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Stability and Instability of Certain Foliations of 4-Manifolds by Closed Orientable Surfaces

By

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§0. Introduction

Let $\mathrm{Fol}_q(M)$ denote the set of codimension q C^∞-foliations of a closed m-manifold M. $\mathrm{Fol}_q(M)$ carries a natural weak C^r-topology $(0 \leq r \leq \infty)$, which is described in [7]. We denote this space by $\mathrm{Fol}_q^r(M)$. We say a foliation F is C^r-stable if there exists a neighborhood V of F in $\mathrm{Fol}_q^r(M)$ such that every foliation in V has a compact leaf. We say F is C^r-unstable if not. A foliation in a small neighborhood of F in $\mathrm{Fol}_q^r(M)$ is said to be a C^r-perturbation of F. It seems to be of interest to determine if F is C^r-stable or not. Let L be a compact leaf of F. Langevin-Rosenberg [8] showed, generalizing the Reeb stability theorem [11], that if $H^1(L; \mathbf{R})=0$, then F is C^1-stable. Let $\pi_1(L) \to GL(q, \mathbf{R})$ be the action determined from the linear holonomy of L, where q is the codimension of F. Then generalizing the results of Hirsch [7] and Thurston [16], Stowe [15] showed that if the cohomology group $H^1(\pi_1(L); \mathbf{R})$ is trivial, then F is C^1-stable. On the other hand, let F be the foliation of an orientable S^1-bundle over a closed surface B by fibres. Seifert [13] showed that F is C^0-stable if $\chi(B) \neq 0$, where $\chi(B)$ is the euler characteristic of B. The result was generalized by Fuller [6] to orientable circle bundles over arbitrary closed manifolds B with $\chi(B) \neq 0$. Langevin-Rosenberg [9] considered a fibration $\pi: M \to B$ with fibre L and showed that the foliation of M by fibres is C^0-stable provided that 1) $\pi_1(L) \cong \mathbf{Z}$, 2) B is a closed surface with $\chi(B) \neq 0$ and 3) $\pi_1(B)$ acts trivially on $\pi_1(L)$. The author [4] generalized the above result to compact codimension two foliations. Plante [10] classified all foliations of closed 3-manifolds by closed orientable surfaces into stable or unstable foliations. The author [5] classified all foliations of closed 3-manifolds by circles into stable or unstable foliations.

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We study here the case M is a closed 4-manifold and F is a foliation of M by closed orientable surfaces. Our main results are as follows. See §1 for definitions.

Theorem A (Theorem 8). *Let F be a foliation with all leaves tori and only reflection leaves as singular leaves. Then we can regard a union of reflection leaves as $T^2\times[0, 1]/h$, where h is a diffeomorphism of T^2. If the induced automorphism $h_*: H_1(T^2; \boldsymbol{Z})\to H_1(T^2; \boldsymbol{Z})$ is equal to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then F is C^1-stable.*

Theorem B (Theorem 17). *Let F be a foliation of M without singular leaves. Then F is C^r-unstable ($r\geq 0$) if one of the following is satisfied;*
(1) *M/F is homeomorphic to the 2-sphere and the genus of a generic leaf ≥ 2,*
(2) *M/F is homeomorphic to the projective plane and the genus of a generic leaf ≥ 4,*
(3) *M/F is neither homeomorphic to the 2-sphere nor the projective plane and the genus of a generic leaf ≥ 6.*

Theorem C (Theorem 25). *Let F be a foliation of M with generic leaf of genus g and $B=M/F$ be the leaf space. Suppose F has m rotation leaves with holonomy groups $\boldsymbol{Z}_{k_i}$ ($i=1, 2, \cdots, m$) and m_j dihedral leaves with holonomy groups $\boldsymbol{D}_{l_{j,k}}$ ($k=1, 2, \cdots, m_j$) which correspond to points of $\partial_j B$ for each $j(1\leq j\leq n')$. If $g\geq \max(3\max(k_i;\ 1\leq i\leq m)+1,\ 8\max(l_{j,k};\ 1\leq j\leq n',\ 1\leq k\leq m_j)+1, 7\varepsilon)$, then F is C^r-unstable ($r\geq 0$), where $\varepsilon=0$ or 1 and F has no reflection leaves if and only if $\varepsilon=0$.*

The paper is organized as follows. In §1, we recall the (local) structure of compact codimension two foliations and prepare some definitions and notations. In §2, we discuss about foliations with all leaves tori and prove Theorem A. In §3, we discuss about foliations with generic leaf of genus≥ 2 and no singular leaves and prove Theorem B. In §4, we discuss about foliations with generic leaf of genus≥ 2 and singular leaves and prove Theorem C. All foliations we consider here are smooth of class C^∞ and of codimension two.

§1. Compact Foliations and Singular Leaves

Let M be a closed manifold and F a compact foliation of codimension two. By the results of Epstein [2] and Edwards-Millett-Sullivan [1], we have a nice picture of the local behavior of F as follows.

Proposition 1 (Epstein [3]). *There is a generic leaf L_0 with property that* *there is an open dense saturated subset of M, where all leaves have trivial holonomy and are diffeomorphic to L_0. Given a leaf L, we can describe a neighborhood $U(L)$ of L, together with the foliation on the neighborhood as follows. There is a finite subgroup $G(L)$ of $O(2)$ such that $G(L)$ acts freely on L_0 on the right and $L_0/G(L) \cong L$. Let D^2 be the unit disk. We foliate $L_0 \times D^2$ with leaves of the form $L_0 \times \{pt\}$. This foliation is preserved by the diagonal action of $G(L)$, defined by $g(x, y) = (x \cdot g^{-1}, g \cdot y)$ for $g \in G(L)$, $x \in L_0$ and $y \in D^2$, where $G(L)$ acts linearly on D^2. So we have a foliation induced on $U = L_0 \times_{G(L)} D^2$. The leaf corresponding to $y=0$ is $L_0/G(L)$. Then there is a C^∞-imbedding $\varphi: U \to M$ with $\varphi(U) = U(L)$, which preserves leaves and $\varphi(L_0/G(L)) = L$.*

Remark 2. $U(L)$ can be considered to be the total space of a normal disk bundle of L in M with structure group $G(L)$.

Since $G(L)$ is a finite subgroup of $O(2)$, $G(L)$ is isomorphic to a rotation group $\mathbb{Z}_k$ $(k>1)$, a dihedral group $\mathbb{D}_l$ $(l>1)$ which consists of l rotations and l reflections or a group $\mathbb{D}$ consisting of only one reflection, which is called a reflection group.

Definition 3. A leaf L is *singular* if $G(L)$ is not trivial. The order of $G(L)$ is called the order of holonomy of L. We say such an L is *a rotation leaf, a reflection leaf* or *a dihedral leaf if* $G(L)$ is isomorphic to $\mathbb{Z}_k$ $(k>1)$, $\mathbb{D}$ or $\mathbb{D}_l$ $(l>1)$ respectively.

Let $B = M/F$ be the leaf space. B is a compact V-manifold of dimension two and is also a topological manifold. The quotient map $\pi: M \to B$ is a V-bundle (see Satake [12] for definitions). Since M is compact, there are finitely many rotation leaves and dihedral leaves in F. Dihedral and reflection leaves correspond to the boundary points of B. Let $n(=n'+n'')$ be the number of boundary components of B. We let $L_i (i=1, 2, \cdots, m)$ be all rotation leaves with holonomy groups $\mathbb{Z}_{k_i}$ and $L_{j,k}$ $(j=1, 2, \cdots, n'; k=1, 2, \cdots, m_j)$ all dihedral leaves with holonomy groups $\mathbb{D}_{l_{j,k}}$ respectively such that $L_{j,k}$ $(k=1, 2, \cdots, m_j)$ correspond to points in the j-th boundary $\partial_j B$ of B $(1 \leq j \leq n')$. All points in other boundaries $\partial_{n'+j} B$ $(1 \leq j \leq n'')$ of B correspond to reflection leaves. Choose saturated neighborhoods $U(L_i)$ as in Proposition 1 to be disjoint and take saturated neighborhoods $U'(L_i)$ such that $U'(L_i) \subset \mathring{U}(L_i)$, where $\mathring{U}$ denotes the interior of U. Let $V_i = \pi(U(L_i))$ and $V'_i = \pi(U'(L_i))$. Let $B_0 = B - \bigcup_{i=1}^{m} \mathring{V}'_i$, $B_1 = B_0 - \partial B$ and $M_1 = \pi^{-1}(B_1)$. The restricted map π: $M_1 \to B_1$ is a fibre bundle with generic leaf L as fibre. We now assume that

M is a closed 4-manifold and F is a foliation of M by closed orientable surfaces of genus≥ 2. Then the bundle π: $M_1 \rightarrow B_1$ is represented as follows. Let B_0 be a compact surface of genus h with $m+n$ boundaries. First we consider the case B is orientable. Take simple closed curves c_i $(i=1, 2, \cdots, 2h)$ and arcs d_j $(j=1, 2, \cdots, m+n)$ on B_0 such that 1) c_i and c_{i+1} intersect at points p_i $(i=1, 2, \cdots, 2h-1)$, 2) d_j $(j=1, 2, \cdots, m)$ join points p_{2h+j-1} of c_{2h} and $\partial V'_j$ respectively and d_{m+j} $(j=1, \cdots, n)$ join points $p_{2h+m+j-1}$ of c_{2h} and points q_j of $\partial_j B$, where $q_j=\pi(L_{j,1})$ for $j=1, \cdots, n'$, and 4) cutting off B_0 along c_i $(i=1, \cdots, 2h)$ and d_j $(j=1, \cdots, m+n)$ yields a compact manifold B_2 which is homeomorphic to a disk (see Fig. 1). c_1 is separated to an arc c'_1 by c_2, c_i $(i=2, \cdots,$

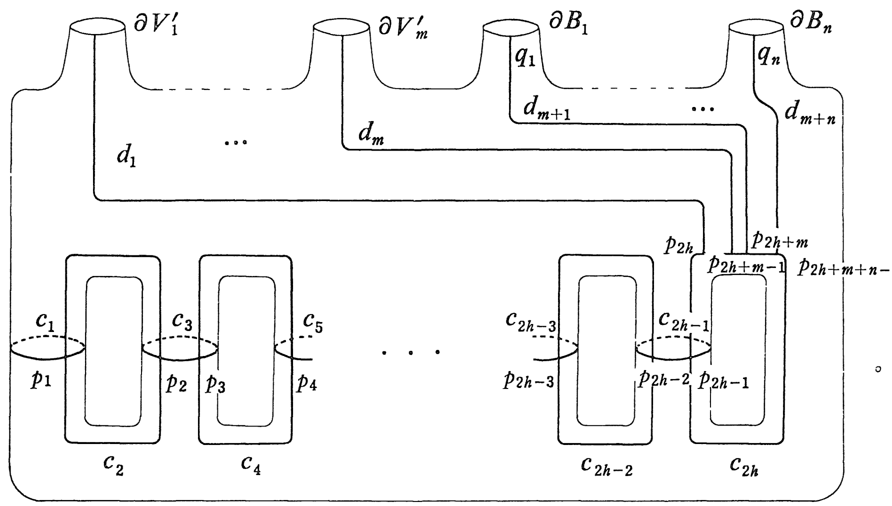

Figure 1

$2h-1)$ are separated to two arcs $c_{i,1}$, $c_{i,2}$ by c_{i-1} and c_{i+1}, and c_{2h} is separated to $m+n+1$ arcs $c_{2h,j}$ $(j=1, \cdots, m+n+1)$ by c_{2h-1} and d_j $(j=1, \cdots, m+n)$. Cutting off B_1 along c_i and d_j yields a subset B_3 of B_2 whose interior is homeomorphic to an open disk. Then we obtain M_1 by making the following identifications in $L \times B_3$ as follows;

$(x, y) \sim (\varphi_1(y)(x), y)$ for $x \in L$, $y \in c'_1$, $(x, y) \sim (\varphi_{ij}(y)(x), y)$ for $x \in L$, $y \in c_{ij}$ $(i=2, \cdots, 2h-1; j=1, 2)$, $(x, y) \sim (\varphi_{2h,j}(y)(x), y)$ for $x \in L$, $y \in c_{2h,j}$ $(j=1, \cdots, m+n+1)$ and $(x, y) \sim (\psi_k(y)(x), y)$ for $x \in L$, $y \in d_k$ $(k=1, \cdots, m+n)$, where φ_1: $c'_1 \times L \rightarrow L$, φ_{ij}: $c_{ij} \times L \rightarrow L$ $\varphi_{2h,j}$: $c_{2h,j} \times L \rightarrow L$ and ψ_k: $d_k \times L \rightarrow L$ are smooth maps such that $\varphi_1(y)$, $\varphi_{ij}(y)$, $\varphi_{2h,j}(y)$ and $\psi_k(y)$ for $y \in c'_1$, c_{ij}, $c_{2h,j}$ and d_k are diffeomorphisms of L respectively. We may assume that $\varphi_1(y)$, $\varphi_{ij}(y)$, $\varphi_{2h,j}(y)$ and $\psi_k(y)$ are constant diffeomorphisms on neighborhoods of the boundaries of c_1', c_{ij}, $c_{2h,j}$ and d_k respectively.

Next we consider the non-orientable case. Let B_0' be a closed surface obtained by pasting disks to B_0 along the boundaries. B_0' is homeomorphic to $\sum_{h_1} \# P^2$ or $\sum_{h_1} \# K^2$ according to $h=2h_1+1$ or $2h_1+2$, where $\sum_{h_1}$ is an orientable surface of genus h_1, P^2 is the projective plane and K^2 is the Klein bottle. We can identify B_0 with $\sum_{h_1} \# P^2 - \bigcup_{i=1}^{m+n} \mathring{D}_i^2$ or $\sum_{h_1} \# K^2 - \bigcup_{i=1}^{m+n} \mathring{D}_i^2$.

Case of $h=2h_1+1$. Take simple closed curves c_i $(i=1, 3, 4, \cdots, h+1)$ and arcs c_2, d_j $(j=1, 2, \cdots, m+n)$ on B_0 such that 1) c_i and c_{i+1} intersect at points p_i $(i=3, 4, \cdots, h)$, 2) c_1 generates the fundamental group of P^2, 3) c_2 joins a point p_1 of c_1 and a point p_2 of c_3, 4) d_j $(j=1, \cdots, m)$ join points p_{h+j} of c_{h+1} and $\partial V_j'$, d_{m+j} $(j=1, \cdots, n)$ join points p_{h+m+j} of c_{h+1} and points q_j of $\partial_j B$, where $q_j=\pi(L_{j,1})$ for $j=1, \cdots, n'$ and 5) cutting off B_0 along c_i $(i=1, \cdots, h+1)$ and d_j $(j=1, \cdots, m+n)$ yields a compact topological manifold B_2 which is homeomorphic to a disk.

Case of $h=2h_1+2$. Take simple closed curves c_1, c_2 on B_0 instead of c_1, c_2 in the case of $h=2h_1+1$ as follows: their homotopy classes $\{c_1\}$ and $\{c_2\}$ are generators of the fundamental group of K^2 with the relation $\{c_1\}\cdot\{c_2\} =\{c_2\}^{-1}\cdot\{c_1\}$ and c_1 and c_2 intersect at p_1. Take other simple closed curves c_i $(i=3, 4, \cdots, h+1)$ and arcs d_j $(j=1, 2, \cdots, m+n)$ on B_0 in the same way as in the case of $h=2h_1+1$. Note that c_2 and c_3 intersect at a point p_2. c_1 is separated to an arc c_1' by c_2, c_i $(i=3, 4, \cdots, 2h_1-1)$ are separated to two arcs $c_{i,1}$, $c_{i,2}$ by c_{i-1} and c_{i+1}, and c_{2h} is separated to $m+n$ arcs $c_{2h,j}(j=1, 2, \cdots, m+n)$ by c_{2h-1} and d_j $(j=1, 2, \cdots, m+n)$. In the case of $h=2h_1+2$, c_2 is separated to two arcs $c_{2,1}$, $c_{2,2}$ by c_1 and c_3. The rest is similar to the orientable case.

Definition 4. Let Y be a subset of a manifold X. We say a diffeomorphism $f\colon X\to X$ satisfies the property $P(Y,r)$ if 1) f is sufficiently C^r-close to 1_X, 2) f is equal to 1_X on the outside of Y and 3) f has no fixed points in $\mathring{Y}$.

We fix the following notations.

Notation 5. B is the leaf space, and B_0, B_1 and B_2 are subsets of B as is stated above. g is the genus of a generic leaf L, and α_i, β_i $(1\leq i\leq g)$ are simple closed curves on L such that $\langle\alpha_i, \alpha_j\rangle=0$, $\langle\beta_i, \beta_j\rangle=0$ and $\langle\alpha_i, \beta_j\rangle=\delta_{ij}$ for any i, j, where $\langle\ ,\ \rangle$ denotes the algebraic intersection number of 1-cycles in L. We denote by $[\alpha_i]$ and $[\beta_i]$ the homology classes of α_i and β_i. Then $[\alpha_i]$, $[\beta_i]$ $(1\leq i\leq g)$ form a canonical symplectic basis for $H_1(L\colon \mathbb{Z})$. We denote by $\{\alpha_i\}$, $\{\beta_i\}$ the homotopy classes of α_i, β_i. Then $\{\alpha_i\}$, $\{\beta_i\}$ $(1\leq i\leq g)$

represent generators of $\pi_1(L)$.

§2. Stability of a Foliation with All Leaves Tori

In this section we study about perturbations of foliations with all leaves tori and singular leaves.

Proposition 6. *If a foliation with all leaves tori has a rotation or dihedral leaf, then the foliation is C^1-stable.*

Proof. This follows from Theorem 1.1 of Hirsch [7] since a certain linear holonomy of such a leaf has not 1 as an eigenvalue.

We consider here the case a foliation F has only reflection leaves as singular leaves. Each connected component of the union $R(F)$ of reflection leaves of F is diffeomorphic to $T^2\times[0, 1]/h$, where $(x, 0)$ and $(y, 1)$ are identified by a diffeomorphism h of T^2. We denote a connected component of $R(F)$ by the same letter. Let Möb be the Möbius band obtained in the product $S^1\times(-1, 1)$ with coordinate (θ, u), $\theta\in S^1=\boldsymbol{R}/\boldsymbol{Z}$, $u\in(-1, 1)$ by identifying (θ, u) and $(\theta+1/2, -u)$. The foliation on $S^1\times(-1, 1)$ with leaves of form $S^1\times\{pt\}$ induces a foliation F_1 on Möb. So we define a foliation F_2 on Möb $\times S^1\times[0, 1]$ with leaves of form $L\times S^1\times\{pt\}$, $L\in F_1$. Let U be a saturated tubular neighborhood of $R(F)$ in M. Then (U, F) is diffeomorphic to (Möb $\times S^1\times[0, 1], F_2)/H$, where H: Möb$\times S^1\rightarrow$ Möb$\times S^1$ is a foliation preserving diffeomorphism extended from h and a point p in Möb$\times S^1$ is assumed to be fixed by H. We take generators α and β of $\pi_1(\text{Möb}\times S^1, p)$ corresponding to generators of $\pi_1(\text{Möb})$ and $\pi_1(S^1)$ respectively. Let h_*: $H_1(T^2)\rightarrow H_1(T^2)$ be the automorphism.

Lemma 7. $h_*=\begin{pmatrix} 2k+1 & l \\ 2m & 2n+1\end{pmatrix}$, *where* $k, l, m, n\in\boldsymbol{Z}$ *and* $(2k+1)(2n+1)-2ml=\pm1$.

Proof. The holonomy along α is non-trivial and of order two and the holonomy along β is trivial. So the holonomy along $h_*(\beta)$ is trivial, hence $h_*(\beta)=2m\alpha+n'\beta$ $(m, n'\in\boldsymbol{Z})$. Since h_* belongs to $GL(2, \boldsymbol{Z})$, diagonal components are odd numbers.

We consider the special case $k=-1$ and $n=-1$.

Theorem 8. *Let F and U be as above. Suppose* $h_*=\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$. *Then* *every sufficiently small C^1-perturbation of F has a compact leaf in U. Hence F is C^1-stable.*

Proof. We may assume that $U=\text{Möb}\times S^1\times[0, 1]/H$ with coordinate (θ, u, φ, t), $(\theta, u)\in\text{Möb}$, $\varphi\in S^1=\mathbf{R}/\mathbf{Z}$ and $t\in[0, 1]$, and $p=(0, 0, 0)\in\text{Möb}\times S^1$ and a segment $E=\{(0, u, 0);\ -1<u<1\}\subset\text{Möb}\times S^1$ are left invariant by H. The set $E\times[0, 1]/H$ can be considered to be the set $E\times S^1$, if necessary, by taking an appropriate double covering $\tilde{U}$ of U. For, let F' be a small C^1-perturbation of F. Then the foliation $\tilde{F}'$ induced on $\tilde{U}$ is also a small C^1-perturbation of the foliation $\tilde{F}$ induced on $\tilde{U}$. If $\tilde{F}'$ has a compact leaf in $\tilde{U}$, then F' has a compact leaf in U.

Let α and β be loops in $L_{p\times\{0\}}$ with base point $p\times\{0\}=(0, 0, 0, 0)$, representing generators of $\pi_1(L_{p\times\{0\}})\cong\mathbf{Z}\oplus\mathbf{Z}$ such that the holonomy along α (resp. β) is non-trivial (resp. trivial). Let $\alpha(t)$ and $\beta(t)$ be translations of α and β along the curve $p\times\{t\}$, $t\in[0, 1]$. Then we can define perturbed holonomy maps $H(F', \alpha(t))$, $H(F', \beta(t))$: $E_\delta\times\{t\}=\{(0, u, 0);\ -\delta<u<\delta\}\times\{t\}\to E\times S^1$ for each t and some $\delta>0$, which are imbeddings (cf. Hirsch [7], Langevin-Rosenberg [9] and Fukui [4]). Note that 1) $H(F', \alpha(t_0))$ and $H(F', \beta(t_0))$ are extended to maps $H(F', \alpha_{t_0})$ and $H(F', \beta_{t_0})$: $E_\delta\times(t_0-r, t_0+r)\to E\times S^1$ for some small r, which are local diffeomorphisms, 2) the extended maps $H(F', \alpha_{t_0})$ and $H(F', \alpha_{t_1})$. $H(F', \beta_{t_0})$ and $H(F', \beta_{t_1})$ coincide on the intersections of their domains respectively if t_0 and t_1 are close, 3) $H(F', \alpha(t))$ and $H(F', \beta(t))$ are C^1-close to the map $R(u, t)=(-u, t)$ and $\text{id}(u, t)=(u, t)$ respectively because F and F' are C^1-close and 4) $H(F', \alpha(1))=H(F', -\alpha(0))$ and $H(F', \beta(1))=H(F', -\beta(0))$ may not coincide with $H(F', \alpha(0))$ and $H(F', \beta(0))$ respectively.

We put $\tilde{S}^1=\mathbf{R}/2\mathbf{Z}$ and let π: $\tilde{S}^1\to S^1$ be the double covering map defined by $\pi(\tilde{t})=\tilde{t}\,(\text{mod } 1)$, $\tilde{t}\in\tilde{S}^1$. Then there exist the maps $H_\alpha(F')$ and $H_\beta(F')$: $E_\delta\times\tilde{S}^1\to E\times\tilde{S}^1$ extended from $H(F', \alpha(t))$ and $H(F', \beta(t))$ (cf. [4], [9]) respectively, such that the following diagram commutes;

$$\begin{array}{ccc} E_\delta\times\tilde{S}^1 & \xrightarrow{\ H_\alpha(F')\ (\text{resp. } H_\beta(F'))\ } & E\times\tilde{S}^1 \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle 1\times\pi} \\ E_\delta\times\{t\} & \xrightarrow{\ H(F', \alpha(t))\ (\text{resp. } H(F', \beta(t)))\ } & E\times S^1\,, \end{array} \qquad \cdots\cdots \qquad (*)$$

where $i((0, u, 0, t))=(0, u, 0, t)$ and $(1\times\pi)\,(0, u, 0, \tilde{t}))=(0, u, 0, \pi(\tilde{t}))$. We put $H_\alpha(F')\,(u, \tilde{t})=(f_1(u, \tilde{t}), f_2(u, \tilde{t}))$ using the coordinate $(u, \tilde{t})$ of $E_\delta\times\tilde{S}^1$. Then there exists a unique $u(\tilde{t})$ for each $\tilde{t}\in\tilde{S}^1$ such that $u(\tilde{t})=f_1(u(\tilde{t}), \tilde{t})$. The set $\tilde{l}=\{(u(\tilde{t}), \tilde{t});\ \tilde{t}\in\tilde{S}^1\}$ is a loop in $E\times\tilde{S}^1$, so $(1\times\pi)\,(\tilde{l})$ is a loop in $E\times S^1$, which rotates twice around S^1. Therefore there exists a $t_0(t_0\in S^1)$ with $u(t_0)=u(t_0+1)$. We may assume $t_0=0$. So the set $\{(u(t), t);\ t\in S^1\}$ in $E\times S^1$ is a loop l, which may have a corner at $(u(0), 0)$. By the same argument as in the proof of Theorem 4 of Fukui [5], there exists a point $q=(u(t_1), t_1)\in l$ such that $H(F', \alpha(t_1))(q)=q$, that is, q is a fixed point of $H_\alpha(F')$.

We consider the behavior of $H_\beta^n(F')(q)$ $(n\in\boldsymbol{Z})$ for a fixed point q of $H_\alpha(F')$.

Lemma 9. *$H_\beta(F')(q)$ is a fixed point of $H_\alpha(F')$ and belongs to $\tilde{l}$.*

Proof. $H_\alpha(F')\circ H_\beta(F')=H(F', \alpha\cdot\beta)=H(F', \beta\cdot\alpha)=H_\beta(F')\circ H_\alpha(F')$. Thus $H_\alpha(F')(H_\beta(F')(q))=H_\beta(F')(H_\alpha(F')(q))=H_\beta(F')(q)$. Hence $H_\beta(F')(q)$ is a fixed point of $H_\alpha(F')$. Since we have a unique $u(\tilde{t})$ with $u(\tilde{t})=f_1(u(\tilde{t}), \tilde{t})$ for each $\tilde{t}\in\tilde{S}^1$, $H_\beta(F')(q)$ lies in $\tilde{l}$.

For the simplicity, we put $\varphi'=H_\beta(F')$. We denote by $\mathrm{Fix}(H_\alpha)$ the fixed point set of $H_\alpha(F')$. Note that $\varphi'(\mathrm{Fix}(H_\alpha))=\mathrm{Fix}(H_\alpha)$ and $\varphi'(\mathrm{Fix}(H_\alpha))\subset\tilde{l}$ by Lemma 9.

Let $\tilde{\pi}$: $E\times\tilde{S}^1\to\tilde{l}$ be the map defined by $\tilde{\pi}(u, \tilde{t})=(u(\tilde{t}), \tilde{t})$ and $\varphi=\tilde{\pi}\circ\varphi'|\tilde{l}$: $\tilde{l}\to\tilde{l}$. Then we easily see that φ is a diffeomorphism of $\tilde{l}$ and φ and φ' coincide on $\mathrm{Fix}(H_\alpha)$.

Proposition 10. *There exists a point p in $\tilde{l}$ such that p is a fixed point of $H_\alpha(F')$ and a periodic point of $H_\beta(F')$.*

We prove Proposition 10 as follows. We suppose that φ has no periodic points on $\mathrm{Fix}(H_\alpha)$. We introduce on $\tilde{l}$ a fixed orientation. If a and b are different points of $\tilde{l}$, the $\widehat{ab}$ denotes the oriented simple arc connecting a with b, and the formula $a<c<b$ means that the point c lies on the arc $\widehat{ab}$. Since φ is C^1-close to the identity, φ preserves the orientation.

Then we can prove Lemma 11 and use it to prove Lemma 12 similarly as in the proofs of Lemmas 1 and 2 of Siegel [14].

Lemma 11. *Let q be a point in $\mathrm{Fix}(H_\alpha)$. For a non-zero integer m be given, then there exists an integer h such that $q<\varphi^h(q)<\varphi^m(q)$.*

We suppose that a point $p\in\mathrm{Fix}(H_\alpha)$ is not ergodic, that is, the orbit set $O(p)=\{\varphi^n(p);\ n\in\boldsymbol{Z}\}$ is not dense in $\tilde{l}$.

$\tilde{l}-\overline{O(p)}$ is an open and non-empty set. Choose in $\tilde{l}-\overline{O(p)}$ an open arc $\widehat{ab}$ whose end points belong to $\overline{O(p)}$. The end points a_n, b_n of all images arcs $\widehat{a_nb_n}=\varphi^n(\widehat{ab})$ $(n\in\boldsymbol{Z})$ lie in $\overline{O(p)}$ and the inner points of these arcs lie in $\tilde{l}-\overline{O(p)}$, hence $\widehat{a_n b_n}(n \in \mathbb{Z})$ are disjoint.

Lemma 12. *Let a_n, $b_n(n \in \mathbb{Z})$ be as above. For an arbitrarily large natural number N, there exists an integer $m > N$ such that either the m-arcs $\widehat{a_{-k} b_{m-k}}$ or $\widehat{a_{m-k} b_{-k}}(k=1, 2, \cdots, m)$ are disjoint.*

By the similar argument as in Siegel [14], Lemma 12 leads us to a contradiction. Hence $O(p)$ is dense in $\tilde{l}$. This implies $\mathrm{Fix}(H_\alpha)=\tilde{l}$. We put $\varphi'(u, \tilde{t})=(g_1(u, \tilde{t}), g_2(u, \tilde{t}))$ for $(u, \tilde{t}) \in E_\delta \times \tilde{S}^1$. From (*), $H(F', \beta(t))(u, t)=(g_1(u, t), g_2(u, t))$. Since we suppose that φ' has no periodic points on $\mathrm{Fix}(H_\alpha)$, the point $(u(0), 0)$ is not a fixed point of φ' and if $g_2(u(0), 0)>0$, then $g_2(u(1), 1) >1$. On the other hand, $g_2(u(1), 1)<1$ because $H(F', \beta(1))=H(F', -\beta(0))$. This is a contradiction. Hence we have Proposition 10.

The following proposition is proved by the standard argument (cf. Langevin-Rosenberg [8]).

Proposition 13. *If p in $\tilde{l}$ is a fixed point of $H_\alpha(F')$ and a periodic point of $H_\beta(F')$, then L'_p is compact, where L'_p is a leaf of F' through p.*

We complete the proof of Theorem 8 by Propositions 10 and 13.

Remark 14. Theorem 8 holds for C^2-foliations.

Theorem 15. *Let F and U be as above. Suppose $h_* = \begin{pmatrix} \pm 1 & l \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 2m & \pm 1 \end{pmatrix}$ $(l, m \in \mathbb{Z})$. Then there is a foliation F' such that F' is C^r-close to F and F' has no compact leaves in U.*

Proof. We prove the case $h_* = \begin{pmatrix} \pm 1 & l \\ 0 & 1 \end{pmatrix}$. It is proved similarly for the case $h_* = \begin{pmatrix} 1 & 0 \\ 2m & \pm 1 \end{pmatrix}$. We consider the product $\text{Möb} \times S^1 \times \mathbb{R}$ with coordinate (θ, u, φ, t), $(\theta, u) \in \text{Möb}$, $\varphi \in S^1$ and $t \in \mathbb{R}$, and define a foliation G on $\text{Möb} \times S^1 \times \mathbb{R}$ to be the set of leaves whose tangent spaces are spanned by $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \varphi}$. Let $\hat{H}$: $\text{Möb} \times S^1 \times \mathbb{R} \to \text{Möb} \times S^1 \times \mathbb{R}$ be a diffeomorphism defined by $\hat{H}(\theta, u, \varphi, t)=(\pm\theta + l \cdot \varphi, \varepsilon u, \varphi, t+1)$, $\varepsilon = \pm 1$. Then (U, F) is diffeomorphic to $(\text{Möb} \times S^1 \times \mathbb{R}, G)/\hat{H}$. Now we define a new foliation G' on $\text{Möb} \times S^1 \times \mathbb{R}$ to be the set of leaves whose tangent spaces are spanned by $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \varphi} + \lambda \frac{\partial}{\partial t}$, where λ is a small irrational number. Then $\hat{H}$ preserves G' because $\hat{H}_*\left(\frac{\partial}{\partial \theta}\right) = \pm \frac{\partial}{\partial \theta}$

and $\hat{H}_*\left(\frac{\partial}{\partial\varphi}+\lambda\frac{\partial}{\partial t}\right)=\frac{\partial}{\partial\varphi}+l\frac{\partial}{\partial\theta}+\lambda\frac{\partial}{\partial t}$. Hence we can define a foliation $F_1=G'/\hat{H}$ on $\text{Möb}\times S^1\times\mathbb{R}/\hat{H}$. We can easily extend F_1 to F' on M such that F and F' are C^r-close ($r\geq 0$). It is easy to see that F' has no compact leaves in U.

§3. Instability of Foliations without Singular Leaves

In this section we consider the case F has no singular leaves, that is, π: $M\to B$ is a fibre bundle. To begin with we show the following which is true for arbitrary closed manifolds B.

Proposition 16. *If the bundle π: $M\to B$ is trivial and $g\geq 2$, then F is C^r-unstable* ($r\geq 0$).

Proof. Take diffeomorphisms f_1 and f_2 of B such that f_1 and f_2 are sufficiently C^r-close to 1_B and the periodic point sets of f_1 and f_2 are disjoint. Then we define a homomorphism Φ: $\pi_1(L)\to \text{Diff}(B)$ by $\Phi(\{\alpha_i\})=f_i$ ($i=1$, 2), $\Phi(\{\alpha_i\})=1_B$ ($i=3$, 4, $\cdots$, g) and $\Phi(\{\beta_i\})=1_B$ ($i=1$, 2, $\cdots$, g). This defines a foliation F' of $M=L\times B$ whose leaves are transverse to the fibres of another fibre bundle π': $M\to L$ with fibre B. From the properties of f_i, we see that F' is sufficiently C^r-close to F and has no compact leaves.

Theorem 17. (1) *If B is homeomorphic to the 2-sphere S^2 and $g\geq 2$, then F is C^r-unstable* ($r\geq 0$).
(2) *If B is homeomorphic to the projective plane P^2 and $g\geq 4$, then F is C^r-unstable* ($r\geq 0$).
(3) *If B is neither homeomorphic to S^2 nor P^2 and $g\geq 6$, then F is C^r-unstable* ($r\geq 0$).

Proof of (1). In this case, it is an immediate consequence of Proposition 16 because that any bundle over S^2 with fibre L of genus≥ 2 is trivial.

Proof of (2). Any bundle over P^2 with fibre L of genus≥ 2 is obtained by making the identifications in $L\times D^2$ as follows: $(x, y)\sim(\varphi(x), -y)$ for $x\in L$, $y\in D^2$, where φ: $L\to L$ is a diffeomorphism with $\varphi^2=1_L$.

Step 1. We perturb F on $\pi^{-1}(\mathring{D}^2)\cong L\times\mathring{D}^2$ as follows. Let G: $D^2\to D^2$ be a diffeomorphism satisfying the property $P(D^2, r)$. Let Q be an open tubular neighborhood of α_1 in L such that its closure $\bar{Q}$ is homeomorphic to $\alpha_1\times[0, 1]$ with coordinate (s, t), $s\in\alpha_1$, $0\leq t\leq 1$. We start with the foliation of $(L-Q)\times\mathring{D}^2$ having leaves of form $(L-Q)\times\{y\}$, $y\in\mathring{D}^2$ and make the identifications $(s, 0, y) \sim (s, 1, G(y))$ to obtain a foliation of $L \times \mathring{D}^2$ having no compact leaves. Replacing F on $\pi^{-1}(\mathring{D}^2)$ by this foliation yields a new foliation F' which has no compact leaves on $\pi^{-1}(\mathring{D}^2)$.

Step 2. Take a point p of $\partial D^2/_\sim$ and a conic tubular neighborhood A of $\partial D^2/_\sim - p$ in $D^2/_\sim$ as in Fig. 2.

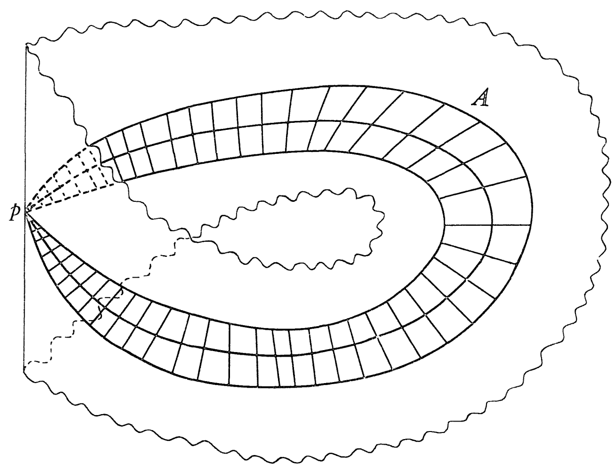

Figure 2

We want to perturb F' on $\pi^{-1}(A) \cong L \times A$ to obtain a foliation F'' such that 1) F'' is sufficiently C^r-close to F, 2) F'' has no compact leaves on $\pi^{-1}(P^2 - p)$ and 3) $L_p = \pi^{-1}(p)$ is the only compact leaf of F''. Since $g \geq 4$, there exists a simple closed curve δ on L in $\pi^{-1}(\partial D^2/_\sim)$ such that $[\delta] \neq 0$ in $H_1(L; \mathbb{Z})$ and $\langle \delta, \alpha_1 \rangle = \langle \delta, \varphi(\alpha_1) \rangle = 0$ (see Plante [10]). In fact $\langle \alpha_2, \varphi(\alpha_1) \rangle \beta_2 - \langle \beta_2, \varphi(\alpha_1) \rangle \alpha_2$ is homologous to a multiple of a simple closed curve. Let γ be an arbitrary closed curve on L. Then the holonomy of L in F' along γ is trivial if and only if $\langle \gamma, \alpha_1 \rangle = \langle \gamma, \varphi(\alpha_1) \rangle = 0$. Hence the holonomy of L in F' along δ is trivial. Take a tubular neighborhood Q_δ of δ in L such that its closure $\bar{Q}_\delta$ is homeomorphic to $\delta \times [0, 1]$ with coordinate (s, t), $s \in \delta$, $0 \leq t \leq 1$. Take a diffeomorphism H: $P^2 \to P^2$ satisfying the property $P(A, r)$. F' restricted to $\bar{Q}_\delta \times A$ has leaves of form $\delta \times [0, 1] \times \{y\}$, $y \in A$. Thus we start with the foliation of $\delta \times [0, 1] \times A - \delta \times (1/3, 2/3) \times A$ having leaves of form $\delta \times [0, 1/3] \cup [2/3, 1] \times \{y\}$, $y \in A$ and make the identifications $(s, 1/3, y) \sim (s, 2/3, H(y))$ to obtain a foliation of $\bar{Q}_\delta \times A$. Replacing F' on $\bar{Q}_\delta \times A$ by this foliation, we obtain a required foliation F''.

Step 3. Finally we perturb F'' on a neighborhood of the compact leaf L_p. Since $g \geq 4$, there is a simple closed curve η on L_p such that $[\eta] \neq 0$ in

$H_1(L_p; \boldsymbol{Z})$ and (1) $\langle\eta, \alpha_1\rangle=\langle\eta, \delta\rangle=0$, (2) $\langle\eta, \varphi(\alpha_1)\rangle=\langle\eta, \varphi(\delta)\rangle=0$. For, put $\eta=\sum_{i=3}^{g} m_i\alpha_i+\sum_{i=3}^{g} n_i\beta_i$ ($m_i, n_i\in\boldsymbol{Z}$). Then the equations (1) are satisfied. So we consider the equations (2). We can solve (2) over integers since $g\geq 4$. Let η_0 be a non-trivial general solution of (2). Then we can choose a simple closed curve η such that η_0 is homologous to a multiple of η because $\eta=\sum_{i=3}^{g} m_i\alpha_i+\sum_{i=3}^{g} n_i\beta_i$ is realizable by a simple closed curve on L_p if $m_3, \cdots, m_g, n_3, \cdots, n_g$ are relatively prime. The holonomy of L_p in F'' along η is trivial. Hence the rest of the proof is done similarly as in Step 2.

Proof of (3). We prove the case B is orientable. It is proved similarly for the non-orientable case. Note that B_1 is a closed surface in this case.

Lemma 18. *There exists a foliation F_1 of M such that* 1) *F_1 is sufficiently C^r-close to F,* 2) *F_1 has no compact leaves on $\pi^{-1}(B-\{p_1, \cdots, p_{2h-1}\})$ and* 3) *$L_{p_i}=\pi^{-1}(p_i)$ ($i=1, 2, \cdots, 2h-1$) are the compact leaves of F_1.*

Proof. First we perturb F on $\pi^{-1}(\mathring{B}_2)\cong L\times\mathring{B}_2$ as in Step 1 in the proof of (2) using α_1 on L and a diffeomorphism of B satisfying the property $P(B_2, r)$. We let F' be the resulting foliation. Next take conic tubular neighborhoods A_1, A_{ij} ($i=2, 3, \cdots, 2h-1; j=1, 2$) and $A_{2h,1}$ of c_1', c_{ij} and $c_{2h,1}$ such that they are disjoint and the closures of A_1 and A_{2j} have the only point p_1 in common ($j=1, 2$) and the closures of A_{i1} and A_{i2} ($i=2, \cdots, 2h-1$) have the points p_{i-1} and p_i in common and the closures of $A_{2h-1,j}$ and $A_{2h,1}$ have the only point p_{2h-1} in common ($j=1, 2$) (see Fig. 3).

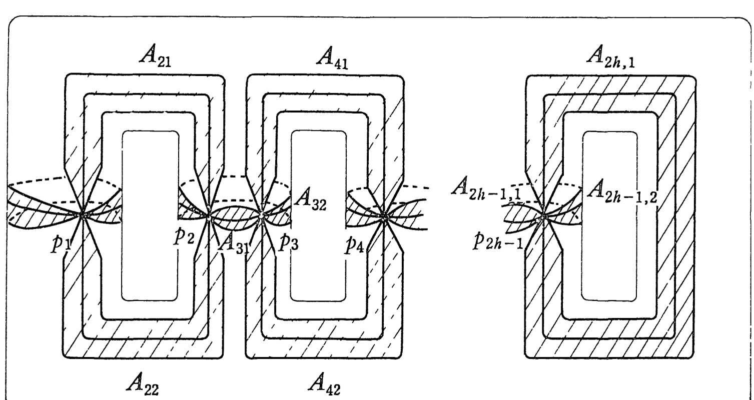

Figure 3

There exist simple closed curves δ_1, δ_{ij} ($i=2, \cdots, 2h-1; j=1, 2$) and $\delta_{2h,1}$

on the compact leaves L of F' in $\pi^{-1}(A_1)$, $\pi^{-1}(A_{ij})$ and $\pi^{-1}(A_{2h,1})$ respectively such that $[\delta_1]\neq 0$, $[\delta_{ij}]\neq 0$ and $[\delta_{2h,1}]\neq 0$ in $H_1(L; \mathbf{Z})$ and $\langle\delta_1, \alpha_1\rangle=\langle\delta_1, \varphi_1(\alpha_1)\rangle=0$, $\langle\delta_{ij}, \alpha_1\rangle=\langle\delta_{ij}, \varphi_{ij}(\alpha_1)\rangle=0$ $(i=2, 3, \cdots, 2h-1;\ j=1, 2)$ and $\langle\delta_{2h,1}, \alpha_1\rangle=\langle\delta_{2h,1}, \varphi_{2h,1}(\alpha_1)\rangle=0$ (see Plante [10]). $[\delta_1]$, $[\delta_{ij}]$ and $[\delta_{2h,1}]$ can be considered to be expressed as linear combinations of $[\alpha_2]$ and $[\beta_2]$. Take diffeomorphisms G_1, G_{ij} $(i=2, 3, \cdots, 2h-1;\ j=1, 2)$ and $G_{2h,1}\colon B\to B$ satisfying the properties $P(A_1, r)$, $P(A_{ij}, r)$ and $P(A_{2h,1}, r)$ respectively. Then, by the similar argument to Step 2 in the proof of (2), we can perturb F' on $\pi^{-1}(A_1)\cong L\times A_1$, $\pi^{-1}(A_{ij})\cong L\times A_{ij}$ and $\pi^{-1}(A_{2h,1})\cong L\times A_{2h,1}$ using δ_1, δ_{ij} and $\delta_{2h,1}$ to obtain a required foliation F_1.

Lemma 19. *There exist simple closed curves η_i on L_{p_i} $(i=1, 2, \cdots, 2h-1)$ such that $[\eta_i]\neq 0$ in $H_1(L_{p_i}; \mathbf{Z})$ and*

(1—1) $\langle\eta_1, \alpha_1\rangle=\langle\eta_1, \delta_1\rangle=\langle\eta_1, \delta_{2j}\rangle=0$ $(j=1, 2)$

(1—2) $\langle\eta_1, \varphi_1(\delta_1)\rangle=\langle\eta_1, \varphi_{2j}(\delta_{2j})\rangle=0$ $(j=1, 2)$

(1—3) $\langle\eta_1, \varphi_1(\alpha_1)\rangle=\langle\eta_1, \varphi_{22}\circ\varphi_1(\alpha_1)\rangle=\langle\eta_1, \varphi_1^{-1}\circ\varphi_{22}\circ\varphi_1(\alpha_1)\rangle=0$

(i—1) $\langle\eta_i, \alpha_1\rangle=\langle\eta_i, \delta_{ij}\rangle=\langle\eta_i, \delta_{i+1,j}\rangle=0$ $(j=1, 2)$

(i—2) $\langle\eta_i, \varphi_{ij}(\delta_{ij})\rangle=\langle\eta_i, \varphi_{i+1,j}(\delta_{i+1,j})\rangle=0$ $(j=1, 2)$

(i—3) $\langle\eta_i, \varphi_{i1}(\alpha_1)\rangle=\langle\eta_i, \varphi_{i+1,1}\circ\varphi_{i1}(\alpha_1)\rangle=\langle\eta_i, \varphi_{i2}^{-1}\circ\varphi_{i+1,1}\circ\varphi_{i1}(\alpha_1)\rangle=0$
$(i=2, \cdots, 2h-2)$,

($2h-1$—1) $\langle\eta_{2h-1}, \alpha_1\rangle=\langle\eta_{2h-1}, \delta_{2h-1,j}\rangle=\langle\eta_{2h-1}, \delta_{2h,1}\rangle=0$ $(j=1, 2)$

($2h-1$—2) $\langle\eta_{2h-1}, \varphi_{2h-1,j}(\delta_{2h-1,j})\rangle=\langle\eta_{2h-1}, \varphi_{2h,1}(\delta_{2h,1})\rangle=0$ $(j=1, 2)$

($2h-1$—3) $\langle\eta_{2h-1}, \varphi_{2h-1,1}(\alpha_1)\rangle=\langle\eta_{2h-1}, \varphi_{2h,1}\circ\varphi_{2h-1,1}(\alpha_1)\rangle$
$=\langle\eta_{2h-1}, \varphi_{2h-1,1}^{-1}\circ\varphi_{2h}\circ\varphi_{2h-1,1}(\alpha_1)\rangle=0$

Proof. We prove for each i. Put $\eta=\sum_{i=3}^{g} m_i\alpha_i+\sum_{i=3}^{g} n_i\beta_i$ $(m_i, n_i\in\mathbf{Z})$. Then the equations (i—1) are satisfied. So we consider the system equations (i—2) and (i—3). If $g\geq 6$, we can solve the equations (i—2) and (i—3) over integers. Let η_0 be a non-trivial general solution of (i—2) and (i—3). Then we can choose a simple closed curve η such that η_0 is homologous to a multiple of η because $\eta=\sum_{i=3}^{g} m_i\alpha_i+\sum_{i=3}^{g} n_i\beta_i$ $(m_i, n_i\in\mathbf{Z})$ is realizable by a simple closed curve on L_{p_i} if $m_3, \cdots, m_g, n_3, \cdots, n_g$ are relatively prime.

Proof of Theorem 17 (3) *continued.* By Lemma 18, it is sufficient to perturb the compact leaves L_{p_i} $(i=1, 2, \cdots, 2h-1)$. Take small neighborhoods C_i of p_i such that they are disjoint. From Lemma 19, there exist simple closed curves η_i on L_{p_i} satisfying the conditions (i—1), (i—2) and (i—3). Thus by the similar argument to Step 3 in the proof of (2), we can perturb F_1 on $\pi^{-1}(C_i)$ using η_i to obtain a foliation which has no compact leaves. This completes the proof.

§4. Instability of Foliations with Singular Leaves

In this section we consider the case F has singular leaves.

Proposition 20. *Let F be a foliation of M such that B_0 has $m+n$ boundaries. If $g\geq 2$, then there exists a foliation F_1 of M such that F_1 is sufficiently C^r-close to F and F_1 has no compact leaves on $\pi^{-1}(\mathring{B}_0-\{p_1, \cdots, p_s\})$, where $s=2h+m+n-1$ if B is an orientable surface of genus h and $s=2h_1+m+n$ if B is homeomorphic to $\sum_{h_1}\# P^2-\bigcup_{i=1}^{m+n}\mathring{D}_i^2$ or $\sum_{h_1}\# K^2-\bigcup_{i=1}^{m+n}\mathring{D}_i^2$ for an orientable surface of genus h_1, $\sum_{h_1}$.*

Proof. It is similarly proved as in the proof of Lemma 18. Moreover we can perturb F on $\pi^{-1}(\mathring{B}_2)$ using α_1, $\pi^{-1}(\mathring{c}_1)$ using δ_1, $\pi^{-1}(\mathring{c}_{ij})$ using δ_{ij} $(i=2, \cdots, 2h-2; j=1, 2)$, $\pi^{-1}(\mathring{c}_{2h-1,j})$ using $\delta_{2h-1,j}$ $(j=1, \cdots, m+n+1)$ and $\pi^{-1}(\mathring{d}_j)$ using δ'_j $(j=1, \cdots, m+n)$, where $[\delta]$'s are expressed as linear combinations of $[\alpha_2]$ and $[\beta_2]$.

A) Case of foliations with only rotation leaves as singular leaf.

Note that F is a foliation which satisfies $n=0$ in §1 and B is a V-manifold without boundary.

Theorem 21. *Let F be a foliation of M such that F has m rotation leaves $L_1, \cdots, L_m$ with holonomy order $k_1, \cdots, k_m$ respectively. If $g\geq 3\max(k_i; 1\leq i\leq m)+1$, then F is C^r-unstable $(r\geq 0)$.*

Proof. First we perturb F_1 in Proposition 20 on each $U(L_j)\cong\pi^{-1}(V_j)$. Let $\bar{\alpha}_1=p_j(\alpha_1)$ and $\bar{\delta}'_j=p_j(\delta'_j)$ $(j=1, \cdots, m)$, where p_j: $L\to L_j$ is the covering map. Let τ_j $(j=1, 2, \cdots, m)$ be simple closed curves on L_j representing generators of the holonomy groups $\boldsymbol{Z}_{k_j}$ of L_j. If $g(L_j)\geq 4$, then there exist simple closed curves η_j on L_j such that $[\eta_j]\neq 0$ in $H_1(L_j; \boldsymbol{Z})$ and $\langle\eta_j, \bar{\alpha}_1\rangle=\langle\eta_j, \bar{\delta}'_j\rangle=\langle\eta_j, \tau_j\rangle=0$ respectively. Note that $p_j\circ\psi_j(\alpha_1)=p_j(\alpha_1)$. Take tubular neighborhoods S_j of η_j in L_j which are homeomorphic to $\eta_j\times(0, 1)$ respectively. By Proposition 3 (a) of Vogt [17], the normal disk bundles $U(L_j)$ are trivial. Hence the bundles over S_j $(j=1, 2, \cdots, m)$ are the products $S_j\times D^2$ respectively. Thus by the similar argument to Step 2 in the proof of Theorem 17 (2), we perturb F_1 on $U(L_j)$ to obtain a foliation F_2 which has no compact leaves on $\pi^{-1}(B-\{p_1, \cdots, p_s\})$. We easily see that $g(L_j)\geq 4$ if and only if $g\geq 3k_j+1$ $(j=1, \cdots, m)$. Finally we perturb $L_{p_i}=\pi^{-1}(p_i)$ $(i=1, \cdots, s)$ similarly as in Step 3 of the proof of Theorem 17 (2). For this purpose, it is sufficient to take simple closed curves r_i on L_{p_i} $(i=1, \cdots, s)$ such that the holonomy along each r_i is trivial. This is possible because $3\max(k_i;\ 1\leq i\leq m)+1\geq 6$ (see Lemma 19). This completes the proof.

B) Case of foliations with only reflection leaves as singular leaf.

Note that F is a foliation which satisfies $m=0$, $n=n''$ in §1 and B is a smooth manifold with n boundary components.

Theorem 22. *If $g\geq 7$, then F is C^r-unstable* $(r\geq 0)$.

Proof. We take conic tubular neighborhoods A_j of $\partial_j B-q_j$ $(j=1, \cdots, n)$ in B and want to perturb F_1 in Proposition 20 on each $\pi^{-1}(A_j)$ which is homeomorphic to $L\times_{\mathbb{D}}\mathring{D}^2$, where for $g\in\mathbb{D}$, $g\neq 1$, g: $\mathring{D}^2\to\mathring{D}^2$ is defined by $g(x, y)=(x, -y)$ for $(x, y)\in\mathring{D}^2$. Let p: $L\to L_0(=L\times\{0\}/\mathbb{D})\subset\pi^{-1}(\partial_j B-q_j)$ be the covering map and $\bar{\alpha}_1=p(\alpha_1)$. We put $\bar{\alpha}_1=\bar{\alpha}$ if $p(\alpha_1)$ is homologous to a twice of a simple closed curve $\bar{\alpha}$. We take a simple closed curve δ_j'' on L_0 such that $[\delta_j'']\neq 0$ in $H_1(L_0;\mathbb{Z})$ and $\langle\delta_j'', \bar{\alpha}_1\rangle=0$. We can assume that $[\delta_j'']$ is expressed as a linear combination of $[\bar{\alpha}_2]$ and $[\bar{\beta}_2]$, where $[\bar{\alpha}_i]$ and $[\bar{\beta}_i]$ form a symplectic basis for $H_1(L_0;\mathbb{Z})$. Then taking a diffeomorphism G_j: $\mathring{D}^2\to\mathring{D}^2$ satisfying the property $P(D^2, r)$ and $G_j(x, y)=(G_j^1(x, y), y)$, $G_j^1(x, -y)=G_j^1(x, y)$, we can perturb F_1 on $\pi^{-1}(A_j)$ using δ_j'' to obtain a foliation F_2 which has no compact leaves on $\pi^{-1}(B-\{p_1, \cdots, p_s, q_1, \cdots, q_n\})$. We can perturb the compact leaves $L_{p_i}=\pi^{-1}(p_i)$ $(i=1, \cdots, s)$ in the usual way. Finally we want to perturb the compact leaves $L_{q_j}=\pi^{-1}(q_j)$ $(j=1, \cdots, n)$. It is sufficient to take simple closed curves η_j on L_{q_j} such that $[\eta_j]\neq 0$ in $H_1(L_{q_j};\mathbb{Z})$ and $\langle\eta_j, \bar{\alpha}_1\rangle=\langle\eta_j, p(\delta_j')\rangle=\langle\eta_j, \delta_j''\rangle=\langle\eta_j, \psi_j(\bar{\alpha}_1)\rangle=\langle\eta_j, \psi_j(\delta_j'')\rangle=0$. This is possible because of $g(L_0)\geq 4$. $g(L_0)\geq 4$ if and only if $g\geq 7$. This completes the proof.

C) Case of foliations with dihedral leaves.

We consider the case F has no rotation leaves and some points of each boundary $\partial_j B$ of B correspond to dihedral leaves $L_{j,k}$ with holonomy groups $\mathbb{D}_{l_{j,k}}$.

Theorem 23. *If $g\geq 8\max(l_{j,k};\ 1\leq j\leq n;\ 1\leq k\leq m_j)+1$, then F is C^r-unstable* $(r\geq 0)$.

Proof. We can perturb F_1 in Proposition 20 on each $\pi^{-1}(\partial_j B)-\bigcup_{k=1}^{m_j} L_{j,k}$ using a simple closed curve $\delta''_{j,k}$ on a reflection leaf in $\pi^{-1}(\partial_j B)-\bigcup_{k=1}^{m_j} L_{j,k}$ as in the proof of Theorem 22, where $[\delta''_{j,k}]$ can be considered to be expressed as linear combinations of $[\bar{\alpha}_2]$ and $[\bar{\beta}_2]$. We let F_2 be the resulting foliation. Next we want to perturb F_2 on saturated tubular neighborhoods $U(L_{j,k})$ of $L_{j,k}$. Let $p_{j,k}$: $L\to L_{j,k}$ be the covering map and λ and μ simple closed curves on $L_{j,k}$ which represent generators of the holonomy group of $L_{j,k}$. We need the following lemma.

Lemma 24. *There exist simple closed curves $\eta_{j,k}$ on $L_{j,k}$ such that* 1) $[\eta_{j,k}]\neq 0$ *in $H_1(L_{j,k}; \boldsymbol{Z})$,* 2) $\langle\eta_{j,k}, p_{j,k}(\alpha_1)\rangle=\langle\eta_{j,k}, p_{j,k}(\bar{\alpha}_2)\rangle=\langle\eta_{j,k}, p_{j,k}(\bar{\beta}_2)\rangle=\langle\eta_{j,k}, \lambda\rangle=\langle\eta_{j,k}, \mu\rangle=0$ *and* 3) *$U(L_{j,k})$ restricted to $\eta_{j,k}$ are trivial disk bundles.*

Proof. By the result of Vogt [17], List 1, Propositions 5 and 6, $(U(L_{j,k}), F)$ is represented by the vector $(v, 1, u, 1, \cdots, 1)$ or $(v, u^{l_{j,k}/2}, u, 1, \cdots, 1)$ (see [17] for details). We may assume that $p_{j,k}(\alpha_1)$, $p_{j,k}(\bar{\alpha}_2)$ $p_{j,k}(\bar{\beta}_2)$, λ and μ are represented by $[\bar{\bar{\alpha}}_l]$, $[\bar{\bar{\beta}}_l]$ $(l=1, 2, 3, 4)$ where $[\bar{\bar{\alpha}}_l]$, $[\bar{\bar{\beta}}_l]$ $(l=1, 2, \cdots, g(L_{j,k}))$ form a canonical symplectic basis for $H_1(L_{j,k}; \boldsymbol{Z})$. We can take the simple closed curves $\bar{\bar{\alpha}}_5$ and $\bar{\bar{\beta}}_5$ which satisfy 1) and 2) since $g\geq 8\max(l_{j,k}; 1\leq j\leq n, 1\leq k\leq m_j)+1$ implies $g(L_{j,k})\geq 5$. If $U(L_{j,k})$ restricted to $\bar{\bar{\alpha}}_5$ and $\bar{\bar{\beta}}_5$ are non-trivial respectively, there is a simple closed curve $\bar{\bar{\eta}}$ on $L_{j,k}$ such that $U(L_{j,k})$ restricted to $\bar{\bar{\eta}}$ is trivial and $[\bar{\bar{\eta}}]=[\bar{\bar{\alpha}}_5]+[\bar{\bar{\beta}}_5]$. We put $\eta_{j,k}=\bar{\bar{\eta}}$.

Now we continue the proof of Theorem 23. We perturb F_2 on $U(L_{j,k})$ using $\eta_{j,k}$ in the usual way to obtain a foliation F_3 which has no compact leaves on the $\pi^{-1}(B-\{p_1, \cdots, p_s\})$. It is easy to perturb the compact leaves $L_{p_i}=\pi^{-1}(p_i)$ $(i=1, \cdots, s)$. This completes the proof.

D) General case.

Combining Theorems 21, 22 and 23, we have the following.

Theorem 25. *Let F be a foliation of a closed 4-manifold M by closed orientable surfaces and $B=M/F$ the leaf space. Suppose F has m rotation leaves with holonomy groups $\boldsymbol{Z}_{k_i}$ $(i=1, 2, \cdots, m)$ and m_j dihedral leaves with holonomy groups $\boldsymbol{D}_{l_{j,k}}$ $(k=1, 2, \cdots, m_j)$ which correspond to points of $\partial_j B$ for each j $(1\leq j\leq n')$.*

If $g\geq\max(3\max(k_i; 1\leq i\leq m)+1, 8\max(l_{j,k}; 1\leq j\leq n', 1\leq k\leq m_j)+1, 7\varepsilon)$, then F is C^r-unstable $(r\geq 0)$, where $\varepsilon=0$ or 1 and F has no reflection leaves if *and only if* $\varepsilon=0$.

Remark 26. If $g\geq 2$ and g is even, then F has only rotation leaves as singular leaf. Hence Theorem 25 reduces to Theorem 21.

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