansion

$$D_c(\lambda) = \sum_{h=0}^{l-1} T_c(\lambda)^h + D_{c,l}(\lambda) \, ,$$

(see (4.1)). Referring to (2.3), we then get

(6.1) $$\check{D}_c* = \sum_{h=0}^{l-1} (\check{T}_c*)^h + \check{D}_{c,l}* \, ,$$

provided $\check{T}_c = \mathcal{L}_t^{-1} T_c$ and $\check{D}_{c,l} = \mathcal{L}_t^{-1} D_{c,l}$ exist. Apparently, $\check{T}_c*$ is given as

$$T_c* = \tilde{M}_0 E_c^\infty e * K_c \check{R}_c* \, .$$

Lemma 6.1. $\forall c \in \mathbb{R}^n$, $\forall \beta > n/2$, $\check{T}_c* \in \mathcal{B}(Y^\infty_{\beta;\gamma})$ with $\gamma = n/2$. Here the

operator norm is locally bounded in c.

Proof. In Proposition 3.5 (ii), put $q=1$, $m=\delta=0$ and $r=n/2$. Then $\alpha=\mu=r=n/2$. Use thrice Proposition 4.2 (ii) as in the proof of Lemma 4.4. Then

$$r^{\pm}E_c e * K_c \check{R}_c * \in \boldsymbol{B}(Y_{\beta;\gamma}^{\infty;-}, Y_{\beta;\gamma}^{\infty;\pm}).$$

Hence the lemma follows by the assumption $[M]_2$ (iii) and Remark 1.1 of §I.1.

To prove the existence of $\check{D}_{c,l}(t)$, let $\Sigma(\bar{a}_0, \bar{\sigma}_0)$ be the closed domain of $\boldsymbol{C}$ given in Proposition I.7.5. Owing to Lemma I.7.3 (ii) and Proposition I.7.5 (iii), $D_{c,l}(\lambda)$ is analytic in $\lambda \in \Sigma(\bar{a}_0, \bar{\sigma}_0)\backslash\{0\}$, while, in view of Proposition I.7.4,

(6.2) $$\|D_{c,l}(\lambda)\| \leq C(1+|\lambda|)^{-l\gamma},$$

both in $\boldsymbol{B}(Y^{2,-})$. Choose $l > 1/r$. Then Lemma 2.3 can be applied to $X = \boldsymbol{B}(Y^{2,-})$ and $g(\lambda)=D_{c,l}(\lambda)$, so the integral

(6.3) $$\check{D}_{c,l}(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}D_{c,l}(\lambda)d\lambda, \qquad \sigma>0,$$

converges and $D_{c,l}=\mathcal{L}_l\check{D}_{c,l}$ holds, both in $\boldsymbol{B}(Y^{2,-})$. Thus $\check{D}_{c,l}(t)\in\boldsymbol{B}(Y^{2,-})$ exists for all $t\in\boldsymbol{R}$. But since Propositions I.7.4 and I.7.5 are valid also in $\boldsymbol{B}(Y_{\beta}^{\infty,-})$, $\beta>n/2$, then so is (6.2) with the same constant $r>0$. Hence (6.3) converges also in $\boldsymbol{B}(Y_{\beta}^{\infty,-})$ and $\check{D}_{c,l}(t)\in\boldsymbol{B}(Y_{\beta}^{\infty,-})$.

To study asymptotic behaviors of $\check{D}_{c,l}(t)$, we look again at Lemma I.7.3 and Proposition I.7.5 to note that

$$D_{c,l}(\lambda)\in\mathcal{B}^0(\Sigma(\bar{a}_0, \bar{\sigma}_0); \boldsymbol{B}(Y^{2,-}\cap Y_{\beta}^{\infty,-})).$$

Consequently it is possible by virtue of Cauchy's theorem to shift the path of integration in (6.3) to the boundary $\partial\Sigma(\bar{a}_0, \bar{\sigma}_0)$ of $\Sigma(\bar{a}_1, \bar{\sigma}_0)$, in the topology of $\boldsymbol{B}(Y^{2,-})$. By (6.2), the integral on $\partial\Sigma(\bar{a}_0, \bar{\sigma}_0)$ converges in $\boldsymbol{B}(Y^{2,-})$ as well as in $\boldsymbol{B}(Y_{\beta}^{\infty,-})$.

Put $\tau_0=(\bar{a}_0/\bar{\sigma}_0)^{1/2}$ and decompose $\partial\Sigma(\bar{a}_0, \bar{\sigma}_0)$ into two parts

$$\Gamma^{(1)} = \{\lambda\in\boldsymbol{C} \mid \mathrm{Re}\,\lambda = -\bar{\sigma}_0, \ |\mathrm{Im}\,\lambda| \geq \tau_0\},$$

$$\Gamma^{(2)} = \{\lambda\in\boldsymbol{C} \mid \mathrm{Re}\,\lambda = -\bar{a}_0|\mathrm{Im}\,\lambda|^2, \ |\mathrm{Im}\,\lambda| \leq \tau_0\},$$

according to which $\check{D}_{c,l}(t)$ is decomposed as

$$\check{D}_{c,l}(t) = \check{D}^{(1)}(t)+\check{D}^{(2)}(t),$$

$$\check{D}^{(j)}(t) = \frac{1}{2\pi i}\int_{\Gamma^{(j)}} e^{\lambda t}D_{c,l}(\lambda)d\lambda, \qquad j=1, 2.$$

In the rest of this section we write

$$B = \mathcal{B}(Y_\beta^\infty \cdot^-) .$$

Lemma 6.2. $\forall \beta > n/2,\ \exists C > 0;\ \forall t \geq 0,\ \forall c \in B[\bar{c}_0],$

$$\|\check{D}^{(1)}(t)\|_B \leq C e^{-\bar{\sigma}_0 t} .$$

Proof. Write $\lambda = \sigma + i\tau$. Then

$$\check{D}^{(1)}(t) = \frac{1}{2\pi} e^{-\bar{\sigma}_0 t} \int_{|\tau| \geq \tau_0} e^{i\tau t} D_{c,I}(-\bar{\sigma}_0 + i\tau) d\tau .$$

In view of (6.2), the last integral converges and is uniformly bounded for $t \in R$, in the topology of B, whence the lemma follows.

Similarly we can get for $t \geq 0$,

$$(6.4) \qquad \|\check{D}^{(2)}(t)\|_B \leq C \int_0^{\tau_0} e^{-\bar{a}_0 \tau^2 t} d\tau \leq C(1+t)^{-1/2} .$$

This can be improved as follows. Let $\alpha_2 > 0$ be that of Theorem 4.1.

Lemma 6.3. $\forall \theta \in [0, 1),\ \forall \beta > n/2,\ \exists C \geq 0;\ \forall t \geq 0,\ \forall c \in B[\bar{c}_0],$

$$\|D^{(2)}(t)\|_B \leq C |c|^{-\theta}(1+t)^{-\alpha_2} .$$

Before proving this lemma, we note that Lemmas 6.1, 6.2, 6.3 prove Proposition 4.4. Therefore, the remaining part of this section is devoted to the proof of Lemma 6.3.

Lemma 6.4. $\forall k \in N_+,\ \forall c \in R^n,\ \forall \beta \in R,\ \exists C \geq 0;\ \forall \lambda \in C_+(-\nu_0),$
  (i) $\|R_c^{(k)}(\lambda)\|_{\mathcal{B}(Y_\beta^\infty \cdot^-,\ L_\beta^\infty)} \leq C(\mathrm{Re}\ \lambda + \nu_0)^{-k},$
  (ii) $\|K_c R_c^{(k)}(\lambda)\|_{\mathcal{B}(Y_\beta^\infty \cdot^-,\ L_{\beta-\gamma}^b)} \leq C(\mathrm{Re}\ \lambda + \nu_0)^{-k-1/p},\ r > \dfrac{1}{p} + \dfrac{1}{r}.$

Here $R_c^{(k)}(\lambda) = d^k R_c(\lambda)/d\lambda^k$, and the constant C is locally bounded in c.

Proof. Differentiate (I.7.4) with respect to λ. Then

$$R_c^{(k)}(\lambda)h = (-t^-(x, \xi))^k e^{-(\lambda+\nu_c(\xi))t^-(x,\xi)} h(x - t^-(x, \xi)\xi, \xi)$$

for $(x, \xi) \in Q^-$ and $= 0$ otherwise. Note that $\sup_{t \geq 0} t^k e^{-t/2} < \infty$ and proceed as in the proof of Lemma I.7.2 to conclude the lemma.

Let $\Sigma(a_0, \sigma_0)$ be that defined by (I.6.8).

Lemma 6.5. $\forall k \in N_+,\ \forall c_0 > 0,\ \forall \beta \geq 0,\ \forall \theta \in [0, 1),\ \exists C \geq 0;\ \forall \lambda \in \Sigma(a_0, \sigma_0),$

$\forall c \in B[c_0]$,

$$\|(\lambda - B_c^\infty)^{-k} u\|_{L_\beta^\infty} \le C \, |c|^{-\theta} |\tau|^{\min(0,\gamma)} (\|u\|_{L_\beta^\infty} + \|u\|_{Z^1}) ,$$

where $\gamma = n + \theta - 2k$.

Proof. Set, as in the proof of Theorem I.6.7,

$$R = (\lambda - B_c^\infty)^{-1}, \quad R' = (\lambda - A_c^\infty)^{-1}, \quad G = R' K_c .$$

As a property of the resolvent, we have

(6.5) $d^k R / d\lambda^k = (-1)^k R^{k+1} ,$

and similarly for R'.   Hence differentiation of (I.6.14) yields

(6.6) $R^k = A_0 + \sum_{h=1}^{k} \binom{k-1}{h-1} A_{k-h} R^h ,$

where

$$A_0 = (-1)^k \sum_{h=0}^{l-1} d^{k-1}(G^h R')/d\lambda^{k-1} ,$$

$$A_h = (-1)^{k-h} d^h G^l / d\lambda^h .$$

In view of (6.5) for R', A_h's are expressed in linear combinations of products of R' and K_c.   Therefore, similarly to Lemma I.5.4 and (I.6.15), we see that if $l > 0$ is choosen large enough,

$$A_0 \in \mathcal{B}(L_\beta^\infty) , \quad A_h \in \mathcal{B}(L_\beta^\infty) \cap \mathcal{B}(L^2, L_\beta^\infty)$$

the norms being uniformly bounded for $(\lambda, c) \in \Sigma(a_0, \sigma_0) \times B[c_0]$.   Recall that the decomposition (I.6.11) is orthogonal.   Hence

$$R^k = \sum_{j=0}^{\bar{m}} U_j(\lambda, c)^k$$

holds.   Substitute this into the last sum in (6.6) to deduce

$$\|R^k u\|_{L_\beta^\infty} \le C[\|u\|_{L_\beta^\infty} + \sum_{h=1}^{k} \{\|U_0(\lambda, c)^h u\|_{L^2} + \sum_{j=1}^{\bar{m}} \|U_j(\lambda, c)^h u\|_{L_\beta^\infty}\}] .$$

Now the lemma readily follows from Propositions I.6.4 and I.6.6 for $p = \infty$, $q = 1$.

   Lemma 6.6. $\forall k \in N_+$, $\forall \theta \in [0, 1)$, $\exists C \ge 0$; $\forall \lambda \in \Gamma^{(2)}$, $\forall c \in B[\bar{c}_0]$,

$$\|T_c^{(k)}(\lambda)\|_B \le C \, |c|^{-\theta} |\tau|^{\min(0,\gamma)}$$

Here $T_c^{(k)} = d^k T_c / d\lambda^k$ *and* $\gamma = n + \theta - 2k - 2$.

   Proof.   In virtue of (6.5),

$$T_c^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \tilde{M}_0 R^{h+1} e K_c R_c^{(k-h)}(\lambda).$$

Since $\tilde{M}_0$ satisfies $[M]_2(\text{iii})$, we have only to evaluate $r^{\pm} R^{h+1} e K_c R_c^{(k-h)}(\lambda)$. And this can be done by combining Lemmas 6.4 and 6.5 with (3.8) for $p = \infty$.

Lemma 6.7. $D_{c,l}(\lambda)$ *also enjoys the conclusion of Lemma 6.6.*

Proof. For $D_c(\lambda) = (I - T_c(\lambda))^{-1}$, we have

$$dD_c(\lambda)/d\lambda = -D_c(\lambda)(dT_c(\lambda)/d\lambda)D_c(\lambda).$$

Compute $D_{c,l}^{(k)}(\lambda) = d^k (D_c(\lambda) T_c(\lambda)')/d\lambda^k$ by this and Leibniz' rule, and evaluate the resulting sum by Proposition I.7.5 and the above lemma to conclude the lemma.

Proof of Lemma 6.3. By integration by parts,

$$(6.7) \quad 2\pi i \check{D}_{c,l}(t) = -\sum_{h=1}^{k} (-t)^{-h} [e^{\lambda t} D_{c,l}^{(h-1)}(\lambda)]_{\lambda=\bar{\lambda}_0}^{\lambda=\lambda_0} + (-t)^{-k} \int_{\Gamma^{(2)}} e^{\lambda t} D_{c,l}^{(k)}(\lambda) d\lambda,$$

where $\lambda_0 = -\bar{\sigma}_0 + i\tau_0$. Put $\lambda = \lambda_0$, $\bar{\lambda}_0$ in Lemma 6.7. Then the sum from $h = 1$ to k of (6.7), denoted as $I_1(t)$, satisfies

$$\|I_1(t)_B\| \le C |c|^{-\theta} t^{-1} e^{-\bar{\sigma}_0 t}, \qquad t \ge 1.$$

Denote the last integral in (6.7) as $I_2(t)$ and put $\mu = \min(0, n + \theta - 2k - 2)$. By Lemma 6.7 it follows that if $\mu > -1$, then

$$\|I_2(t)\|_B \le C |c|^{-\theta} \int_0^{\tau_0} e^{-a_0 \tau^2 t} \tau^\mu d\tau \le C |c|^{-\theta} (1+t)^{-(1+\mu)/2}.$$

We should choose k as large as possible under the restriction $\mu > -1$. Therefore $k = (n-1)/2$ when n is odd and $k = (n-2)/2$ when n is even. Then $\alpha_2 = k + (1 + \mu)/2$ is identical with that of Theorem 4.1, and Lemma 6.3 follows if (6.4) is taken into account.

§7. Proof of Proposition I.7.4

In what follows, L_β^p will stand for either $L_\beta^p(Q^\infty)$ or $L_\beta^p(Q)$ with β dropped when $\beta = 0$. No confusions will arise. For each $c_0 > 0$ and $\delta > 0$, put

$$\Sigma_1 = \bar{C}_+(-\nu_0 + \delta) \times B[c_0],$$
$$\Sigma_2 = \Sigma(a_0, \sigma_0) \times B[c_0],$$

where $\Sigma(a_0, \sigma_0)$ was defined by (I.6.8). All the conclusions in the below are

valid for arbitrarily fixed positive c_0 and δ. Write $G=(\lambda-A_c^\infty)^{-1}K_c$ as before.

Lemma 7.1. $\exists \gamma>0$; $\forall \beta \geq 0$, $\forall p \in [2, \infty]$, $\exists C \geq 0$; $\forall (\lambda, c) \in \Sigma_1$,

(i) $\|G\|_{\mathbf{B}(L_\beta^p)} \leq C(1+|\lambda|)^{-\gamma/p}$,

(ii) $\|\tilde{M}_0 G\|_{\mathbf{B}(L_\beta^p, Y_\beta^{p,-})} \leq C(1+|\lambda|)^{-\gamma/(2p)}$.

Proof. By Proposition I.3.5 and Lemma I.5.2 (i),

$$(7.1) \qquad \|G\|_{\mathbf{B}(L_\beta^p)} \leq C(\mathrm{Re}\,\lambda+\nu_0)^{-1},$$

while by (3.6) and Parseval,

$$\|G\|_{\mathbf{B}(L_\beta^2)} \leq C(1+|\mathrm{Im}\,\lambda|)^{-\gamma 2/(1+\gamma)},$$

which proves (i) for $p=2$ if $\gamma^2/(1+\gamma)$ is rewritten as $\gamma/2$. Note that the proof of Lemma I.6.12, and subsequently that of (3.6), remain valid in L_β^2 if $\beta \neq 0$. Now (i) for $p>2$ follows by the interpolation between the result for $p=2$ and (7.1) for $p=\infty$. In order to prove (ii), we first repeat the proofs of (I.4.4)$_+$ and (I.4.6)$_-$ with u replaced by $|u|^2$, noting that $\partial|u(X\pm t\xi, \xi)|^2/\partial t = 2\,\mathrm{Re}\,\overline{u(X\pm t\xi, \xi)}\partial u(X\pm t\xi, \xi)/\partial t$. Then for $u \in W_{\nu_c}^{2,\pm}(Q^\infty)$,

$$\|\gamma^\pm u\|_{Y_\beta^2,\pm}^2 \leq 2\|u\|_{L_\beta^2}\|(\lambda-A_c^\infty)u\|_{L_\beta^2}.$$

Replace u by Gu and substitute (i) to conclude

$$\|\gamma^\pm G\| \leq C(1+|\lambda|)^{-\gamma/4} \qquad \text{in} \quad \mathbf{B}(L_\beta^2, Y_\beta^{2,\pm}).$$

Then (ii) for $p=2$ is verified by $[M]_2$ (i). Note that

$$(7.2) \qquad \|\gamma^\pm u\|_{Y_\beta^\infty,\pm} \leq \|u\|_{L_\beta^\infty}.$$

Put Gu for u. Then $\tilde{M}_0 G \in \mathbf{B}(L_\beta^\infty, Y_\beta^{\infty,-})$ by (7.1) and $[M]_2$ (iii). Hence (ii) for $p>2$ follows again by the interpolation.

Proof of Proposition I.7.4. (i). What we shall prove is that

$$(7.3) \qquad \|T_c(\lambda)\| \leq C(1+|\lambda|)^{-\gamma/(2p)} \qquad \text{in} \quad \mathbf{B}(Y_\beta^{p,-})$$

holds for all $(\lambda, c) \in \Sigma_2$. In view of Proposition I.7.3, however, it is enough for this to hold for $|\lambda| \geq r_0$ with some $r_0>0$. Rewrite $T_c(\lambda)$ as

$$T_c(\lambda) = \tilde{M}_0 G S_c(\lambda),$$

which follows with substitution of the second resolvent equation (I.6.14) for $l=1$ into the definition of $T_c(\lambda)$ (see (4.1)). Let $R=(\lambda-B_c^\infty)^{-1}$ and evaluate

the right hand side of (I.6.14) with sufficiently large l by the aid of Lemma I.5.4. Then

$$\|Ru\|_{L_\beta^p} \le C(\|u\|_{L_\beta^p} + \|Ru\|_{L^2})$$

for $p \ge 2$. By Proposition I.6.4, $\|Ru\| \le C\|u\|$ in L^2 if $|\lambda| \ge r_0$ with any $r_0 > 0$, the constant $C \ge 0$ being dependent on r_0 but not on λ. Consequently by Lemma I.7.2,

$$(7.4) \qquad \|S_c(\lambda)h\|_{L_\beta^p} \le C(\|h\|_{Y_\beta^{p,-}} + \|h\|_{Y_\gamma^{p,-}}),$$

for $p \ge 2$, $\beta \ge 0$ and $\gamma > \left(\frac{1}{2} - \frac{1}{p}\right)(n+1) - 1$. Now Lemma 7.1 (ii) proves (7.3) for $(\lambda, c) \in \Sigma_2$, $|\lambda| \ge r_0$, $r_0 > 0$ being arbitrary.

We proceed to *the proof of Proposition* I.7.4 (ii). Similarly to the above, it suffices to show that

$$(7.5) \qquad \|T_c(\lambda)^3\| \le C(1 + |\lambda|)^{-\gamma} \qquad \text{in} \quad \mathcal{B}(Y_\beta^{\infty,-})$$

for $(\lambda, c) \in \Sigma_2$, $|\lambda| \ge r_0$ with some $r_0 > 0$. To this end, we use

$$(7.6) \qquad T_c(\lambda) = \tilde{M}_0 Ge R_c(\lambda) + \tilde{M}_0 G^2 S_c(\lambda) \equiv T_1 + T_2,$$

which is obtained by (I.6.14) for $l = 2$.

Lemma 7.2.   (i) $\forall p \in [2, \infty]$, $\forall \beta \ge 0$, $\exists C \ge 0$; $\forall(\lambda, c) \in \Sigma_1$,

$$\|T_1\| \le C \qquad \text{in} \quad \mathcal{B}(Y_\beta^{p,-}, Y_{\beta+1}^{p,-}).$$

(ii) $\forall p \in (n, \infty]$, $\forall \beta > \left(\frac{1}{2} - \frac{1}{p}\right)(n+1) - 1$, $\exists C \ge 0$; $\exists \delta > 0$; $\forall(\lambda, c) \in \Sigma_2$,

$$\|T_2\| \le C \qquad \text{in} \quad \mathcal{B}(Y_\beta^{p,-}, Y_{\beta+\delta}^{\infty,-}).$$

Proof. (i) follows by Proposition I.3.5, Lemmas I.5.5 and I.7.2. By Lemma I.5.4 and (7.2),

$$\|T_2 h\|_{Y_\beta^{\infty,-}} \le C\|S_c(\lambda)h\|_{Y_{\beta-\delta}^p},$$

if $p > n$ and $\delta \le 1 - \frac{1}{p}$. Then (7.4) proves (ii).

From (7.6) it follows that

$$T_c(\lambda)^2 = T_1^2 + T_3$$

with $T_3 = T_1 T_2 + T_2 T_1 + T_2^2$. By the above lemma,

$$\|T_3\| \leq C \qquad \text{in} \quad \mathcal{B}(Y_{\beta-\delta}^{p,-}, Y_\beta^{\infty,-})$$

with $\delta \leq 2\left(1 - \dfrac{1}{p}\right)$, $p > n$. Let $p > n+1$ be fixed. In view of (I.7.6), (7.3) holds even if $\mathcal{B}(Y_\beta^{p,-})$ is replaced by $\mathcal{B}(Y_\beta^{\infty,-}, Y_{\beta-\delta}^{p,-})$ with $\delta > (n+1)/p$. Therefore $T_3 T_c(\lambda)$ enjoys the same inequality of (7.5).

It remains to evaluate T_1^2, and it is at this stage that the type of boundary operator M_0 turns to be crucial and that the assumptions $[M]_1$ (iv) and $[M]_2$ (iv) are required. Estimates of different types are available as seen from the

Proposition 7.3. *Let* $\beta \geq 0$.

(i) *Suppose* M_0 *be given by* (I.1.5) *and fulfill* $[M]_1$. *Then for each* $p \in (n, \infty]$, *there is a constant* $C \geq 0$ *such that for any* $\varepsilon > 0$ *and* $(\lambda, c) \in \Sigma_1$

$$\|T_1^2 h\|_{Y_\beta^{\infty,-}} \leq C \left(\varepsilon \|h\|_{Y_{\beta-1}^{\infty,-}} + \frac{1}{\varepsilon} \|h\|_{Y_{\beta-1}^{p,-}} \right).$$

(ii) *Suppose* M_0 *fulfill* $[M]_2$ *and let* p_0 *be that of* $[M]_2$ (iv). *There is a constant* $C \geq 0$ *and for any* $p \in [p_0, \infty]$ *and* $(\lambda, c) \in \Sigma_1$,

$$\|T_1^2\| \leq C \qquad \text{in} \quad \mathcal{B}(Y_{\beta-1}^{p,-}, Y_\beta^{\infty,-}).$$

Before proving this, we complete the

Proof of Proposition I.7.4 (ii). As stated already, it suffices to show that $T_1^2 T_c(\lambda)$ enjoys (7.5). Recall that (7.3) holds with $\mathcal{B}(Y_\beta^{p,-})$ replaced by $\mathcal{B}(Y_\beta^{\infty,-}, Y_{\beta-\delta}^{p,-})$, $\delta < (n+1)/p$. Choose again $p > n+1$. When M_0 is given by (I.1.5), put $T_c(\lambda)h$ for h in Proposition 7.3 (i) and obtain

$$\|T_1^2 T_c(\lambda) h\|_{Y_\beta^{\infty,-}} \leq C(\varepsilon + \varepsilon^{-1}(1+|\lambda|)^{-\gamma/(2p)}) \|h\|_{Y_\beta^{\infty,-}}.$$

Then the desired estimate follows with the choice $\varepsilon = (1+|\lambda|)^{-\gamma/(4p)}$. For the case of $[M]_2$, it suffices to combine Proposition 7.3 (ii) with (7.3).

The rest of this section is to be devoted to *the proof of Proposition* 7.3. Write $R' = (\lambda - A_c^\infty)^{-1}$ and define

$$V_- = R_c(\lambda) r^- R', \qquad W_- = \tilde{M}_c R' e K_c V_- e K_c R_c(\lambda).$$

Lemma 7.4. $\forall p > n$, $\forall \beta \geq 0$, $V_- \in L^\infty(\Sigma_1; \mathcal{B}(L_\beta^{p,\infty}, L_\beta^{\infty,p}))$.

Proof. (I.5.5) and (I.7.4) yield an explicit expression

$$(V_- u)(x, \xi) = \int_{t^-(x,\xi)}^\infty e^{-(\lambda + \nu_c(\xi))t} u(x - t\xi, \xi) dt$$

for $(x, \xi) \in Q^-$ and $= 0$ otherwise. Then the lemma can be proven exactly in the same way as Lemma I.5.2 (ii).

In the above the space L cannot be replaced by $\tilde{L}$.

Corollary 7.5. *Under the same situation of Lemma 7.4,*

$$W_- \in L^\infty(\Sigma_1; \mathcal{B}(Y^{p,-}_{\beta-\delta}, Y^{\infty,\cdot-}_\beta)), \quad \delta < 2\left(1 - \frac{1}{p}\right).$$

Proof. By Proposition I.3.5, $K_c \in L^\infty(\Sigma_1; \mathcal{B}(L^p_{\beta-\delta}, L^{p,\infty}_\beta))$ if $p > n/2$, $\delta < 1 - 1/p$, so by Lemma I.7.2 (i), $K_c R_c(\lambda) \in L^\infty(\Sigma_1; \mathcal{B}(Y^{p,\cdot-}_\beta, L^{p,\infty}_\beta))$. Similarly, $\tilde{M}_0 R' e K_c \in L^\infty(\Sigma_1; \mathcal{B}(L^{\infty,p}_\beta, Y^{\infty,-}_{\beta+\delta}))$ by the aid of Lemma I.5.5 (i) and $[M]_2$ (iii). Combine these with Lemma 7.4.

We should also define

$$V_+ = R_c(\lambda)M_0 r^+ R', \quad W_+ = \tilde{M}_0 R' e K_c V_+ e K_c R_c(\lambda).$$

Apparently $T_1^2 = W_+ - W_-$.

Proof of Proposition 7.3 (ii). We have only to show that W_+ enjoys the conclusion of Corollary 7.5. Similarly to (3.8), $r^+ \in \mathcal{B}(L^{\infty,p}_\beta, Y^{\infty,p,+}_\beta)$. Then by Lemma I.5.2 (ii) with $p = \infty$, $r^+ R' \in L^\infty(\Sigma_1; \mathcal{B}(L^{p,\infty}_\beta, L^{p,p,+}_\beta))$ if $p > n$. Now we shall use $[M]_2$ (iv), in which it may be assumed that $p_0 > n$ (if not, take an interpolation with $[M]_2$ (iii)). Then $M_0 r^+ R' \in L^\infty(\Sigma_1; \mathcal{B}(L^{p_0,\infty}_\beta, Y^{\infty,\cdot-}_\beta))$, and by Lemma I.7.2 (i), $V_+ \in L^\infty(\Sigma_1; \mathcal{B}(L^{p_0,\infty}_\beta, L^\infty_\beta))$. A slight modification of the proof of Corollary 7.5 then leads to the desired result.

The proof of Proposition 7.3 (i) is far more complicated. Since M_0 is given by (I.1.5) and by virtue of (I.5.5) (ii) and (I.7.4), V_+ can be written as

(7.7) $$V_+ u = \int_0^\infty e^{-\psi(t)} u(X - t\tilde{m}, \tilde{m}) dt$$

for $(x, \xi) \in Q^-$ and $= 0$ for $(x, \xi) \in Q \backslash Q^-$, where

$$\psi(t) = (\lambda + \nu_c(\xi))t^-(x, \xi) + (\lambda + \nu_c(\tilde{m}))t,$$
$$X = X(x, \xi) = x - t^-(x, \xi)\xi,$$
$$\tilde{m} = \tilde{m}(x, \xi) = m(X(x, \xi), \xi),$$

$m(x, \xi)$ being subject to $[M]_1$. Write $\tilde{\xi} = \xi/|\xi|$. The following homogeneity properties come from $[M]_1$ (iv) and (I.7.3) (iv).

$$t^-(x, \xi) = t^-(x, \tilde{\xi})/|\xi|,$$
(7.8)
$$X(x, \xi) = X(x, \tilde{\xi}),$$
$$\tilde{m}(x, \xi) = |\xi|\tilde{m}(x, \tilde{\xi}),$$

all for $(x, \xi) \in Q^-$. Put

$$\Xi(x) = \{\xi \in \boldsymbol{R}^n \,|\, (x, \xi) \in Q^-\},$$

and define the map $y = y(\xi)$ by

$$\Xi(x) \ni \xi \rightarrow y = X(x, \xi) - \tilde{m}(x, \xi) \in \boldsymbol{R}^n,$$

where $x \in \Omega$ is arbitrarily fixed.   Define for each t_0, b, $\delta > 0$,

$$\Xi_1(x) = \{\xi \in \Xi(x) \,|\, t^-(x, \tilde{\xi}) \leq t_0, \ \delta \leq |\xi| \leq b, \ \rho(X(x, \tilde{\xi}), \tilde{\xi}) \geq \delta\}.$$

Let y_ξ denote Jacobi's matrix $(\partial y_j/\partial \xi_k)$.

Lemma 7.6. $\forall \delta > 0, \ \exists t_0 > 0; \ \exists m_0 > 0; \ \forall b > 0, \ \forall x \in \Omega, \ \forall \xi \in \Xi_1(x),$

$$|\det y_\xi| \geq m_0.$$

Proof.   The assumption $[\mathcal{O}]$ of §I.1 implies that $\partial \Omega$ is represented locally by the equation $\phi(x) = 0$, ϕ being a piecewise smooth function.   Then $t = t^-(x, \xi)$ is a unique solution to $\phi(x - t\xi) = 0$.   Differentiate this and deduce

$$\nabla_\xi t^-(x, \xi) = t^-(x, \xi)\rho(X, \xi)^{-1}n(X),$$

where $n(X)$ is the unit outward normal to $\partial \Omega$ and $\rho(X, \xi) = |n(X) \cdot \xi|$ at $X = X(x, \tilde{\xi})$.   Consequently Jacobi's matrix X_ξ is given as

(7.9)
$$X_\xi(x, \xi) = -t^-(x, \xi)B(X(x, \tilde{\xi}), \tilde{\xi}),$$
$$B(X, \tilde{\xi}) = I_n + \rho(X, \tilde{\xi})^{-1}(\tilde{\xi}_j n_k(X)),$$

I_n being a unit matrix of order n.   The $n \times n$ matrix $B(X, \xi)$ has, in view of $[\mathcal{O}]$ and $[M]_1$ (i), piecewise smooth and uniformly bounded entries as far as $\rho(X, \tilde{\xi}) \geq \delta > 0$.   By (7.8) and (7.9),

(7.10) $\tilde{m}_\xi(x, \xi) = m_X X_\xi + m_\xi = -t^-(x, \tilde{\xi})m_X(X, \tilde{\xi})B(X, \tilde{\xi}) + m_\xi(X, \tilde{\xi})$

is homogeneous of degree 0 with piecewise smooth and uniformly bounded entries if $\rho(X, \tilde{\xi}) \geq \delta$, $t^-(X, \tilde{\xi}) \leq t_0$.   Then

(7.11) $y_\xi = X_\xi - \tilde{m}_\xi = -t^-(x, \xi)\left(\dfrac{1}{|\xi|}I_n - m_x(X, \tilde{\xi})\right) \times B(X, \tilde{\xi}) - m_\xi(X, \tilde{\xi}).$

Replace $t^-(x, \tilde{\xi})$ by t in the last expression and denote its determinant by

$J(t)=J(t, X, \tilde{\xi}, |\xi|)$.   Then $J(t)$ is a polynomial of t of order at most n, whose coefficients are all piecewise smooth and uniformly bounded if $\rho(X, \tilde{\xi})\geq\delta$ and $|\xi|\geq\delta$.   Therefore $J(t)\to J(0)$ as $t\to 0$ uniformly, and

$$|J(0)| = |\det m_\xi(X, \tilde{\xi})| \geq m_0$$

by $[M]_1$ (iii), so $|J(t)|\geq m_0/2$ for $0\leq t\leq t_0$ with some $t_0>0$.   Since $\det y_\xi = J(t^-(x, \tilde{\xi}))$, this proves the lemma.

Next, define $\Xi_2(x)=\Xi_2(x, t_1, b, \delta)$ by

$$\Xi_2(x) = \{\xi\in\Xi(x) \,|\, t^-(x, \tilde{\xi})\geq t_1, \, |\xi|\leq b, \, \rho(X(x, \tilde{\xi}), \tilde{\xi})\geq\delta\} .$$

Lemma 7.7. $\exists m_0>0; \forall\delta>0, \forall x\in\varOmega, \forall t_1>0, \forall b>0, \exists\Xi_2'(x)\subset\Xi_2(x);$
(i)   mes $\Xi_2'(x)\leq m_0 b^{n-1}\delta,$
(ii) $|\det y_\xi| \geq m_0(t_1-\delta/b)^{n-1}, \, \xi\in\Xi_2(x)\backslash\Xi_2'(x).$

Proof.   Rewrite (7.11) as

$$(7.12) \qquad y_\xi = -t^-(x, \xi)B(X, \tilde{\xi})-\tilde{m}_\xi(x, \xi) .$$

For each fixed $X\in\partial\varOmega$, we may choose the coordinates such that $n(X)=(0, 0, \cdots, 0, 1)$.   Then for $(X, \xi)\in S^-$ (see §I.4),

$$(7.13) \qquad \rho(X, \xi) = -\xi_n .$$

Let $\omega'=-\tilde{\xi}_n^{-1}{}^t(\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{n-1})$ be a column vector of order $n-1$.   Then in our choice of coordinates,

$$B(X, \xi) = \begin{pmatrix} I_{n-1} & \omega' \\ 0 & 0 \end{pmatrix}.$$

Write $\tilde{m}_\xi=(\tilde{m}_{jk})$, $1\leq j, k\leq n$, and put $m^1=(\tilde{m}_{j,k})$, $1\leq j, k\leq n-1$, (a matrix of order $n-1$), $m^2={}^t(m_{1n}, m_{2n}, \cdots, m_{n-1n})$ (a column vector of order $n-1$), $m^3=(\tilde{m}_{n1}, \tilde{m}_{n2}, \cdots, \tilde{m}_{nn-1})$ (a law vector of order $n-1$) and $m^4=\tilde{m}_{nn}$.   Then in the same coordinates,

$$(7.14) \qquad \begin{aligned} \det y_\xi &= (-t^-(x, \xi))^{n-1}J_0(x, \tilde{\xi}, |\xi|) , \\ J_0(x, \xi, t) &= \begin{vmatrix} I_{n-1}+t\mu m^1 & \omega'+t\mu m^2 \\ m^3 & m^4 \end{vmatrix} \end{aligned}$$

where $\mu=t^-(x, \tilde{\xi})^{-1}$.   Obviously J_0 is a polynomial of t of order at most $n-1$,

and its coefficients are all piecewise smooth and uniformly bounded for $x \in \Omega$, $\xi \in \Xi_2(x)$.   Noting (7.13), we see

$$J_0(x, \tilde{\xi}, 0) = m^4 - m^3 \cdot \omega' = \tilde{\xi}_n^{-1}(\tilde{\xi} \cdot \nabla_\xi m_n)(X, \tilde{\xi}) \, .$$

By (7.8), $\tilde{\xi} \cdot \nabla_\xi m_n = \partial m_n(X, |\xi|\tilde{\xi})/\partial|\xi| = m_n(X, \tilde{\xi})$, and in our choice of coordinates, $\tilde{\xi}_n^{-1} m_n(X, \tilde{\xi}) = -\rho(X, \tilde{\xi})^{-1}\rho(X, \tilde{m})$.   Hence by virtue of $[M]_1$ (i) (iii), there holds with some constant $m_0 > 0$,

$$(7.15) \qquad |J_0(x, \tilde{\xi}, 0)| = |\det m_\xi(X(x, \tilde{\xi}), \tilde{\xi})|^{-1} \geq m_0 \, ,$$

for all $(x, \tilde{\xi}) \in Q^-$.   Obviously this result is free from the choice of coordinates. As a consequence, the polynomial J_0 of t does not vanish identically, and thereby it has at most $n-1$ distinct real zeros $t = t_j(x, \tilde{\xi})$, $1 \leq j \leq j_0$, where the number j_0 of the zeros may of course vary with x and $\tilde{\xi}$.   Let us show that

$$\Xi_2'(x) = \{\xi \in \Xi_2(x) \mid |\,|\xi| - t_j(x, \xi)| \leq \delta, \ 1 \leq j \leq j_0\}$$

is the desired subset stated in the lemma.   Since t_j's are independent of $|\xi|$, (i) of the lemma is evident.   Since $t_j \neq 0$ by (7.15), we can write

$$(7.16) \qquad J_0(x, \tilde{\xi}, |\xi|) = J_0(x, \tilde{\xi}, 0) \prod_{j=1}^{j_0} \left(1 - \frac{|\xi|}{t_j(x, \xi)}\right)^{k_j},$$

k_j being the multiplicity of t_j.   Let $\xi \in \Xi_2(x) \backslash \Xi_2'(x)$.   It $|t_j| \leq 2b$, then

$$\left|1 - \frac{|\xi|}{t_j}\right| \geq |\,|\xi| - t_j| / |t_j| \geq \delta/(2b) \, ,$$

while if $|t_j| \geq 2b$, then

$$\left|1 - \frac{|\xi|}{t_j}\right| \geq 1 - |\xi|/(2b) \geq 1/2 \, .$$

This, together with (7.15) and (7.16), gives $|J_0| \geq m_0(\delta/(2b))^{n-1}$, whence (ii) of the lemma follows in virtue of (7.14).

Proposition 7.8. $\forall p > n$, $\forall \beta \geq 0$, $\exists C \geq 0$; $\forall a > 0$, $\forall \varepsilon > 0$,

$$\|\chi(|\xi| < a)V_+ u\|_{L_\beta^{\infty, p}} \leq Ca^n(\varepsilon \|u\|_{L_\beta^\infty} + \varepsilon^{-1}\|u\|_{L_\beta^{p, \infty}}) \, .$$

Proof.   Recall (7.7), and put $v(x) = \|u(x, \cdot)\|_{L^\infty(R_\xi^n)}$ and

$$w(x, t, a) = \|\chi(|\xi| < a)v(\chi(x, \xi) - t\tilde{m}(x, \xi))\|_{L^p(\mathbf{R}_\xi^n)} \, .$$

The proof of the proposition will be carried out only for $\beta = 0$ because the

case $\beta \neq 0$ can be dealt with exactly in the same way. Let $(\lambda, c) \in \Sigma_1$, but in the below we change the notation δ by δ_0 in the definition of Σ_1. Since Re $\psi(t) \geq \delta_0 t$, (7.7) leads to

$$\| \chi(|\xi| < a) V_+ u) \|_{L^p(R_\xi^n)} \leq \int_0^\infty e^{-\delta_0 t} w(x, t, a) dt .$$

Denote the last integral as I. Given an $\varepsilon > 0$, put $a' = |\log \varepsilon| / \delta_0$ and split the integral I as

$$I = \int_{a'}^\infty + \int_0^{a'} \equiv I_1 + I_2 .$$

Apparently $\sup_{t \geq 0} w(x, t, a) \leq C a^{n/p} \|u\|_{L^\infty}$, independently of x, so

$$I_1 \leq C \varepsilon a^{n/p} \|u\|_{L^\infty} ,$$

while since $w(x, t, a) \leq t^{-n/p} w(x, 1, at)$ by (7.8) and by the change of variables $t\xi \to \xi$, and since w is monotone increasing in a, then

$$I_2 \leq C w(x, 1, aa')$$

holds for $p > n$. In Lemmas 7.6 and 7.7, put

$$\delta = \varepsilon^p , \quad b = aa' = a |\log \varepsilon| / \delta_0 , \quad t_1 = t_0 ,$$

where t_0 is the constant of Lemma 7.6. Define

$$\Xi_3(x) = \{ \xi \in \Xi(x) \cap B[b] \, | \, |\xi| \leq \delta \text{ or } \rho(X(x, \tilde{\xi}), \tilde{\xi}) \leq \delta \} .$$

Then $B[b] = \{ \xi \in R^n \, | \, |\xi| \leq b \}$ can be expressed as the union of $\Xi_j(x), j = 1, 2, 3$ and recalling the map $y = y(\xi)$,

$$w(x, 1, b)^p = \int_{B[b]} |v(y(\xi))|^p d\xi = \sum_{j=1}^3 \int_{\Xi_j(x)} \equiv \sum_{j=1}^3 I_{2j} .$$

By Lemma 7.6,

$$I_{21} = \int_{\Xi_1(x)} |v(y)|^p |\det y_\xi|^{-1} dy \leq C \int_{R^n} |v(y)|^p dy = C \|u\|_{L^{p,\infty}}^p ,$$

while by Lemma 7.7,

$$I_{22} = \int_{\Xi_2'(x)} + \int_{\Xi_2(x) \setminus \Xi_2'(x)}$$

$$\leq (\text{mes } \Xi_2'(x)) \|v\|_{L^\infty}^p + C \left(\frac{b}{\delta t_0} \right)^{n-1} \|v\|_{L^p}^p$$

$$\leq C (\varepsilon^{p+1} \|u\|_{L^\infty}^p + (a\varepsilon^{-p})^{n-1} \|u\|_{L^{p,\infty}}^p) .$$

Finally, it is easily checked that mes $\varXi_3(x) \leq C(\delta^n + b\delta^{n-1})$ holds uniformly for $x \in \varOmega$, so that

$$I_{23} \leq Ca\varepsilon^{pn} \|u\|_{L^\infty}^p .$$

Combining all the estimates so far obtained completes the proof of the proposition.

Proof of Proposition 7.3 (i). Owing to Corollary 7.5, it suffices to show that W_+ enjoys Proposition 7.3 (i). Use Proposition I.3.5 twice. Then the above proposition yields

$$\|K_c\chi(|\xi| < a) V_+ K_c u\|_{L_\beta^\infty} \leq Ca^n \left(\varepsilon \|u\|_{L_{\beta-2}^\infty} + \frac{1}{\varepsilon} \|u\|_{L_{\beta-\gamma}^p} \right)$$

for $\gamma < 2\left(1 - \dfrac{1}{p}\right)$, while since $\|V_+\| \leq C$ in $\boldsymbol{B}(L_\beta^\infty)$ as seen from $[M]_2$ (iii), Remark I.1.1 and Lemmas I.5.5 and I.7.2, then by the use of (I.3.7),

$$\|K_c\chi(|\xi| > a) V_+ K_c u\|_{L_\beta^\infty} \leq C(1+a)^{-1} \|u\|_{L_{\beta-1}^\infty} .$$

Choose $a = \varepsilon^{-1/(n+1)}$ and replace $\varepsilon^{1/(n+1)}$ by ε. Then

$$\|K_c V_+ K_c u\|_{L_\beta^\infty} \leq C\left(\varepsilon \|u\|_{L_{\beta-1}^\infty} + \frac{1}{\varepsilon} \|u\|_{L_{\beta-1}^p} \right) .$$

Now the desired estimate for W_+ follows readily from Lemmas I.5.5 and I.7.2.

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