

On the Structure of the State Space of Maximal Op^* -algebras

By

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§ 1. Introduction

In this paper we continue the investigation of the structure of the state space of $\mathcal{L}^+(\mathcal{D})$ begun in [16]. The paper is organized as follows. Section 2 contains the necessary definitions, notations and auxiliary results. In Section 3 we prove the main results about the pure state space and the vector state space of $\mathcal{L}^+(\mathcal{D})$. These two sets of states coincide. Moreover there is given a representation theorem for the pure (and vector) state space analogous to the bounded case.

Furthermore Section 3 contains several results concerning sequences of states. It appears that there are some differences to the bounded case.

In Section 4 we investigate the state space of general Op^* -algebras. Among other things it is proved that the state space is the w^* -closed convex hull of the vector states. If the Op^* -algebra is selfadjoint and topologically irreducible (cf. Definition 4.3) then the state space is the w^* -closed convex hull of pure states.

§ 2. Preliminaries

For a dense linear manifold \mathcal{D} in a separable Hilbert space \mathcal{H} the set $\mathcal{L}^+(\mathcal{D}) = \{A: A\mathcal{D} \subset \mathcal{D}, A^*\mathcal{D} \subset \mathcal{D}\}$ is a $*$ -algebra with respect to the usual operations and the involution $A \rightarrow A^+ = A^*|_{\mathcal{D}}$. An Op^* -algebra $\mathcal{A}(\mathcal{D})$ is a $*$ -subalgebra of $\mathcal{L}^+(\mathcal{D})$ containing the identity operator I . The graph topology $t_{\mathcal{A}}$ on \mathcal{D} induced by $\mathcal{A}(\mathcal{D})$ is given by the family of seminorms $\mathcal{D} \ni \varphi \rightarrow \|A\varphi\|$

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for all $A \in \mathcal{A}(\mathcal{D})$. Denote $t_{\mathcal{L}^+(\mathcal{D})}$ simply by t . An Op*-algebra $\mathcal{A}(\mathcal{D})$ is called:

closed if $\mathcal{D} = \bigcap_{A \in \mathcal{A}} \mathcal{D}(\bar{A})$ or equivalently if $\mathcal{D}[t]$ is complete;

selfadjoint if $\mathcal{D} = \mathcal{D}_* \equiv \bigcap_{A \in \mathcal{A}} \mathcal{D}(A^*)$.

Among the many possible topologies on Op*-algebras (cf. e.g. [10], [11], [12], [19], [20] and the references there) we mention only those needed here:

the *uniform topology* $\tau_{\mathcal{D}}$ given by the family of seminorms

$$\mathcal{A}(\mathcal{D}) \ni A \rightarrow \|A\|_{\mathcal{U}} = \sup_{\varphi, \psi \in \mathcal{U}} |\langle \varphi, A\psi \rangle|$$

where \mathcal{U} runs over all $t_{\mathcal{A}}$ -bounded subsets of \mathcal{D} ; the topology $\tau_{\mathcal{D}}^c$ given by the family of seminorms $\| \cdot \|_{\mathcal{U}}$ as above Remark that $\tau_{\mathcal{D}}$, $\tau_{\mathcal{D}}^c$ are also defined on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$, hence on $\mathcal{B}(\mathcal{H})$, too [12], [8]. Here \mathcal{D}' means the strong dual of $\mathcal{D}[t]$.

In the remainder of this section and in Section 3 we always assume that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint and $\mathcal{D}[t]$ is an (F)-space. To simplify notations let us denote a bounded operator $A \in \mathcal{L}^+(\mathcal{D})$ and its closure $\bar{A} \in \mathcal{B}(\mathcal{H})$ by the same letter A . The following two-sided *-ideals of $\mathcal{L}(\mathcal{D})$ play an important role in the description of $\tau_{\mathcal{D}}$ and $\tau_{\mathcal{D}}^c$ [8], [11], [12], [16], [21], [22]:

$$\mathcal{B}(\mathcal{D}) = \{T: T\mathcal{H} \subset \mathcal{D}, T^*\mathcal{H} \subset \mathcal{D}\} = \{T: AT, AT^* \text{ bounded for all } A \in \mathcal{L}^+(\mathcal{D})\}$$

$$\mathcal{S}_{\infty}(\mathcal{D}) = \{T: T \in \mathcal{S}_{\infty}(\mathcal{H}) \cap \mathcal{B}(\mathcal{D})\} = \{T: AT, AT^* \in \mathcal{S}_{\infty}(\mathcal{H}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}.$$

Here $\mathcal{S}_{\infty}(\mathcal{H})$ denotes the *-ideal of compact operators on \mathcal{H} . Properties of $\mathcal{B}(\mathcal{D})$ needed in our context were collected in [16]. The ideal $\mathcal{S}_{\infty}(\mathcal{D})$ gives a description of the relatively t -compact sets in \mathcal{D} [22]. For completeness we include a proof. It is based on the following fact which seems to be not included in standard books on Hilbert spaces (the authors are grateful to K.D. Kürsten for suggesting a short proof):

Let \mathcal{H} be a separable Hilbert space, then the family
 (1) $\{C\mathcal{H}: C \geq 0, C \in \mathcal{S}_{\infty}(\mathcal{H}), \mathcal{R}(C) \text{ dense in } \mathcal{H}\}$ is a
 fundamental system of relatively $\| \cdot \|$ -compact sets in \mathcal{H} .

Proposition 2.1. *The family $\{S\mathcal{K}: S \geq 0, S \in \mathcal{S}_{\infty}(\mathcal{D}), \mathcal{R}(S) \text{ } t\text{-dense in } \mathcal{D}, S^{-1} \text{ exists}\}$ is a fundamental system of relatively t -compact sets in \mathcal{D} .*

Proof: It is easy to see that any set of the family is relatively t -compact.

To prove the other direction we first of all remark that the proof of (1) (or some other considerations) gives immediately: if $\mathcal{N} \subset \mathcal{H}$ is relatively $\| \cdot \|$ -compact and $\|\varphi\| \leq a$ for all $\varphi \in \mathcal{N}$, then C can be chosen to satisfy $a \leq \|C\| \leq 2a$. Now let $\mathcal{U} \subset \mathcal{D}$ be relatively t -compact, hence t -bounded. By [8] there is a $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$ with $B\mathcal{K} \supset \mathcal{U}$. Without restriction of generality let us assume that $\|B\| \leq 1$. Let $B = \int_0^1 \lambda dE_\lambda$ and put $P_n = \int_{J_n} dE_\lambda$ with $J_n = (2^{-2n}, 2^{-2(n-1)}]$, $n=1, 2, \dots$, $\mathcal{H}_n = P_n \mathcal{H} \subset \mathcal{D}$. Then $P_n \in \mathcal{B}(\mathcal{D})$ and $B_n = P_n B P_n = B P_n \in \mathcal{L}^+(\mathcal{D})$ (even $\in \mathcal{B}(\mathcal{D})$), $I = \sum \oplus P_n$.

Since \mathcal{U} is relatively t -compact, $\mathcal{U}_n = P_n \mathcal{U} \subset \mathcal{H}_n$ is relatively $\| \cdot \|$ -compact and from $B\mathcal{K} \supset \mathcal{U}$ it follows that $B_n \mathcal{K} = B_n \mathcal{K}_n \supset \mathcal{U}_n$, where \mathcal{K}_n is the unit ball in \mathcal{H}_n .

Consequently for all $\varphi \in \mathcal{U}_n$ it is $\|\varphi\| \leq \|B_n\| \leq 2^{-2n+2}$. Applying (1) to $\mathcal{U}_n, \mathcal{H}_n$ we get a compact $S'_n \in \mathcal{B}(\mathcal{H}_n)$ with $S'_n \mathcal{K}_n \supset \mathcal{U}_n$ and

$$(2) \quad 2^{-2n+2} \leq \|S'_n\| \leq 2 \cdot 2^{-2n+2}.$$

Clearly, S'_n can be considered as an element of $\mathcal{B}(\mathcal{H})$. Put $S_n = 2^{3/2} \cdot \|S'_n\|^{1/2} \cdot S'_n$. Then $\|S_n\| = \|S'_n\|^{1/2} \cdot 2^{3/2}$ and $S_n(2^{-3/2} \cdot \|S'_n\|^{1/2})\mathcal{K} = S'_n \mathcal{K}_n \supset \mathcal{U}_n$.

The operator $S = \sum \oplus S_n$ has the following properties:

- i) $S = S^*$ is compact because $S_n = S_n^*$ and the series $\sum \oplus S_n$ is norm-convergent as can be seen from the estimation $\|S\| \leq \sum \|S_n\| = 2^{3/2} \sum \|S'_n\|^{1/2} \leq 2^3 \sum 1/2^n < \infty$.
- ii) $S \mathcal{K} \supset \mathcal{U}$. Indeed, let $\varphi \in \mathcal{U}$, $\varphi = \sum \varphi_n$, $\varphi_n \in \mathcal{U}_n$. Then there exist $\psi_n \in (2^{-3/2} \|S'_n\|^{1/2})\mathcal{K}$ with $S_n \psi_n = \varphi_n$. Because $\|\psi_n\| \leq 2^{-n}$ it follows that $\psi = \sum \psi_n$ has norm $\|\psi\| \leq \sum 2^{-n} = 1$, i.e. $\psi \in \mathcal{K}$ and $S\psi = \varphi \in \mathcal{U}$.
- iii) $S \in \mathcal{S}_\infty(\mathcal{D})$. Because $S \in \mathcal{S}_\infty(\mathcal{H})$ it is enough to show that $S \in \mathcal{B}(\mathcal{D})$. Let $A \in \mathcal{L}^+(\mathcal{D})$ be arbitrary, then

$$\|AS\varphi\| = \|AB^{1/2}B^{-1/2}S\varphi\| \leq \|AB^{1/2}\| \cdot \|B^{-1/2}S\varphi\|.$$

The first factor is bounded because $B^{1/2} \in \mathcal{B}(\mathcal{D})$, for the second factor remark that

$$\begin{aligned} \|B^{-1/2}S\varphi\|^2 &= \sum \|B_n^{-1/2}S_n\varphi_n\|^2 \leq \sum \|B_n^{-1/2}S_n\|^2 \cdot \|\varphi_n\|^2 \leq \\ &= C \|\varphi\|^2 \text{ where } C = \sup_n \|B_n^{-1/2}S_n\|^2 = \sup_n 2^{2n} 2^{-2n+2} \|S'_n\| \leq 16. \end{aligned}$$

Moreover, S^{-1} exists, so S is the desired operator. The assertion about $\mathcal{R}(S)$ can be proved as in the $\mathcal{B}(\mathcal{D})$ -case using if necessary a larger t -relatively compact set $\mathcal{N} \supset \mathcal{U}$. q.e.d.

Next we consider linear functionals on Op^* -algebras $\mathcal{A}(\mathcal{D})$. We restrict

ourselves to $\tau_{\mathcal{D}}$ -continuous functionals. Let $E(\mathcal{A}) = \{\omega \in \mathcal{A}(\mathcal{D})[\tau_{\mathcal{D}}]': \omega \geq 0, \omega(I) = 1\}$ be the state space of $\mathcal{A}(\mathcal{D})$. Here, $\omega \geq 0$ means $\omega(A) \geq 0$ for all $A \in \mathcal{A}(\mathcal{D})$ with $\langle \varphi, A\varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}$ (strongly positive functionals). Further let $\sigma_0(\mathcal{A}) = \sigma(\mathcal{A}', \mathcal{A})$ be the w^* -topology in $\mathcal{A}(\mathcal{D})[\tau_{\mathcal{D}}]'$. We need the following subsets of $E(\mathcal{A})$:

vector states: $V_0(\mathcal{A}) = \{\omega \in E(\mathcal{A}) : \omega(A) = \langle \varphi, A\varphi \rangle \text{ for some } \varphi \in \mathcal{D}, \|\varphi\| = 1\}$;

vector state space: $V(\mathcal{A}) = \sigma_0(\mathcal{A})$ -closure of $V_0(\mathcal{A})$;

pure states: $P_0(\mathcal{A}) = \{\omega \in E(\mathcal{A}) : \omega_1 \in \mathcal{A}', \omega_1 \geq 0, \omega_1 \leq \omega \text{ implies } \omega_1 = \lambda\omega \text{ for some } \lambda \in [0, 1]\}$,

pure state space: $P(\mathcal{A}) = \sigma_0(\mathcal{A})$ -closure of $P_0(\mathcal{A})$.

Clearly, $\omega \in P_0(\mathcal{A})$ if and only if ω cannot be represented as a convex combination (non-trivial) of $\omega_1, \omega_2 \in E(\mathcal{A})$. The corresponding subsets of $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]'$ are simply denoted by E, V_0, V, P_0, P respectively, and $\sigma_0(\mathcal{L}^+(\mathcal{D}))$ we denote by σ_0 . To define normal and singular functionals on $\mathcal{L}^+(\mathcal{D})$ we introduce the two-sided $*$ -ideals in $\mathcal{L}^+(\mathcal{D})$ [13], [9], [14], [19]:

$$S_1(\mathcal{D}) = \{T \in \mathcal{L}^+(\mathcal{D}) : AT, AT^* \in S_1(\mathcal{A}) \text{ for all } A \in \mathcal{L}^+(\mathcal{D})\}$$

$$\mathcal{F}(\mathcal{D}) = \{F \in \mathcal{L}^+(\mathcal{D}) : \dim F\mathcal{D} < \infty\}$$

$$\mathcal{C}(\mathcal{D}) = \tau_{\mathcal{D}}\text{-closure of } \mathcal{F}(\mathcal{D}).$$

Here $S_1(\mathcal{A})$ stands for the $*$ -ideal of nuclear operators on \mathcal{A} .

Definition 2.2. *A linear functional ω on $\mathcal{L}^+(\mathcal{D})$ is said to be normal if $\omega(A) = \text{Tr } AT$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and some $T \in S_1(\mathcal{D})$; singular if ω is $\tau_{\mathcal{D}}$ -continuous and $\omega(C) = 0$ for all $C \in \mathcal{C}(\mathcal{D})$.*

Let us remark that while normal functionals are automatically $\tau_{\mathcal{D}}$ -continuous (even $\tau_{\mathcal{D}}^c$ -continuous [19]) we have included the $\tau_{\mathcal{D}}$ -continuity in the definition of singular functionals. Further, the notion of singularity used here is a direct generalization of that from the bounded case and has nothing to do with that used by Inoue [6].

In [16] we described a procedure relating $\tau_{\mathcal{D}}$ -continuous functionals on $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{B}(\mathcal{A})$ based on the $\tau_{\mathcal{D}}$ -density of $\mathcal{B}(\mathcal{D})$. Let $\omega \in \mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]'$ ($\in \mathcal{B}(\mathcal{A})[\tau_{\mathcal{D}}]'$), by restriction to $\mathcal{B}(\mathcal{D})$ and extension to $\mathcal{B}(\mathcal{A})$ (to $\mathcal{L}^+(\mathcal{D})$) one gets a unique $\tau_{\mathcal{D}}$ -continuous functional on $\mathcal{B}(\mathcal{A})$ (on $\mathcal{L}^+(\mathcal{D})$) which we denote by $\tilde{\omega}(\omega)$. Some properties of this procedure were collected in [16].

§ 3. The Vector State Space and the Pure State Space of $\mathcal{L}^+(\mathcal{D})$

This section is devoted to the description of P and V . In [5] Glimm

proved among other things the following result about $V(\mathcal{B}(\mathcal{A}))$ and $P(\mathcal{B}(\mathcal{A}))$.

Theorem 3.1. $P(\mathcal{B}(\mathcal{A}))=V(\mathcal{B}(\mathcal{A}))=Z(\mathcal{B}(\mathcal{A}))\equiv \{\omega \in E(\mathcal{B}(\mathcal{A})): \omega = \lambda\omega_1 + (1-\lambda)\omega_2, 0 \leq \lambda \leq 1, \omega_1\text{-vector state, } \omega_2\text{-singular state on } \mathcal{B}(\mathcal{A})\}$.

Our aim is to generalize this result to the unbounded case, namely to $\mathcal{L}^+(\mathcal{D})$. To do this we will use a theorem of Wils [23] which was generalized by Anderson [2].

Theorem 3.2. *There is a fixed sequence $(\psi_n) \subset \mathcal{A}$, $\|\psi_n\|=1$ so that any $\omega \in P(\mathcal{B}(\mathcal{A}))=Z(\mathcal{B}(\mathcal{A}))$ can be represented by*

$$\omega(A) = \lim_{\mathcal{U}} \langle \psi_n, A\psi_n \rangle$$

with an appropriate ultrafilter \mathcal{U} on \mathbb{N} .

It is enough to take (ψ_n) to be $\|\cdot\|$ -dense in the unit sphere of \mathcal{A} , so we suppose $(\psi_n) \subset \mathcal{D}$. Call such a sequence a *Wils sequence*.

In analogy to the bounded case we define

$$Z = \{\omega \in E: \omega = \lambda\omega_1 + (1-\lambda)\omega_2, 0 \leq \lambda \leq 1, \omega_1 \in V_0, \omega_2\text{-singular}\}.$$

Theorem 3.3. *Let (ψ_n) be a fixed Wils sequence. For every $\omega \in Z$ there is a $C \in \mathcal{B}(\mathcal{D})$ and an ultrafilter \mathcal{U} on \mathbb{N} so that*

$$\omega(A) = \lim_{\mathcal{U}} \langle C\psi_n, AC\psi_n \rangle$$

Proof. Let $\omega \in Z$, $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\omega_1 = \langle \varphi, \cdot \varphi \rangle$, $\|\varphi\|=1$, and ω_2 -singular.

Then ω , ω_1 , ω_2 and the corresponding states $\tilde{\omega}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$ on $\mathcal{B}(\mathcal{A})$ can be estimated by some seminorm $\|B \cdot B\|$, $B \geq 0$, $B \in \mathcal{B}(\mathcal{D})$. Let $B = \int_0^b \lambda dE_\lambda$. From [16] we know that $\tilde{\omega}_1 = \langle \varphi, \cdot \varphi \rangle$, $\tilde{\omega}_2$ -singular, i.e. $\tilde{\omega} = \lambda\tilde{\omega}_1 + (1-\lambda)\tilde{\omega}_2 \in Z(\mathcal{B}(\mathcal{A}))$. For $n \in \mathbb{N}$ and fixed $0 < \alpha < 1$ put $P_n = \int_{1/n}^b dE_\lambda$, $B_n^{-\alpha} = \int_{1/n}^b \lambda^{-\alpha} dE_\lambda$. Then $B_n^{-\alpha} B = B^{1-\alpha} P_n$, $B_n^{-\alpha} \in \mathcal{B}(\mathcal{A})$. In [16] it was shown that $\tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha})$ is a Cauchy sequence for all $A \in \mathcal{B}(\mathcal{A})$ and $\rho(A) = \lim_{n \rightarrow \infty} \tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha})$ defines a positive linear functional on $\mathcal{B}(\mathcal{A})$. Moreover

$$(1) \quad \rho(B^\alpha A B^\alpha) = \tilde{\omega}(A).$$

The same can be done for $\tilde{\omega}_1$ and $\tilde{\omega}_2$ which leads to ρ_1 and ρ_2 respectively. Furthermore

$$\begin{aligned} \rho(I) &= \lim_{n \rightarrow \infty} \tilde{\omega}(B_n^{-2\alpha}) = \lambda \lim_{n \rightarrow \infty} \tilde{\omega}_1(B_n^{-2\alpha}) + (1-\lambda) \lim_{n \rightarrow \infty} \tilde{\omega}_2(B_n^{-2\alpha}) \\ &= \lambda \rho_1(I) + (1-\lambda) \rho_2(I). \end{aligned}$$

Now let us distinguish some special cases. $\rho(I)=0$ and the Cauchy-Schwarz inequality give $\rho \equiv 0$, hence $\tilde{\omega} \equiv 0$ (by (1)), which is a contradiction. If $\rho_1(I)=0$ or $\rho_2(I)=0$, then $\tilde{\omega}_1 \equiv 0$ or $\tilde{\omega}_2 \equiv 0$. But this means that ω is a singular state or a vector state on $\mathcal{L}^+(\mathcal{D})$. Then we are done. For singular states this representation theorem was given in [16]. For vector states the representation is obtained in the following simple way. Take any $B \in \mathcal{B}(\mathcal{D})$ with $B\psi_k = \varphi$ for some k (having in mind that the vector state is generated by φ) and the ultrafilter \mathcal{U} fixed at k .

So we may suppose that $\tilde{\omega}(B_n^{-2\alpha})$, $\tilde{\omega}_1(B_n^{-2\alpha})$ and $\tilde{\omega}_2(B_n^{-2\alpha})$ are larger than c for $n \geq n_0$ and some $c > 0$.

Now we define states on $\mathcal{B}(\mathcal{H})$ by

$$\rho^n(A) = \tilde{\omega}(B_n^{-\alpha} A B_n^{-\alpha}) / \tilde{\omega}(B_n^{-2\alpha}) \quad \text{for } n \geq n_0.$$

These states belong to $Z(\mathcal{B}(\mathcal{H}))$ as can be seen by the decomposition

$$\begin{aligned} \rho^n(A) &= [\lambda \tilde{\omega}_1(B_n^{-\alpha} A B_n^{-\alpha}) + (1-\lambda) \tilde{\omega}_2(B_n^{-\alpha} A B_n^{-\alpha})] / \tilde{\omega}(B_n^{-2\alpha}) \\ &= \mu_n \rho_1^n(A) + (1-\mu_n) \rho_2^n(A) \end{aligned}$$

with $\mu_n = \lambda \tilde{\omega}_1(B_n^{-2\alpha}) / \tilde{\omega}(B_n^{-2\alpha})$ and $\rho_i^n(A) = \tilde{\omega}_i(B_n^{-\alpha} A B_n^{-\alpha}) / \tilde{\omega}_i(B_n^{-2\alpha})$, $i=1, 2$.

From the properties of $\tilde{\omega}$ it follows that ρ_i^n are vector states and ρ_2^n are singular states on $\mathcal{B}(\mathcal{H})$, hence $\rho^n \in Z(\mathcal{B}(\mathcal{H}))$.

Moreover

$$\lim_{n \rightarrow \infty} \rho^n(A) = \rho(A) / \rho(I) \equiv \sigma(A) \quad \text{for all } A \in \mathcal{B}(\mathcal{H}),$$

so $\sigma \in Z(\mathcal{B}(\mathcal{H}))$ because this set is w^* -closed. By Theorem 3.2 there is an ultrafilter \mathcal{U} so that

$$\sigma(A) = \lim_{\mathcal{U}} \langle \psi_k, A \psi_k \rangle \quad \text{for all } A \in \mathcal{B}(\mathcal{H}).$$

From (1) it is seen that

$$\tilde{\omega}(A) = \rho(B^\alpha A B^\alpha) = \rho(I) \sigma(B^\alpha A B^\alpha) = \lim_{\mathcal{U}} \langle C \psi_k, A C \psi_k \rangle$$

with $C = B^\alpha \cdot (\rho(I))^{1/2} \in \mathcal{B}(\mathcal{D})$. The standard estimation (cf. [16]) for $X \in \mathcal{L}^+(\mathcal{D})$:

$$\begin{aligned} |\omega(X) - \lim_{\mathcal{U}} \langle C \psi_k, X C \psi_k \rangle| &\leq |\omega(X) - \tilde{\omega}(P_n X P_n)| + |\tilde{\omega}(P_n X P_n) \\ &\quad - \lim_{\mathcal{U}} \langle C \psi_k, X C \psi_k \rangle| \rightarrow 0 \quad \text{for } n \rightarrow \infty \end{aligned}$$

leads to the desired result

$$\omega(X) = \lim_{\mathcal{U}} \langle C \psi_k, X C \psi_k \rangle. \quad \text{q.e.d.}$$

Corollary 3.4. *The following inclusions are valid:*

$$P_0 \subset Z \subset V.$$

Epecially, every pure state on $\mathcal{L}^+(\mathcal{D})$ has a representation

$$\omega(A) = \lim_{\mathcal{C}J} \langle C\psi_k, AC\psi_k \rangle$$

for some ultrafilter \mathcal{U} and $C \in \mathcal{B}(\mathcal{D})$, (ψ_k) as in Theorem 3.3.

Proof. The first inclusion follows from [16], where it was proved that pure states are either vector states or pure singular states. The second inclusion is an immediate consequence of the representation theorem. The proof is the same as that of Corollary 4.6 in [16]. q.e.d.

Lemma 3.5. *The vector state space and the pure state space of $\mathcal{L}^+(\mathcal{D})$ coincide, i.e. $V=P$.*

Proof. Since any vector state is pure, $V \subset P$ follows (cf. [16]). On the other hand, let $\omega \in P$. For given $A_1, \dots, A_n \in \mathcal{L}^+(\mathcal{D})$, $\epsilon > 0$ there is a pure state ω' so that $|\omega(A_i) - \omega'(A_i)| < \epsilon/2$ for all i . If ω' is a vector state we are done. If not, ω' must be a singular state. But the singular states are contained in V ([16], Corollary 4.6), so there is a vector state ω'' with $|\omega'(A_i) - \omega''(A_i)| < \epsilon/2$ for all i , i.e. $|\omega(A_i) - \omega''(A_i)| < \epsilon$. This means $P \subset V$, hence $V=P$. q.e.d.

Now we prove the main result of the paper.

Theorem 3.6. *For $\mathcal{L}^+(\mathcal{D})$ one has $V=P=Z$.*

Proof. In view of Corollary 3.4 and Lemma 3.5 it remains to prove that $V \subset Z$. The proof uses the idea of Glimm [5]. Let $\omega \in V$ and $\omega = \lambda\omega_n + (1-\lambda)\omega_s$ the corresponding decomposition in normal and singular states ([16], Theorem 3.4). If $\lambda = \omega_n(I) = 0$, then $\omega_n = 0$ and we are done. So let $\lambda \neq 0$. It remains to prove that ω_n is a vector state. By [19] ω_n is $\tau_{\mathcal{D}}^c$ -continuous, i.e.

$$|\omega_n(A)| \leq \|CAC\| \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}) \text{ and some } C \in \mathcal{S}_{\infty}(\mathcal{D}).$$

Let $C = \int_0^c \lambda dE_{\lambda}$ and put $P_n = \int_{1/n}^c dE_{\lambda}$ (without restriction of generality suppose $C \geq I$). The P_n are finite dimensional and $P_n \mathcal{H} \subset \mathcal{D}$. Since $\omega \in V$, there is a net of unit vectors $\{\varphi_{\alpha}, \alpha \in J\} \subset \mathcal{D}$ so that $\omega = \sigma_0 - \lim \omega_{\alpha}$ with $\omega_{\alpha} = \langle \varphi_{\alpha}, \cdot \varphi_{\alpha} \rangle$. The first step in the proof is to construct a sequence $(\psi_n) \subset \mathcal{D}$ with

- i) $\omega_{\psi_k} = \lambda \omega_n (P_k \circ P_k)$
- ii) $P_k \psi_{k+1} = \psi_k$.

The set $\{P_1 \varphi_{\alpha}, \alpha \in J\}$ is $\|\cdot\|$ -bounded and contained in the finite dimensional subspace $P_1 \mathcal{H} \subset \mathcal{D}$. So there is a subnet $\{\varphi_{j(\alpha)}, \alpha \in J\}$ and $\psi_1 \in \mathcal{D}$ with $P_1 \varphi_{j(\alpha)} \xrightarrow{\|\cdot\|} \psi_1 = P_1 \psi_1$. Let us remark that $\{P_1 \varphi_{j(\alpha)}\}$ is a t -bounded set

and $P_1AP_1 \in \mathcal{S}_\infty(\mathcal{D}) \subset \mathcal{C}(\mathcal{D})$ for all $A \in \mathcal{L}^+(\mathcal{D})$. It is

$$\begin{aligned} \omega(P_1 \circ P_1) &= \lambda \omega_n(P_1 \circ P_1) + (1 - \lambda) \omega_s(P_1 \circ P_1) \\ &= \lambda \omega_n(P_1 \circ P_1) = \sigma_0 - \lim \omega_{j(\omega)}(P_1 \circ P_1) \\ &= \sigma_0 \lim \langle P_1 \varphi_{j(\omega)}, \circ P_1 \varphi_{j(\omega)} \rangle = \langle \psi_1, \circ \psi_1 \rangle. \end{aligned}$$

To avoid complicated notations let us denote the subnets again by $\{\varphi_\alpha, \alpha \in J\}$. Let $\psi_k \in \mathcal{D}$ be chosen so that $P_k \varphi_\alpha \xrightarrow{\parallel \parallel} \psi_k$, then the same considerations as above show that in view of the boundedness of $\{P_{k+1} \varphi_\alpha\}$ we find a $\psi_{k+1} \in \mathcal{D}$ with $P_{k+1} \varphi_\alpha \xrightarrow{\parallel \parallel} \psi_{k+1}$. Moreover from $P_k P_{k+1} = P_k$ we see that $P_k \psi_{k+1} = P_k (\lim P_{k+1} \varphi_\alpha) = \psi_k$. Thus the existence of the sequence (ψ_k) is established.

The second step is to prove that $\psi_k \xrightarrow{t} \psi \in \mathcal{D}$. Let $k > l$, then in view of $P_k \psi_k = \psi_k, P_l \psi_k = \psi_l, P_k P_l = P_l$:

$$\begin{aligned} \|A(\psi_k - \psi_l)\|^2 &= \|A(P_k - P_l)\psi_k\|^2 = \omega_{\psi_k}((P_k - P_l)A^+A(P_k - P_l)) \\ &= \lambda \omega_n(P_k((P_k - P_l)A^+A(P_k - P_l)P_k)) = \lambda \omega_n((P_k - P_l)A^+A(P_k - P_l)) \\ &\leq \lambda \|C(P_k - P_l)A^+A(P_k - P_l)C\| \leq 2\lambda \cdot \|C(P_k - P_l)\| \cdot \|A^+AC\| \rightarrow 0 \\ &\text{for } k, l \rightarrow \infty. \end{aligned}$$

This means (ψ_k) is a t -Cauchy sequence, so there is a $\psi \in \mathcal{D}$ $\psi_k \xrightarrow{t} \psi$. Moreover,

$$\|\psi\|^2 = \lim_{k \rightarrow \infty} \|\psi_k\|^2 = \lim \omega_{\psi_k}(I) = \lambda \cdot \lim \omega_n(P_k) = \lambda.$$

Now it is easy to see that $\omega_n = \omega_\varphi$ with $\varphi = \psi / \|\psi\|$. Indeed using $\omega_n(P_k A P_k) \rightarrow \omega_n(A)$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and $\psi_k \xrightarrow{t} \psi$ it follows that

$$\begin{aligned} |\omega_n(A) - \omega_\varphi(A)| &\leq |\omega_n(A) - \omega_n(P_k A P_k)| + |\omega_n(P_k A P_k) - (1/\lambda) \langle \psi, A \psi \rangle| \\ &= |\omega_n(A) - \omega_n(P_k A P_k)| + |(1/\lambda) \langle \psi_k, P_k A P_k \psi_k \rangle - \langle \psi, A \psi \rangle| \end{aligned}$$

which goes to zero for $k \rightarrow \infty$. Thus ω_n is a vector state and therefore $\omega \in Z$.
q.e.d.

In the second part of this section we add some results concerning σ_0 -sequentially completeness and closedness of some sets of functionals. Since the σ_0 -topology is not metrizable one has to work with nets to consider closedness or completeness. In the bounded case it is known that the set of normal functionals is weakly sequentially complete [1], [3], [18]. This result is not valid for $\mathcal{L}^+(\mathcal{D})$ as we shall see. Let us start with a lemma which is a weaker

variant of Theorem III.1 of [1].

Lemma 3.7. *Let $\omega_k = \omega_k^n + \omega^s$ be a sequence of $\tau_{\mathcal{D}}$ -continuous functionals on $\mathcal{L}^+(\mathcal{D})$, so that ω_k^n are normal and ω^s is fixed singular. Suppose $\omega_k(A) \rightarrow 0$ for all $A \in \mathcal{L}^+(\mathcal{D})$. Then $\omega^s \equiv 0$ and clearly $\omega_k^n \xrightarrow{\sigma_0} 0$.*

Proof. It is enough to prove that $\omega^s(P) = 0$ for all projections $P \in \mathcal{B}(\mathcal{D})$. Indeed, the linear space generated by these projections is $\tau_{\mathcal{D}}$ -dense in $\mathcal{B}(\mathcal{D})$ hence in $\mathcal{L}^+(\mathcal{D})$. Let $P \in \mathcal{B}(\mathcal{D})$, $P\mathcal{H} = \mathcal{H}_1 \subset \mathcal{D}$. Consider $\mathcal{B}(\mathcal{H}_1)$ as a *-subalgebra of $\mathcal{L}^+(\mathcal{D})$ and denote the restrictions of the functionals above by $\bar{\omega}_k, \bar{\omega}_k^n, \bar{\omega}^s$. Then they fulfil the conditions of [1], Theorem III.1. Therefore $\bar{\omega}^s \equiv 0$, especially $\bar{\omega}^s(I_{\mathcal{H}_1}) = \omega^s(P) = 0$. q.e.d.

This lemma can be used to prove a result which demonstrates in a nice way the difference between the convergence of nets and sequences.

Lemma 3.8. *The set V_0 of vector states on $\mathcal{L}^+(\mathcal{D})$ is σ_0 -sequentially complete.*

Proof. Let $(\omega_{\psi_n}) \subset V_0$ be a σ_0 -Cauchy sequence. Then considering $\omega_{\psi_n}(A^+A)$ it is seen that $\sup_n \|A\psi_n\| < \infty$ for all $A \in \mathcal{L}^+(\mathcal{D})$, i.e. (ψ_n) is t -bounded. Again by [8] there is a $B \in \mathcal{B}(\mathcal{D})$ and a sequence $(\varphi_n) \subset \mathcal{H}$, $\|\varphi_n\| \leq 1$ so that $B\varphi_n = \psi_n$. This leads to

$$|\omega_{\psi_n}(A)| = |\langle \psi_n, A\psi_n \rangle| = |\langle B\varphi_n, AB\varphi_n \rangle| \leq \|BAB\| \quad \text{for all } A \in \mathcal{L}^+(\mathcal{D}),$$

i.e. (ω_{ψ_n}) is an equicontinuous sequence. Because (ω_{ψ_n}) is a weak Cauchy sequence, by $\omega(A) = \lim_n \omega_{\psi_n}(A)$ there is defined a positive normed linear functional on $\mathcal{L}^+(\mathcal{D})$ which is $\tau_{\mathcal{D}}$ -continuous in view of the equicontinuity mentioned above. Therefore $\omega \in V = P = Z$ and $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, $\omega_1 \in V_0$, ω_2 -singular. The sequence $\omega_n = \omega - \omega_{\psi_n} = (\lambda \cdot \omega_1 - \omega_{\psi_n}) + (1-\lambda)\omega_2$ fulfils the assumptions of Lemma 3.7. Hence $(1-\lambda)\omega_2 = 0$ and so $\omega = \omega_1 = \omega_{\psi}$ for some $\psi \in \mathcal{D}$, $\|\psi\| = 1$. q.e.d.

Let us add some simple remarks.

Remarks 3.9. i) The proof of Lemma 3.8 can be a little bit modified to show that the set of all positive vector functionals $\{\omega_{\varphi} = \langle \varphi, \cdot \varphi \rangle, \varphi \in \mathcal{D}\}$ is σ_0 -sequentially complete.

ii) It is trivial that $\psi_n \xrightarrow{t} \psi$ implies $\omega_{\psi_n} \xrightarrow{\sigma_0} \omega_{\psi}$. What about the converse? Let $\|\psi_n\| = \|\psi\| = 1$, $\omega_{\psi_n} \rightarrow \omega_{\psi}$ as in Lemma 3.8. Then (ψ_n) is t -bounded. The

weak compactness of the unit ball in \mathcal{H} implies the existence of a subsequence (ψ_{n_k}) which is weakly convergent, say to $\chi \in \mathcal{H}$, i.e.

$$\langle \varphi_{n_k}, \varphi \rangle \rightarrow \langle \chi, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{H}, \text{ hence } \psi_{n_k} \xrightarrow{\|\cdot\|} \chi.$$

This implies $\|\chi\|=1$. Moreover, put $\varphi = \psi$, then $|\langle \psi_{n_k}, \psi \rangle|^2 \rightarrow |\langle \chi, \psi \rangle|^2$ and at the same time $|\langle \psi_{n_k}, \psi \rangle|^2 = \omega_{\psi_{n_k}}(P_\psi) \rightarrow \omega_\psi(P_\psi) = \|\psi\|^2 = 1$. This means $|\langle \chi, \psi \rangle| = 1$ and consequently $\chi = \lambda\psi$ with $|\lambda| = 1$.

This can be interpreted as follows: $\omega_{\psi_n} \xrightarrow{\sigma_0} \omega_\psi$ implies that any $\|\cdot\|$ -convergent subsequence of (ψ_n) converges to the same element $[\psi]$ in the projective space $[\mathcal{H}]$ associated with \mathcal{H} , i.e. $[\mathcal{H}] = \mathcal{H}/\sim$, where \sim is the equivalence relation $\psi \sim \chi$ if and only if $\chi = \lambda\psi$ for some λ with $|\lambda| = 1$. This is quite natural because ω_ψ on $\mathcal{L}^+(\mathcal{D})$ determines only $[\psi]$ but not ψ uniquely.

Lemma 3.10. *The set of normal functionals on $\mathcal{L}^+(\mathcal{D})$ is σ_0 -sequentially closed but not σ_0 -sequentially complete.*

Proof. The first part follows from Lemma 3.7. Indeed, if $\omega_n \xrightarrow{\sigma_0} \omega$ and $\omega = \omega_1 + \omega_2$, ω_1 -normal, ω_2 -singular, then $\omega'_n = \omega - \omega_n = (\omega_1 - \omega_n) + \omega_2$ fulfils the conditions of Lemma 3.7, hence $\omega_2 = 0$ and ω is normal. To see the second part, consider $\psi \neq 0$, $\psi \in \mathcal{D}$, $\varphi \in \mathcal{H} \setminus \mathcal{D}$. Then there is a sequence $(\varphi_n) \subset \mathcal{D}$. $\varphi_n \xrightarrow{\|\cdot\|} \varphi$. The vector functionals $\omega_{\varphi_n, \psi}(A) = \langle \varphi_n, A\psi \rangle$ are $\tau_{\mathcal{D}}$ -continuous, $\omega_{\varphi_n, \psi}(A) \rightarrow \omega_{\varphi, \psi}(A) = \langle \varphi, A\psi \rangle$ for all $A \in \mathcal{L}^+(\mathcal{D})$. Because $\varphi \notin \mathcal{D}$ the functional $\omega_{\varphi, \psi}$ is not $\tau_{\mathcal{D}}$ -continuous [16] and hence not normal. q.e.d.

Remark 3.11. In the proof of Lemma 3.8 the equicontinuity of the set $\{\omega_{\psi_n}\}$ was important. In the bounded case the equicontinuity is automatically fulfilled for w^* -Cauchy sequences of normal functionals. This is not the case for $\mathcal{L}^+(\mathcal{D})$ as the sequence $(\omega_{\varphi_n, \psi})$ above shows. Indeed, suppose $|\langle \varphi_n, A\psi \rangle| \leq \|BAB\|$. This would imply that (φ_n) is t -bounded since for $P_{\psi, \varphi_n} \in \mathcal{L}^+(\mathcal{D})$: $|\langle \varphi_n, A^+AP_{\psi, \varphi_n}\psi \rangle| = \|\psi\|^2 \|A\varphi_n\|^2 \leq \|BA^+AP_{\psi, \varphi_n}B\| \leq \|BA^+A\| \cdot \|B\varphi_n\| \cdot \|\psi\|$. But (φ_n) is $\|\cdot\|$ -bounded, so $\|B\varphi_n\| \leq C$ because $B \in \mathcal{B}(\mathcal{D})$. Thus (φ_n) is t -bounded and $\varphi_n \xrightarrow{\|\cdot\|} \varphi$ which implies $\varphi_n \xrightarrow{t} \varphi$ by [14]. This is a contradiction.

Now one could give several conditions which would imply that σ_0 -Cauchy sequences of normal functionals have a normal functional as limit. But we will not push this further. Let us only give a corollary to Lemma 3.10 for the case where $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ is a bornological space. A large class of examples

of such (F)-domains \mathcal{D} was given in [20].

Corollary 3.12. *Let $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ be a bornological space. Then the set of positive normal functionals is σ_0 -sequentially complete. Especially the set of normal states is σ_0 -sequentially complete.*

Proof. Under the assumptions above $\tau_{\mathcal{D}}$ coincides with the order topology and every positive functional on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$ -continuous [20]. On the other hand, if $\omega_n(A) \rightarrow \omega(A)$ for all $A \in \mathcal{L}^+(\mathcal{D})$ and $\omega_n \geq 0$, normal, then ω is positive, hence $\tau_{\mathcal{D}}$ -continuous and the assertion follows from Lemma 3.10.

q.e.d.

We close this section with two results about singular functionals. The first one is analogously to [1], Theorem III,5 for $\mathcal{L}^+(\mathcal{D})$.

Lemma 3.13. *Let (ω_k) be a sequence of singular functionals on $\mathcal{L}^+(\mathcal{D})$. Then any weak limit point of (ω_k) is singular.*

Proof. Let ω be a weak limit point of (ω_k) and let (ω_α) be a subnet so that $\omega_\alpha \xrightarrow{\sigma_0} \omega = \omega^n + \omega^s$. Again (cf. Lemma 3.7) it is enough to prove that $\omega^n(P) = 0$ for all projections $P \in \mathcal{B}(\mathcal{D})$. Let $\bar{\omega}, \bar{\omega}^n, \bar{\omega}^s$ be the restrictions of the functionals above to $\mathcal{B}(\mathcal{A}_1) \subset \mathcal{L}^+(\mathcal{D})$, $\mathcal{A}_1 = P\mathcal{A} \subset \mathcal{D}$. Then these functionals fulfil the assumptions of [1], Theorem III.5, so $\bar{\omega}^n \equiv 0$, i.e. $\omega^n(P) = 0$. q.e.d.

Corollary 3.14. *The set of singular functionals on $\mathcal{L}^+(\mathcal{D})$ is σ_0 -sequentially closed.*

§ 4. The State Space of Op*-algebras

In this section we start the investigation of the state space of general Op*-algebras.

Remember that in the C*-theory the fact that the unit ball in the dual space is w^* -compact allows to apply the Krein-Milman theorem. This leads to the well-known result that the state space of a C*-algebra is the w^* -closed convex hull of pure states.

In contrast to this in the unbounded case the state space is not w^* -compact, even not w^* -bounded if the Op*-algebra under consideration contains unbounded operators. This can be seen by the following simple example. Let $A = A^+ \in \mathcal{A}(\mathcal{D})$ be unbounded. Then there is a sequence $(\varphi_n) \subset \mathcal{D}$, $\|\varphi_n\| = 1$ with $\|A\varphi_n\| \rightarrow \infty$. By $\omega_n = \langle \varphi_n, \cdot \varphi_n \rangle$ we get a sequence of well-defined vector states, but $\omega_n(A^+A) = \|A\varphi_n\|^2 \rightarrow \infty$. So (ω_n) is not w^* -bounded.

Nevertheless there can be derived some results which correspond to those in the bounded case. But one has to take into account some refinements. On the one hand the topology $\tau_{\mathcal{D}}$ does not play an exceptional role in what follows. On the other hand in the proofs it is essential that states are strongly positive functionals. For $\mathcal{L}^+(\mathcal{D})$ the positive and strongly positive $\tau_{\mathcal{D}}$ -continuous functionals coincide [16]. But this is in general not the case for arbitrary Op^* -algebras or other topologies. So in what follows on $\mathcal{L}^+(\mathcal{D})$ or $\mathcal{A}(\mathcal{D})$ there can be taken any locally convex topology τ so that the vector states are τ -continuous and states are supposed to be strongly positive, normed and τ -continuous functionals.

The first proposition we are going to prove is Lemma 3.4.1. ii) of [4] for Op^* -algebras. $E(\mathcal{A})$ is now thought to be in the context of a topology τ just mentioned.

Proposition 4.1. *Let $\mathcal{A}(\mathcal{D})$ be an Op^* -algebra, $Q \subset E(\mathcal{A})$ a subset with the property: if $A \in \mathcal{A}(\mathcal{D})_h$ (hermitean part) and $\omega(A) \geq 0$ for all $\omega \in Q$, then $A \geq 0$. Under these assumptions the w^* -closed convex hull of Q coincides with $E(\mathcal{A})$.*

Proof. The proof is the same as in [4]. Since $E(\mathcal{A})$ is convex and w^* -closed, the w^* -closed convex hull Q_1 of Q is contained in $E(\mathcal{A})$. Now let Q^0 be the polar of Q in $\mathcal{A}(\mathcal{D})_h$. If $A \in \mathcal{A}(\mathcal{D})_h$, then

$$\begin{aligned} A \in Q^0 &\text{ if and only if } \omega(A) \leq 1 \text{ for all } \omega \in Q \\ &\text{ if and only if } \omega(I-A) \geq 0 \text{ for all } \omega \in Q \\ &\text{ if and only if } I-A \geq 0 \text{ if and only if } \omega(A) \leq 1 \text{ for} \\ &\text{ all } \omega \in E(\mathcal{A}) \text{ if and only if } A \in E(\mathcal{A})^0. \end{aligned}$$

Hence $Q^0 = E(\mathcal{A})^0 = Q_1^0$. Then by the bipolar theorem:

$$Q^{00} = E(\mathcal{A}^{00}) = Q_1^{00} = \overline{\text{co}(Q_1 \cup \{0\})} = \overline{\text{co}(E(\mathcal{A}) \cup \{0\})}.$$

Since Q_1 and $E(\mathcal{A})$ are convex and w^* -closed this implies

$$\text{co}(Q_1 \cup \{0\}) = \text{co}(E(\mathcal{A}) \cup \{0\}), \quad \text{hence } Q_1 = E(\mathcal{A}).$$

To see the last implication suppose $\omega \in E(\mathcal{A})$. Then using the convexity of Q_1 there are $\rho \in Q_1$, $\lambda \in [0, 1]$ so that $\omega = \lambda\rho + (1-\lambda) \cdot 0 = \lambda\rho$. Since $\omega(I) = \rho(I) = 1$ it follows that $\lambda = 1$ hence $\rho = \omega \in Q_1$ q.e.d.

An example of such a set Q is $V_0(\mathcal{A})$ the set of vector states. Therefore we obtain as a conclusion:

Corollary 4.2. *Let $\mathcal{A}(\mathcal{D})$ be an Op^* -algebra. Then $E(\mathcal{A})$ is the w^* -closed*

convex hull of the set of vector states on $\mathcal{A}(\mathcal{D})$.

In the C^* -theory the following fact is well-known [4]. Let \mathcal{A} be a C^* -algebra, say with unit, which acts irreducible on \mathcal{H} , then the vector states are pure. In the C^* -case the notion of irreducibility is unambiguous. In contrast to this for general Op^* -algebras many different notions of irreducibility can be given. The weakest seems to be the triviality of the *weak commutant* [17]:

$$\mathcal{A}(\mathcal{D})' = \{T \in \mathcal{B}(\mathcal{H}) : \langle \varphi, TA\psi \rangle = \langle A^+\varphi, T\psi \rangle \text{ for all } \varphi, \psi \in \mathcal{D}, A \in \mathcal{A}(\mathcal{D})\}.$$

If $\mathcal{A}(\mathcal{D})$ is selfadjoint the weak commutant coincides with the strong commutant $\mathcal{A}(\mathcal{D})'_s = \{T \in \mathcal{B}(\mathcal{H}) : AT\varphi = TA\varphi \text{ for all } \varphi \in \mathcal{D} \text{ and } A \in \mathcal{A}(\mathcal{D})\}$. Moreover, in this case both sets are von Neumann algebras (cf. e.g. [17]. Clearly, $\mathcal{A}(\mathcal{D})' \subset \mathcal{L}^+(\mathcal{D})$. We will use here the following somewhat stronger notion of irreducibility [15]:

Definition 4.3. An Op^* -algebra $\mathcal{A}(\mathcal{D})$ is said to be *topologically irreducible* if the set $\mathcal{D}_\varphi = \{A\varphi : A \in \mathcal{A}(\mathcal{D})\}$ is $t_{\mathcal{A}}$ -dense in \mathcal{D} for all non-zero $\varphi \in \mathcal{D}$ (i.e. any such φ is a strongly cyclic vector for $\mathcal{A}(\mathcal{D})$).

It is not our intention to analyse here the whole hierarchy of possible irreducibility notions. Let us only remark that topological irreducibility implies the triviality of the weak commutant for selfadjoint Op^* -algebras. This follows immediately from the fact that a non-trivial commutant contains non-trivial projections P . But than the non-zero vectors from $P\mathcal{D}$ can not be strongly cyclic.

Proposition 4.4. Let $\mathcal{A}(\mathcal{D})$ be a selfadjoint, topologically irreducible Op^* -algebra. Then every vector state on $\mathcal{A}(\mathcal{D})$ is pure.

Proof. Suppose $0 \leq \omega \leq \omega_\varphi$, that is $\omega(X) \leq \langle \varphi, X\varphi \rangle$ for all $X \geq 0, X \in \mathcal{A}(\mathcal{D})$. On $\mathcal{D}_\varphi \times \mathcal{D}_\varphi$ consider the sesquilinear form

$$(\psi, \chi) = \omega(B^+A) \quad \text{for } \psi = B\varphi, \chi = A\varphi, A, B \in \mathcal{A}(\mathcal{D}).$$

It is easy to check that $(,)$ is correctly defined, positive and moreover

$$|(\psi, \psi)| = |\omega(B^+B)| \leq |\omega_\varphi(B^+B)| = \|\psi\|^2.$$

This estimation can be continued onto $\mathcal{H} \times \mathcal{H}$ because \mathcal{D}_φ is $\|\cdot\|$ -dense in \mathcal{H} . Thus there exists a positive operator $T \in \mathcal{B}(\mathcal{H})$ with

$$(\psi, \chi) = \langle \psi, T\chi \rangle \quad \text{for all } \psi, \chi \in \mathcal{H}$$

Now let $C \in \mathcal{A}(\mathcal{D})$. Then the two equalities

$$\begin{aligned} \langle \chi, TC\psi \rangle &= \langle A\varphi, TCB\varphi \rangle = (A\varphi, CB\varphi) = \omega(A^+CB) \quad \text{and} \\ \langle C^+\chi, T\psi \rangle &= \langle C^+A\varphi, TB\varphi \rangle = (C^+A\varphi, B\varphi) = \omega(A^+CB) \end{aligned}$$

imply that

$$\langle \chi, TC\psi \rangle = \langle C^+\chi, T\psi \rangle \quad \text{for all } \psi, \chi \in \mathcal{D}_\varphi.$$

From the $t_{\mathcal{L}}$ -density of \mathcal{D}_φ we conclude that this last equality is valid for all $\psi, \chi \in \mathcal{D}$. Hence $T \in \mathcal{A}(\mathcal{D})' = \mathcal{A}(\mathcal{D})'_s$. The topological irreducibility implies that $T = \lambda I$, and moreover $0 \leq \lambda \leq 1$. This leads to the desired result since $\omega(X) = (\varphi, X\varphi) = \langle \varphi, TX\varphi \rangle = \lambda \langle \varphi, X\varphi \rangle = \lambda \omega_\varphi(X)$, $\forall X \in \mathcal{A}(\mathcal{D})$. q.e.d.

- Corollary 4.5.** i) *Let $\mathcal{A}(\mathcal{D})$ be a selfadjoint, topologically irreducible Op^* -algebra. Then $E(\mathcal{A})$ is the w^* -closed convex hull of pure states.*
 ii) *Let $\mathcal{L}^+(\mathcal{D})$ be selfadjoint, the E is the w^* -closed convex hull of pure states.*

Remark 4.6. By quite other methods Corollary 4.5.ii) was obtained in [16] for the case that $\mathcal{D}[t]$ is an (F) -space and the topology under consideration is $\tau_{\mathcal{D}}$.

Let us further remark that the representation of positive functionals dominated by vector functionals (cf. Proposition 4.4) is not new. It is well-known in the bounded case and also used in the unbounded case in several versions (for one possibility see [24]).

At the end let us summarize some of the structure properties of the state space E of $\mathcal{L}^+(\mathcal{D})$ in the case that $\mathcal{L}^+(\mathcal{D})$ is selfadjoint, $\mathcal{D}[t]$ is an (F) -space and the topology on $\mathcal{L}^+(\mathcal{D})$ is $\tau_{\mathcal{D}}$.

- Proposition 4.7.** i) *E is the w^* -closed convex hull of V_0 . V_0 is w^* -sequentially complete. If $\mathcal{D}[t]$ is a Montel space, then V_0 is w^* -closed.*
 ii) *The normal states are w^* -sequentially closed and at the same time w^* -dense in E .*

Proof. The first two assertions of i) are already proved. The last assertion follows from Theorems 3.3 and 3.6 and the fact that the Montel property implies that there are no singular states on $\mathcal{L}^+(\mathcal{D})$.

The first part of ii) is the content of Lemma 3.10, while the second part follows from i). q.e.d.

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