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An MU-analogue of the Lambda Algebra

In memory of the late Professor Ryôji Shizuma

By

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§0. Introduction

The so-called lambda algebra Λ is the differential (bi-)graded augmented algebra over $\mathbb{Z}/p$, p a prime number, introduced by Bousfield, Curtis, Kan et al. [4]. It is originally given as the E_1-term of a spectral sequence of Adams type derived from a filtration by the mod p lower central series of the Kan loop group of the simplicial sphere spectrum. There has been found since many interesting applications of the lambda algebra to homotopy theory [5], [7], [10] etc.

It is characterizing that Λ is a quotient of the reduced cobar construction $\bar{C}(A_*)$ ([1]), where A_* means the dual Hopf algebra [12] of the mod p Steenrod algebra A, and itself being a considerably small cochain complex with cohomology group $H^{s,t}(\Lambda, d)$ isomorphic to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$. In such a view point there appeared similar constructions of resolutions over (Hopf) algebras [15], [9] and [17].

The purpose of this paper is to construct a differential graded algebra Λ^{MU} in the MU-cohomology theory ([3], [2]), similarly as the lambda algebra Λ in the ordinary cohomology theory.

Our method is also available for the case of BP-theory, which will be dealt with in a subsequent paper [18].

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§1. Hopf Algebroid Associated to the MU-Homology Theory

Recall first the definition of Hopf algebroid [2], [11].

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Definition 1.1. *A Hopf algebroid over a commutative ground ring R consists of a pair of graded-commutative R-algebras* (A, Γ) *such that there are morphisms of graded R-algebras (called structure morphisms):*

$\eta_L: A \to \Gamma$ *(left unit)*

$\eta_R: A \to \Gamma$ *(right unit)*

$\varepsilon : A \to \Gamma$ *(augmentation)*

$\Delta : \Gamma \to \Gamma \otimes_A \Gamma$ *(diagonal)*

$\mu : \Gamma \otimes_A \Gamma \to \Gamma$ *(composition-multiplication)*

$c : \Gamma \to \Gamma$ *(conjugation),*

where Γ *is regarded as a left A-module by* η_L *and also a right A-module by* η_R*, and the tensor product* $\Gamma \otimes_A \Gamma$ *is that of A-bimodules. Moreover* ε*,* Δ *and* μ *are morphisms of A-bimodules,* Δ *and* μ *are associative, and the above morphisms subject to the following commutation rules and diagrams:*

$$\varepsilon\eta_L = \varepsilon\eta_R = 1\,, \quad \varepsilon c = \varepsilon, \quad c^2 = 1\,, \quad c\eta_L = \eta_R\,, \quad c\eta_R = \eta_L\,,$$

$$\begin{array}{ccccc} A\otimes_A \Gamma & \xleftarrow{\sim} & \Gamma & \xrightarrow{\sim} & \Gamma\otimes_A A \\ & {\scriptstyle \varepsilon\otimes 1}\nwarrow & \downarrow{\scriptstyle \Delta} & \nearrow{\scriptstyle 1\otimes\varepsilon} & \\ & & \Gamma\otimes_A \Gamma & & \end{array} \qquad \begin{array}{ccccc} A & \xleftarrow{\varepsilon} & \Gamma & \xrightarrow{\varepsilon} & A \\ {\scriptstyle \eta_R}\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \eta_L} \\ \Gamma & \xleftarrow[\mu(c\otimes 1)]{} & \Gamma\otimes_A\Gamma & \xrightarrow[\mu(1\otimes c)]{} & \Gamma \end{array}$$

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Delta} & \Gamma\otimes_A\Gamma \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle C} \\ \Gamma & \xrightarrow{\Delta} & \Gamma\otimes_A\Gamma\,, \end{array} \qquad (C(\gamma\otimes\delta) = (-1)^{\deg\gamma\cdot\deg\delta} c(\delta)\otimes c(\gamma))\,.$$

The R-algebra A is called the coefficient ring and Γ is called the cooperation algebra of the Hopf algebroid (A, Γ). If $A=R$ and $\eta_L=\eta_R$, then Γ is the usual Hopf algebra over A. Examples of Hopf algebroid are provided by $R=\pi_0(E)$, $A=\pi_*(E)$ and $\Gamma=E_*E=\pi_*(E\wedge E)$ for each ring spectrum E, usually with the assumption Γ being left A-flat (and therefore simultaneously right A-flat). This condition is known to be satisfied in many important cases: $E=KO$, K, MO, MU, MSp, S, $H\mathbf{Z}/p$, BP etc. ([2], [3]).

In the below, we shall only deal with the case $E=MU$. As is well known its associated Hopf algebroid $(A, \Gamma)=(MU_*, MU_*MU)$ consists of $A=MU_*=\mathbf{Z}[x_1, x_2, \cdots]$, the complex cobordism ring isomorphic to a polynomial algebra over $\mathbf{Z}$ with generators x_i of even degree ($\deg x_i=2i$), and $\Gamma\cong A\otimes S$, where $S=\mathbf{Z}[b_1, b_2, \cdots]$ is the dual Hopf algebra of the algebra of the Landweber-Novikov cohomology operations ([8], [2]). S is presented as a polynomial algebra over $\boldsymbol{Z}$ with generators b_i of even degree ($\deg b_i=2i$), and its diagonal is given by

$$\psi(b) = \sum_{i\geqq 0} b^{i+1}\otimes b_i\,, \tag{1.2}$$

where $b=1+b_1+b_2+\cdots$ and $b_0=1$.

The Hopf algebroid $(A, A\otimes S)$ is thus of split type [11] with structure morphisms:

$$\begin{aligned}
&\eta_L\colon A\to\Gamma = A\otimes S\,,\quad \eta_L(a) = a\otimes 1\\
&\varepsilon: \Gamma = A\otimes S\to A\,,\quad \varepsilon(1\otimes s) = 0 \qquad \text{for}\quad \deg s>0\\
&\Delta = 1\otimes\psi\colon \Gamma = A\otimes S\to\Gamma\otimes_A\Gamma = A\otimes S\otimes S\\
&\mu = 1\otimes m\colon \Gamma\otimes_A\Gamma = A\otimes S\otimes S\to\Gamma = A\otimes S\,,
\end{aligned}$$

where m is the multiplication in S.

It is convenient, for our purpose, to take another set of generators for the polynomial algebra S.

Put

$$\begin{aligned}
&s = 2-b^{-1}\,,\\
&s = 1+s_1+s_2+\cdots\,,\quad s_0 = 1\,.
\end{aligned} \tag{1.3}$$

Then we have

$$\begin{aligned}
&2b = 1+bs\,,\\
&b_k = \sum_{i=0}^{k-1} b_i s_{k-i} \qquad (k\geqq 1)\,.
\end{aligned} \tag{1.4}$$

We can express b_k as polynomials of $s_i's$ and vice versa. Note that b_k are polynomials of $s_i's$ with positive coefficients, for example:

$$\begin{aligned}
b &= \frac{1}{1-(s-1)} = \sum_{n\geqq 0}\bar{s}^n\,,\quad \bar{s} = s-1\,.\\
b_1 &= s_1\\
b_2 &= s_2+s_1^2\\
b_3 &= s_3+2s_1s_2+s_1^3\\
b_4 &= s_4+2s_1s_3+3s_1^2s_2+s_2^2+s_1^4 \qquad \text{etc.}
\end{aligned} \tag{1.5}$$

Lemma 1.6. *The diagonal of the Hopf algebra $S=\boldsymbol{Z}[s_1, s_2, \cdots]$, $\deg s_i=2i$, is given by*

$$\psi(s) = s\otimes 1+\sum_{i\geqq 1} b^{i-1}\otimes s_i\,,$$

or more explicitly:

$$\psi(s_1) = s_1\otimes 1+1\otimes s_1$$
$$\psi(s_2) = s_2\otimes 1+1\otimes s_2$$
$$\psi(s_3) = s_3\otimes 1+s_1\otimes s_2+1\otimes s_3$$
$$\psi(s_4) = s_4\otimes 1+(s_2+s_1^2)\otimes s_2+2s_1\otimes s_3+1\otimes s_4 \qquad \text{etc.}$$

Thus only s_1 and s_2 are primitive elements among the generators s_i.

Proof of Lemma 1.6. We see, from (1.2) and (1.3),

$$\psi(b^{-1}) = \psi(b)^{-1} = b^{-1}\otimes 1-\sum_{j\geqq 1} b^{j-1}\otimes s_j .$$

Then, from (1.4), it follows

$$\psi(s) = 2(1\otimes 1)-\psi(b^{-1}) = s\otimes 1+\sum b^{j-1}\otimes s_j .$$

§2. DGA-Algebra Λ^S

Consider the primitively generated Hopf subalgebra $S_2=\boldsymbol{Z}[s_1, s_2]$ of $S=\boldsymbol{Z}[s_1, s_2, \cdots]$. Let $p: S\to S_2$ be the canonical projection, and $\varepsilon: S\to \boldsymbol{Z}$ be the augmentation. We shall use the following notations:

$$\text{(2.1)}\qquad \begin{aligned} &\bar{S} = \operatorname{Ker}\varepsilon\,, \quad L = S_2\,, \quad \bar{L} = \bar{S}_2 = S_2\cap\bar{S}\,,\\ &\theta = (1-\varepsilon)\circ p\colon S\to L\to\bar{L}\,,\\ &\lambda_{ij} = \theta(s_1^i s_2^j)\in\bar{L} \qquad (i, j\geqq 0,\ i+j>0)\,. \end{aligned}$$

Thus, θ is a linear map such that

$$\text{(2.2)}\qquad \begin{aligned} &\theta(\text{monomial containing } s_k \text{ for some } k\geqq 3) = 0\\ &\theta(1) = 0 \end{aligned}$$

and $\bar{L}$ is a free $\boldsymbol{Z}$-module with basis $\{\lambda_{ij}\,|\,i, j\geqq 0 \text{ and } i+j>0\}$. $\bar{L}$ is bigraded by $\deg\lambda_{ij}=(1, 2i+4j)$.

Let $T(\bar{L})$ be the tensor algebra on $\bar{L}$:

$$\text{(2.3)}\qquad T(\bar{L}) = \boldsymbol{Z}+\bar{L}+\bar{L}\otimes\bar{L}+\bar{L}\otimes\bar{L}\otimes\bar{L}+\cdots$$

and define a linear map

$$\text{(2.4)}\qquad \theta\cup\theta = (\theta\otimes\theta)\circ\psi\colon S\to\bar{L}\otimes\bar{L}\subset T(\bar{L})\,.$$

Definition-Proposition 2.5. *Define a quotient algebra*

$$\Lambda^S = T(\bar{L})/I$$

of the tensor algebra $T(\bar{L})$ *by the two-sided ideal* I *generated by elements of*

$\theta\cup\theta(\mathrm{Ker}\,\theta)$. *This will be a DGA-algebra (differential, bigraded, augmented algebra)over the ring Z of integers, with differential d, of degree* $(1, 0)$, *being a derivation such that* $d\theta=-\theta\cup\theta$.

To show that this definition is well defined, we first consider linear map

$$d=-(\theta\cup\theta)\circ\iota\colon \bar{L}\to\bar{L}\otimes\bar{L}\subset T(\bar{L}), \tag{2.6}$$

where $\iota\colon\bar{L}\to S$ is defined by $\iota(\lambda_{ij})=s_1^i s_2^j$. Extending d onto $T(\bar{L})$ as a derivation, we have

$$\begin{aligned} d\theta &= -(\theta\cup\theta)\circ\iota\theta = \{(\theta\cup\theta)\circ(1-\iota\theta)\}-\theta\cup\theta\equiv-\theta\cup\theta \pmod{I} \\ d(\theta\cup\theta) &= d\theta\cup\theta-\theta\cup d\theta\equiv-(\theta\cup\theta)\cup\theta+\theta\cup(\theta\cup\theta) = 0 \pmod{I}, \end{aligned} \tag{2.7}$$

in virtue of the associativity of the diagonal ψ, where the $\cup$-product is defined similarly as in (2.4).

Thus we have, in particular, $d(I)\subset I$ and $d\circ d\equiv 0 \pmod{I}$, and d induces a differential d_A on Λ^S, which we will denote by the same notation d for simplicity. And we have shown that Λ^S is a DGA-algebra over $\mathbb{Z}$, generated by $\{\lambda_{ij};\, i,j\geqq 0,\, i+j>0\}$. The differential d is given explicitly by

$$d\lambda_{ij} = -\sum_{k,l\geqq 0}\binom{i}{k}\binom{j}{l}\lambda_{kl}\circ\lambda_{i-k,j-l}\,. \tag{2.8}$$

As for relations between the generators λ_{ij}, there are first basic relations

$$R_k=\theta\cup\theta(s_k)=\sum_{i+2j=k-2}\binom{i+j}{i}\lambda_{ij}\circ\lambda_{01}=0 \qquad (k\geqq 3,\; i,j\geqq 0), \tag{2.9}$$

where $\binom{i+j}{i}$ means binomial coefficient, and $\lambda\circ\lambda'$ product in Λ^S. And more generally

$$D_1^i D_2^j(R_{k_1}*\cdots*R_{k_n}) = \theta\cup\theta(s_1^i s_2^j s_{k_1}\cdots s_{k_n}) = 0 \qquad (i,j\geqq 0,\; 3\leqq k_1\leqq\cdots\leqq k_n), \tag{2.10}$$

where D_1 and D_2 are derivations of Λ^S, defined by

$$\begin{aligned} D_1(\lambda_{ij}) &= \lambda_{i+1,j} \qquad (\text{or } D_1(\theta(s_1^i s_2^j)) = \theta(s_1^{i+1}s_2^j)) \\ D_2(\lambda_{ij}) &= \lambda_{i,j+1} \qquad (\text{or } D_2(\theta(s_1^i s_2^j)) = \theta(s_1^i s_2^{j+1})), \end{aligned} \tag{2.11}$$

and $*\colon(\bar{L}\otimes\bar{L})\otimes(\bar{L}\otimes\bar{L})\to\bar{L}\otimes\bar{L}$ is the product induced from the usual one in the tensor product $S_2\otimes S_2$, explicitly given by

$$(\lambda_{i_1j_1}\circ\lambda_{k_1l_1})*(\lambda_{i_2j_2}\circ\lambda_{k_2l_2}) = \lambda_{i_1+i_2,j_1+j_2}\circ\lambda_{k_1+k_2,l_1+l_2}\,.$$

The left hand sides of (2.10) will generate the whole of the ideal I. Here are some examples of the relators (2.10) (notation $(ijkl)=\lambda_{ij}\cdot\lambda_{kl}$, for simplicity):

$$
(2.12)\qquad
\begin{aligned}
R_3 &= (1001)\\
R_4 &= (2001)+\underline{(0101)}\\
D_1R_3 &= (2001)+(1011)\\
R_5 &= (3001)+2\underline{(1101)}\\
D_1R_4 &= (3001)+(2011)+\underline{(1101)}+\underline{(0111)}\\
D_1^2R_3 &= (3001)+2(2011)+(1021)\\
D_2R_3 &= \underline{(1101)}+(1002)\\
R_6 &= (4001)+3\underline{(2101)}+\underline{(0201)}\\
D_1R_5 &= (4001)+(3011)+2\underline{(2101)}+2\underline{(1111)}\\
D_1^2R_4 &= (4001)+2(3011)+\underline{(2101)}+2\underline{(1111)}+(2021)+\underline{(0121)}\\
D_1^3R_3 &= (4001)+3(3011)+3(2021)+(1031)\\
D_2R_4 &= \underline{(2101)}+(2002)+\underline{(0201)}+(0102)\\
D_1D_2R_3 &= \underline{(2101)}+\underline{(1111)}+(2002)+(1012)\\
R_3*R_3 &= (2002)\\
\text{etc.} &\qquad \text{(Cf. (2.19), for the meaning of the underline)}
\end{aligned}
$$

We remark that the map $\theta\colon S\to\underline{L}\to\Lambda^S$ (we made a confusing use of the notation) gives a *twisting cochain*, in the sense of E.H. Brown [6].

On the additive structure of Λ^S, we have

Theorem 2.13. *Λ^S is a free $\boldsymbol{Z}$-module.*

To prove this, we prepare the following lemma.

Lemma 2.14. *In the n-th power tensor product $S^{\otimes n}=S\otimes\cdots\otimes S$ of the Hopf algebra S, the submodule*

$$\psi_n(S)=\sum_{i=0}^{n-2}\bar{S}^{\otimes i}\otimes\psi(S)\otimes\bar{S}^{\otimes(n-i-2)}\qquad(n\geqq 2)$$

is a direct summand (of $S^{\otimes n}$), where ψ is the diagonal of S and $\bar{S}$ is the kernel of the augmentation $\varepsilon\colon S\to\boldsymbol{Z}$.

Proof. Induction on n. In the case $n=2$, from

$$S\underset{\psi}{\overset{\varepsilon\otimes 1}{\rightleftarrows}}S\otimes S\,,\qquad(\varepsilon\otimes 1)\circ\psi=id.\,,$$

we have a direct sum decomposition

$$S\otimes S=\psi(S)\oplus(\bar{S}\otimes S)\,.$$

For the case $n=3$, using the above decomposition, we have

$$\begin{aligned}S^{\otimes 3}&=\{\psi(S)\oplus(\bar{S}\otimes S)\}\otimes S=(\psi(S)\otimes S)\oplus(\bar{S}\otimes(S\otimes S))\\&=(\psi(S)\otimes \boldsymbol{Z})\oplus(\psi(S)\otimes\bar{S})\oplus(\bar{S}\otimes\psi(S))\oplus(\bar{S}\otimes\bar{S}\otimes S)\end{aligned}$$

Thus $\psi_3(S)=\psi(S)\otimes\bar{S}+\bar{S}\otimes\psi(S)$ is a direct summand of $S^{\otimes 3}$. Iterating this process inductively, we obtain the lemma.

Proof of Theorem 2.13. Recall that, in the beginning of this section, we put $S=\boldsymbol{Z}[s_1, s_2, \cdots]$, $L\approx S_2=\boldsymbol{Z}[s_1, s_2]$, $\bar{L}=\mathrm{Ker}\,(\varepsilon\colon S_2\to\boldsymbol{Z})$, $p\colon S\to L$ the projection, and $\theta\colon S\to L\to\bar{L}$ the composition $(1-\varepsilon)\circ p$. Consider the map $p^{\otimes n}$: $S^{\otimes n}\to L^{\otimes n}$ and the image of the submodule $\psi_n(S)$ (in 2.14) by $p^{\otimes n}$:

$$p^{\otimes n}(\psi_n(S))=\sum\bar{L}\otimes\cdots\otimes(p\cup p(S))\otimes\cdots\otimes\bar{L}\,. \tag{2.15}$$

This is a submodule of the $\boldsymbol{Z}$-free module $L^{\otimes n}$ and itself $\boldsymbol{Z}$-free. Since $p^{\otimes n}$ is a surjection, it follows that $p^{\otimes n}(\psi_n(S))$ is a direct summand of $\psi_n(S)$. By lemma 2.14, we have

(2.16) $\quad\sum\bar{L}\otimes\cdots\otimes(p\cup p(S))\otimes\cdots\otimes\bar{L}$ is a direct summand of $S^{\otimes n}$.

On the other hand, it is easy to see that

$$\begin{aligned}&(\sum\bar{L}\otimes\cdots\otimes(p\cup p(S))\otimes\cdots\otimes\bar{L})\cap\bar{L}^{\otimes n}\\&\quad=\sum\bar{L}\otimes\cdots\otimes(\theta\cup\theta(\mathrm{Ker}\,\theta))\otimes\cdots\otimes\bar{L}=I_{(n)}=I\cap\bar{L}^{\otimes n}\,.\end{aligned}\tag{2.17}$$

Therefore we have

$$\begin{aligned}\Lambda^S_{(n)}&=\bar{L}^{\otimes n}/I_{(n)}\subset L^{\otimes n}/\sum\bar{L}\otimes\cdots\otimes(p\cup p(S))\otimes\cdots\otimes\bar{L}\\&\subset S^{\otimes n}/\sum\bar{L}\otimes\cdots\otimes(p\cup p)(S)\otimes\cdots\otimes\bar{L}\,.\end{aligned}\tag{2.18}$$

The last module is $\boldsymbol{Z}$-free by (2.16), so the part $\Lambda^S_{(n)}$ of Λ^S of tensor-grade n is proved to be $\boldsymbol{Z}$-free. q.e.d.

In concluding this section we add a conjecture on a free basis of Λ^S. Let us call a monomial $\lambda_{i_1j_1}\circ\lambda_{i_2j_2}\cdots\lambda_{i_kj_k}$ in Λ^S *admissible*, if the following condition

$$\min\,(1, j_1)\geqq j_2\geqq\cdots\geqq j_k \tag{2.19}$$

is satisfied.

Conjecture 2.20. *The set of admissible monomials would constitute a $\boldsymbol{Z}$-basis of Λ^S.*

In the list (2.12), we underlined admissible terms.

§3. Comodule Resolutions

Let (A, Γ) be a Hopf algebroid (See §1).

Definition 3.1. *A left A-module and an A-map $\psi_M: M \to \Gamma \otimes_A M$ define a left (A, Γ)-comodule (M, ψ_M) (or simply, a left Γ-comodule), if the following conditions are satisfied:*

i) $M \xrightarrow{\psi_M} \Gamma \otimes_A M \xrightarrow{\varepsilon \otimes 1} M = id.$,

ii) *there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\psi_M} & \Gamma \otimes_A M \\ {\scriptstyle \psi_M}\downarrow & & \downarrow{\scriptstyle \Delta \otimes 1} \\ \Gamma \otimes_A M & \xrightarrow{1 \otimes \psi_M} & \Gamma \otimes_A \Gamma \otimes_A M . \end{array}$$

For example, A is a left Γ-comodule via $\psi_A = \eta_L: A \to \Gamma$ (left unit), and so is Γ itself via $\psi_\Gamma = \Delta$. A morphism (or Γ-comodule map) $f: M \to N$, between left Γ-comodules M, N, is defined to be an A-map compatible with ψ_M and ψ_N. Denote by $\mathrm{Hom}_\Gamma(M, N)$ the totality of morphisms of M into N. Then we have [11]

$$\begin{aligned} &\mathrm{Hom}_\Gamma (A, N) \cong \{n \in N;\ \psi_N(n) = 1 \otimes n\} \\ (3.2)\quad &\mathrm{Hom}_\Gamma (A, A) \cong \{a \in A;\ \eta_L(a) = \eta_R(a)\} \\ &\mathrm{Hom}_\Gamma (A, \Gamma) = \{f: A \to \Gamma;\ f(1) = \eta_R(a) \quad \text{for some } a \in A\} \cong A . \end{aligned}$$

Now consider the twisted tensor product $S \otimes_\theta \Lambda^S$ (which we denote by $S \otimes \Lambda^S$ for simplicity, See §2). This is a bigraded cochain complex with diffferential d such that

$$(3.3) \qquad \begin{aligned} &d(\alpha \otimes \lambda) = d\alpha \cdot \lambda + \alpha \otimes d\lambda \qquad (\alpha \in S,\ \lambda \in \Lambda^S) \\ &d\alpha = (1 \otimes \theta)\psi(\alpha)\ , \quad \mathrm{bideg}\, \alpha = (0, \deg \alpha)\ . \end{aligned}$$

Theorem 3.4. *The cohomology of $S \otimes \Lambda^S$ is given by*

$$\mathrm{H}^{s,t}(S \otimes \Lambda^S, d) \cong \begin{cases} \mathbf{Z} & \text{for} \quad (s, t) = (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the unnormalized cobar construction $C(S, \mathbf{Z}) = S \otimes T(S)$ over the coalgebra S. $T(S)$ denotes the tensor algebra on S and the differential is given by

$$d(\alpha[\alpha_1|\alpha_2|\cdots|\alpha_n]) = \sum \alpha'[\alpha''|\alpha_1|\cdots|\alpha_n]+\sum_{i=1}^{n}(-1)^i\alpha[\alpha_1|\cdots|\psi(\alpha_i)|\cdots|\alpha_n] \\ +(-1)^{n+1}\alpha[\alpha_1|\cdots|\alpha_n|1]$$

where $\psi(\alpha)=\sum \alpha'\otimes\alpha''$. It is well known that $H^{s,t}(S\otimes T(S))=\mathbf{Z}$ for $(s, t)=(0, 0)$, $=0$ otherwise.

Let π be the natural projection: $T(S)\to\Lambda^S$, which factors through $T(\theta)$: $T(S)\to T(\bar{S}_2)$. Let J be the kernel of π, a two-sided ideal of $T(S)$, and J^c be a direct summand of $T(S)$ complementary to J $(T(S)=J\oplus J^c, J^c\overset{\pi}{\approx}\Lambda^S)$. We may choose and fix $J^c\subset T(\bar{S}_2)$.

To prove Theorem 3.4, it is sufficient to prove, in the exact sequence of complexes:

$$0\to S\otimes J\to S\otimes T(S)\to S\otimes\Lambda^S\to 0\,,$$

the following

Lemma 3.5. $H^{**}(S\otimes J)=0$.

For this purpose consider the following subcomplexes of J:

$$J' = S\cdot J+K\cdot J^c+\psi(K_\theta)\circ J^c\,, \\ J''= \bar{S}\cdot J\oplus[1]\circ J'\,, \tag{3.6}$$

where $K=\mathrm{Ker}\,(p\colon S\to S_2)$, $K_\theta=\mathrm{Ker}\,(\theta\colon S\to\bar{S}_2)$ and $\cdot$ means the (tensor) product in $T(S)$. We can verify that

$$J = J'\oplus[1]\cdot J^c\,, \\ dJ'\subset J'\,, \\ dJ''\subset J''\,. \tag{3.7}$$

Lemma 3.8. *In the exact sequence of complexes*:

$$0\to S\otimes J''\overset{i}{\to} S\otimes J\to S\otimes J/J''\to 0\,,$$

1) *the induced map* $i_*\colon H^{**}(S\otimes J'')\to H^{**}(S\otimes J)$ *is the zero map,*
2) $H^{**}(S\otimes J/J'')=0$.

Proof of Lemma 3.8. Defining the chain homotopy $\sigma\colon S\otimes J''\to S\otimes J$ by

$$\sigma(\alpha\otimes|\beta|\circ u) = \varepsilon(\alpha)\beta\otimes u \quad \text{for} \quad \alpha\in S, \quad \begin{cases}\beta\in\bar{S} \text{ and } u\in J\,, & \text{or}\\ \beta=1 \text{ and } u\in J'\,,\end{cases} \tag{3.9}$$

we can easily verify that $d\sigma+\sigma d=i$ on $S\otimes J''$. This proves 1). For the proof of 2), we note the following direct-sum decomposition

(3.10) $$J/J'' \cong K_\theta \cdot J^c \oplus \psi(K_\theta) \cdot J^c .$$

Then, defining the following contracting homotopy on $S \otimes J/J''$ by

(3.11) $$\begin{aligned} &\sigma(\alpha \otimes |k| \cdot v) = 0 \quad \text{for} \quad \alpha \in S, \quad k \in K_\theta \quad \text{and} \quad v \in J^c, \\ &\sigma(\alpha \otimes |\psi(k)| \cdot v) = \begin{cases} -\alpha \otimes |k| \circ v & \text{for} \quad k \in K \\ \alpha \otimes |1| \circ v & \text{for} \quad k = 1, \end{cases} \end{aligned}$$

we can directly calculate $d\sigma + \sigma d = id.$ on $S \otimes J/J''$. This calculations would be straightforward but somewhat tedious. For example, we remark that

(3.12) $$\begin{aligned} &\pi \circ d = d_A \circ \pi \\ &dv \equiv \pi^{-1} \circ d_A \circ \pi(v) \pmod{J} \quad \text{for} \quad v \in J^c, \end{aligned}$$

so that $|k| \cdot dv$ (resp. $|\psi(k)| \cdot dv$) (mod J'') would be identified to $|k| \cdot (\pi^{-1} d_A \pi)(v)$ (resp. $|\psi(k)| \cdot (\pi^{-1} d_A \pi)(v)$. Thus we have proved Lemma 3.8 and therefore Lemma 3.5, which proves Theorem 3.4.

In the conclusion, we have an acyclic, injective (left) S-comodule resolution of $\boldsymbol{Z}$:

(3.13) $$\begin{aligned} &\boldsymbol{Z} \xrightarrow{\eta} S \otimes \Lambda^S, \quad \text{or} \\ &0 \to \boldsymbol{Z} \xrightarrow{\eta} S \xrightarrow{d} S \otimes \Lambda^S_{(1)} \xrightarrow{d} S \otimes \Lambda^S_{(2)} \to \cdots . \end{aligned}$$

Tensoring $A = MU_*$ to this from the left, we obtain an acyclic, injective (left) Γ-comodule resolution of A:

(3.14) $$A \xrightarrow{\eta_L} \Gamma \otimes \Lambda^S \qquad (\Gamma = A \otimes S = MU_* MU) .$$

§4. Λ^{MU} and the Adams-Novikov Spectral Sequence

Applying the Hom-functor in the category of left Γ-comodules to (3.14), we have a cochain complex

(4.1) $$\mathrm{Hom}_\Gamma (A, \Gamma \otimes \Lambda^S) \cong A \otimes \Lambda^S ,$$

with differential d of the form

(4.2) $$d(a \otimes \lambda) = (1 \otimes \theta) \eta_R(a) \cdot \lambda + a \otimes d\lambda .$$

We shall denote the complex (4.1) by Λ^{MU} and call it the *MU-lambda algebra*, as it will be justified in the following theorem.

Theorem 4.3. (i) *the cohomology group* $\mathrm{H}^{s,t}(\Lambda^{MU}, d)$ *of the cochain complex* $\Lambda^{MU}=MU_*\otimes\Lambda^S$ *is isomorphic to* $\mathrm{Ext}_{MU_*MU}^{s,t}(MU_*, MU_*)$, (ii) Λ^{MU} *has a canonical structure of free* MU_**-bimodule, and thereby becomes a DGA-algebra over* $\mathbb{Z}$, (iii) *there exists a spectral sequence of Adams-Novikov type, of which* E_1*-term is* Λ^{MU}.

Proof. (i) is clear, because $\mathrm{Ext}_\Gamma(A,\ )$ is the derived functor of $\mathrm{Hom}_\Gamma(A,\ )$. To prove (ii), we let MU_* act on Λ^{MU} from the right as follows:

$$(4.4)\qquad \begin{aligned}&(a\otimes\lambda)\cdot b \overset{\text{def}}{\equiv} \sum a\cdot b_{ij}\otimes D_1^i D_2^j\lambda\,,\quad a,\, b\in MU_*\,,\quad \lambda\in\Lambda^S\,,\\ &\eta_R(b) = \sum_{i,j\geq 0} b_{ij}\otimes s_1^i s_2^j + (\text{terms containing } s_k \text{ for some } k\geqq 3)\,,\end{aligned}$$

here D_1, D_2 are the derivations of Λ^S defined in (2.11).

This right action is well-defined:

$$(4.5)\qquad \begin{aligned}&(\lambda\circ\mu)\circ b = \lambda\circ(\mu\circ b)\\ &\lambda\circ(ab) = (\lambda\circ a)\circ b\,,\quad \lambda,\,\mu\in\Lambda^S\,,\quad a,\, b\in MU_*\,.\end{aligned}$$

Moreover we have

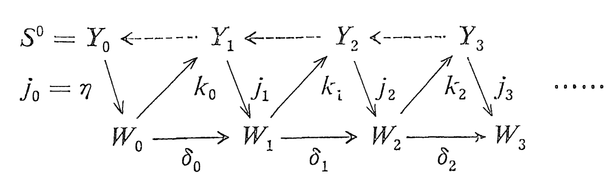
$$(4.6)\qquad d(\lambda_{ij}\circ a) = (d\lambda_{ij})\circ a - \lambda_{ij}\circ da\,,\qquad a\in MU_*\,.$$

By these properties, Λ^{MU} has a free MU_*-bimodule structure and becomes a DGA-algebra over $\mathbb{Z}$. Next, to prove (iii) we follow Adams' method [3]. Consider a sequence of cofibrations of spectra:

$$\begin{aligned}&Y_0 = S_0 \xrightarrow{\eta} W_0 = MU \to Y_1\,,\quad Y_1 \to W_1 = MU\wedge S^{\Lambda^S_{(1)}} \to Y_2, \cdots\\ &Y_n \to W_n = MU \underset{I}{\wedge} S^{\Lambda^S_{(n)}} \to Y_{n+1}, \cdots\end{aligned}$$

which will be defined inductively on n such that

$$(4.7)\qquad S^{\Lambda^S_{(n)}} = \bigvee_I S^{t(\lambda_I)}\quad (\text{wedge sum, } \{\lambda_I\} \text{ a basis of } \Lambda^S_{(n)},\ \text{bideg}\ \lambda_I = (n,\, t(\lambda_I)))$$



$$\begin{array}{ccccccccc} 0 \to MU_*(S^0) & \overset{\eta}{\to} & MU_*(W_0) & \overset{\delta_0}{\to} & MU_*(W_1) & \overset{\delta_1}{\to} & MU_*(W_2) & \to \cdots\cdots \\ \| & & \| & & \| & & \| & \\ 0 \longrightarrow MU_* & \underset{\eta_L}{\longrightarrow} & \Gamma & \underset{d_0}{\longrightarrow} & \Gamma\otimes A^S_{(1)} & \underset{d_1}{\longrightarrow} & \Gamma\otimes A^S_{(2)} & \to \cdots\cdots \end{array}$$

To define these maps of spectra, we start from the unit $\eta: S^0 \to MU = W_0$ and its cofiber $k_0: W_0 \to Y_1 = MU/S^0$.

Lemma 4.8. (Proposition 13.5 of Adams [3], Part III). *Let E be one of ring spectra listed in the beginning of* § 1, *and F an E-module spectrum. Then, if E_*X is a projective E_*-module, we have*

$$F^*X \simeq \mathrm{Hom}_{E_*}(E_*X, F_*) .$$

By applying this lemma to the case: $E=MU$, $X=W_{n-1}=MU\wedge S^{A^S_{(n-1)}}$ and $F=MU\wedge W_n$, we get a map $\delta_{n-1}: W_{n-1}\to MU\wedge W_n \overset{\varphi}{\to} W_n$ which corresponds to $d_{n-1}: MU_*(W_{n-1})\to MU_*(W_n)$ for all n. Now from the induction hypothesis:

$$(4.9)_{n-1}\qquad \begin{array}{l} Y_{n-1} \xrightarrow{j_{n-1}} W_{n-1} \xrightarrow{k_{n-1}} Y_n \quad \text{is a cofibration,} \\ E_*(j_{n-1}) \quad \text{is monic,} \\ E_*(k_{n-1}) \quad \text{is epic,} \\ \mathrm{Ker}\, E_*(k_{n-1}) = \mathrm{Im}\, E_*(j_{n-1}) = \mathrm{Ker}\, E_*(\delta_{n-1}) , \end{array}$$

it follows, by Lemma 4.8, that $\delta_{n-1}\circ j_{n-1}\sim 0$. Since (j_{n-1}, k_{n-1}) is a cofibration, there is a map $j_n: Y_n \to W_n$ such that $\delta_{n-1}\sim j_n\circ k_{n-1}$. Since $E_*(k_{n-1})$ is epic, $\mathrm{Im}\, E_*(j_n)=\mathrm{Im}\, E_*(\delta_{n-1})=\mathrm{Ker}\, E_*(\delta_n)$, and $E_*(j_n)$ is monic. If we form the cofibration $Y_n \overset{j_n}{\to} W_n \overset{k_n}{\to} Y_{n+1}$, all the conditions of the n-th step $(4.9)_n$ are satisfied and the induction is completed.

Applying the homotopy group functor π_* to the cofibrations diagram in (4.7), we have an exact couple

$$(4.10)\qquad \begin{array}{ccc} \sum_n \pi_*(Y_{n+1}) & \xrightarrow{i_*} & \sum_n \pi_*(Y_n) \\ & k_* \nwarrow \quad \swarrow j_* & \\ & E_1 = \sum_n \pi_*(W_n) \simeq \sum_n MU_*\otimes A^S_{(n)} = A^{MU} , & \end{array}$$

which will give rise to a spectral sequence of Adams-Novikov type converging to the stable homotopy groups of the sphere. This completes the proof of

Theorem 4.3.

Remark 4.11. Direct computation of $H^{s,t}(\Lambda^{MU}, d)$ is rather difficult except for the cases of smaller s and t, because the differential d is related to the right unit η_R (cf. 4.2). Here are a few examples:

$$\begin{aligned} &H^{s,\mathrm{odd}}=0\,, \quad H^{0,t}=0 \quad (t>0)\,, \quad H^{0,0}=\mathbb{Z} \\ &H^{1,2}\cong \mathbb{Z}/2=\{\lambda_{10}\} \\ (4.12)\qquad &H^{1,4}\cong \mathbb{Z}/12=\{\lambda_{01}\} \\ &H^{1,6}\cong \mathbb{Z}/2=\{4\lambda_{30}+6a_{11}\lambda_{20}+3a_{11}^2\lambda_{10}\} \\ &H^{1,8}\cong \mathbb{Z}/240=\{a_{12}\lambda_{01}-a_{11}\lambda_{11}-(a_{22}-a_{11}a_{12}-a_{13})\lambda_{10}\} \end{aligned}$$

where $\{\alpha\}$ means that the cocycle α represents a generator, and a_{ij} are the coefficients of the universal formal group law regarded as elements of MU_* (cf. [3], [13], [14], [16], [19]).

Remark 4.12. Putting $L=A[s_1, s_2]$, we have a Hopf algebroid (η'_L, η'_R: $A\to L$), where $\eta'_L=\eta_L$ and $\eta'_R=p\circ\eta_R$ ($p: \Gamma\to L$ the canonical projection). Put $\bar{L}=\mathrm{Ker}(\varepsilon: L\to A)$, $\theta: \Gamma\to\bar{L}$, $\theta=(1-\varepsilon)\circ p$, and $T_A(\bar{L})$ the tensor algebra on $\bar{L}$ over A (i.e. $T_A(\bar{L})=\sum \bar{L}\otimes_A\bar{L}\otimes_A\cdots\otimes_A\bar{L}$). Then a direct definition of Λ^{MU} will be given as the quotient $T_A(\bar{L})/(\theta\cup\theta\,(\mathrm{Ker}\,\theta))$ (cf. [18]).

References

[1] Adams, J.F., On the cobar construction, *Colloque de Topologie Algebrique*, Louvain, Paris (1956).

[2] ———. Lectures on generalized cohomology, *Lecture Notes in Mathematics* **99**, Springer-Verlag, Berlin, 1969.

[3] ———, Stable homotopy and generalised homology, *Chicago Lectures in Mathematics*, The Univ. of Chicago Press. 1974.

[4] Bousfield, A.K., Curtis, E.B., Kan, D.M., Quillen, D.G., Rector, D.L., and Schlesinger, J.W., The mod-p lower central series and the Adams spectral sequence, *Topology*, **5** (1966), 331–342.

[5] Bousfield, A.K. and Kan,, D.M., The homotopy spectral sequence of a space with coefficients in a ring, *Topology*, **11** (1972), 79–106.

[6] Brown, E.H., Twisted tensor product I, *Ann. of Math.* **69** (1959), 223–246.

[7] Brown, E.H. and Gitler, S., A spectrum whose cohomology is a cyclic module over the Steenrod algebra, *Topology* **12** (1973), 283–295.

[8] Landweber, P.S., Cobordism operations and Hopf algebras, *Trans. Amer. Math. Soc.* **129** (1967), 94–110.

[9] Lellmann, W. and Mahowald, M., A generalization of the lambda algebra, *to appear*.

[10] M. Mahowald, A new infinite family in ${}_2\pi_s^*$, *Topology*, **16** (1977), 249–256.

[11] Miller, H., Some algebraic aspects of the Adams-Novikov spectral sequence, *Dissertation*, Princeton Univ. 1974.

[12] Milnor, J.W., The Steenrod algebra and its dual, *Ann. of Math.* **67** (1958),1 50–171.

[13] Milnor, J.W., On the cobordism ring Ω^* and a complex analogue, *Amer. J. of Math.*, **82** (1960), 505–521.
[14] Novikov, S.P., The method of algebraic topology from the view point of cobordism theories (Russian), *Izvestija Akad. Nauk SSSR, Ser. Mat.*, **31** (1967), 855–951.
[15] Priddy, S.B.. Koszul resolutions, *Trans. Amer. Math. Soc.*, **152** (1970), 39–60.
[16] Quillen, D., On the formal group laws of unoriented and complex cobordism theory, *Bull. Amer. Math. Soc.* **75** (1969), 1293–1298.
[17] Shimada, N. and Iwai, A., On the cohomology of some Hopf algebras, *Nagoya Math. J.* **30** (1967), 103–111.
[18] N. Shimada and Katsumi Shimomura, On BP-lambda algebra, *to appear*.
[19] Zahler, R., The Adams-Novikov spectral sequence for the spheres, *Ann. of Math.* **96** (1972), 480–504.

Note added in proof: It turns out that the definition (3.6) of the subcomplex J' of J has to be modified as follows. K is to be replaced dy K', which is defined by the decomposition $K_\theta = K_0 \oplus K'$ where

$$K_\theta = \mathrm{Ker}\,(\theta : S \to \bar{L}),$$
$$K_0 = \mathrm{Ker}\,(\theta \cup \theta : K_\theta \to I_{(2)}).$$

Then $K' \approx I_{(2)}$.

As a consequence, $[1] \cdot J^c$ in (3.7) is to be replaced by $K_0 \cdot J^c$.