Completely Positive Linear Maps in a Concept of Majorization on Certain Operator Algebras

By

Peter M. ALBERTI*

Abstract

Some concepts of Majorization in the state spaces of certain operator algebras are discussed and the interrelations among them are established. The main case considered deals with the class of non-finite AFD-factor vN-algebras.

§ 1. Some Majorization Concepts on Operator Algebras

Let $\mathfrak{A}$ be a unital C^*-algebra, with unitary group $\mathcal{U}(\mathfrak{A})$ and dual space $\mathfrak{A}^*$. Let $\mathfrak{B}(\mathfrak{A}^*)$ be the set of all bounded linear operators over $\mathfrak{A}^*$ acting into $\mathfrak{A}^*$. For $u \in \mathfrak{A}$, let us define $T_u \in \mathfrak{B}(\mathfrak{A}^*)$ through the setting $(T_u \mu)(x) = \mu(u^* x u) = \mu^u(x)$, for all $x \in \mathfrak{A}$, $\mu \in \mathfrak{A}^*$. We put $C_u(\mathfrak{A}) =$ point-w-*closure of conv $\{T_v : v \in \mathcal{U}(\mathfrak{A})\}$, where the point-$w^*$-topology refers to the system of semi-norms $p_{v,a}$ over $\mathfrak{B}(\mathfrak{A}^*)$ given by $p_{v,a}(T) = |(T\nu)(a)|$, $\nu \in \mathfrak{A}^*$, $a \in \mathfrak{A}$. Then, by well-known facts, $C_u(\mathfrak{A})$ is a convex and compact subset of $\mathfrak{B}(\mathfrak{A}^*)$ with respect to the topology mentioned. Suppose, $S(\mathfrak{A})$ denotes the w^*-compact convex set of states over $\mathfrak{A}$. Throughout this paper we let n be a fixed natural number. Whenever $\nu_1, \cdots, \nu_n \in S(\mathfrak{A})$ are given, the finite sequence $\{\nu_1, \cdots, \nu_n\} \subset S(\mathfrak{A})$ will be referred to as a n-tuple of states, which will be abbreviated by $\{\nu\}$. In all what follows we shall have to work in the set of all n-tuples of states over $\mathfrak{A}$. Within this set of all n-tuples of states we are going to introduce a preordering relation $\gg_u$. A preordering in our context is supposed to mean a reflexive and transitive relation: $\{\omega\} \gg_u \{\omega\}$, $\forall \{\omega\}$, and $\{\omega\} \gg_u \{\nu\}$ together with $\{\nu\} \gg_u \{\sigma\}$ implies always $\{\omega\} \gg_u \{\sigma\}$.

Definition 1. *For n-tuples* $\{\omega\}$, $\{\sigma\} \subset S(\mathfrak{A})$ *we let* $\{\omega\} \gg_u \{\sigma\}$ *mean that* $\omega_k = \Phi(\sigma_k)$, $\forall k$, *with* $\Phi \in C_u(\mathfrak{A})$.

In this situation, let us say that "$\{\omega\}$ is more unitarily mixed than $\{\sigma\}$" or "$\{\omega\}$ arises from $\{\sigma\}$ through unitary mixing".

It is easy to see that the concept of unitary mixing for n-tuples of states as introduced above is a straightforward extension of the concept of G-majorization as given in [1, Chap. 14, B]. Namely, what we are doing is to consider the n-fold direct product C^*-algebra $\mathfrak{A}_n$ (i.e. the set of n-tuples $\{a\} = \{a_1, \cdots, a_n\}$ of elements $a_j \in \mathfrak{A}$). Then, the set of all n-tuples of states over $\mathfrak{A}$ can be interpreted as a convex, w^*-compact subset of elements of $\mathfrak{A}_{n+}^*$ in an obvious way. Within $\mathfrak{B}(\mathfrak{A}_n^*)$, let us define a group G of transformations by $G = \{T \in \mathfrak{B}(\mathfrak{A}_n^*):$ $T = \{T_v, T_v, \cdots, T_v\}, \ v \in \mathcal{U}(\mathfrak{A})\}$, where the action of $S = \{S_1, \cdots, S_n\}$ on the n-tuple $\{\nu\}$ is given via $S(\{\nu\}) = \{S_1(\nu_1), \cdots, S_n(\nu_n)\}$. Then, the relation $\{\omega\} \gg_u \{\sigma\}$ between the n-tuples mentioned is equivalent to saying that the $\{\omega\}$-element of $\mathfrak{A}_{n+}^*$ is in the w^*-closed convex hull of the orbit of the $\{\sigma\}$-element under the action of the group G. The relations to the concept of G-majorization should be clear now by remembering that the original concept as represented in [1] can also be interpreted in terms of C^*-algebras. Indeed, $\mathbb{R}^N$ might be thought of as the hermitian part of $\mathfrak{A}^*$ when $\mathfrak{A} = \mathbb{C}^N$ is chosen, cf. Section 2, Example 1, and $n = 1$ is considered.

In the following, let us introduce and discuss another preordering relation $\gg$ over the set of n-tuples of states on $\mathfrak{A}$. A linear map $\Phi: \mathfrak{A}^* \to \mathfrak{A}^*$ is referred to as a c.p.-stochastic linear map, and the set of all these mappings be $ST(\mathfrak{A}^*)$, if Φ carries states into states and the adjoint $\Phi^\sharp$ is a completely positive, unital linear transformation over the second dual $\mathfrak{A}^{**}$ (for generalities on C^*- and W^*-algebras the reader is referred to the standard textbooks, e.g. [2], [3]). Note that $ST(\mathfrak{A}^*)$ is a point-w^*-compact convex subset of transformations of $\mathfrak{B}(\mathfrak{A}^*)$, which also forms a semigroup and contains the identity map.

Definition 2. For *n-tuples* $\{\omega\}$, $\{\sigma\} \subset S(\mathfrak{A})$ *put* $\{\omega\} \gg \{\sigma\} \Leftrightarrow \exists \Phi \in ST(\mathfrak{A}^*)$ *with* $\omega_k = \Phi(\sigma_k)$, $\forall k$.

In case we are given a vN-algebra $\mathfrak{A} = \mathfrak{M}$ acting over some Hilbert-space $\mathfrak{H}$, a third preordering relation $\gg_*$ will be of further interest. Let $\mathfrak{M}_*$ be the pre-dual of $\mathfrak{M}$, and $S_0(\mathfrak{M})$ the set of normal states over $\mathfrak{M}$. Then, a linear map $\psi: \mathfrak{M}_* \to \mathfrak{M}_*$ is said to be c.p.-stochastic over $\mathfrak{M}_*$, and we collect all these mappings into the set $ST(\mathfrak{M}_*)$, if the adjoint $\psi^\sharp$ over $\mathfrak{M}$ is completely positive and unital. Note that in the latter case $\psi^\sharp$ is a normal positive linear map in

the usual sense. We shall denote by $CP_1(\mathfrak{M})$ and $CP_1^0(\mathfrak{M})$ the set of unital c.p.-linear maps over $\mathfrak{M}$ and the normal unital c.p.-mappings over $\mathfrak{M}$, respectively.

Definition 3. For n-tuples $\{\omega\}$, $\{\sigma\} \subset S_0(\mathfrak{M})$ of normal states we define $\{\omega\} \gg_* \{\sigma\} \Leftrightarrow \exists \Phi \in ST(\mathfrak{M}_*)$ with $\omega_k = \Phi(\sigma_k)$, $\forall k$.

Note that $\gg_*$ is a preordering relation since $ST(\mathfrak{M}_*)$ is a semigroup in $\mathfrak{B}(\mathfrak{M}_*)$ containing the identity map.

Remark. Whereas by Def. 1 the concept of G-majorization has been extended, Def. 2 and Def. 3 relate to the extension of the concept of a *stochastic matrix* or *stochastic operator* from the commutative case (either $\mathfrak{M} = \boldsymbol{C}^N$, or more generally $\mathfrak{M} = L^\infty(\Omega, \mu)$, $\mathfrak{M}_* = L^1(\Omega, \mu)$, with some localizable measure space (Ω, μ)) to the general non-commutative situation of a unital C^*-algebra or W^*-algebra, cf. [4] and references quoted there. Note that $\{\omega\} \gg_* \{\sigma\}$ is equivalent with the existence of $T \in CP_1^0(\mathfrak{M})$ such that $\omega_k = \sigma_k \circ T$, $\forall k$. The convex semigroups $CP_1(\mathfrak{M})$ and $CP_1^0(\mathfrak{M})$ both contain the identity map and extend to the non-commutative situation the notions of *Markovian operators* and *normal Markovian operators,* respectively, which are known from the abelian cases.

Majorization theory as one of its basic goals requires to establish functional characterizations of the preordering relations concerned. By this the following is meant. Assume a preordering $\gg$ on our set of n-tuples of states (resp. normal states) is given. A realvalued function F over the set of n-tuples of states will be said to be *isotonic* with respect to the preordering in question, whenever the implication $\{\omega\} \gg \{\sigma\} \Rightarrow F(\{\omega\}) \geq F(\{\sigma\})$ takes place, and F is said to be $\gg$-*anti-isotonic,* if the implication $\{\omega\} \gg \{\sigma\} \Rightarrow F(\{\omega\}) \leq F(\{\sigma\})$ is always true. Aiming at a functional characterization of $\gg$ then means to meet two lines:

(1)   Impose conditions on a realvalued function F which imply that F is isotonic (resp. anti-isotonic) and which do not read in terms of the very definition of the preordering.

(2)   Extract a *sufficient* system $(\mathfrak{F}, \gg)$ of isotonic (resp. anti-isotonic) functions such that always $F(\{\omega\}) \geq F(\{\sigma\})$, $\forall F \in \mathfrak{F}, \Rightarrow \{\omega\} \gg \{\sigma\}$ is true (resp. with $\geq$ substituted by $\leq$ in the anti-isotonic case).

Clearly, a theory of majorization tacitly confines to cases where such a sufficient system of functions exists and reflects the non-trivial character of the preordering considered. Besides this fact, however, there might be a variety of sufficient systems which might differ from each other. It is of an obvious use to keep

the sufficient systems as "small" as possible.   Speaking in words of (1) one is tempted to look for conditions on the functions which are as "stringent" as possible when imposed on.

We shall try to meet this task of a functional characterization in case of $\gg_*$ in the set of normal states over a properly infinite, approximately finite-dimensional factor on a separable Hilbert-space.

§ 2.   Classical Examples

In all what follows we shall strictly adhere on interpreting the majorization concepts in an algebraic language, even if these concepts do not refer to this context in their original formulation.

Example 1.   Let $\mathfrak{A}$ be a finite-dimensional (say N-dimensional) abelian C^*-algebra, and let Aut $\mathfrak{A}$ be the full group of linear *-automorphisms over $\mathfrak{A}$. Define G as a group of actions of Aut $\mathfrak{A}$ on $\mathfrak{A}^*$ by defining $T_g \in G$ through $(T_g \nu)(x) = \nu(g(x))$, $\forall x \in \mathfrak{A}$, $\nu \in \mathfrak{A}^*$, for each $g \in$ Aut $\mathfrak{A}$.   An element $\nu \in S(\mathfrak{A})$ can be characterized by the values it takes on the atoms (call them spectral values).   Let $\nu^1 \geq \nu^2 \geq \cdots \geq \nu^N$ be the decreasingly ordered sequence of these spectral values (one has to take care to the multiplicity of the spectral values). Let ω, σ be states, and let $\omega \gg_G \sigma$ mean that the state ω is in the convex hull of the orbit of σ under G.

This then will give the starting point for one of the classical examples majorization theory deals with: $\omega \gg_G \sigma$ is equivalent with the following conditions,

$\Leftrightarrow$ $$\sum_{j \leq k} \omega^j \leq \sum_{j \leq k} \sigma^j \, , \ \forall k; \tag{2-1}$$

$\Leftrightarrow$ $$F(\omega) \leq F(\sigma) \tag{2-2}$$

for any quasi-convex (see [1]), G-invariant function F over $S(\mathfrak{A})$;

$\Leftrightarrow$ $$\sum_j f(\omega^j) \leq \sum_j f(\sigma^j) \tag{2-3}$$

for any convex, continuous function f on [0, 1].

In this example, in which we refer to the fundamental result of the theory of inequalities in [5] (Muirhead's Theorem), Theorems of Rado and Birkhoff (see [1], [4]), we have given three different functional characterizations of the G-majorization in question in terms of anti-isotonic functions.   Whereas the system of functions in (2-2) certainly is the largest of the given three, the one of (2-1) is the smallest under comparison (it contains $n-1$ functions, the n-th

being constant on $S(\mathfrak{A})$). This provides a good example to illustrate the discussion below (1), (2) in the previous section.

In the next example we refer to Def. 1 with $n=1$, and $\mathfrak{A}$ is an arbitrary vN-algebra (we omit the parantheses in case of 1-tuples).

Example 2.   For states ω, $\sigma \in S(\mathfrak{A})$, $\omega \gg_u \sigma$ is equivalent with the following two conditions:

$$\Leftrightarrow \qquad\qquad F(\omega) \leq F(\sigma) \qquad\qquad\qquad\qquad (2\text{-}4)$$

for every quasi-convex, w^*-lower semicontinuous, unitarily invariant function F over $S(\mathfrak{A})$;

$$\Leftrightarrow \qquad\qquad \sup_{u \in \mathcal{U}(\mathfrak{A})} \omega^u(p) \leq \sup_{u \in \mathcal{U}(\mathfrak{A})} \sigma^u(p) \qquad\qquad (2\text{-}5)$$

for each of the orthoprojection of $\mathfrak{A}$, $p \in \mathfrak{A}$.

This example (unitary mixing on vN-algebras) has been investigated extensively by several authors, and is represented in [6], cf. the list of references there. The original concept is due to A. Uhlmann and refers to the case of $\mathfrak{A}=M_N$, the full algebra of complex $N \times N$-matrices, see [7].   In connection with its applications in Statistical Physics, the whole subject for density operators is referred to as "Uhlmann's Theory", and "order structure of states on W^*-algebras" in the general case, respectively, see [8], [9].   Once more again, the structure of the example reflects the significance of the discussion at the end of Section 1.

Note that because of the Murray-von Neumann theory of factors a sufficient set of functions which figures in (2–5) in case of factors over a separable Hilbert-space can be parametrized conveniently by the relative dimension τ.

Indeed, let us define functions $\{e_\tau : \tau \in \varDelta\}$ over $S(\mathfrak{A})$, with the range $\varDelta$ of the relative dimension given by $\varDelta = \{0, 1, \cdots, N\}$ for type I_N, $\varDelta = \mathcal{N} \cup \{0, \infty\}$ for type I_∞, $\varDelta = [0, 1]$ for type II_1, $\varDelta = [0, \infty]$ for type II_∞, and $\varDelta = \{0, \infty\}$ in the remaining type III cases of factors.   The function e_τ reads $e_\tau(\nu) = \sup\limits_{u \in \mathcal{U}(\mathfrak{A})} \nu^u(p)$, for an orthoprojection p with relative dimension $\tau \in \varDelta$, $\forall \nu$.   Then, (2–5) is equivalent to

$$\Leftrightarrow \qquad\qquad e_\tau(\omega) \leq e_\tau(\sigma), \qquad \forall \tau \in \varDelta. \qquad\qquad (2\text{-}6)$$

Hence, $\{e_\tau : \tau \in \varDelta\}$ is a set of unitarily invariant, convex functions over $S(\mathfrak{A})$ which forms a sufficient system of $\gg_u$-anti-isotonic functions in case of a factor, by (2–6).

Example 3. Suppose $\mathfrak{M}$ is a vN-factor of finite type. Let us look on the case $n=2$ (the case of tuples) and restrict $\gg_*$ (see Def. 3) to all pairs of normal states the second state of which is the canonical trace-state τ. Then

$$\{\omega, \tau\} \gg_* \{\sigma, \tau\} \Leftrightarrow \omega \gg_u \sigma$$

in the sense of Example 2. Therefore, the sufficient systems of anti-isotonic functions (2–4), (2–5) and (2–6) characterize $\gg_u$ as well as $\gg_*$ on the subset considered (see [6]).

Example 4. Let $\mathfrak{M}=L^\infty([0, 1])$ (the unit interval with Lebesgue measure), and suppose $n=2$. Then, for normal states ω_1, ω_2, σ_1, σ_2 (which correspond to probability distributions)

$$\{\omega_1, \omega_2\} \gg_* \{\sigma_1, \sigma_2\} \Leftrightarrow \|\omega_1 - t\,\omega_2\|_1 \le \|\sigma_1 - t\sigma_2\|_1 , \qquad \forall t \in \boldsymbol{R}_+ \qquad (2\text{–}7)$$

This has been proved in [10]. Note that the assertion remains true also in the non-abelian case $\mathfrak{M}=M_2$, see [11], [6].

The next example to be considered is a very instructive one and deals with the general n-tuple situation over an abelian vN-algebra.

Example 5. Let $\mathfrak{M}$ be a commutative W^*-algebra. Then, $\mathfrak{M}$ is isomorphic with $L^\infty(\varOmega, \mu)$ for some suitable localizable measure space $(\varOmega, \mu)$. Let $\{\omega\}$, $\{\sigma\} \subset S_0(\mathfrak{M})$ be n-tuples of normal states ovre $\mathfrak{M}$. Considering $\mathfrak{M} \simeq L^\infty(\varOmega, \mu)$, a normal state ν corresponds to a probability distribution

$$\nu \in L^1(\varOmega, \mu), \quad \text{and} \quad \nu(x) = \int_\varOmega \nu(t) x(t) d\mu(t), \qquad \text{for} \quad x \in L^\infty(\varOmega, \mu) .$$

With all this notations adopted, we have:

$$\{\omega\} \gg_* \{\sigma\} \Leftrightarrow \{\omega\} \gg \{\sigma\} \Leftrightarrow$$

$$\int_\varOmega \omega_n(t) G(\omega_1(t)/\omega_n(t), \cdots, \omega_{n-1}(t)/\omega_n(t)) d\mu(t) \ge \int_\varOmega \sigma_n(t) G(\sigma_1(t)/\sigma_n(t), \cdots,$$

$$\sigma_{n-1}(t)/\sigma_n(t)) d\mu(t) \qquad (2\text{–}8)$$

for every non-negative, concave function G over $\boldsymbol{R}_+^{n-1}$.

This means, the homogeneous and superadditive functionals

$$H_G(\nu_1, \cdots, \nu_n) = \int_\varOmega \nu_n G(\nu_1/\nu_n, \cdots, \nu_{n-1}/\nu_n) d\mu \qquad (2\text{–}9)$$

over the convex set of n-tuples of normal states of $\mathfrak{M}$ provide a sufficient system

of isotonic functions in both the $\gg$ and the $\gg_*$ case. But note that (2–9) has to be defined with some care (at the boundary of the cone $\mathbb{R}_+^{n-1}$ one has to adopt some convention concerning the function under the integral; we omit the details, but refer to [13]).

Note that for $n=2$, $\Omega=[0, 1]$, μ-Lebesgue measure, the given equivalences mean that (2–8)$\Leftrightarrow$(2–7), which fact is known since [10]. Basic results relating the equivalences given have been derived in [12] and considered systematically in [14], especially in the case where the measure space is finite and $\nu_n=\omega_n=\sigma_n=\mu(\Omega)^{-1}$, which means doubly stochasticity of all transformations involved. The complete set of equivalences in the general case is shown in [13]. Obviously, the example reads in the classical tradition of majorization theory, theory of stochastic and doubly stochastic transformations on L^1-spaces, and so has much to do with rearrangement considerations, see references given in [13], and [15] for related topics.

Note that the functions (2–9) form only one specific sufficient system of isotonic functions. A sufficient system of anti-isotonic functions is given by

$$S_{\{x_1, \cdots x_n\}}(\nu_1, \cdots, \nu_n) = \int_\Omega \|\nu_1(t)x_1 + \cdots + \nu_n(t)x_n\|_\infty \, d\mu(t)$$

for any n-tuple $\{x\} \subset L^\infty(\Omega, \mu)_+$, see [13], [14].

§3. Notations, the Problem of This Paper

The objective of this paper is to show the intimate relations among the preorderings $\gg$, $\gg_*$, $\gg_u$ introduced in Section 1. In this connection, we will find that such relations can only exist in cases of certain classes of non-commutative vN-algebras and related to them C^*-algebras. That means, we restrict the subject of our investigations from the very beginning to those structures which seem most promising. In line with the examples of the previous section, and following the discussion of majorization concepts in Section 1, what will be done is to construct sufficient systems of isotonic resp. anti-isotonic functions, and to investigate their properties.

The central class of algebras under consideration consists of the *approximately finite-dimensional factors* (AFD-factors) over some separable Hilbert-space $\mathfrak{H}$. Let $\mathfrak{M}$ be such a factor. Then, there exists an increasing sequence $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}$ of finite-dimensional subfactors of $\mathfrak{M}$ with $\mathfrak{M}=(\cup \mathfrak{M}_N)''$. Let $\mathfrak{A}_0 = \cup \mathfrak{M}_N$, and define $\mathfrak{A} =$ norm closure $\mathfrak{A}_0$. The unital C^*-algebra $\mathfrak{A}$ is referred to as a generating U.H.F.-C^*-subalgebra of $\mathfrak{M}$ (U.H.F. means

uniformly hyperfinite). AFD-factors can be characterized by several conditions all of which are equivalent, see [16], and [17] for a new and instructive proof of these facts. The non-finite AFD-factors - these are all AFD-factors with exclusion of the hyperfinite factor (which is the type II_1 case) - will be of special interest in our investigations. Let us call a vN-algebra $\mathfrak{N}$ over $\mathfrak{H}$ *admissible* if there is a non-finite AFD-factor $\mathfrak{M}$ over $\mathfrak{H}$ such that $\mathfrak{N} \subset \mathfrak{M}$ and a *normal* conditional expectation E projecting from $\mathfrak{M}$ onto $\mathfrak{N}$ exists. Note that both the finite dimensional factor case and the hyperfinite factor case are admissible in our sense. Of course, there are much more admissible cases than the mentioned ones.

§4. An Account of Basic Results

Let $\mathfrak{M}$ be a AFD-factor over the separable Hilbert-space $\mathfrak{H}$, and suppose $\mathfrak{A}$ is a generating U.H.F.-C^*-subalgebra of $\mathfrak{M}$. For n-tuples of normal states $\{\nu\} \subset S_0(\mathfrak{M})$ and positive elements $\{x\} \subset \mathfrak{M}_+$ let us define

$$K_n(\{\nu\}, \{x\}) = \sup_T \{\textstyle\sum \nu_k \circ T(x_k): T \in CP_1^0(\mathfrak{M})\} , \qquad (4\text{--}1)$$

$$K_n'(\{\nu\}, \{x\}) = \sup_u \{\textstyle\sum \nu_k(u^* x_k u): u \in \mathcal{U}(\mathfrak{M})\} , \qquad (4\text{--}2)$$

$$K_n(\mathfrak{A}; \{\rho\}, \{y\}) = \sup_u \{\textstyle\sum \rho_k(u^* y_k u): u \in \mathcal{U}(\mathfrak{A})\} \qquad (4\text{--}3)$$

if $\{\rho\} \subset S(\mathfrak{A})$ and $\{y\} \subset \mathfrak{A}_+$. In [18], the following two results have been derived for AFD-factors.

Proposition 1. *To every generating $\mathfrak{A}$, there exists a net $S_\lambda: \mathfrak{M} \to \mathfrak{A}_0$ of completely positive, σ-weakly continuous linear mappings into $\mathfrak{A}_0$ such that*

$$K_n(\{\nu\}, \{x\}) = \lim_\lambda K_n(\{\nu\}, \{S_\lambda(x)\}) = \sup_\lambda K_n(\{\nu\}, \{S_\lambda(x)\}) \qquad (4\text{--}4)$$

for each n-tuple of normal states and any $\{x\} \subset \mathfrak{M}_+$.

Remark. The proof uses the condition "semidiscreteness" which holds for AFD-factors. The right hand side equality in (4–4) is a consequence of the proof in [18, Lemma 2].

Proposition 2. *For a non-finite AFD-factor $\mathfrak{M}$ one has*

$$K_n(\{\nu\}, \{y\}) = K_n'(\{\nu\}, \{y\}) = K_n(\mathfrak{A}; \{\nu_{/\mathfrak{A}}\}, \{y\}) \qquad (4\text{--}5)$$

for all $\{y\} \subset \mathfrak{A}_+$ and every n-tuple $\{\nu\} \subset S_0(\mathfrak{M})$ of normal states ($\nu_{/\mathfrak{A}}$ indicates the restriction of ν onto $\mathfrak{A}$).

These two facts together give the key for a proof of the following basic result.

Theorem 1. *Let $\mathfrak{M}$ be a non-finite AFD-factor, and $\mathfrak{A}$ be an arbitrary generating U.H.F.-C^*-subalgebra. Assume $\{\omega\}$, $\{\sigma\}$ are n-tuples of normal states over $\mathfrak{M}$. The following conditions are equivalent to each other*:

(1) $\{\omega_{/\mathfrak{A}}\} \gg_u \{\sigma_{/\mathfrak{A}}\}$ (cf. Def. 1);

(2) $\{\omega\} \gg \{\sigma\}$ (cf. Def. 2);

(3) $\{\omega\} \gg_* \{\sigma\}$ (cf. Def. 3);

(4) $K_n(\{\omega\}, \{x\}) \leq K_n(\{\sigma\}, \{x\})$, $\forall \{x\} \subset \mathfrak{M}_+$;

(5) $K_n'(\{\omega\}, \{x\}) \leq K_n'(\{\sigma\}, \{x\})$, $\forall \{x\} \subset \mathfrak{A}_{0+}$; (!)

(6) $K_n(\mathfrak{A}; \{\omega_{/\mathfrak{A}}\}, \{x\}) \leq K_n(\mathfrak{A}; \{\sigma_{/\mathfrak{A}}\}, \{x\})$, $\forall \{x\} \subset \mathfrak{A}_{0+}$.

For a detailed proof see [18]. The remarkable feature in the result lies in the fact that all problems around $\gg$ or $\gg_*$ between normal states may be handled equivalently as unitary mixing problems on the subset $S_0(\mathfrak{M})_{/\mathfrak{A}}$ of the state space of an arbitrarily given generating U.H.F.-C^*-subalgebra $\mathfrak{A}$. Moreover, the system of functions

$$\mathfrak{F}_0 = \{K_n(\{\cdot\}, \{x\}): \{x\} \subset \mathfrak{M}_+\}, \quad \mathfrak{F}'(\mathfrak{A}) = \{K_n'(\{\cdot\}, \{x\}): \{x\} \subset \mathfrak{A}_{0+}\},$$

$\mathfrak{F}(\mathfrak{A}) = \{K_n(\mathfrak{A}; \{\cdot_{/\mathfrak{A}}\}, \{x\}): \{x\} \subset \mathfrak{A}_{0+}\}$ provide us with sufficient systems of anti-isotonic functions in the sense discussed in Section 1 in either case. By (4-1)-(4-3) and our Propositions, it is easy to see that $\mathfrak{F}_0 \supset \mathfrak{F}'(\mathfrak{A})$, and $\mathfrak{F}'(\mathfrak{A}) = \mathfrak{F}(\mathfrak{A})$ for every generating U.H.F.-C^*-algebra $\mathfrak{A}$ holds, provided $\mathfrak{M}$ is properly infinite (i.e. $\mathfrak{M}$ is non-finite, since we are in the factor case).

§5. Properties of Some Convex and Concave Functions

Let us adopt all the notations of the previous sections. Throughout, assume $\mathfrak{M}$ is a non-finite AFD-factor, with an arbitrarily chosen, but throughout fixed, generating U.H.F.-C^*-subalgebra $\mathfrak{A}$, and $\mathfrak{A}_0 = \cup \mathfrak{M}_N$. We are going to introduce some further notations. Let $\{x\} \subset \mathfrak{M}$ be a n-tuple of elements of $\mathfrak{M}$. $\{x\}$ will be referred to as an *algebraically finite-dimensional system, a.f.d.-system* for short, if there is a finite-dimensional subfactor F of $\mathfrak{M}$ such that $\{x\} \subset F \subset \mathfrak{M}$. By $\Lambda_n(\mathfrak{M})$ let us denote the set of all n-tuples in $\mathfrak{M}$ which form a.f.d.-systems, and $\Lambda_n(\mathfrak{M})_+$ be the subset which consists of n-tuples of positive elements. Let us introduce two further sets of functions:

$$\mathfrak{F}_n^+(\mathfrak{M}) = \{f \in R_+^{S_0(\mathfrak{M}) \times \cdots \times S_0(\mathfrak{M})}: f(\{\omega\}) = \sup_{u \in \mathcal{U}(\mathfrak{M})} \sum \omega_k^u(x_k),$$

$$\text{for some } \{x\} \in \Lambda_n(\mathfrak{M})_+\},$$

$$\mathfrak{J}_n^-(\mathfrak{M}) = \{ f \in \mathcal{R}_+^{S_0(\mathfrak{M}) \times \cdots \times S_0(\mathfrak{M})} : f(\{\omega\}) = \inf_{u \in \mathcal{U}(\mathfrak{M})} \sum \omega_k^u(x_k),$$

$$\text{for some } \{x\} \in \Lambda_n(\mathfrak{M})_+ \} .$$

Both sets of functions are intimately related families. Namely, let $f \in \mathfrak{J}_n^+(\mathfrak{M})$ be generated by $\{x\} \in \Lambda_n(\mathfrak{M})_+$, and $g \in \mathfrak{J}_n^-(\mathfrak{M})$ be generated by the a.f.d.-system $\{y\}$. Then, there are constants $\alpha, \beta \in \mathcal{R}_+$ and $g' \in \mathfrak{J}_n^-(\mathfrak{M}), f' \in \mathfrak{J}_n^+(\mathfrak{M})$ such that

$$f = \alpha - g' , \tag{5-1}$$

$$g = \beta - f' . \tag{5-2}$$

In fact, let $\alpha = \sum \|x_k\|$, and g' be the element of $\mathfrak{J}_n^-(\mathfrak{M})$ generated by the a.f.d.-system $\{y'\}$ with $y'_k = \|x_k\| 1 - x_k, \forall k$; for β take $\beta = \sum \|y_k\|$, and let f' be the element of $\mathfrak{J}_n^+(\mathfrak{M})$ generated by the a.f.d.-system $\{x'\}$ given through $x'_k = \|y_k\| 1 - y_k, \forall k$. Then, a verification of (5-1) and (5-2) is a matter of easy inserting, hence the details are omitted. Owing to (5-1) and (5-2), the properties of the two systems of functions are dual to each other. We shall investigate the basic properties of the family $\mathfrak{J}_n^+(\mathfrak{M})$.

Lemma 1. *Let $f \in \mathfrak{J}_n^+(\mathfrak{M})$. There is $\{y\} \subset \mathfrak{A}_{0+}$ such that*

$$f(\{\omega\}) = K_n'(\{\omega\}, \{y\}), \forall \{\omega\} \subset S_0(\mathfrak{M}) . \tag{5-3}$$

Proof. Let f be generated by $\{x\} \in \Lambda_n(\mathfrak{M})_+$. Hence, there is a finite-dimensional subfactor F of $\mathfrak{M}$ which contains the a.f.d.-system $\{x\}$. Let us chose the subscript N such that $\dim \mathfrak{M}_N \geq \dim F$ (dim means the relative dimension of the subfactors under discussion). We are going to show that $v \in \mathfrak{B}(\mathfrak{H})$, with $v^*v = 1$, exists such that $v^* \mathfrak{M}_N v = F$. In fact, we find unitaries $u_1, u_2 \in \mathcal{U}(\mathfrak{H})$ with the property that $u_1 \mathfrak{M}_N u_1^* = \mathfrak{B}(\mathfrak{H}_1) \otimes 1, u_2 F u_2^* = \mathfrak{B}(\mathfrak{H}_1') \otimes 1$, with Hilbert-spaces $\mathfrak{H}_1, \mathfrak{H}_1', \mathfrak{H}_2, \mathfrak{H}_2'$ such that $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$, $\mathfrak{H} = \mathfrak{H}_1' \otimes \mathfrak{H}_2'$, and $\dim \mathfrak{H}_1 = \dim \mathfrak{M}_N$, $\dim \mathfrak{H}_1' = \dim F$. Let $p' \in \mathfrak{B}(\mathfrak{H}_1)$ be an orthoprojection with $\dim p' = \dim \mathfrak{H}_1'$. Let $\{\Psi_k\}, \{\Psi_k'\}$ be complete orthonormal systems of $\mathfrak{H}_2, \mathfrak{H}_2'$, respectively, and $\{\varphi_j\}, \{\varphi_j'\}$ complete orthonormal systems in $p'\mathfrak{H}_1, \mathfrak{H}_1'$, respectively.

We might define an isometry $w \in \mathfrak{B}(\mathfrak{H})$ through the settings $w\varphi_k' \otimes \Psi_j' = \varphi_k \otimes \Psi_j$, $k = 1, 2, \cdots, \dim F, j \in N$. Then, $w^*w = 1$, $ww^* = p' \otimes 1$. Moreover, one easily recognizes that $w\mathfrak{B}(\mathfrak{H}_1') \otimes 1 w^* = p'\mathfrak{B}(\mathfrak{H}_1)p' \otimes 1$ is fulfilled. Therefore, in putting $v = u_1^* w u_2 \in \mathfrak{B}(\mathfrak{H})$, and using $w^* p' \otimes 1 = w^*$, we easily see from the facts above that $F = v^* \mathfrak{M}_N v$, with $v^*v = 1$, and $vv^* = p \in \mathfrak{M}_N$. Let T be defined by $T(x) = v^* xv, \forall x \in \mathfrak{M}_N$. Now, the map $T': p\mathfrak{M}_N p \to F$, with $T'(x) = T(x)$ for all $x \in p\mathfrak{M}_N p$, is a *-isomorphism between $p\mathfrak{M}_N p$ and F. In fact, T' is linear,

and $T'(ab)=v^*abv=v^*apbv=v^*avv^*bv=T'(a)T'(b)$ for all a, b in $p\mathfrak{M}_N p$, and $T'(a^*)=T'(a)^*$ trivially. Moreover, $c\in p\mathfrak{M}_N p$ and $T'(c)=0$ implies $pcp=0$, by $vv^*=p$, i.e. $c=0$.

Let $J=T'^{-1}$, and E_F a normal conditional expectation from $\mathfrak{M}$ onto F, and let $g\in S_0(\mathfrak{M})$ fixed. Let us define a map R acting from $\mathfrak{M}$ into $\mathfrak{M}_N$ as

$$R = J\circ E_F + g(\cdot)p^{\perp}. \qquad (5\text{-}4)$$

Then, R is normal and completely positive. Since $J(1)=p$, and g is a state, $R(1)=1$ follows, i.e. $R\in CP_1^0(\mathfrak{M})$. By (5-4), for $x\in F$ we see:

$$T\circ R(x) = v^*J(x)v+g(x)v^*p^{\perp}v = v^*J(x)v = T'\circ J(x) = x. \qquad (5\text{-}5)$$

Therefore, if we define $y_k=R(x_k)$, for $k=1, 2, \cdots, n$, then $\{y\}$ is a n-tuple of non-negative elements in $\mathfrak{M}_N$ such that $T(y_k)=x_k$ for all k. Both maps T, R are in $CP_1^0(\mathfrak{M})$. Hence

$$K_n(\{\omega\}\,;\,\{x\}) = \sup_s \sum_k \omega_k\cdot S\cdot T(y_k)\leq K_n(\{\omega\}\,;\,\{y\})$$
$$K_n(\{\omega\}\,;\,\{y\}) = \sup_s \sum_k \omega_k\cdot S\cdot R(x_k)\leq K_n(\{\omega\}\,;\,\{x\}),$$

with S extending over $CP_1^0(\mathfrak{M})$. Taking together these inequalities gives

$$K_n(\{\omega\}\,;\,\{y\}) = K_n(\{\omega\}\,;\,\{x\}) \qquad (5\text{-}6)$$

for all n-tuples $\{\omega\}\subset S_0(\mathfrak{M})$. By Proposition 2 (cf. (4–5)) this gives

$$K_n(\{\omega\}\,;\,\{x\}) = K_n'(\{\omega\}\,;\,\{y\}). \qquad (5\text{-}7)$$

Since F is a finite dimensional subfactor of $\mathfrak{M}$, with $\mathfrak{M}$ properly infinite, we may apply the result of [17, Proposition 2.1] to see that for each given $V\in CP_1^0(\mathfrak{M})$ there is an $a\in\mathfrak{M}$, with $a^*a=1$, such that $V(x)=a^*xa$ for all $x\in F$. Therefore

$$K_n(\{\omega\}\,;\,\{x\}) = \sup\{\sum_k \omega_k(a^*x_k a)\colon a^*a = 1, a\in\mathfrak{M}\}. \qquad (5\text{-}8)$$

Since $\mathfrak{M}$ is properly infinite, every isometry in $\mathfrak{M}$ can be strongly approximated by unitaries. Hence, (5-8) means that $K_n(\{\omega\}\,;\,\{x\})=f(\{\omega\})$. The latter, together with (5-7), proves the assertion. $\square$

Assume, $E_N\colon \mathfrak{M}\to\mathfrak{M}_N$ is a normal conditional expectation from $\mathfrak{M}$ onto $\mathfrak{M}_N$. Let us investigate what happens to $f(\{\omega\circ E_N\})$, $f\in\mathfrak{F}_n^+(\mathfrak{M})$, when $N\to\infty$ is considered.

Lemma 2. *For every n-tuple $\{\omega\}\subset S_0(\mathfrak{M})$ the following is true:*

$$f(\{\omega\}) = \lim_{N\to\infty} f(\{\omega\circ E_N\}) = \sup_N f(\{\omega\circ E_N\}), \qquad (5\text{-}9)$$

$\forall f \in \mathfrak{F}_n^+(\mathfrak{M})$.

Proof. For given f, by Lemma 1, $\exists \{y\} \subset \mathfrak{A}_{0+}$ with

$$f(\{\nu\}) = K_n'(\{\nu\}, \{y\}), \qquad \forall \{\nu\} \subset S_0(\mathfrak{M}). \tag{5-10}$$

We know $E_N \in CP_1^0(\mathfrak{M})$, and $E_N \circ E_M = E_M$, for $N \geq M$. Hence, (5-10) allows us to apply Theorem 1, especially (3)$\Leftrightarrow$(5), twice with the result

$$f(\{\omega \circ E_M\}) = f(\{\omega \circ E_N \circ E_M\}) \leq f(\{\omega \circ E_N\}) \leq f(\{\omega\}), \tag{5-11}$$

which proves the existence of the expression on the right hand side of (5-9). Let $\varepsilon > 0$. Then, to each $\{\omega\}$ of $S_0(\mathfrak{M})$, an subscript N_0 is easily found such that

$$\exists u \in \mathcal{U}(\mathfrak{M}_{N_0}) \Rightarrow K_n'(\{\omega\}, \{y\}) - \varepsilon \leq \sum \omega_k(u^* y_k u). \tag{5-12}$$

Indeed, Proposition 2, (4-5) makes that this is possible. Let N_1 such that $\{y\} \subset \mathfrak{M}_{N_1}$. If $N \geq \max\{N_0, N_1\}$, we have $E_N(u^* y_k u) = u^* y_k u, \forall k$, for, $u^* y_k u \in \mathfrak{M}_N$ holds and E_N projects onto $\mathfrak{M}_N$. Therefore,

$$\sum \omega_k^u(y_k) \leq K_n'(\{\omega \circ E_N\}, \{y\}) = f(\{\omega \circ E_N\}), \tag{5-13}$$

for all $N \geq \max\{N_0, N_1\}$, where we have been using the representation (5-10). Taking together (5-12), (5-13), and making use of (5-10) once more again, yields

$$f(\{\omega\}) - \varepsilon \leq f(\{\omega \circ E_N\}), \tag{5-14}$$

for N sufficiently large. Since $\varepsilon > 0$ could have been chosen at will, and the information of (5-11) is available, (5-14) implies (5-9). $\qquad\square$

Lemma 3. *Let* $f_{\{x\}} \in \mathfrak{F}_n^+(\mathfrak{M})$ *be generated by* $\{x\} \in \Lambda_n(\mathfrak{M})_+$. *Then,*

$$\|\sum x_k\| \leq f_{\{x\}} \leq \sum \|x_k\|, \text{ over the } n\text{-tuples}, \tag{5-15}$$

moreover, the following equivalences take place:

$$f_{\{x\}}(\{\omega\}) = \|\sum x_k\|, \forall \{x\} \in \Lambda_n(\mathfrak{M})_+, \Leftrightarrow \tag{5-16}$$

$\{\omega\}$ *is an orthogonal system of states:*

$$f_{\{x\}}(\{\omega\}) = \|\sum x_k\|, \forall \{x\} \in \Lambda_n(\mathfrak{M})_+, \Leftrightarrow \omega_1 = \cdots = \omega_n. \tag{5-17}$$

Proof. The right hand part of the estimation in (5-15) is obviously valid. To see the other one, let us look on the linear map T given by $T(x) = \nu(x)1$, with an arbitrarily chosen $\nu \in S_0(\mathfrak{M})$. Then, $\omega_k \circ T = \nu, \forall k$. By Lemma 1,

$\exists\ \{y\}\subset\mathfrak{A}_{0+}$ such that (we suppose f is generated by $\{x\}$)

$$f(\{\mu\}) = K'_n(\{\mu\},\ \{y\}),\ \forall\ \{\mu\}\subset S_0(\mathfrak{M})\ .\qquad(5\text{--}18)$$

Since T is unital and completely positive, normal, Theorem 1 says that for $\{\nu\}=\{\nu,\ \cdots,\ \nu\}$

$$f(\{\nu\})\leq f(\{\omega\})\ .\qquad(5\text{--}19)$$

But then, $f(\{\nu\})=\sup\limits_{u\in\mathcal{U}(\mathfrak{M})}\nu^u(\sum x_k)\leq||\sum x_k||$. To see equality to hold, let $p\in\mathfrak{M},\ p\neq\mathcal{O}$, an orthoprojection with $p(\sum x_k)p=||\sum x_k||p$. Since $\{x\}\subset\mathfrak{A}_{0+}$, such p exists. Let us now for ν specify a normal state with support on p, i.e. $\nu(p)=1$. Then, by the previously clarified things, $\nu(\sum x_k)=f(\{\nu\})=||\sum x_k||$. Together with (5–19) we get the left hand side estimation of (5–15). Note that, when taking $T(x)=\omega(x)1,\ \forall x\in\mathfrak{M}$ for an arbitrary other normal state ω, by the same as above reasoning with Theorem 1 we see $f(\{\omega,\ \cdots,\ \omega\})\leq f(\{\nu,\ \cdots,\ \nu\})=||\sum x_k||$, which together with (5–15) (which has been proved above) implies the back-part of (5–17) to be true. On the other hand, assume $f_{\{x\}}(\{\omega\})=||\sum x_k||$, $\forall\ \{x\}\in\Lambda_n(\mathfrak{M})_+$. Because of (5–18) and (5–15), (5) of Theorem 1 happens to be true for any other n-tuple $\{\sigma\}\subset S_0(\mathfrak{M})$. Especially, the n-tuple $\{\nu,\ \cdots,\ \nu\}$ might be chosen. Theorem 1 then asserts that $\varPhi\in ST(\mathfrak{M}_*)$ exists with $\omega_k=\varPhi(\nu),\ \forall k$, which obviously can only be true if $\omega_1=\cdots=\omega_n$. This proves the implication $\Rightarrow$ of (5–17).

To see (5–16), assume the left hand part to hold. Then, (5–15) together with (5–18) and Theorem 1 implies for each other n-tuple $\{\nu\}\subset S_0(\mathfrak{M})$ the existence of $T\in CP_1^0(\mathfrak{M})$ with $\nu_k=\omega_k\circ T,\ \forall k$. Hence, $||\nu_k-\nu_j||_1=||(\omega_k-\omega_j)\circ T||_1\leq||\omega_k-\omega_j||_1,\ \forall k,j$. Since the n-tuple $\{\nu\}$ might be chosen at will, assume $\{\nu\}$ is a system of mutually orthogonal, normal states (since $\mathfrak{M}$ is infinite dimensional, to every n such a system can be chosen). For $k\neq j$ we have $||\nu_k-\nu_j||_1=2$. Therefore, by the inequality above, $||\omega_k-\omega_j||_1=2,\ \forall k\neq j$, i.e. $\{\omega\}$ is an orthogonal system of states. This proves the $\Rightarrow$-implication of (5–16).

To see the other way around, assume $\{\nu\}$ is a n-tupel of mutually orthogonal normal states over $\mathfrak{M}$. Let us denote by $s_1,\ \cdots,\ s_n$ the support projections of the ν_k's. To each n-tuple $\{\omega\}$ of normal states let us associate a linear map $\varPhi:\mathfrak{M}_*\to\mathfrak{M}_*$ defined through the settings

$$\varPhi(\rho) = \sum\limits_{j\leq n-1}\rho(s_j)\omega_j+\rho(1-\sum\limits_{j\leq n-1}s_j)\omega_n,\qquad\forall\rho\in\mathfrak{M}_*\ .$$

By construction, $\varPhi\in ST(\mathfrak{M}_*)$. Therefore, Theorem 1 applies and yields

$$K'_n(\{\omega\},\ \{z\})\leq K'_n(\{\nu\},\ \{z\})\ ,\qquad\forall\ \{\omega\}\subset S_0(\mathfrak{M})\ ,\qquad(5\text{--}20)$$

for every $\{z\} \subset \mathfrak{A}_{0+}$. Given such a $\{z\}$, we can provide us with orthoprojections $p_1, \cdots, p_n$ such that $p_j z_j p_j = ||z_j|| p_j$, $\forall j$. Since $\{z\} \subset \mathfrak{M}_N$ for a suitable subscript N, all p_k may be supposed dimensionally infinite (with respect to $\mathfrak{B}(\mathfrak{H})$). Then, a result of [19, 4.3 Lemma] applies and gives the existence of a n-tuple $\{\xi\} \subset \mathfrak{H}$ of mutually orthogonal unit vectors ξ_k in $\mathfrak{H}$ such that $p_k \xi_k = \xi_k$, $\forall k$. Defining $\omega_k(x) = \langle \xi_k, x\xi_k \rangle$, $\forall x \in \mathfrak{M}$, i.e. the vector state over $\mathfrak{M}$ generated by ξ_k, we see that $\omega_k(z_k) = \omega_k(p_k z_k p_k) = ||z_k||$, $\forall k$, by assumptions on the choice of the p_k's. By (5–15), we have to follow that $K'_n(\{\omega\}, \{z\}) = \sum ||z_k||$. Owing to (5–20), and using (5–15) once more again, we see

$$K'_n(\{\nu\}, \{z\}) = \sum ||z_k|| . \tag{5–21}$$

The latter holds for each $\{z\} \subset \mathfrak{A}_{0+}$. Suppose $\{x\} \in \Lambda_n(\mathfrak{M})_+$. Then, (5–18) reads as

$$f_{\{x\}}(\{\nu\}) = K'_n(\{\nu\}, \{y\}) , \quad \text{with} \quad y_k = R(x_k) \in \mathfrak{A}_{0+}, \ \text{Rc.p., unital} \tag{5–22}$$

where Lemma 1 and the special form of the y_k from the proof there has been referred to. Since also $x_k = v^* y_k v$, $v \in \mathfrak{M}_1$, we see $||y_k|| = ||x_k||$, $\forall k$, and (5–21), (5–22) imply the desired equality. This closes the proof. $\qquad \square$

§ 6. The Main Results

Theorem 2. *Let $\mathfrak{M}$ be a non-finite AFD-factor over some separable Hilbert-space $\mathfrak{H}$. Let $\{\omega\}$, $\{\sigma\} \subset S_0(\mathfrak{M})$ be n-tuples of normal states. The following conditions are equivalent to each other*:

(1) $\{\omega\} \gg \{\sigma\}$, *i.e. $\exists \Phi \in ST(\mathfrak{M}^*)$ with $\omega_k = \Phi(\sigma_k)$, $\forall k$;*

(2) $\{\omega\} \gg_* \{\sigma\}$, *i.e. $\exists \Phi \in ST(\mathfrak{M}_*)$ with $\omega_k = \Phi(\sigma_k)$, $\forall k$;*

(3) $\exists T \in CP_1(\mathfrak{M})$ *with $\omega_k = \sigma_k \circ T$, $\forall k$;*

(4) $\exists T \in CP_1^0(\mathfrak{M})$ *with $\omega_k = \sigma_k \circ T$, $\forall k$;*

(5) $f(\{\omega\}) \leq f(\{\sigma\})$ *for all $f \in \mathfrak{F}_n^+(\mathfrak{M})$;*

(6) $g(\{\omega\}) \geq g(\{\sigma\})$ *for all $g \in \mathfrak{F}_n^-(\mathfrak{M})$.*

Thus, $\mathfrak{F}_n^-(\mathfrak{M})$ and $\mathfrak{F}_n^+(\mathfrak{M})$ provide sufficient systems of $\gg$-isotonic and $\gg$-anti-isotonic functions, respectively.

Proof. Follows at once from Theorem 1 and Lemma 1. The latter telling us that $\mathfrak{F}(\mathfrak{A}) = \mathfrak{F}_n^+(\mathfrak{M})$, for *every* generating U.H.F.-C^*-subalgebra $\mathfrak{A}$ of $\mathfrak{M}$. The equivalence (5)$\Leftrightarrow$(6) is due to (5–1), (5–2). $\qquad \square$

Remark. Note that Lemma 2, Lemma 3 have their counterparts in case of $\mathfrak{F}_n^-(\mathfrak{M})$. By Lemma 1, $\mathfrak{F}(\mathfrak{A}) = \mathfrak{F}_n^+(\mathfrak{M})$ for any generating U.H.F.-C^*-sub-

algebra $\mathfrak{A}$ of $\mathfrak{M}$. Hence, we arrived at an *intrinsic* characterization for a particular family of $\gg$-isotonic resp. anti-isotonic functions without to have need to refer to the (non-unique) substructure $\mathfrak{A}$.

For convenience, let us list up some of the properties of the functions collected into $\mathfrak{F}_n^+(\mathfrak{M})$ (an analogous dual list hold in case of $\mathfrak{F}_n^-(\mathfrak{M})$). Let $f \in \mathfrak{F}_n^+(\mathfrak{M})$. Then:

f is uniformly continuous, (6-1)

f is (relatively) w^*-lower semicontinuous, (6-2)

f is (simultaneously) $\mathcal{U}(\mathfrak{M})$-invariant, (6-3)

i.e. $f(\{\omega^u\}) = f(\{\omega\})$, with $\{\omega^u\} = \{\omega_1^u, \cdots, \omega_n^u\}$,

f is (jointly) convex (6-4)

over the set of all n-tuples of normal states on $\mathfrak{M}$. Moreover, if $f_{\{x\}}$ is generated by the a.f.d.-system $\{x\}$, then

$$\| \sum x_k \| \leq f_{\{x\}} \leq \sum \| x_k \| , \qquad (6\text{-}5)$$

and

$$f_{\{x\}}(\{\omega\}) = \sum \| x_k \|, \ \forall \ \{x\} \in \varLambda_n(\mathfrak{M})_+, \ \Leftrightarrow \{\omega\} \qquad (6\text{-}6)$$

is an orthogonal system of normal states;

$$f_{\{x\}}(\{\omega\}) = \sum \| x_k \|, \ \forall \ \{x\} \in \varLambda_n(\mathfrak{M})_+, \ \Leftrightarrow \omega_1 = \cdots = \omega_n . \qquad (6\text{-}7)$$

Furthermore, each $f \in \mathfrak{F}_n^+(\mathfrak{M})$ has the isotony-property, i.e. if $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}$ is a sequence of finite-dimensional subfactors of $\mathfrak{M}$ which generates $\mathfrak{M}$, and $E_N; \mathfrak{M} \to \mathfrak{M}_N$ is a sequence of associated normal conditional expectations, then

$$f(\{\omega \circ E_1\}) \leq f(\{\omega \circ E_2\}) \leq \cdots \leq f(\{\omega\}) \qquad (6\text{-}8)$$

and

$$\lim_{N \to \infty} f(\{\omega \circ E_N\}) = f(\{\omega\}) . \qquad (6\text{-}9)$$

These properties are either obvious, or they are collected together from all we have been deriving in the previous sections.

§ 7. The Case of the Admissible vN-Algebras

Let $\mathfrak{N}$ be a vN-algebra that is admissible in our sense (see Section 2). Thus, there is a properly infinite AFD-factor $\mathfrak{M}$ which contains $\mathfrak{N}$, and there is a normal conditional expectation E projecting from $\mathfrak{M}$ onto $\mathfrak{N}$. In these notations, the result of [18, Proposition] and our Theorem 2 imply the following to be valid:

Corollary. *For n-tuples $\{\omega\}$, $\{\sigma\} \subset S_0(\mathfrak{N})$ the subsequently listed conditions are mutually equivalent*

(1) $\{\omega\} \gg \{\sigma\}$;

(2) $\{\omega\} \gg_* \{\sigma\}$;

(3) $f(\{\omega \circ E\}) \leq f(\{\sigma \circ E\})$ *for all* $f \in \mathfrak{F}_n^+(\mathfrak{M})$; (!)

(4) $g(\{\omega \circ E\}) \geq g(\{\sigma \circ E\})$ *for all* $g \in \mathfrak{F}_n^-(\mathfrak{M})$. (!)

Therefore, the set of functions $\{f(\{(\cdot) \circ E\}) : f \in \mathfrak{F}_n^-(\mathfrak{M})\}$ forms, e.g., a sufficient family of $\gg$-isotonic functions.

As an application, let us consider the case $\mathfrak{M} = \mathfrak{B}(\mathfrak{H})$, and $\mathfrak{N} \simeq l^\infty$. Let $\{\xi_j : j \in N\}$ be a complete orthonormal system in $\mathfrak{H}$, and denote by q_j the orthoprojection onto $[\xi_j]$. We might assume that $\mathfrak{N}$ is generated by the q_j's. Then, $E(\cdot) = \sum q_j(\cdot) q_j$ is the normal conditional expectation onto $\mathfrak{N}$. Let $\{x\} \in \Lambda_n(\mathfrak{B}(\mathfrak{H}))_+$. Let $\{\omega\} \subset S_0(\mathfrak{N})$. We find non-negative reals ω^{kl}, with $\sum_1 \omega^{kl} = 1$, $\forall k$, such that $\omega_k \circ E = \sum_1 \omega^{kl} q_l$. Therefore, we have $f_{\{x\}}(\{\omega \circ E\}) = \sup_u \sum \omega_k \circ E(u^* x_k u) = \sup_u \sum_1 \langle u\xi_l, y_l u\xi_l \rangle = f_{\{y\}}(\{\nu\})$, with $y_l = \sum_k \omega^{kl} x_k$, $\nu_k(\cdot) = \langle \xi_k, (\cdot)\xi_k \rangle$. The system $\{\nu\}$ being an orthogonal one, and since $\sum \|y_l\| < \infty$, makes that (6–6) (with a little modification) applies with the result:

$$f_{\{x\}}(\{\omega \circ E\}) = \sum_l \|y_l\| = \sum_l \left\| \sum_k \omega^{kl} x_k \right\|.$$

This, however, gives Example 5 in case $\Omega = N$, μ-the counting measure, essentially (compare this to the sufficient system given at the end of Section 2).

Acknowledgements

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