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Criteria for Hypoellipticity of Differential Operators

By

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Introduction and Main Results

In the previous paper [12] (see also [10]), the author has given a sufficient condition for the hypoellipticity of differential operators of second order. It was also proved there that the sufficient condition is necessary for a special class of differential operators. The result about the necessity was extended to operators of higher order. The main purpose of the present paper is to extend the sufficient condition given in [12] to be applicable for differential operators of higher order.

Let $P=p(x, D_x)$ be a differential operator of order $m\geqq 1$ with coefficients in $\mathscr{B}^\infty(\mathbb{R}^n)$. Here $\mathscr{B}^\infty(\mathbb{R}^n)$ denotes the set of $C^\infty(\mathbb{R}^n)$-functions whose derivatives of any order are all bounded in $\mathbb{R}^n$. Let Λ and $\log \Lambda$ denote pseudodifferential operators with symbols $\langle\xi\rangle$ and $\log\langle\xi\rangle$, respectively, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. Set $\|u\|_s=\|\Lambda^s u\|$ for real s and $u\in C_0^\infty(\mathbb{R}^n)$, where $\|\cdot\|$ denotes the L^2 norm. We write $p_{(\beta)}^{(\alpha)}(x, \xi)=\partial_\xi^\alpha D_x^\beta p(x, \xi)$ for multi-indices α and β. About the other notations used in the present paper we refer to Kumano-go [5].

Theorem 1. *Assume that for any $\varepsilon>0$ and any compact set K of $\mathbb{R}^n$ there exists a constant $C_{\varepsilon,K}$ such that*

$$(1)\qquad \|(\log \Lambda)^m u\|+\sum_{0<|\alpha+\beta|<m}\|(\log \Lambda)^{|\alpha+\beta|}P_{(\beta)}^{(\alpha)}u\|_{-|\beta|} \leqq \varepsilon\|Pu\|+C_{\varepsilon,K}\|u\|, \qquad u\in C_0^\infty(K),$$

where m is the order of P and $P_{(\beta)}^{(\alpha)}=p_{(\beta)}^{(\alpha)}(x, D_x)$. Then P is hypoelliptic in $\mathbb{R}^n$. Furthermore we have

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$$(2) \qquad \mathrm{WF}\,Pv = \mathrm{WF}\,v \qquad \textit{for} \quad v \in \mathcal{D}'(\boldsymbol{R}^n)\,.$$

As a corollary to our main theorem, we have Theorem 1 of [12].

Corollary 2. *Let P be a differential operator of second order, that is,*

$$P = \sum_{j,k} a_{jk}(x) D_j D_k + \sum_j i b_j(x) D_j + c(x)\,, \quad D_j = -i\partial_{x_j}\,.$$

We assume that

$$(*) \qquad \begin{cases} a_{jk} \text{ and } b_j \text{ are real valued,} \\ \sum a_{jk}(x)\xi_j\xi_k \geqq 0 \qquad \text{for any} \quad (x, \xi) \in \boldsymbol{R}^{2n}\,. \end{cases}$$

If for any $\varepsilon>0$ and any compact set K of $\boldsymbol{R}^n$ the estimate

$$(3) \qquad \|(\log \Lambda)^2 u\| \leqq \varepsilon \|Pu\| + C_{\varepsilon,K} \|u\|\,, \qquad u \in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$ then we have (2).

Theorem 1 seems to be applicable to a large class of hypoelliptic operators with finite degeneracy because for such operators we generally have the subelliptic estimate

$$\|u\|_\kappa + \sum_{0<|\alpha+\beta|<m} \|P_{(\beta)}^{(\alpha)} u\|_{\kappa-|\beta|} \leqq C_K(\|Pu\| + \|u\|)\,, \qquad u \in C_0^\infty(K)$$

for some $\kappa>0$. (cf. [2], [13]). Theorem 1 also applies to a class of elliptic operators with infinite degeneracy. In fact, as discussed in [12] and [11], Theorem 1 (Corollary 2) is applicable to show the hypoellipticity of a second order differential operator

$$L_1 = D_{x_1}^2 + D_{x_2}^2 + \exp(-1/|x_1|^\delta) D_y^2\,, \qquad 0<\delta<1\,.$$

Now we have:

Proposition 3. *Let L be a differential operator*

$$D_{x_1}^{2l} + D_{x_2}^{2l} + \exp(-1/|x_1|^\delta) D_y^{2l} \qquad \textit{in} \quad \boldsymbol{R}^3\,,$$

where $\delta>0$ and $l=1, 2, \cdots$. Then L is hypoelliptic in $\boldsymbol{R}^3$ if (and only if)

$$(4) \qquad \delta < 1\,.$$

Remark 1. The proposition for the case $l=1$ was first proved by Kusuoka-Strook [6] (see Theorem 8.41 of [6]), by using the Malliavin calculus. About the Malliavin calculus we refer to Malliavin [7] and Ikeda-Watanabe [4].

Remark 2. The necessity of (4) follows from Theorem 3 of [9]. For the proof of the necessity of (4) we can also employ Theorem 3 of [12] and Theorem 2 of Hoshiro [3].

Remark 3. It should be noted that a differential operator $\mathcal{A}=D_{x_1}^{2l}+\exp(-1/|x_1|^{\delta})D_y^{2l}$ is hypoelliptic in $\mathbb{R}^2$ for any $\delta>0$ (see Theorem 1.1 of [8], cf. Fediĭ [1]).

Unfortunately, we can not apply Theorem 1 directly to the proof of Proposition 3, because it is quite hard to check the hypothesis (1) for L, more precisely, to show for the case $l\geqq 2$

$$\|(\log \Lambda)D_{x_1}^{2l-1}u\|\leqq\varepsilon\|Lu\|+C_{\varepsilon,K}\|u\|\,, \qquad u\in C_0^\infty(K)\,.$$

So, we need the following amelioration of Theorem 1 under an additional assumption.

Theorem 4. *Assume that the principal symbol $p_m(x, \xi)$ of P satisfies*

$$(5)\qquad p_m(x, \xi)\neq 0 \qquad \textit{for}\quad x'\neq 0,\ \textit{where}\ x=(x', x'')\,.$$

Then the conclusion (2) *of Theorem* 1 *still holds when for any $\varepsilon>0$ and any compact set K of $\mathbb{R}^n$ the estimate*

$$(6)\qquad \|(\log \Lambda)^m u\|+\sum_{\substack{0<|\alpha+\beta|<m\\ \alpha=(0,\alpha'')}}\|(\log \Lambda)^{|\alpha+\beta|}P_{(\beta)}^{(\alpha)}u\|_{-|\beta|} \leqq\varepsilon\|Pu\|+C_{\varepsilon,K}\|u\|\,, \qquad u\in C_0^\infty(K)$$

holds with a constant $C_{\varepsilon,K}$.

The hypoelliptic operator $\mathcal{A}$ with $\delta\geqq 1$ in Remark 3 is not covered by Theorem 4, because the estimate (6) does not holds for some small $\varepsilon>0$ (see [9] and Remark 3.1 of [12]). To cover this exceptional example we give another criterion of hypoellipticity.

Theorem 5. *Assume that the principal symbol $p_m(x, \xi)$ of P satisfies* (5). *If for any compact set K of $\mathbb{R}^n$ there exist a $\kappa_0>0$ and a constant C_K such that*

$$(7)\qquad \|u\|+\sum_{\substack{0<|\alpha+\beta|\leqq m\\ \alpha=(0,\alpha'')}}\|P_{(\beta)}^{(\alpha)}u\|_{\kappa_0-|\beta|} \leqq C_K(\|Pu\|+\|u\|_{-1})\,, \qquad u\in C_0^\infty(K)\,,$$

then we have (2).

This paper consists of three sections. In Section 1 we prove Theorem 1 and Corollary 2. Section 2 is devoted to proofs of Theorems 4 and 5. In

Section 3 we give the proof of Proposition 3 and show that the condition (7) is satisfied for $\mathcal{A}$ given in Remark 3.

§ 1. Proofs of Theorem 1 and Corollary 2

First we shall prove Corollary 2, admitting Theorem 1. By Theorem 1 it suffices to show that $\|(\log \Lambda)P^{(\alpha)}u\|$ $(|\alpha|=1)$ and $\|(\log \Lambda)P_{(\beta)}u\|_{-1}$ $(|\beta|=1)$ are estimated above by the right hand side of (1). As proved in [13], it follows from the assumption (*) that for any compact set K we have

$$\sum_{|\alpha|=1}\|P^{(\alpha)}u\|^2\leqq C_K(\mathrm{Re}\,(Pu, u)+\|u\|^2)\,, \qquad u\in C_0^\infty(K)\,.$$

Indeed, this estimate follows from (2.6.6) and (2.6.9) of [13]. For $u\in C_0^\infty(K)$, take ϕ, $\psi\in C_0^\infty(R^n)$ such that $\phi=1$ in a neighborhood of K and $\phi\subset\subset\psi$ (, that is, $\psi=1$ in a neighborhood of supp ϕ). Replace u in the above estimate by $\psi(\log \Lambda)\phi u\in C_0^\infty(K_0)$, where $K_0=\mathrm{supp}\,\psi$. Then we have

$$\sum_{|\alpha|=1}\|P^{(\alpha)}(\log \Lambda)u\|^2\leqq C'_K(\mathrm{Re}\,(P(\log \Lambda)u,\,(\log \Lambda)u)$$
$$+\|(\log \Lambda)u\|^2+\|u\|^2)\,, \qquad u\in C_0^\infty(K)\,,$$

because $(1-\psi)(\log \Lambda)\phi$ belongs to $S^{-\infty}$ (see Chapter 2 of [5]). Since the principal symbol of $[P, \log \Lambda]$ is purely imaginary we have

$$\mathrm{Re}([P, \log \Lambda]u,\,(\log \Lambda)u)\leqq C(\|(\log \Lambda)^2u\|^2+\|u\|^2)\,, \qquad u\in C_0^\infty(K)\,.$$

Above two estimates, together with Schwartz's inequality, show that the estimate

$$\sum_{|\alpha|=1}\|(\log \Lambda)P^{(\alpha)}u\|^2\leqq C''_K(\mathrm{Re}(Pu,\,(\log \Lambda)^2u)+\|(\log \Lambda)^2u\|^2+\|u\|^2)$$
$$\leqq\mu\|Pu\|^2+C_{\mu,K}(\|(\log \Lambda)^2u\|^2+\|u\|^2)\,, \qquad u\in C_0^\infty(K)$$

holds for any small $\mu>0$. Set $\varepsilon=(\mu/2C_{,\mu K})^{1/2}$ in (3). Then we obtain

$$\sum_{|\alpha|=1}\|(\log \Lambda)P^{(\alpha)}u\|^2\leqq 2\mu\|Pu\|^2+C'_{\mu,K}\|u\|^2\,, \qquad u\in C_0^\infty(K)\,.$$

Since it follows from (2.6.14) of [13] that we have

$$\sum_{|\beta|=1}\|\Lambda^{-1}P_{(\beta)}u\|^2\leqq C(\sum_{j=1}\mathrm{Re}(\Lambda^{-1}D_{x_j}Pu,\,\Lambda^{-1}D_{x_j}u)+\|u\|^2)\,, \qquad u\in C_0^\infty(K)\,,$$

the similar discussion as above shows that $\|(\log \Lambda)P_{(\beta)}u\|_{-1}$ $(|\beta|=1)$ is estimated above by the right hand side of (1). We have completed the proof of Corollary 2.

From now on we shall prove Theorem 1. Let $h(x)$ be a C_0^∞-function such that $h=1$ on $|x|\leqq 1/5$ and $\operatorname{supp} h\subset\{|x|<7/24\}$. Write

$$p(x, \xi) = \sum_{k=0}^{m} p_k(x, \xi),$$

where p_k is positively homogeneous in ξ of degree k. For $\gamma\equiv(x_0, \bar{\xi}_0)\in \mathbb{R}^n\times S^{n-1}$ we consider a microlocalized pseudodifferential operator of P as follows:

$$P_\gamma = p_\gamma(y, D_y; \lambda) = \sum_{k=0}^{m} p_k(x_0+\lambda y, \bar{\xi}_0+\lambda D_y)h(\lambda D_y/3)\lambda^{-2k} \tag{1.1}$$

where λ is a positive small parameter (see Hörmander [2] and Section 1 of [12]). It is clear that the symbol of P_γ, $p_\gamma(y, \eta; \lambda)$ $(\equiv\sigma(P_\gamma))$ satisfies for any α and β

$$|\partial_\eta^\alpha D_y^\beta p_\gamma(y, \eta; \lambda)|\leqq C_{\alpha\beta}\lambda^{-2m+|\alpha+\beta|}, \tag{1.2}$$

where $C_{\alpha\beta}$ is a constant independent of λ. From the estimate (1) we have:

Lemma 1.1. *For any $\varepsilon>0$ and any $\gamma\equiv(x_0, \bar{\xi}_0)\in\mathbb{R}^n\times S^{n-1}$ there exists a constant $C(\varepsilon, \gamma)$ such that*

$$\begin{aligned}(\log\lambda^{-1})^m\|Hv\|+\sum_{0<|\alpha+\beta|<m}\lambda^{|\alpha+\beta|}(\log\lambda^{-1})^{|\alpha+\beta|}\|HP_{\gamma(\beta)}^{(\alpha)}v\| \\ \leqq\varepsilon\|H_0P_\gamma v\|+C(\varepsilon, \gamma)\Big(\sum_{0<|\alpha+\beta|<m}\lambda^{|\alpha+\beta|}\|H_0P_{\gamma(\beta)}^{(\alpha)}v\|+\|v\|\Big), \\ v\in\mathcal{S}_y, \quad \text{if} \quad 0<\lambda\leqq 1,\end{aligned} \tag{1.3}$$

where $H=h(\lambda D_y)h(\lambda y)$ and $H_0=h(\lambda D_y/2)h(\lambda y/2)$. Here $P_{\gamma(\beta)}^{(\alpha)}$ are pseudodifferential operators with symbols $\sigma(P_{\gamma(\beta)}^{(\alpha)})\equiv\partial_\eta^\alpha D_y^\beta p_\gamma(y, \eta: \lambda)$.

Proof. Note that $\{h(\lambda^2\xi-\bar{\xi}_0);\ 0<\lambda\leqq 1\}$ is a bounded set in $S_{1,0}^1$, as a pseudodifferential operator in R_x^n, because $\lambda^2<(4/3)|\xi|^{-1}$ on $\operatorname{supp} h(\lambda^2\xi-\bar{\xi}_0)$. Replace u in (1) by $h(x-x_0)h(\lambda^2D_x-\bar{\xi}_0)w(=h(x-x_0)h(\lambda^2D_x-\bar{\xi}_0)h((\lambda^2D_x-\bar{\xi}_0)/3)w)$, $w\in\mathcal{S}_x$. Then, by means of the symbolic calculus of pseudodifferential operators we have

$$\begin{aligned}&\|(\log\Lambda)^m h(\lambda^2D_x-\bar{\xi}_0)h(x-x_0)w\| \\ &\quad+\sum_{0<|\alpha+\beta|<m}\|\Lambda^{-|\beta|}(\log\Lambda)^{|\alpha+\beta|}h(\lambda^2D_x-\bar{\xi}_0)h(x-x_0)P_{(\beta)}^{(\alpha)}h((\lambda^2D_x-\bar{\xi}_0)/3)w\| \\ &\leqq\varepsilon\|h((\lambda^2D_x-\bar{\xi}_0)/2)h((x-x_0)/2)Ph((\lambda^2D_x-\bar{\xi}_0)/3)w\| \\ &\quad+C_\varepsilon\Big(\sum_{0<|\alpha+\beta|<m}\|\Lambda^{-|\beta|}h((\lambda^2D_x-\bar{\xi}_0)/2)h((x-x_0)/2)P_{(\beta)}^{(\alpha)} \\ &\qquad h((\lambda^2D_x-\bar{\xi}_0)/3)w\|+\|w\|\Big), \qquad w\in\mathcal{S}_x,\end{aligned} \tag{1.4}$$

because $(\log\Lambda)^j\Lambda^{-1}\in S_{1,0}^0$ for $j=1, \cdots, m$ and $h(x/2)=1$ on $\operatorname{supp} h(x)$. Set

$w(x)=(\exp(i\lambda^{-2}x\cdot\bar{\xi}_0)v(\lambda^{-1}(x-x_0))$ for $v(y)\in\mathcal{S}_y$. Then we have

$$\exp(-i\lambda^{-2}x\cdot\bar{\xi}_0)p(x, D_x)h((\lambda^2 D_x-\bar{\xi}_0)/3)w(x) = (P_\gamma v)(\lambda^{-1}(x-x_0)) \tag{1.5}$$

by noting the change of variables.

$$x-x_0=\lambda y\,,\quad \xi-\lambda^{-2}\bar{\xi}_0=\lambda^{-1}\eta\,.$$

Similarly we have for any real s

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\bar{\xi}_0)\Lambda^s h(\lambda^2 D_x-\bar{\xi}_0)w(x)\\ &\quad=\lambda^{-2s}(q(D_y;\lambda)^s h(\lambda D_y)v)(\lambda^{-1}(x-x_0))\,,\end{aligned}\tag{1.6}$$

where $q(\eta;\lambda)=(\lambda^4+|\lambda\eta+\bar{\xi}_0|^2)^{1/2}$. It is clear that $q(\eta;\lambda)^s h(\lambda\eta)\geqq(2/3)^s h(\lambda\eta)$ and $\{q(\eta;\lambda)^s h(\lambda\eta/2);\ 0<\lambda\leqq 1\}$ is a bounded set in $S^0_{0,0}$, as a pseudodifferential operator in $\boldsymbol{R}^n_y$. Furthermore we have

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\bar{\xi}_0)(\log\Lambda)^j h(\lambda^2 D_x-\bar{\xi}_0)w(x)\\ &\quad=((\log\lambda^{-4}+r(D_y;\lambda))^j h(\lambda D_y)v)(\lambda^{-1}(x-x_0))\,,\end{aligned}\tag{1.7}$$

where $r(\eta;\lambda)=\log q(\eta;\lambda)$. Since $\{r(\eta;\lambda)^j h(\lambda\eta/2);\ 0<\lambda\leqq 1\}$ $(j=1,\cdots,m)$ are bounded sets in $S^0_{0,0}$ we have

$$||r(D_y;\lambda)^j h(\lambda D_y)v||\leqq m^{-2m}(\log\lambda^{-1})^j||h(\lambda D_y)v||\,,\qquad v\in\mathcal{S}_y\,,\tag{1.8}$$

if $0<\lambda\leqq\lambda_1$ for a sufficiently small λ_1. In view of (1.5)–(1.8), it follows from (1.4) that

$$\begin{aligned}&(\log\lambda^{-1})^m||Hv||+\sum_{0<|\alpha+\beta|<m}\lambda^{2|\beta|}(\log\lambda^{-1})^{|\alpha+\beta|}||H(P^{(\alpha)}_{(\beta)})_\gamma v||\\ &\quad\leqq m\varepsilon||H_0P_\gamma v||+C'_\varepsilon(\sum_{0<|\alpha+\beta|<m}\lambda^{2|\beta|}||H_0(P^{(\alpha)}_{(\beta)})_\gamma v||+||v||)\,,\\ &\quad v\in\mathcal{S}_y\,,\ \text{if}\ 0<\lambda\leqq\lambda_1\,,\end{aligned}\tag{1.9}$$

where $(P^{(\alpha)}_{(\beta)})_\gamma$ is defined by the same formula as (1.1) with $p(x,\xi)$ replaced by $p^{(\alpha)}_{(\beta)}(x,\xi)$. Note that

$$\sigma(P_{\gamma}{}^{(\alpha)}_{(\beta)})\equiv\partial^\alpha_\eta D^\beta_y p_\gamma(y,\eta;\lambda)=\lambda^{-|\alpha|+|\beta|}\sigma((P^{(\alpha)}_{(\beta)})_\gamma)\qquad\text{if}\quad|\lambda\eta|\leqq 3/5\,.$$

Since $\operatorname{supp} h(\lambda\eta)$ and $\operatorname{supp} h(\lambda\eta/2)$ are contained in $\{\eta;\ |\lambda\eta|\leqq 3/5\}$, the symbolic calculus of pseudodifferential operators shows that (1.3) follows from (1.9) if $0<\lambda\leqq\lambda_1$. The estimate (1.3) for $\lambda_1<\lambda\leqq 1$ is trivial. Q.E.D.

For a real $\kappa>0$ and an integer $k>0$ we denote by $\Lambda_{\kappa,k}$ a pseudodifferential operator with a symbol $(1+\kappa\langle\xi\rangle)^{-k}$. It is easy to check that for any α the estimate

(1.10) $$|\partial_\xi^\alpha((1+\kappa\langle\xi\rangle)^{-k})|\leqq C_\alpha\langle\xi\rangle^{-|\alpha|}(1+\kappa\langle\xi\rangle)^{-k}$$

holds with a constant C_α independent of κ. Set

(1.11) $$k_\kappa(\eta;\lambda)=(1+\kappa\langle\lambda^{-2}\xi_0+\lambda^{-1}\eta\rangle)^{-k}h(\lambda\eta)\,.$$

Then it follows from (1.10) that for any α the estimate

(1.12) $$|\partial_\eta^\alpha k_\kappa(\eta;\lambda)|\leqq C'_\alpha\lambda^{|\alpha|}k_\kappa(\eta;\lambda)\,,\qquad \lambda|\eta|\leqq 1/5$$

holds with another constant C'_α independent of κ and λ.

Set $h_\delta(x)=h(x/\delta)$ for a small $0<\delta\leqq 1/10$. Fix an integer $N>m$. Take a sequence $\{h_\delta^j\}_{j=1}^{N-m+1}\subset C_0^\infty(\boldsymbol{R}_x^n)$ such that

(1.13) $$h_\delta=h_\delta^1\subset\subset h_\delta^2\subset\subset\cdots\subset\subset h_\delta^{N-m+1}=h_{2\delta}$$

and for any α the estimate

(1.14) $$|D_x^\alpha h_\delta^j(x)|\leqq C''_\alpha N^{|\alpha|}\,,$$

holds with a constant C''_α independent of N and j ($C''_0=1$).

Lemma 1.2. *Write*

(1.15) $$\begin{aligned}&h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)h_\delta^{j+1}(\lambda D_y)h_\delta^{j+1}(\lambda y)\\&\quad=h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)+r(y,D_y;\lambda)\,.\end{aligned}$$

Then for any integer $l>0$ there exists a constant C_l independent of λ, κ and N such that

(1.16) $$\|r(y,D_y;\lambda)v\|\leqq C_l\lambda^{2l}N^{2l+2n+2}\|v\|\,,\qquad v\in\mathcal{S}_y\,.$$

Proof. Note (1.12) and (1.14). Then (1.16) follows from the symbolic calculus of pseudodifferential operators and the Carderon-Vaillancourt theorem (See Chapter 7 of [5]). Q.E.D.

As in [12] we state the following simple proposition:

Proposition 1.3. *Let N be a fixed positive integer and let λ satisfy $0<\lambda\leqq 1$. For any finite sequence of positive numbers $\{C_j\}_{j=1}^l$ there exists a constant C'_l such that*

(1.17) $$\sum_{j=1}^{l}C_j(N\lambda)^{2j}\leqq 1+C'_l(N\lambda)^{2l}\,.$$

Proof is omitted. (See [12]).

Lemma 1.4. *Set $\tilde{H}_\delta^j=h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)$ for $j=1,\cdots,N-m+1$. Then* *for any* $s>0$ *and any* $\gamma=(x_0, \bar{\xi}_0)\in \boldsymbol{R}^n\times S^{n-1}$ *there exists a constant* $C(s, \gamma)$ *independent of* λ, κ, j *and* N *such that*

$$(1.18)\qquad \begin{aligned}&(\log \lambda^{-s})^m\|\tilde{H}_\delta^j v\|+\sum_{0<|\alpha+\beta|<m}(\log \lambda^{-s})^{|\alpha+\beta|}\lambda^{|\alpha+\beta|}\|P_{\gamma(\beta)}^{(\alpha)}\tilde{H}_\delta^j v\|\\ &\leqq\|P_\gamma\tilde{H}_\delta^j v\|+C(s, \gamma)\lambda^s N^{s+2n+3m}\|v\|,\\ &\quad v\in\mathcal{S}_y\,,\ \textit{if}\ 0<\lambda\leqq\lambda_0(s, \gamma)\,,\end{aligned}$$

where $\lambda_0(s, \gamma)$ *is a sufficiently small positive number.*

Proof. Note that $\|v\|\leqq\|Hv\|+\|(1-H)v\|$ and that there exists a small $\lambda_0(\varepsilon, \gamma)$ ($\leqq 1/3$) such that

$$(1.19)\qquad (\log \lambda^{-1})\geqq C(\varepsilon, \gamma)/2 \qquad \text{if}\quad 0<\lambda\leqq\lambda_0(\varepsilon, \gamma)\,,$$

where $C(\varepsilon, \gamma)$ is the same constant as in (1.3). If $0<\lambda\leqq\lambda_0(\varepsilon, \gamma)$ we have the estimate (1.3) with ε, $C(\varepsilon, \gamma)$ and $\|v\|$ in the right hand side replaced by 2ε, $2C(\varepsilon, \gamma)$ and $\|(1-H)v\|$, respectively. Substitute $\tilde{H}_\delta^j v$ into this modified estimate. Then, in view of (1.19) we obtain (1.18) by setting $\varepsilon=s^{-m}/2$. Indeed, as in the proof of Lemma 1.2, it is easy to see that for any real $s>0$ there exists a C_s such that

$$\begin{aligned}&\|(1-H)\tilde{H}_\delta^j v\|+\|(1-H)P_{\gamma(\beta)}^{(\alpha)}\tilde{H}_\delta^j v\|+\|(1-H_0)P_{\gamma(\beta)}^{(\alpha)}\tilde{H}_\delta^j v\|\\ &+\|(1-H_0)P_\gamma\tilde{H}_\delta^j v\|\leqq C_s\lambda^s N^{s+2n+3m}\|v\|\,, \qquad v\in\mathcal{S}_y\,.\end{aligned}$$

Q.E.D.

Lemma 1.5. *For any* $s>0$ *there exists a constant* M *independent of* s, κ, λ *and* N *such that for* $j=1, \cdots, N-m$

$$(1.20)\qquad \begin{aligned}&\|P_\gamma\tilde{H}_\delta^j v\|\leqq M\|\tilde{H}_{2\delta}P_\gamma v\|+MN(\log \lambda^{-s})^{-1}\|P_\gamma\tilde{H}_\delta^{j+1} v\|\\ &\quad+C_s\lambda^s N^{s+2n+3m}\|v\|\,, \qquad v\in\mathcal{S}_y\,,\\ &\qquad \textit{if}\ \log\lambda^{-s}\geqq MN\ \textit{and}\ 0<\lambda\leqq\lambda_0(s, \gamma)\,,\end{aligned}$$

where C_s *is a constant independent of* λ, κ *and* N, *and* $\lambda_0(s, \gamma)$ *is the same as in Lemma* 1.4. *Here* $\tilde{H}_{2\delta}=h_{2\delta}(\lambda D_y)k_k(D_y;\lambda)h_{2\delta}(\lambda y)$.

Proof. It follows that

$$(1.21)\qquad \|P_\gamma\tilde{H}_\delta^j v\|\leqq\|\tilde{H}_\delta^j P_\gamma v\|+\|[P_\gamma, \tilde{H}_\delta^j]v\|\,.$$

Noting $h_\delta^j(x)=h_\delta^j(x)h_{2\delta}(x)$ and considering the expansion formula of the simplified symbol of $\tilde{H}_\delta^j$, we have

$$||\tilde{H}_\delta^j P_\gamma v|| = ||\tilde{H}_\delta^j h_{2\delta}(\lambda y) P_\gamma v||$$
$$\leqq (1+\sum_{q=1}^{[s/2]+m} C_q'(N\lambda)^{2q}) ||h_{2\delta}(\lambda D_y) k_\kappa h_{2\delta}(\lambda y) P_\gamma v|| + C_s \lambda^s N^{s+2n+3m} ||v||$$

for some constants C_q' and C_s. Using Proposition 1.3 we have

$$||\tilde{H}_\delta^j P_\gamma v|| \leqq 2||\tilde{H}_{2\delta} P_\gamma v|| + C_s \lambda^{s+2n+3m} ||v|| .$$

Here and in what follows we denote by the same notation C_s different constant independent of λ, κ and N (, depending on s). We shall estimate the second term of the right hand side of (1.21). In view of Lemma 1.2 it suffices to estimate $||[P_\gamma, \tilde{H}_\delta^j] H_\delta^{j+1} v||$, where $H_\delta^{j+1} = h_\delta^{j+1}(\lambda D_y) h_\delta^{j+1}(\lambda y)$. Write

$$[P_\gamma, \tilde{H}_\delta^j] = [P_\gamma, h_\delta^j(\lambda D_y) k_\kappa] h_\delta^j(\lambda y) + k_\kappa h_\delta^j(\lambda D_y)[P_\gamma, h_\delta^j(\lambda y)] .$$

Note that the expansion formula

$$[P_\gamma, h_\delta^j(\lambda D_y) k_\kappa] = \sum_{0<|\beta|\leqq[s/2]+m} (-1)^{|\beta|} (h_\delta^j(\lambda D_y) k_\kappa)^{(\beta)} P_{\gamma(\beta)}/\beta! + R(y, D_y; \lambda) ,$$

where R is an operator negligible, in the sense of

$$||Rv|| \leqq C_s \lambda N^{s+2n+3m} ||v|| .$$

In view of (1.2), (1.12) and (1.14), we see that there exists a constant M_1 independent of s, κ, λ and N such that

$$\begin{aligned}(1.22)\qquad & ||[P_\gamma, h_\delta^j(\lambda D_y) k_\kappa] h_\delta^j(\lambda y) H_\delta^{j+1} v|| \\ & \leqq \sum_{0<|\beta|<m} (M_1 N)^{|\beta|} \lambda^{|\beta|} ||k|_\kappa P_{\gamma(\beta)} h_\delta^j(\lambda y) H_\delta^{j+1} v|| \\ & \quad + (M_1 N)^m (1+\sum_{q=1}^{[s/2]} C_q'' N^q \lambda^{2q}) ||k_\kappa h_\delta^j(\lambda y) H_\delta^{j+1} v|| \\ & \quad + C_s \lambda^s N^{s+2n+3m} ||v||\end{aligned}$$

holds with some constants C_q''. Consider the expansion formula of the simplified symbol of $k_\kappa h_\delta^j(\lambda y)$ and use Proposition 1.3. Then the second term of the right hand side of (1.22) is estimated above by

$$(1.23)\qquad 2(M_1 N)^m ||\tilde{H}_\delta^{j+1} v|| + C_s \lambda^s N^{s+2n+3m} ||v|| .$$

Together with Proposition 1.3, the symbolic calculus shows that for any β $(0<|\beta|<m)$ the estimate

$$\begin{aligned}(1.24)\qquad & \lambda^{|\beta|} ||k_\kappa P_{\gamma(\beta)} h_\delta^j(\lambda y) H_\delta^{j+1} v|| \leqq \sum_{0\leqq|\tilde{\alpha}+\tilde{\beta}|<m-|\beta|} \lambda^{|\beta+\tilde{\alpha}+\tilde{\beta}|} (M_2 N)^{|\tilde{\alpha}+\tilde{\beta}|} \\ & \times ||P_{\gamma(\beta+\tilde{\beta})}^{(\tilde{\alpha})} k_\kappa H_\delta^{j+1} v|| + (M_2 N)^{m-|\beta|} ||\tilde{H}_\delta^{j+1} v|| + C_s \lambda^s N^{s+2n+3m-|\beta|} ||v||\end{aligned}$$

holds with a constant M_2 independent of s, κ, λ and N. Note (1.22)–(1.24) and use (1.18) with $j=j+1$. Then, taking a suitable M larger than M_1 and M_2, we see that $\|[P_\gamma, h_\delta^j(\lambda D_y)k_\kappa]h_\delta^j(\lambda y)H_\delta^{j+1}v\|$ is estimated above by the right hand side of (1.20) because $(MN)^j \leqq (\log \lambda^{-s})^j MN(\log \lambda^{-s})^{-1}$, $j=1, \cdots, m$ if $\log \lambda^{-s} \geqq MN$. Noting $\|k_\kappa h_\delta^j(\lambda D_y)\tilde{v}\| \leqq \|k_\kappa h_{2\delta}(\lambda D_y)\tilde{v}\|$ for $\tilde{v} \in \mathcal{S}_y$, we can also estimate $\|k_\kappa h_\delta^j(\lambda D_y)[P_\gamma, h_\delta^j(\lambda y)]H_\delta^{j+1}v\|$ by means of (1.18). We have estimated the second term of the right hand side of (1.21). So we obtain the desired estimate (1.20). Q.E.D.

By Lemmas 1.4 and 1.5 we have

Lemma 1.6. *For any real $s>0$, $\kappa>0$ and any integer $N>m$ there exists a constant M independent of s, κ, λ and N such that*

$$(1.25)\qquad (\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \leqq (\log \lambda^{-s})^N \|\tilde{H}_{2\delta} P_\gamma v\| + 2(MN)^N \lambda^{-2m}\|v\| + C_s N! N^{s+2n+3m}\|v\|, \qquad v \in \mathcal{S}_y, \text{ if } 0<\lambda \leqq \lambda_0(s, \gamma),$$

where C_s is a constant independent of λ, κ and N. Here $\tilde{H}_\delta = h_\delta(\lambda D_y)k_\kappa(D_y; \lambda) \times h_\delta(\lambda y)$ and $\lambda_0(s, \gamma)$ is the same as in Lemma 1.4.

Proof. In view of $\tilde{H}_\delta = \tilde{H}_\delta^1$ it follows from (1.18) that

$$(\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \leqq (\log \lambda^{-s})^{N-m}(\|P_\gamma \tilde{H}_\delta^1 v\| + C_s \lambda^s N^{s+2n+3m}\|v\|)\,.$$

Applying (1.20) to the first term of the right hand side. Then we have

$$(\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \leqq M(\log \lambda^{-s})^{N-m}\|\tilde{H}_{2\delta} P_\gamma v\| + MN(\log \lambda^{-s})^{N-m-1}\|P_\gamma \tilde{H}_\delta^2 v\| + 2C_s(\log \lambda^{-s})^{N-m}\lambda^s N^{s+2n+3m}\|v\| \qquad \text{if} \quad \log \lambda^{-s} \geqq MN\,.$$

Use (1.20) for the second term of the right hand side and moreover use (1.20) repeatedly $(N-m-2)$ times. Then we obtain

$$\begin{aligned}(1.26)\quad &(\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \\ &\leqq M \sum_{j=0}^{N-m-1} (\log \lambda^{-s})^{N-j-m}(MN)^j \|\tilde{H}_{2\delta} P_\gamma v\| \\ &\quad + (MN)^{N-m}\|P_\gamma \tilde{H}_\delta^{N-m+1} v\| \\ &\quad + (\log \lambda^{-s})^{N-m}\Big(1 + \sum_{j=0}^{N-m-1}(\log \lambda^{-s})^{-j}(MN)^j\Big) C_s \lambda^s N^{s+2n+3m}\|v\|\,, \\ &\qquad\qquad \text{if} \quad \log \lambda^{-s} \geqq MN\,.\end{aligned}$$

Since $h_\delta^{N-m+1} = h_{2\delta}$ and $\{\lambda^{2m}\sigma(P_\gamma \tilde{H}_\delta^{N-m+1});\ 0<\lambda \leqq 1\}$ is a bounded set in $S_{0,0}^0$, we have

$$\|P_\gamma \tilde{H}_\delta^{N-m+1} v\| \leqq M\lambda^{-2m}\|v\|\,,$$

by taking another large M if necessary. If $\log \lambda^{-s} \geqq MN$ it follows from (1.26) that

$$(\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \leqq (\log \lambda^{-s})^{N-m+1} \|\tilde{H}_{2\delta} P_\gamma v\| \\ +(MN)^N(\lambda^{-2m}+1)\|v\| + C_s(\log \lambda^{-s})^{N-m+1} \lambda^s N^{s+2n+3m}\|v\| .$$

When $\log \lambda^{-s} \leqq MN$ this estimate still holds because of the second term of the right hand side. Noting

$$(\log \lambda^{-s})^N \lambda^s = (\log \lambda^{-s})^N \exp(-\log \lambda^{-s}) \leqq N! , \quad \text{we obtain (1.25).}$$

Q.E.D.

By means of (1.18) with $j=1$ we see that there exists a constant M' independent of s, κ and λ such that for any $N=0, 1, \cdots, m$

$$(\log \lambda^{-s})^N \|\tilde{H}_\delta v\| \leqq M' \lambda^{-2m}\|v\| + C^s N^{s+2n+3m}\|v\| .$$

Therefore, by taking another large M if necessary, we may assume that (1.25) holds for any $N=0, 1, \cdots$. Let τ be a small parameter chosen later on. Multiply $\tau^N/N!$ by both sides of (1.25) and sum up with respect to $N=0, 1, \cdots$. Then we obtain

$$\lambda^{-s\tau}\|\tilde{H}_\delta v\| \leqq \lambda^{-s\tau}\|\tilde{H}_{2\delta} P_\gamma v\| + (2\lambda^{-2m} \sum_{N=0}^{\infty} (MN\tau)^N/N! + C_s \sum_{N=0}^{\infty} \tau^N N^{s+2n+3m})\|v\|$$

because $\sum(\tau \log \lambda^{-s})^N/N! = \lambda^{-s\tau}$. Choose τ such that $Me\tau<1$ and $0<\tau<1$. Then, by means of the stirling formula $N^N/N! \leqq e^N$ we have

$$\lambda^{-s\tau}\|\tilde{H}_\delta v\| \leqq \lambda^{-s\tau}\|\tilde{H}_{2\delta} P_\gamma v\| + C'_s \lambda^{-2m}\|v\| , \qquad v \in \mathcal{S}_y , \tag{1.27}$$

for another constant C'_s. Note that τ is independent of s because M is so. Hence we can replace $s\tau$ in (1.27) by $2s'+2s''+2m$ for any real s', $s''>0$. Multiply $\lambda^{2s''+2m}$ by (1.27) with $s\tau$ replaced by $2s'+2s''+2m$. Then we see that there exists a constant $C_0=C_0(s', s'', \gamma)$ independent of κ and λ such that

$$\lambda^{-2s'}\|\tilde{H}_\delta v\| \leqq \lambda^{-2s'}\|\tilde{H}_{2\delta} P_\gamma v\| + C_0 \lambda^{2s''}\|v\| , \\ v \in \mathcal{S}_y \text{ if } \lambda \text{ is sufficiently small.} \tag{1.28}$$

Taking another large C_0 if necessary, we may assume that (1.28) holds for $0<\lambda\leqq 1$.

Note that for any $\bar{\xi}_0 \in S^{n-1}$, any $0<\delta'\leqq 1$ and any real $\tilde{s}$ the estimate

$$C^{-1}\|h_{\delta'}(\lambda D_y)v\| \leqq \|q(D_y; \lambda)^{\tilde{s}} h_{\delta'}(\lambda D_y)v\| \\ \leqq C\|h_{\delta'}(\lambda D_y)v\| , \qquad v \in \mathcal{S}_y \tag{1.29}$$

holds for some constant $C=C(\tilde{s}, \delta')$ because

$$C^{-1}\leqq q(\eta;\lambda)^{\tilde{s}}\leqq C \qquad \text{on} \quad \operatorname{supp} h_{\delta'}(\lambda\eta)\,.$$

Substitute $v(y)=h(\lambda D_y)\tilde{v}(y)$ into (1.28) for $\tilde{v}(y)=\exp(-i\lambda^{-2}x\cdot\bar{\xi}_0)w(x)|_{x=\lambda y+x_0}$, $w\in\mathcal{S}_x$. Then in view of (1.5), (1.6) and (1.11) we see that there exists a constant C_0' such that

$$\begin{aligned}(1.30)\qquad &\|h_\delta(\lambda^2D_x-\bar{\xi}_0)\Lambda^{s'}\Lambda_{\kappa,k}h_\delta(x-x_0)w\|^2\\ &\leqq C_0'(\|h_{2\delta}(\lambda^2D_x-\bar{\xi}_0)\Lambda^{s'}\Lambda_{\kappa,k}h_{2\delta}(x-x_0)P(x,D_x)w\|^2\\ &\quad+\|h(\lambda^2D_x-\bar{\xi}_0)\Lambda^{-s''}w\|^2+\lambda\|w\|^2_{-s''})\,,\\ &\qquad w\in\mathcal{S}_x \qquad \text{if} \quad 0<\lambda\leqq 1\,.\end{aligned}$$

Here we used the fact that

$$\{\lambda^{-1/2}h_\delta(\lambda^2D_x-\bar{\xi}_0)h_\delta(x-x_0)(1-h(\lambda^2D_x-\bar{\xi}_0));\ 0<\lambda\leqq 1\}\,,$$
$$\{\lambda^{-1/2}h_{2\delta}(\lambda^2D_x-\bar{\xi}_0)h_{2\delta}(x-x_0)P(x,D_x)(1-h((\lambda^2D_x-\bar{\xi}_0)/3));\ 0<\lambda\leqq 1\}$$

are contained in a bounded set of $S^{-s''}_{1,0}$.

To complete the proof of Theorem 1 we define the following:

Definition 1.7. For $\delta>0$ and $\bar{\xi}_0\in S^{n-1}$ we say that a function $\psi(\xi)\in C^\infty(R^n)$ belongs to $\Psi_{\delta,\bar{\xi}_0}$ if ψ satisfies

$$\begin{cases}0\leqq\psi\leqq 1\,, \quad \psi(\xi)=1 \qquad \text{for} \quad |\xi/|\xi|-\bar{\xi}_0|\leqq\delta/12 \text{ and } |\xi|\geqq 1/2\,,\\ \psi(\xi)=0 \qquad \text{for} \quad |\xi/|\xi|-\bar{\xi}_0|\geqq\delta/10 \text{ or } |\xi|\leqq 1/3\,,\\ \psi(t\xi)=\psi(\xi) \qquad \text{for} \quad t\geqq 1 \text{ and } \xi\in S^{n-1}\,.\end{cases}$$

Devide both sides of (1.30) by λ and integrate with respect to λ from 0 to 1. Then, we see that for any $\gamma=(x_0,\bar{\xi}_0)\in\boldsymbol{R}^n\times S^{n-1}$, any real s', $s''>0$, any integer $k>0$ and any $\kappa>0$ there exists a constant $C''=C''(\gamma,s',s'',k)$ independent of κ such that

$$\begin{aligned}(1.31)\qquad &\|\psi_\delta(D_x)\Lambda_{\kappa,k}h_\delta(x-x_0)w\|^2_{s'}\\ &\leqq C''(\|\tilde{\psi}_\delta(D_x)\Lambda_{\kappa,k}h_{2\delta}(x-x_0)P(x,D_x)w\|^2_{s'}+\|w\|^2_{-s''})\,,\\ &\qquad w\in\mathcal{S}_x\,,\end{aligned}$$

if $\psi_\delta(\xi)\in\Psi_{\delta,\bar{\xi}_0}$ and $\tilde{\psi}_\delta(\xi)\in\Psi_{14\delta,\bar{\xi}_0}$. (See Proposition 1.7 of [12]).

We shall prove (2). Let $(x_0,\xi_0)\in T^*(\boldsymbol{R}^n)\backslash 0$ and let $u\in\mathcal{D}'(\boldsymbol{R}^n)$. Set $\bar{\xi}_0=\xi_0/|\xi_0|$. Suppose that $(x_0,\xi_0)\notin \mathrm{WF}\,Pu$. Then there exists a $\delta>0$ such that $\tilde{\psi}_\delta(D_x)h_{2\delta}(x-x_0)Pu\in H_{s'}$ for any real $s'>0$ if $\tilde{\psi}_\delta(\xi)\in\Psi_{14\delta,\bar{\xi}_0}$. Since $h_{4\delta}(x-x_0)u\in\mathcal{E}'$ we have $h_{4\delta}(x-x_0)u\in H_{-s''}$ for some $s''>0$. Choose $k>0$ in (1.31) such that $k\geqq s'+s''+m$. Then, by taking a sequence $\{w_j\}_{j=1}^\infty\subset\mathcal{S}_x$ such that

$$w_j \to h_{4\delta}(x-x_0)u \qquad \text{in} \quad H_{-s''},$$

from (1.31) we see that

$$\begin{aligned}(1.32)\qquad &\|\Lambda_{\kappa,k}\psi_\delta(D_x)h_\delta(x-x_0)u\|_{s'}^2\\ &\leqq C''(\|\tilde{\psi}_\delta(D_x)h_{2\delta}(x-x_0)Pu\|_{s'}^2+\|h_{4\delta}(x-x_0)u\|_{-s''}^2),\end{aligned}$$

if $\psi_\delta(\xi)\in\Psi_{\delta,\bar{\xi}_0}$ and $\tilde{\psi}_\delta(\xi)\in\Psi_{14\delta,\bar{\xi}_0}$. Here we used the fact that $\|\Lambda_{\kappa,k}w\|\leqq\|w\|$ for $w\in L^2$. Letting κ tend to 0 in (1.32), we have $\psi_\delta(D_x)h_\delta(x-x_0)u\in H_{s'}$. Since s' is arbitrary, we have $(x_0, \bar{\xi}_0)\notin \mathrm{WF}\, u$. Now the proof of Theorem 1 has been completed.

§ 2. Proofs of Theorems 4 and 5

Throughout this section, P satisfies the assumption (5). It is clear that for $(x_0, \xi_0)\in T^*R^n\backslash 0$ and $u\in\mathcal{D}'(R^n)$

$$(2.1)\qquad (x_0, \xi_0)\notin \mathrm{WF}\, Pu \Rightarrow (x_0, \xi_0)\notin \mathrm{WF}\, u$$

if $x_0\neq(0, x_0'')$. So, we shall consider the case $x_0=(0, x_0'')$. Since our consideration is local near x_0 we may assume that for any $\mu>0$ there exists a constant $c_\mu>0$ such that

$$(2.2)\qquad |p_m(x, \xi)|\geqq c_\mu \qquad \text{for all} \quad (x, \xi)\in \boldsymbol{R}^n\times S^{n-1} \quad \text{if } |x'|\geqq\mu .$$

Let $h(x)\in C_0^\infty(\boldsymbol{R}^n)$ be the same as in Section 1. Noting $\sqrt{2}/5<7/24$, we may assume that $h(x)$ is written as

$$(2.3)\qquad h(x) = h_1(x')h_2(x'')$$

for C_0^∞-functions h_1 and h_2 satisfying

$$\begin{cases} h_1(x') = 1 & \text{on} \quad |x'|\leqq 1/5 \\ h_2(x'') = 1 & \text{on} \quad |x''|\leqq 1/5 . \end{cases}$$

First we shall prove Theorem 4. We prepare four lemmas. If $P_\gamma=p_\gamma(y, D_y; \lambda)$ is the same as in Section 1, from (6) we have:

Lemma 2.1. *For any $\varepsilon>0$ and any $\gamma=(x_0, \bar{\xi}_0)\in\boldsymbol{R}^n\times S^{n-1}$ with $x_0=(0, x_0'')$ there exists a constant $C(\varepsilon, \gamma)$ such that*

$$\begin{aligned}(2.4)\quad & (\log \lambda^{-1})^m\|Hv\|+\textstyle\sum' \lambda^{|\alpha+\beta|}(\log \lambda^{-1})^{|\alpha+\beta|}\|HP_{\gamma(\beta)}^{(\alpha)}v\| \\ & \leqq \varepsilon\|h(\lambda D_y/2)P_\gamma v\|+C(\varepsilon, \gamma)(\textstyle\sum' \lambda^{|\alpha+\beta|}\|h(\lambda D_y/2)P_{\gamma(\beta)}^{(\alpha)}v\|+\|v\|)\,, \\ & v\in\mathcal{S}_y\,,\ \textit{if}\ 0<\lambda\leqq 1\,,\end{aligned}$$

where $H=h(\lambda D_y)h(\lambda y)$. *Here* $\sum'$ *means the summation which is taken over all* $\alpha=(\alpha', \alpha'')$, β *satisfying* $0<|\alpha+\beta|<m$ *and* $\alpha'=0$.

Proof. Set $q^0(x, \xi)=(1-h_1(2x'))/p(x, \xi)$. Then it follows from (2.2) and the symbolic calculus that for $Q^0=q^0(x, D_x)$ we have

$$Q^0P = 1-h_1(2x')-R\,, \qquad R\in S_{1,0}^{-1}\,.$$

Setting $Q=\sum_{j=0}^{m-1} R^jQ^0$ we have

$$\begin{aligned}QP &= 1-\sum_{j=0}^{m-1} R^jh_1(2x')-R^m \\ &\equiv 1+R_1+R_2\,, \qquad R_2\in S_{1,0}^{-m}\,.\end{aligned}$$

For $0\leqq|\alpha+\beta|<m$, note

$$(2.5)\quad [P_{(\beta)}^{(\alpha)}, h(x-x_0)] = [P_{(\beta)}^{(\alpha)}, h_1(x')]h_2(x''-x_0'')+h_1(x')[P_{(\beta)}^{(\alpha)}, h_2(x''-x_0'')]\,,$$

and

$$(2.6)\quad \begin{cases}[P_{(\beta)}^{(\alpha)}, h(x')]Q\,, \quad [P_{(\beta)}^{(\alpha)}, h_1(x')]R_2\in S_{1,0}^{-1} \\ [P_{(\beta)}^{(\alpha)}, h_1(x')]R_1\in S^{-\infty}\end{cases}$$

because $\operatorname{supp} h_1(2x')\cap\operatorname{supp} D_{x'}^{\alpha'}h_1(x')=\emptyset$ for $\alpha'\neq 0$. It follows from (2.6) that the estimate

$$\begin{aligned}(2.7)\quad \|[P_{(\beta)}^{(\alpha)}, h_1(x')]w\| &\leqq C(\|Pw\|_{-1}+\|w\|_{-1}) \\ &\leqq \varepsilon\|Pw\|+C_\varepsilon\|w\|\,, \qquad w\in\mathcal{S}_x\end{aligned}$$

holds for any $\varepsilon>0$ and some C_ε. As in the proof of Lemma 1.1, replace u in (6) by $h(x-x_0)h(\lambda^2D_x-\bar{\xi}_0)w(=h(x-x_0)h(\lambda^2D_x-\bar{\xi}_0)\ h((\lambda^2D_x-\bar{\xi}_0)/3)w)$, $w\in\mathcal{S}_x$. Then, using (2.5) and (2.7) together with symbolic calculus, we obtain

$$\begin{aligned}(2.8)\quad & \|(\log \Lambda)^m h(\lambda^2D_x-\bar{\xi}_0)h(x-x_0)w\| \\ & \quad +\textstyle\sum'\|\Lambda^{-|\beta|}(\log \Lambda)^{|\alpha+\beta|}h(\lambda^2D_x-\bar{\xi}_0)h(x-x_0)P_{(\beta)}^{(\alpha)}h((\lambda^2D_x-\bar{\xi}_0)/3)w\| \\ & \leqq \varepsilon\|h((\lambda^2D_x-\bar{\xi}_0)/2)Ph((\lambda^2D_x-\bar{\xi}_0)/3)w\| \\ & \quad +C_\varepsilon(\textstyle\sum'\|\Lambda^{-|\beta|}h((\lambda^2D_x-\bar{\xi}_0)/2)P_{(\beta)}^{(\alpha)}h((\lambda^2D_x-\bar{\xi}_0)/3)w\|+\|w\|)\,, \\ & \qquad w\in\mathcal{S}_x\,.\end{aligned}$$

Set $w(x)=(\exp i\lambda^{-2}x\cdot\bar{\xi}_0)v(\lambda^{-1}(x-x_0))$ for $v(y)\in\mathcal{S}_y$. From (2.8) together with (1.5)–(1.8) we obtain (2.4), similarly as in the proof of Lemma 1.1 Q.E.D.

For $h(x)=h_1(x')h_2(x'')$ and a small $0<\delta\leqq 1/10$ set $h_\delta(x)=h(x/\delta)$ (, $h_{1,\delta}(x')=h_1(x'/\delta)$ and $h_{2,\delta}(x'')=h_2(x''/\delta)$). Fix an integer $N>m$. Take a sequence $\{h_\delta^j(x)=h_{1,\delta}^j(x')h_{2,\delta}^j(x'')\}_{j=1}^{N-m+1}\subset C_0^\infty(\boldsymbol{R}^n)$ such that

$$\text{(2.9)}\qquad \begin{cases} 0\leqq h_{k,\delta}^j\leqq 1\,, \\ h_{k,\delta}=h_{k,\delta}^1\subset\subset h_{k,\delta}^2\subset\subset\cdots\subset\subset h_{k,\delta}^{N-m+1}=h_{k,2\delta}\,, \qquad k=1,2 \end{cases}$$

and for any α' and α'' estimates

$$\text{(2.10)}\qquad |D_{x'}^{\alpha'}h_{1,\delta}^j(x')|\leqq C'_{\alpha'}N^{|\alpha'|}\,,\quad |D_{x''}^{\alpha''}h_{2,\delta}^j(x'')|\leqq C''_{\alpha''}N^{|\alpha''|}$$

hold with constants $C'_{\alpha'}$ and $C''_{\alpha''}$ independent of N and j.

Lemma 2.2. *Set $\tilde{H}_\delta^j=h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)$ for $j=1,\cdots,N-m+1$. Then for any $s>0$ and any $\gamma=(x_0,\xi_0)\in\boldsymbol{R}^n\times S^{n-1}$ with $x_0=(0,x_0'')$ there exists a constant $C(s,\gamma)$ independent of λ, κ, j and N such that*

$$\text{(2.11)}\qquad \begin{aligned}&(\log\lambda^{-s})^m\|\tilde{H}_\delta^j v\|+\textstyle\sum'(\log\lambda^{-s})^{|\alpha+\beta|}\lambda^{|\alpha+\beta|}\|P_{\gamma(\beta)}^{(\alpha)}\tilde{H}_\delta^j v\|\\ &\quad\leqq\|P_\gamma\tilde{H}_\delta^j v\|+C(s,\gamma)\lambda^s N^{s+2n+3m}\|v\|\,,\qquad v\in\mathcal{S}_y\,,\\ &\qquad \text{if } 0<\lambda\leqq\lambda_0(s,\gamma)\,,\end{aligned}$$

where $\lambda_0(s,\gamma)$ is a sufficiently small positive number. Here $\sum'$ means the same summation as in Lemma 2.1.

Proof. The estimate (2.11) follows from (2.4), by the quite same way as in the proof of Lemma 1.4. Q.E.D.

Lemma 2.3. *Set $\tilde{h}_\delta(x')=1-h_1(2x'/\delta)$ for $0<\delta\leqq 1/10$. Then for any $s>0$ and any $\delta=(x_0,\xi_0)\in\boldsymbol{R}^n\times S^{n-1}$ with $x_0=(0,x_0'')$ there exist a constant $C'(s,\gamma)$ such that*

$$\text{(2.12)}\qquad \begin{aligned}&\lambda^{-2m}\|\tilde{h}_\delta(\lambda y')h(\lambda D_y)v\|\leqq C'(s,\gamma)(\|P_\gamma v\|+\lambda^s\|v\|)\,,\\ &\qquad v\in\mathcal{S}_y\,,\ 0<\lambda\leqq 1\,.\end{aligned}$$

Proof. Set $q_\gamma^0(y,\eta;\lambda)=(1-h_1(3\lambda y'/\delta))h(\lambda\eta/2)/p_\gamma(y,\eta;\lambda)$. Then $\{\lambda^{-2m}q_\gamma^0(y,\eta;\lambda);\ 0<\lambda\leqq 1\}$ is a bounded set in $S_{0,0}^0$. The symbolic calculus shows that, if $Q_\gamma^0=q_\gamma^0(y,D_y;\lambda)$, we have

$$Q_\gamma^0 P_\gamma=(1-h_1(3\lambda y'/\delta))h(\lambda D_y/2)-R$$

for $R=r(y,D_y;\lambda)$ such that $\{\lambda^{-2}r(y,\eta;\lambda);\ 0<\lambda\leqq 1\}$ is a bounded set in $S_{0,0}^0$. Take an integer l satisfying $2l\geqq s+2m$ for a fixed $s>0$. Set $Q_\gamma=\sum_{j=0}^{l-1}R^jQ_\gamma^0$. Then we obtain

$$Q_\gamma P_\gamma = 1+\sum_{j=0}^{l-1} R^j((1-h_1(3\lambda y'/\delta))h(\lambda D_y/2)-1)-R^l \equiv 1+R_1+R_2\,. \tag{2.13}$$

Note that $\{\lambda^{-2l}\sigma(R_2);\ 0<\lambda\leqq 1\}$ is a bounded set in $S^0_{0,0}$ and that, for any $s'>0$, $\{\lambda^{-s'}\sigma(\tilde{h}_\delta(\lambda y')h(\lambda D_y)R_1);\ 0<\lambda\leqq 1\}$ is a bounded set in $S^0_{0,0}$ because $(1-h_1(3x'/\delta))h(\xi/2)-1=0$ on supp $\tilde{h}_\delta(x')h(\xi)$. From (2.13) we obtain (2.12) because $\{\lambda^{-2m}\sigma(\tilde{h}_\delta(\lambda y')h(\lambda D_y)Q);\ 0<\lambda\leqq 1\}$ is a bounded set in $S^0_{0,0}$. Q.E.D.

Lemma 2.4. *For any $s>0$ there exists a constant M independent of s, κ, λ and N such that for $j=1, \cdots, N-m$*

$$\begin{aligned}||P_\gamma\tilde{H}^j_\delta v|| &\leqq M||\tilde{H}_{2\delta}P_\gamma v||+MN(\log\lambda^{-s})^{-1}||P_\gamma\tilde{H}^{j+1}_\delta v|| \\ &\quad +C_s\lambda^sN^{s+2n+3m}||v||\,, \qquad v\in\mathcal{S}_y\,, \\ &\quad \text{if } \log\lambda^{-s}\geqq MN \text{ and } 0<\lambda\leqq\tilde{\lambda}_0(s,\gamma)\,,\end{aligned} \tag{2.14}$$

where C_s is a constant independent of λ, κ and N, and $\tilde{\lambda}_0(s, \gamma)$ is a positive number smaller than $\lambda_0(s, \gamma)$ in Lemma 2.2. Here $\tilde{H}_{2\delta}=h_{2\delta}(\lambda D_y)k_\kappa(D_y;\ \lambda)h_{2\delta}(\lambda y)$ and $\gamma=(x_0, \xi_0)\in\boldsymbol{R}^n\times S^{n-1}$ with $x_0=(0, x_0'')$.

Proof. It follows that

$$||P_\gamma\tilde{H}^j_\delta v||\leqq||\tilde{H}^j_\delta P_\gamma v||+||[P_\gamma, \tilde{H}^j_\delta]v||\,. \tag{2.15}$$

The estimation of the first term of the right hand side of (2.15) is the quite same as in the proof of Lemma 1.5. We shall estimate the second term. In view of Lemma 1.2 it suffices to estimate $||[P_\gamma, \tilde{H}^j_\delta]H^{j+1}_\delta v||$, where $H^{j+1}_\delta= h^{j+1}_\delta(\lambda D_y)h^{j+1}_\delta(\lambda y)$. Note

$$\begin{aligned}&||[P_\gamma, \tilde{H}^j_\delta]H^{j+1}_\delta v|| \\ &\leqq||[P_\gamma, h^j_\delta(\lambda D_y)k_\kappa]h^j_\delta(\lambda y)H^{j+1}_\delta v||+||k_\kappa h^j_\delta(\lambda D_y)[P_\gamma, h^j_\delta(\lambda y)]H^{j+1}_\delta v||\,.\end{aligned} \tag{2.16}$$

We shall consider the first term of the right hand side of (2.16). As in the proof of Lemma 1.5 we have (1.22). The estimation of the second term of the right hand side of (1.22) is the quite same as there. About the first term, instead of (1.24), we see that for any β $(0<|\beta|<m)$ the estimate

$$\begin{aligned}\lambda^{|\beta|}||k_\kappa P_{\gamma(\beta)}h^j_\delta(\lambda y)H^{j+1}_\delta v|| &\leqq \sum_{\substack{0\leqq|\tilde{\alpha}+\tilde{\beta}|<m-|\beta| \\ \tilde{\alpha}=(0,\tilde{\alpha}'')}} \lambda^{|\beta+\tilde{\alpha}+\tilde{\beta}|} \\ &\quad\times(M_2N)^{|\tilde{\alpha}+\tilde{\beta}|}||P^{(\tilde{\alpha})}_{\gamma(\beta+\tilde{\beta})}\tilde{H}^{j+1}_\delta v||+(M_2N)^{m-|\beta|}||\tilde{H}^{j+1}_\delta v|| \\ &\quad+\sum_{0<q<m-|\beta|}\lambda^{2|\beta|-2m+2q}(M_2N)^q||\tilde{h}_\delta(\lambda y')\tilde{H}^{j+1}_\delta v||+C_s\lambda^sN^{s+2n+3m-|\beta|}||v||\end{aligned} \tag{2.17}$$

holds with a constant M_2 independent of s, κ, λ and N, where $\tilde{h}_\delta(x')$ is the same as in Lemma 2.3. In fact, the estimate (2.17) follows from the symbolic calculus if we note

$$(2.18)\qquad [P_{\gamma(\beta)}, h^j_\delta(\lambda y)] = [P_{\gamma(\beta)}, h^j_{1,\delta}(\lambda y')]h^j_{2,\delta}(\lambda y'')+h^j_{1,\delta}(\lambda y')[P_{\gamma(\beta)}, h^j_{2,\delta}(\lambda y'')]$$

and $\tilde{h}_\delta(x')=1$ on $\operatorname{supp} D^{\alpha'}_{x'}h^j_{1,\delta}(x')$ for $\alpha'\neq 0$. It follows from (2.11) with $j=j+1$ that the first and the second terms of the right hand side of (2.17) multiplied by $(M_1N)^{|\beta|}$ are estimated above by the right hand side of (2.14). We shall estimate the third term of the right hand side of (2.17) multiplied by $(M_1N)^{|\beta|}$, that is,

$$J\equiv \sum_{0<q<m-|\beta|}\lambda^{2|\beta|-2m+2q}(M_3N)^{|\beta|+q}\|\tilde{h}_\delta(\lambda y')H^{j+1}_\delta v\|,$$

where $M_3=\max(M_1, M_2)$. In view of $\tilde{H}^{j+1}_\delta=h(\lambda D_y)\tilde{H}^{j+1}_\delta$, it follows from (2.12) that

$$J\leqq C'(s, \gamma)\sum_{0<q<m-|\beta|}\lambda^{2(|\beta|+q)}(M_3N)^{|\beta|+q}(\|P_\gamma\tilde{H}^{j+1}_\delta v\|+\lambda_s\|v\|).$$

Take $\tilde{\lambda}_0(s, \gamma)$ small enough so that

$$\lambda^2\max(C'(s, \gamma), 1)\leqq(\log\lambda^{-s})^{-1}\qquad\text{if}\quad 0<\lambda\leqq\tilde{\lambda}_0(s, \gamma).$$

Then J is estimated above by the right hand side of (2.14) when $0<\lambda\leqq\tilde{\lambda}_0(s, \gamma)$ and $\log\lambda^{-s}\geqq MN$ with a suitable $M>M_3$. We have estimated the first term of the right hand side of (2.16). Noting (2.18) with $\beta=0$, by means of (2.11) and (2.12) we can also estimate the second term of the right hand side of (2.16). The estimation of the right hand side of (2.15) is completed. So we obtain (2.14). Q.E.D.

From (2.11) and (2.14), we obtain (1.25) in Lemma 1.6. In fact, in the proof of Lemma 1.6 we did not use the fact that the second term of the left hand side of (1.18) is estimated above by its right hand side. By mean of (1.25) we can also prove (2.1) for $(x_0, \xi_0)\in T^*R^n\backslash 0$ with $x_0=(0, x_0'')$, by the same way as the one after Lemma 1.6. The proof of Theorem 4 is completed.

In the rest of this section we shall prove Theorem 5. We state three lemmas. By the same way as in the proof of Lemma 2.1, from (7) we have:

Lemma 2.5. *For any* $\gamma=(x_0, \bar{\xi}_0)\in R^n\times S^{n-1}$ *with* $x_0=(0, x_0'')$ *there exist a* $0<\kappa_0<1$ *and a constant* C_γ *such that*

$$(2.19)\quad \|Hv\|+\sum'' \lambda^{-2\kappa_0+|\alpha+\beta|}\|HP_{\gamma(\beta)}^{(\alpha)}v\| \leqq C_\gamma(\|h(\lambda D_y/2)P_\gamma v\|+\sum'' \lambda^{|\alpha+\beta|}\|h(\lambda D_y/2)P_{\gamma(\beta)}^{(\alpha)}v\|+\lambda^2\|v\|)\,, \quad v\in\mathcal{S}_y\,,\ \ \text{if}\ \ 0<\lambda\leqq 1\,,$$

where $H=h(\lambda D_y)h(\lambda y)$. *Here* $\sum''$ *means the summation which is taken over all* $\alpha=(\alpha', \alpha'')$, *satisfying* $0<|\alpha+\beta|\leqq m$ *and* $\alpha'=0$.

Proof. We also have (2.5)–(2.7) for $0<|\alpha+\beta|\leqq m$. By means of (2.5) and the first inequality of (2.7), it follows from (7) that

$$\begin{aligned}&\|h(\lambda^2 D_x-\bar{\xi}_0)h(x-x_0)w\| \\ &\quad +\sum'' \|\Lambda^{\kappa_0-|\beta|}h(\lambda^2 D_x-\bar{\xi}_0)h(x-x_0)P_{(\beta)}^{(\alpha)}h((\lambda^2 D_x-\bar{\xi}_0)/3)w\| \\ &\leqq C(\|h((\lambda^2 D_x-\bar{\xi}_0)/2)Ph((\lambda^2 D_y-\bar{\xi}_0)/3)w\| \\ &\quad +\sum'' \|\Lambda^{-|\beta|}h((\lambda^2 D_x-\bar{\xi}_0)/2)P_{(\beta)}^{(\alpha)}h((\lambda^2 D_x-\bar{\xi}_0)/3)w\|+\|\Lambda^{-1}w\|)\,, \\ &\qquad\qquad w\in\mathcal{S}_x\,.\end{aligned}$$

In view of (1.5)–(1.8), the estimate (2.19) immediately follows from the above estimate. Q.E.D.

Let N be a sufficiently large integer chosen later on. If $\{h_\delta^j(x)\}_{j=1}^{N-m+1}$ is the same as the one in the proof of Theorem 4, from (2.19) we have

Lemma 2.6. *Set* $\tilde{H}_\delta^j=h_\delta^j(\lambda D_y)k_\kappa(D_y;\ \lambda)h_\delta^j(\lambda y)$ *for* $j=1, \cdots, N-m+1$. *Then for any* $s>0$ *and any* $\gamma=(x_0, \xi_0)\in \boldsymbol{R}^n\times S^{n-1}$ *with* $x_0=(0, x_0'')$ *there exists a constant* $C''(s, \gamma)$ *independent of* λ *and* κ *such that*

$$(2.20)\quad \|\tilde{H}_\lambda^j v\|+\sum'' \lambda^{-2\kappa_0+|\alpha+\beta|}\|P_{\gamma(\beta)}^{(\alpha)}\tilde{H}_\delta^j v\|\leqq C''(s, \gamma)(\|P_\gamma \tilde{H}_\delta^j v\|+\lambda^s\|v\|)\,, \quad v\in\mathcal{S}_y\,,\ \ \text{if}\ \ 0<\lambda\leqq\lambda_1(\gamma)\,,\ \ j=1, \cdots, N-m+1\,,$$

where $\lambda_1(\gamma)$ *is a sufficiently small positive number. Here* κ_0 *and* $\sum''$ *are the same as in Lemma* 2.5.

Proof. Take a small $\lambda_1(\gamma)$ such that

$$1/2\geqq\lambda^{2\kappa_0}C_\gamma \qquad \text{if}\ \ 0<\lambda\leqq\lambda_1(\gamma)\,,$$

where C_γ is the same constant as in (2.19). Then we have (2.20) from (2.19), by the similar way as in the proof of Lemma 1.4. Q.E.D.

The above lemma together with Lemma 2.3 leads us to the following:

Lemma 2.7. *For any* $s>0$ *there exists a constant* C_s *independent of* κ *and* λ *such that for* $j=1, \cdots, N-m$

$$(2.21)\qquad ||P_\gamma \tilde{H}_\delta^j v|| \leqq C_s(||\tilde{H}_{2\delta} P_\gamma v|| + \lambda^{2\kappa_0}||P_\gamma \tilde{H}_\delta^{j+1} v|| + \lambda^s||v||)\,,$$
$$v \in \mathcal{S}_y\,,\quad if\quad 0<\lambda \leqq \lambda_1(\gamma)\,,$$

where $\gamma=(x_0, \bar{\xi}_0) \in \boldsymbol{R}^n \times S^{n-1}$ *with* $x_0=(0, x_0'')$.

Proof. As in the proof of Lemma 2.4, it suffices to estimate $||[P_\gamma, \tilde{H}_\delta^j]H_\delta^{j+1}v||$. We shall consider the first term of the right hand side of (2.16). Noting the expansion formula of $[P_\gamma, h_\delta^j(\lambda D_y)\not{k}_\kappa]$, we can also have

$$(2.22)\qquad ||[P_\gamma, h_\delta^j(\lambda D_y)\not{k}_\kappa]h_\delta^j(\lambda y)H_\delta^{j+1}v||$$
$$\leqq C(\sum_{0<|\beta|\leqq m} \lambda^{|\beta|}||\not{k}_\kappa P_{\gamma(\beta)} h_\delta^j(\lambda y)H_\delta^{j+1}v||$$
$$+\lambda^2 \sum_{0<q\leqq m+[s/2]} \lambda^{2q}||\not{k}_\kappa h_\delta^j(\lambda y)H_\delta^{j+1}v|| + \lambda^s||v||)\,.$$

In the proof of the lemma we denote by the same notation C different constants independent of λ and κ (and j). The second term of the right hand side is estimated above by a constant times of $\lambda^2||\tilde{H}_\delta^{j+1}v|| + \lambda^s||v||$. So, we can estimate it by the right hand side of (2.21) by means of (2.20) with $j=j+1$. About the first term of the right hand side of (2.22), we have

$$(2.23)\qquad \lambda^{|\beta|}||\not{k}_\kappa P_{\gamma(\beta)} h_\delta^j(\lambda y)H_\delta^{j+1}v|| \leqq C(\sum_{\substack{0\leqq|\tilde{\alpha}+\tilde{\beta}|\leqq m-|\beta| \\ \tilde{\alpha}=(0,\tilde{\alpha}'')}} \lambda^{|\beta+\tilde{\alpha}+\tilde{\beta}|}||P_{\gamma(\beta+\tilde{\beta})}^{(\tilde{\alpha})}\tilde{H}_\delta^{j+1}v||$$
$$+\lambda^2||\tilde{H}_\delta^{j+1}v|| + \lambda^{2|\beta|-2m+2}||\tilde{h}_\delta(\lambda y')\tilde{H}_\delta^{j+1}v|| + \lambda^s||v||)\,,$$

where $\tilde{h}_\delta(x')$ is the same as in Lemma 2.3. It follows again from (2.20) with $j=j+1$ that the first and the second terms of the right hand side of (2.23) are estimated above by a constant times of the right hand side of (2.21). Note that Lemma 2.3 still holds because it was obtained only under the hypothesis of (5) ((2.2)). By means of (2.12) we can also estimate the third term of the right hand side of (2.23). Now, the first term of the right hand side of (2.16) has been estimated. The estimation of its second term is similarly performed by using (2.20) and (2.12). So we obtain (2.21). Q.E.D.

For any $s>0$ take an integer N such that

$$(2.24)\qquad N-m+1>>s\kappa_0\,.$$

Then, by means of (2.20) and the repeated use of (2.21) we easily see that for any $s>0$ there exists a constant C_s' independent of λ and κ such that

$$(2.25)\qquad ||\tilde{H}_\delta v|| \leqq C_s'(||\tilde{H}_{2\delta} P_\gamma v|| + \lambda^s||v||)\,,\qquad v\in\mathcal{S}_y$$
$$\text{if}\quad 0<\lambda\leqq 1\,,$$

where $\gamma=(x_0, \bar{\xi}_0)\in\boldsymbol{R}^n\times S^{n-1}$ with $x_0=(0, x_0'')$. Setting $s=2s'+2s''$ for real

s', $s''>0$ we have the similar estimate as (1.28). So we can prove (2.1) for $(x_0, \xi_0)\in T^*\boldsymbol{R}^n\backslash 0$ with $x_0=(0, x_0'')$. Proof of Theorem 5 is completed.

Remark. The assumption (5) in Theorems 4 and 5 can be weakened to Hörmander's condition in the region $\{x=(x', x''); x'\neq 0\}$ as follows: For any $\mu>0$ estimates

$$(2.26)\quad \begin{cases} |p(x, \xi)|\geqq c_\mu>0\,, \\ |p_{(\beta)}^{(\alpha)}(x, \xi)/p(x, \xi)|\leqq C_{\alpha\beta,\mu}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \qquad (0\leqq\delta<1/2<\rho\leqq 1)\,, \end{cases}$$
$$\text{if}\quad |x'|\geqq\mu \quad\text{and}\quad |\xi| \text{ large enough,}$$

hold with constants c_μ and $C_{\alpha\beta,\mu}$. In fact, under this assumption we have

$$\|[P_{(\beta)}^{(\alpha)}, h_1(x')]w\|\leqq C(\|Pw\|_{\delta-\rho}+\|w\|_{\delta-\rho})\,,\qquad w\in\mathcal{S}_x$$

and

$$\lambda^{\sigma|\alpha+\beta|}\|\tilde{h}_\delta(\lambda y')P_{\gamma(\beta)}^{(\alpha)}v\|\leqq C(\|P_\gamma v\|+\lambda^s\|v\|)\,,\qquad v\in\mathcal{S}_y\,,$$
$$(\sigma=\min(1-2\delta, 2\rho-1))$$

instead of (2.7) and (2.12), respectively. Using these we can prove Theorems 4 and 5. The detail is omitted.

§ 3. Proof of Proposition 3

As stated in the introduction, the necessity of (4) was already proved in [9] in a little more general form (, see Theorem 3 of [9]). We can also prove the sufficiency of (4) in a slightly more general form.

Let P_0 be a differential operator of the form

$$(3.1)\qquad P_0=D_{x_1}^{2l}+D_{x_2}^{2l}+g(x_1)D_y^{2l} \qquad\text{in}\quad \boldsymbol{R}^3\,,$$

where $l=1, 2, \cdots$ and $g(x_1)$ is a $\mathscr{B}^\infty$-function such that $g(x_1)>0$ $(x_1\neq 0)$ and $g(0)=0$. When $l\geqq 2$ we assume that for any integer $j>0$

$$(3.2)\qquad |g^{(j)}(x_1)|\leqq C_j g(x_1)^{1-\sigma j} \qquad\text{in a neighborhood of}\quad x_1=0\,,$$

where $g^{(j)}=D_{x_1}^j g$ and σ is a number satisfying

$$(3.3)\qquad 0<\sigma<1/2l^2\,.$$

It is clear that $\exp(-1/|x_1|^\delta)$, $\delta>0$, satisfies (3.2) for any small $\sigma>0$.

Proposition 3.1. *Let* P_0 *be the above operator. When* $l\geqq 2$ *we assume* (3.2) *with* σ *satisfying* (3.3). *If* $g(x_1)$ *satisfies*

(3.4) $$\lim_{|x_1|\to 0} |x_1|\,|\log g(x_1)| = 0\,.$$

then P_0 is hypoelliptic in $\mathbf{R}^3$.

When $0<\delta<1$, $\exp(-1/|x_1|^{\delta})$ satisfies (3.4). So the sufficiency of (4) follows from this proposition. Proposition 3.1 for the case $l=1$ is included in Theorem 1.1 of [11] (, where (3.2) is not assumed). We shall prove Proposition 3.1 when $l\geqq 2$. Without loss of generality, we may assume that for any $\mu>0$ there exists a constant $c_\mu>0$ such that $g(x_1)\geqq c_\mu$ for $|x_1|\geqq\mu$. We prepare the following:

Lemma 3.2. i) *It follows that*

(3.5) $$||D_{x_2}^{2l}u||\leqq||P_0u||\,,\quad u\in\mathcal{S}\,.$$

ii) *For any $\varepsilon>0$ there exists a constant C_ε such that*

(3.6) $$||(\log\Lambda)^{2l}u||\leqq\varepsilon||P_0u||+C_\varepsilon||u||\,,\qquad u\in\mathcal{S}\,.$$

iii) *Let κ be $(1-\delta_0)/(l+1)$, where $\delta_0=2l^2\sigma$. There exists a constant C_0 such that*

(3.7) $$||g(x_1)u||_{2l-1+\kappa}\leqq C_0(||P_0u||+||u||)\,,\qquad u\in\mathcal{S}\,,$$

and moreover for any $j=1,\cdots,2l$

(3.8) $$||g^{(j)}(x_1)u||_{2l-j+\kappa}\leqq C_0(||P_0u||+||u||)\,,\qquad u\in\mathcal{S}\,.$$

Remark. It follows from (3.3) that $\delta_0<1$ and hence $0<\kappa<1/(l+1)$.

Admitting this lemma for a while, we shall apply Theorem 4 to the proof of Proposition 3.1. We shall check the assumption (6). Set $\varepsilon=\varepsilon'^{2l}$ in (3.6) for a small $\varepsilon'>0$. Then we have

(3.9) $$||((\log\Lambda)/\varepsilon')^{2l}u||\leqq||P_0u||+\tilde{C}_{\varepsilon'}||u||\,,\qquad u\in\mathcal{S}\,.$$

From (3.5) and (3.9) we have

(3.10) $$||((\log\Lambda)/\varepsilon')^{j}D_{x_2}^{2l-j}u||\leqq 2||P_0u||+(\tilde{C}_{\varepsilon'}+1)||u||\,,\qquad u\in\mathcal{S}$$

for $j=1,\cdots,2l-1$. It follows from (3.7) and (3.8) that for any $\varepsilon>0$

(3.11) $$||(\log\Lambda)^{j+k}g^{(j)}(x_1)D_y^{2l-k}u||_{-j}\leqq\varepsilon||P_0u||+C'_\varepsilon||u||\,,\qquad u\in\mathcal{S}$$
$$(0<j+k<2l)$$

holds with a constant C'_ε, because the left hand side of (3.11) is estimated above by

$$(\varepsilon/C_1)\|g^{(j)}(x_1)u\|_{2l-k-j+\kappa}+C''_\varepsilon\|u\|\,.$$

Summing up (3.6), (3.10) and (3.11) we have (6) for P_0. By means of Theorem 4 we have the hypoellipticity of P_0.

For the proof of Proposition 3.1 it remains to prove Lemma 3.2. Note that

$$\|D^l_{x_1}u\|^2+\|D^l_{x_2}u\|^2+\|g(x_1)^{1/2}D^l_y u\|^2=(P_0u,\,u)\,,\qquad u\in\mathcal{S}\,. \tag{3.12}$$

Replacing u by $D^l_{x_2}u$, we have

$$\begin{aligned}\|D^{2l}_{x_2}u\|^2&\leqq(P_0D^l_{x_2}u,\,D^l_{x_2}u)=(P_0u,\,D^{2l}_{x_2}u)\\&\leqq(\|P_0u\|^2+\|D^{2l}_{x_2}u\|^2)/2\,,\qquad u\in\mathcal{S}\,,\end{aligned}$$

so that we have (3.5). Let $f(\xi,\,\eta)$ be a symbol in $S^0_{1,0}$ such that

$$\begin{cases}f=1 & \text{on}\quad\{|\xi|\leqq|\eta|\}\cap\{|\xi|+|\eta|\geqq1\}\,,\\ \operatorname{supp}f\subset\{|\xi|\leqq2|\eta|\cap\{|\xi|+|\eta|\geqq1/2\}\,.\end{cases}$$

Since P_0 is microlocally elliptic on $\{|\xi|\geqq|\eta|\}$ it is easy to see

$$\|(1-f)u\|_{2l}+\|[P_0,f]u\|_1\leqq C(\|P_0u\|+\|u\|)\,,\qquad u\in\mathcal{S}\,. \tag{3.13}$$

Hence, it suffices to show (3.6)–(3.8) with u replaced by fu. Since $\{|\xi|\leqq2|\eta|\}$ on supp f, instead of (3.6) it suffices to show

$$\|(\log\langle D_y\rangle)^{2l}u\|\leqq\varepsilon\|P_0u\|+C_\varepsilon\|u\|\,,\qquad u\in\mathcal{S}\,. \tag{3.14}$$

If $f_1(\xi,\,\eta)$ is a symbol in $S^0_{1,0}$ such that $f_1=1$ on supp f and supp $f_1\subset\{|\xi|\leqq 3|\eta|\}$, we have

$$f_1(D_x,\,D_y)g^{(j)}(x_1)f(D_x,\,D_y)\equiv g^{(j)}(x_1)f(D_x,\,D_y)\ \text{mod.}\ S^{-\infty}\,.$$

Since $|\xi|\leqq3|\eta|$ on supp f_1, instead of (3.7) and (3.8), it also suffices to show

$$\|g(x_1)\langle D_y\rangle^{2l-1+\kappa}u\|\leqq C(\|P_0u\|+\|u\|)\,,\qquad u\in\mathcal{S} \tag{3.15}$$

and

$$\|g^{(j)}(x_1)\langle D_y\rangle^{2l-j+\kappa}u\|\leqq C(\|P_0u\|+\|u\|)\,,\qquad u\in\mathcal{S} \tag{3.16}$$

for $0<j\leqq2l$, respectively. We shall prove (3.14)–(3.16), by using the similar method as in Section 5 of [8].

Let $\phi_k(t)$ ($k=0,\,1,\,2,\,3$) be C^∞_0-functions in $\boldsymbol{R}^1$ such that

$$\begin{cases} \operatorname{supp} \phi_0 \subset \{|t|<1\}, & \phi_0 = 1 \quad \text{on} \quad |t| \leqq 1/2\,, \\ \operatorname{supp} \phi_1 \subset \{|t|<2\}, & \phi_1 = 1 \quad \text{on} \quad |t| \leqq 1\,, \\ \operatorname{supp} \phi_2 \subset \{|t|<1\}, & \phi_2 = 1 \quad \text{on} \quad |t| \geqq 2\,, \\ \operatorname{supp} \phi_3 \subset \{|t|<3\}, & \phi_3 = 1 \quad \text{on} \quad |t| \leqq 2 \end{cases}$$

and $\phi_1+\phi_2=1$ in $\boldsymbol{R}^1$. Let (ξ, η) denote the dual variables of (x, y) and set $\lambda(\xi, \eta)=(|\xi|^{2(l+1)}+\eta^2+1)^{1/2(l+1)}$. Then $\lambda(\xi, \eta)$ satisfies (1.1) of Chapter 7 of [5], so it is a basic weight function.

Lemma 3.3. *Set $h(\xi, \eta)=\lambda(\xi, \eta)^{2l^2}$ and set $\chi_k(x_1, \xi, \eta)=\phi_k(g(x_1)h(\xi, \eta))$ $(k=0, 1, 2)$. Then*

$$\chi_1(x_1, D_x, D_y)+\chi_2(x_1, D_x, D_y) = I$$

and $\chi_k(x_1, D_x, D_y)$ $(k=0, 1, 2)$ belongs to $S^0_{\lambda,1,\delta_0}$, that is, for any α and j

$$|(\partial_\xi \partial_\eta)^\alpha D^j_{x_1} \chi_k(x_1, \xi, \eta)| \leqq C_{\alpha,j} \lambda(\xi, \eta)^{-|\alpha|+\delta_0 j}\,, \tag{3.17}$$

where $\delta_0=2l^2\sigma<1$

Proof. By means of (3.2) we have (3.17) by the quite same way as in the proof of Proposition 5.1 of [8], if we set $\tau=1/(l+1)$ there. The detail is omitted. Q.E.D.

Set $v_k=\chi_k(x_1, D_x, D_y)u$ $(k=1, 2)$ for $u\in\mathcal{S}$. Let $\tilde{v}_1(x, \eta)$ be the Fourier transform of $v_1(x, y)$ with respect to y. Setting $\chi_3(x_1, \eta)=\phi_3(g(x_1)h(0, \eta))$, we have $\tilde{v}_1(x, \eta)=\chi_3(x, \eta)\tilde{v}_1(x, \eta)$ because $\chi_3=1$ on supp $\chi_1(x_1, \xi, \eta)$. It follows from (3.12) that

$$\begin{aligned} &||g(x_1)\langle D_y\rangle^{2l-1+\kappa} v_1||^2 \\ &= ||g(x_1)^{1/2}\langle\eta\rangle^{l-1+\kappa}\chi_3(x_1, \eta)g(x_1)^{1/2}\langle\eta\rangle^l \tilde{v}_1||^2 \\ &\leqq 3(P_0 v_1, v_1)+C||v_1||^2 \\ &\leqq C'(||P_0 v_1||^2+||v_1||^2) \end{aligned} \tag{3.18}$$

because $\kappa=(1-\delta_0)/(l+1)$ and $g(x_1)\langle\eta\rangle^{2l^2/(l+1)}<3$ on supp χ_3. By means of (3.2), for $0<j\leqq 2l$ we have furthermore

$$\begin{aligned} &||g^{(j)}(x_1)\langle D_y\rangle^{2l-j+\kappa} v_1|| \\ &\leqq C_j||g(x_1)^{1/2-\sigma j}\langle\eta\rangle^{l-j+\kappa}\chi_3(x_1, \eta)g(x_1)^{1/2}\langle\eta\rangle^l\tilde{v}_1|| \\ &\leqq C(||P_0 v_1||+||v_1||)\,. \end{aligned} \tag{3.19}$$

It follows from the hypothesis (3.3) of Proposition 3.1 that for any $\varepsilon>0$ there exists a constant $c_\varepsilon>0$ such that

$$|x_1|\leqq \varepsilon(\log\langle\eta\rangle)^{-1} \quad \text{on} \quad \operatorname{supp}\chi_3(x_1, \eta) \quad \text{if} \quad \langle\eta\rangle\geqq c_\varepsilon, \tag{3.20}$$

because $(x_1, \eta)\in\operatorname{supp}\chi_3$ implies $g(x_1)\langle\eta\rangle^{2l^2/(l+1)}<3$. By means of Poincaré's inequality it follows from (3.20) that for a constant C_0 independent of ε we have

$$||(\log\langle\eta\rangle)^l\tilde{v}_1||^2_{L^2(R^2_x)}\leqq C_0\varepsilon^{2l}||D^l_{x_1}v_1||^2_{L^2(R^2_x)} \quad \text{if} \quad \langle\eta\rangle\geqq c_\varepsilon, \tag{3.21}$$

because $\chi_3\tilde{v}_1=\tilde{v}_1$ and so $\operatorname{supp}\tilde{v}_1(x, \eta)\cap\{\langle\eta\rangle\geqq c_\varepsilon\}$ is contained in $\{|x_1|\leqq \varepsilon(\log\langle\eta\rangle)^{-1}\}$. Integrating (3.21) with respect to η we have

$$||(\log\langle D_y\rangle)^l v_1||^2\leqq\varepsilon||D^l_{x_1}v_1||^2+C_\varepsilon||v_1||^2, \tag{3.22}$$

by taking another small $\varepsilon>0$. Estimates (3.22) and (3.12) with u replaced by v_1 give

$$||(\log\langle D_y\rangle)^l v_1||^2\leqq\varepsilon(P_0v_1, v_1)+C_\varepsilon||v_1||^2.$$

Replace v_1 by $(\log\langle D_y\rangle)^l v_1$. Then, for any $0<\varepsilon\leqq 1/2$ we have

$$||(\log\langle D_y\rangle)^{2l} v_1||^2\leqq\varepsilon||P_0v_1||^2+C'_\varepsilon||v_1||^2 \tag{3.23}$$

because for some C'_ε we have

$$C_\varepsilon||(\log\langle D_y\rangle)^l v_1||^2\leqq(1/4)||(\log\langle D_y\rangle)^{2l}v_1||^2+(C'_\varepsilon/2)||v_1||^2.$$

To obtain estimates for $v_2=\chi_2(x_1, D_x, D_y)u$ we consider an operator $\tilde{P}_0=\tilde{p}_0(x_1, D_x, D_y)$ which is obtained by modifying P_0 in "a neighborhood of $x_1=0$" as follows: Set

$$\tilde{p}_0(x_1, \xi, \eta)=\xi_1^{2l}+\xi_2^{2l}+(g(x_1)h(\xi, \eta)+\chi_0(x_1, \xi, \eta))h(\xi, \eta)^{-1}\eta^{2l}.$$

Then we have

Lemma 3.4. *$\tilde{p}_0(x_1, \xi, \eta)$ belongs to $S^{2l(l+1)}_{\lambda,1,\delta_0}$ and satisfies Hörmander's condition as follows*: i) *There exists a constant $c_0>0$ such that*

$$|\tilde{p}_0(x_1, \xi, \eta)|\geqq c_0\lambda(\xi, \eta)^{2l} \quad \textit{for large} \quad |\xi|+|\eta|. \tag{3.24}$$

ii) *For any α and β there exists a constant $C_{\alpha\beta}$ such that*

$$|\tilde{p}^{(\alpha)}_{0(\beta)}(x_1, \xi, \eta)/\tilde{p}_0(x_1, \xi, \eta)|\leqq C_{\alpha\beta}\lambda(\xi, \eta)^{\delta_0|\beta|-|\alpha|} \quad \textit{for large} \quad |\xi|+|\eta|, \tag{3.25}$$

where $\delta_0=2l^2\sigma<1$.

The proof is similar as the one of Proposition 5.3 of [8]. The detail is omitted.

In the consequence of this lemma, we have a parametrix $Q\in S^{-2l}_{\lambda,1,\delta_0}$ such that

$$I = Q\tilde{P}_0+K, \qquad K\in S^{-\infty}, \tag{3.26}$$

furthermore

$$\begin{cases} Q = Q_0Q_1, & Q_0\in S^{-2l}_{\lambda,1,\delta_0} \quad Q_1\in S^{0}_{\lambda,1,\delta_0} \\ \sigma(Q_0) = \tilde{p}_0(x_1, \xi, \eta)^{-1} & \text{for large} \quad |\xi|+|\eta|. \end{cases} \tag{3.27}$$

Note that $\Lambda^{2l/(l+1)}Q\in S^0_{\lambda,1\ \delta_0}$ and $\tilde{P}_0\chi_2\equiv P_0\chi_2$ mod. $S^{-\infty}$. Recall $v_2=\chi_2(x_1, D_x, D_y)u$ for $u\in\mathcal{S}$. It follows from (3.26) that

$$||v_2||_{2l/(l+1)}\leqq C(||P_0 v_2||+||u||). \tag{3.28}$$

In view of (3.27) and (3.25) it is easy to see $g^{(j)}(x_1)D_y^{2l-j}Q\in S^{(\delta_0-1)j}_{\lambda,1,\delta_0}$. Noting $\lambda(D_x, D_y)^{-1}\in S^{-1/(l+1)}_{1/(l+1),0}$, we have

$$||g^{(j)}(x_1)D_y^{2l-j}v_2||_{j\kappa}\leqq C(||P_0 v_2||+||u||) \tag{3.29}$$

for $0\leqq j\leqq 2l$. Note that, for $k=1, 2$, $[P_0, \chi_k]Q\in S^{\delta_0-1}_{\lambda,1,\delta_0}$ and

$$[P_0, \chi_k]\equiv[P_0, \chi_k]Q\tilde{P}_0\equiv[P_0, \chi_k]QP_0 \qquad \text{mod. } S^{-\infty}.$$

Then we have

$$||[P_0, \chi_k]u||_\kappa\leqq C(||P_0u||+||u||) \qquad (k = 1, 2). \tag{3.30}$$

Since $\chi_1+\chi_2=I$, from (3.23), (3.28) and (3.30) we have (3.14). Furthermore, (3.15) follows from (3.18) and (3.29) together with (3.30), and we obtain (3.16) from (3.19), (3.29) and (3.30). Now, we have proved Lemma 3.2. So the proof of Proposition 3.1 for the case $l\geqq 2$ is completed.

We remark that the proof of Proposition 3.1 for the case $l=1$ also follows from the above discussion. It follows from the nonnegativeness of $g(x_1)$ that (3.2) holds with $\sigma=1/2$. Lemmas 3.3 and 3.4 still hold with $\delta_0=1/2$ if we set $\lambda(\xi, \eta)=(|\xi|^6+\eta^4+1)^{1/6}$ and $h(\xi, \eta)=\lambda(\xi, \eta)$. We have (3.22) by the same way as in the case $l\geqq 2$. We also have (3.28) with $2l/(l+1)$ replaced by $2/3$ and (3.30) with κ replaced by $1/3$ because $\lambda(D_x, D_y)^{-1}$ belongs to $S^{-2/3}_{2/3,0}$. Hence we have (3.6) with $l=1$, which is nothing but (3) of Corollary 2. By Corollary 2 we see that Proposition 3.1 also holds when $l=1$.

We finally remark that the condition (7) is satisfied for

$$\mathcal{A} = D_{x_1}^{2l} + \exp(-1/|x_1|^{\delta}) D_y^{2l}, \qquad \delta > 0, \quad \text{in} \quad \boldsymbol{R}^2.$$

Indeed, we have (3.7) and (3.8) with P_0 replaced by $\mathcal{A}$ because we did not use the assumption (3.4) in the derivation of (3.18), (3.19) and (3.28)–(3.30). So, we have (7) if we show

$$\|u\| \leqq C(\|\mathcal{A}u\| + \|u\|_{-1}), \qquad u \in \mathcal{S}. \tag{3.31}$$

It follows from Poincaré's inequality that

$$\begin{aligned} \|u\|^2 &\leqq \|D_{x_1}^l u\|^2 \\ &\leqq (\mathcal{A}u, u) \leqq (\|\mathcal{A}u\|^2 + \|u\|^2)/2 \end{aligned} \tag{3.32}$$

if $u \in \mathcal{S}$ and if supp $u \subset \{|x_1| \leqq \varepsilon\}$ for a sufficiently small $\varepsilon > 0$. The estimate (3.31) easily follows from (3.32) and the elliptic estimate in the region $\{x_1 \neq 0\}$.

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