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On the Dirichlet Problem For Degenerate Elliptic Equations

By

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§ 1. Introduction

The purpose of this paper is to study the Dirichlet problem with L^2-boundary data for degenerate elliptic equations of the form (1) (see Section 1). The degeneracy of the ellipticity is controlled by a function m satisfying conditions (2) and (3). Degenerate elliptic equations, with m satisfying (2), have been widely examined by Murthy and Stampacchia [5]. Further extensions of their results can be found in Trudinger [6]. In particular, the Dirichlet problem in the above mentioned papers, was solved in the case when a boundary data is a trace of a function from a suitable Sobolev space. Here we discuss more general situations when a boundary data belongs to L^2. For uniformly elliptic equations this problem was solved in [1], [2] and [3] (all historical references can be found in [1] and [2]). To solve the Dirichlet problem with L^2-boundary data we impose on m an additional condition (3), which allows us to recover a boundary function in the sense of L^2-convergence. Therefore the equation (1) is uniformly elliptic in a neighbourhood of a boundary and degenerates in an interior part of a set.

The plan of this paper is as follows. Section 1 contains some preliminary work. In Section 2 we examine traces of solutions in $H^{1,2}_{loc}(m,m)$. In particular we obtain a sufficient condition for a solution in $H^{1,2}_{loc}(m,m)$ to have an L^2-trace on the boundary (see Theorem 2). This theorem justifies our approach to the Dirichlet problem adopted in this work. In Section 3 we solve the Dirichlet problem for a boundary data in L^∞. In the final part of the paper, Section 4, we

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solve the Dirichlet problem for a boundary data in L^2, under more restrictive condition on m, namely $m\in L^\infty$.

Finally, we point out that the methods of this paper are not new and have appeared in the author's earlier papers [1] and [2].

§2. Preliminaries

Let $Q\subset \boldsymbol{R}_n$ be a bounded domain with the boundary ∂Q of class C^2. In Q we consider the equation of the form

$$(1)\qquad Lu=-\sum_{j=1}^n D_j[\sum_{i=1}^n a_{ij}(x)D_iu+d_i(x)u]+\sum_{i=1}^n b_i(x)D_iu+c(x)u=f(x),$$

whose coefficients are assumed to be measurable functions on Q.

To formulate further assumptions on the coefficients of L we introduce a non-negative function m on Q such that

$$(2)\qquad m\in L^s(Q) \text{ and } m^{-1}\in L^t(Q) \text{ with } \frac{1}{s}+\frac{1}{t}\leq\frac{2}{n}.$$

We also assume that there exist constants β and β_1 and a neighbourhood N of ∂Q such that

$$(3)\qquad 0<\beta\leq m(x)\leq\beta_1$$

for all $x\in N$.

We denote a distance from $x\in Q$ to ∂Q by $r(x)$.

Throughout this paper we make the following assumptions on the coefficients of L

(A) The operator L is elliptic, that is,

$$m(x)\,|\xi|^2\leq\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j$$

for all $\xi\in\boldsymbol{R}_n$ and almost all $x\in Q$ Moreover we assume that $m^{-1}a_{ij}\in L^\infty(Q)$ and that D_ia_{ij} exist on N with

$$D_ia_{ij}\cdot r^\alpha\in L^\infty(N),\ O<\alpha<1,\ \ (i,j=1,\cdots,n).$$

(B) $b_im^{-1/2}$, $d_im^{-1/2}\in L^{2\bar{q}}(Q)$ $(i=1,\cdots,n)$, $c\in L^{\bar{q}}(Q)$ with $\bar{q}>q$, where q is given by

$$(2q)^{-1}=2^{-1}-(2^\#)^{-1}=n^{-1}-(2t)^{-1}.$$

(C) $fm^{-1/2}r^\theta\in L^2(Q)$, where $2\leq\theta<3$.

We associate with the function m the weighted Sobolev spaces. The completion of the space $C^1(\bar{Q})$ with respect to the norm

$$||u||_{H^{1,2}(Q,m,m)}=||u||_{m,2}+||Du||_{m,2}$$

is denoted by $H^{1,2}(Q,m,m)\equiv H^{1,2}(m,m)$. Here $||\cdot||_{m,2}$ denotes the norm in the space $L^2(Q,m)$ of all functions u such that

$$||u||^2_{m,2}=\int_Q |u(x)|^2 m(x)\,dx<\infty.$$

The closure of the space $C_0^1(Q)$ in $H^{1,2}(m,m)$ is denoted by $H_0^{1,2}(m,m)$.

Throughout this paper we frequently make use of the Sobolev inequality:

there exists a constant $A=A(m,n,t)$ such that

$$||u||_{2^\#}<A||Du||_{m,2} \tag{4}$$

for all $u\in H_0^{1,2}(m,m)$.

The norm in $H_0^{1,2}(m,m)$ is equivalent to $||Du||_{m,2}$, that is, we have

$$||Du||_{m,2}\leq||u||_{H^{1,2}(m,m)}\leq B||Du||_{m,2}, \tag{5}$$

where $B>0$ is a constant independent of u. (For the proofs of (4) and (5) we refer to [5] Theorem 3.2 and Corollaries 3.3 and 3.5).

A function u defined on Q is said to belong to $H^{1,2}_{loc}(m,m)$ if $\zeta u\in H_0^{1,2}(m,m)$ for every $\zeta\in C_0^1(Q)$.

A function $u(x)$ is said to be a weak solution of the equation (1) if $u\in H^{1,2}_{loc}(m,m)$ and u satisfies

$$\int_Q\Big[\sum_{i,j=1}^n a_{ij}D_iuD_jv+\sum_{i=1}^n d_iuD_iv+\sum_{i=1}^n b_iD_iuv+cu\cdot v\Big]dx=\int_Q fv\ dx \tag{6}$$

for every $v\in H^{1,2}(m,m)$ with compact support in Q.

It follows from the regularity of ∂Q that there exists a number $\delta_0>0$ such that for $\delta\in(0,\delta_0]$ the domain

$$Q_\delta=Q\cap\{x;\min_{y\in\partial Q}|x-y|>\delta\}$$

with the boundary ∂Q_δ, possesses the following property: to each $x_0\in\partial Q$ there is a unique point $x_\delta(x_0)\in\partial Q_\delta$ such that $x_\delta(x_0)=x_0-\delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping of $x_0\to x_\delta(x_0)$ is given by the formula $x_0=x_\delta+\delta\nu_\delta(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed $\delta\in(0,\delta_0]$ let

$$A_\varepsilon=\partial Q_\delta\cap\{x;\ |x-x_\delta|<\varepsilon\},$$
$$B_\varepsilon=\{x;\ x=\tilde{x}_\delta+\delta\nu_\delta(\tilde{x}_\delta),\ \tilde{x}_\delta\in A_\varepsilon\},$$

and

$$\frac{dS_\delta}{dS_0}=\lim_{\varepsilon\to 0}\frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the (n-1)-dimensional Hausdorff measure of a set A. Mikhailov [4] proved that there exists a positive number γ such that

$$\gamma^{-2}\leq\frac{dS_\delta}{dS_0}\leq\gamma^2 \tag{7}$$

and

$$\lim_{\delta\to 0}\frac{dS_\delta}{dS_0}=1 \tag{8}$$

uniformly with respect to $x_\delta\in\partial Q_\delta$.

According to Lemma 1 in [3], p. 382, the distance $r(x)$ belongs to $C^2(\bar{Q}-Q_{\delta_0})$, if δ_0 is sufficiently small. Denote by $\rho(x)$ the extension of r into $\bar{Q}$ satisfying the following properties: $\rho(x)=r(x)$ for $x\in\bar{Q}-Q_{\delta_0}$, $\rho\in C^2(Q)$, $\rho(x)\geq\frac{3\delta_0}{4}$ in Q_{δ_0}, $\gamma_1^{-1}\ r(x)\leq\rho(x)\leq\gamma_1\ r(x)$ in Q for some positive constant γ_1, $\partial Q_\delta=\{x;\rho(x)=\delta\}$ for $\delta\in(0,\delta_0]$ and finally $\partial Q=\{x;\rho(x)=0\}$. We may also assume that $\bar{Q}-Q_{\delta_0}\subset N$.

We will use the surface integrals $M_1(\delta)=\int_{\partial Q}|u(x_\delta(x))|^2\,dS_x$ and $M(\delta)=\int_{\partial Q_\delta}|u(x)|^2dS_x$, where $u\in H^{1,2}_{loc}(m,m)$ and the values on ∂Q and ∂Q_δ, respectively, are understood in the sense of traces (see Theorem 3.9 in [4]).

Note that if $M(\delta)$ is bounded on $(0,\delta_0]$ and

$$\int_Q u(x)^2m(x)\,dx<\infty,$$

then for every $0\leq\mu<1$ there exists a constant $C>0$ such that

$$\int_{Q_\delta}\frac{u(x)^2m(x)}{(\rho(x)-\delta)^\mu}dx\leq C \tag{9}$$

for all $\delta\in(0,\delta_0/2]$. Indeed, for $\delta\in(0,\delta_0/2]$ we have

$$\begin{aligned}\int_{Q_\delta}\frac{u(x)^2m(x)}{(\rho(x)-\delta)^\mu}dx&\leq\int_{Q_\delta-Q_{\delta_0}}\frac{u^2m}{(\rho-\delta)^\mu}dx+\int_{Q_{\delta_0}}\frac{u^2m}{(\rho-\delta)^\mu}dx\\&\leq\beta_1\int_\delta^{\delta_0}(s-\delta)^{-\mu}\int_{\partial Q}u^2dS_x+\left(\frac{2}{\delta_0}\right)^\mu\int_{Q_\delta}u^2m\,dx\\&\leq\beta_1\left[\delta_0^{1-\mu}\sup_{0<s<\delta_0}\int_{\partial Q_s}u^2dS_x+\left(\frac{2}{\delta_0}\right)^\mu\int_{Q_{\delta_0}}u^2m\,dx\right]\end{aligned}$$

§ 3. Properties of M and M_1

To study M and M_1 we need the following modification of Lemma 1 in [2].

Lemma 1. *Suppose that* $u \in H^{1,2}_{loc}(m, m)$ *and that*

$$\int_Q |Du(x)|^2 m(x) r(x)\,dx < \infty.$$

Then if $0 \leq \mu < 1$ *and* $0 < \delta_1 \leq \delta_0/2$ *we have for* $\delta \in (0, \delta_1/2]$

$$\int_{Q_\delta} \frac{u(x)^2 m(x)}{(\rho(x)-\delta)^\mu}\,dx \leq K\Big[\delta_1^{-\mu}\int_{Q_{\delta_1}} u(x)^2 m(x)\,dx + \delta_1^{1-\mu}\int_{\partial Q_{\delta_1}} u(x)^2 dS_x + \delta_1^{1-\mu}\int_{Q_\delta - Q_{\delta_1}} |Du(x)|^2(\rho(x)-\delta)\,dx\Big],$$

where K *is a constant independent of* δ_1 *and* δ.

Proof. First we observe that by (3) we have

$$\int_{Q_\delta - Q_{\delta_1}} |Du(x)|^2(\rho(x)-\delta)\,dx \leq \beta^{-1}\int_Q |Du(x)|^2 m(x) r(x)\,dx < \infty.$$

Let $\delta \in (0, \delta_1/2]$ and put

$$\int_{Q_\delta} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx = \int_{Q_\delta - Q_{\delta_1}} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx + \int_{Q_{\delta_1}} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx.$$

Since $\rho(x) \geq \delta_1$ on Q_{δ_1} we have

$$\int_{Q_{\delta_1}} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx \leq \left(\frac{2}{\delta_1}\right)^\mu \int_{Q_{\delta_1}} u^2 m\,dx.$$

Now we note that

$$\int_{Q_\delta - Q_{\delta_1}} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx \leq \beta_1 \int_\delta^{\delta_1} (t-\delta)^{-\mu} \int_{\partial Q} u(x_t(x_0))^2 \frac{dS_t}{dS_0}\,dS_0\,dt$$
$$\leq \beta_1 \gamma^2 \int_\delta^{\delta_1} (t-\delta)^{-\mu} \int_{\partial Q} u(x_t(x_0))^2 dS_0\,dt.$$

As $\int_{\partial Q} u(x_t(x))^2 dS_x$ is absolutely continuous on $[\delta, \delta_1]$ $(\delta > 0)$, integrating by parts and using Young's inequality we obtain

$$\int_{Q_\delta - Q_{\delta_1}} \frac{u^2 m}{(\rho-\delta)^\mu}\,dx \leq \frac{\beta_1 \gamma^2 \delta_1^{1-\mu}}{1-\mu} \int_{\partial Q} u(x_{\delta_1}(x))^2 dS_x$$

$$+\frac{2\gamma^2\beta_1}{1-\mu}\int_\delta^{\delta_1}(t-\delta)^{1-\mu}\int_{\partial Q}|u(x_t(x_0))|\,|Du(x_t(x_0))|\,|\frac{\partial}{\partial t}x_t(x_0)|\,dS_0dt$$
$$\leq\frac{\beta_1\gamma^4\delta_1^{1-\mu}}{1-\mu}\int_{\partial Q_{\delta_1}}u^2dS_x+\frac{2\gamma^4\beta_1}{1-\mu}\int_{Q_\delta-Q_{\delta_1}}|u(x)|\,|Du(x)|\,(\rho-\delta)^{1-\mu}dx$$
$$\leq\frac{\beta_1\gamma^4\delta_1^{1-\mu}}{1-\mu}\int_{\partial Q_{\delta_1}}u^2dS_x+\frac{2\gamma^4\beta_1\varepsilon}{1-\mu}\int_{Q_\delta-Q_{\delta_1}}\frac{u^2}{(\rho-\delta)^\mu}dx$$
$$+\frac{2\gamma^4\beta_1\delta_1^{1-\mu}}{\varepsilon(1-\mu)}\int_{Q_\delta-Q_{\delta_1}}|Du|^2(\rho-\delta)\,dx\quad\text{for all }\varepsilon>0.$$

Now choosing $\frac{2\beta_1\gamma^4\varepsilon}{1-\mu}=\frac{1}{2}$ the result follows.

Lemma 2. *Let $u\in H^{1,2}_{loc}(Q)$ be a solution of* (1) *and let $0<\delta_1<\delta_2<\delta_0$. Then*

$$\int_{Q_{\delta_2}}|Du(x)|^2m(x)\,dx\leq C\Big[\int_{Q_{\delta_1}}u(x)^2m(x)\,dx+\int_{Q_{\delta_1}}f(x)^2m(x)^{-1}dx\Big],$$

where $C>0$ is a constant depending on the norms of the coefficients of L, δ_1 and δ_2.

Proof. We commence with the following observation. Let Ω be a relatively compact subset of Q such that

$$\|\sum_{i=1}^n b_im^{-1/2}\|_{L^{2q}(\Omega)},\ \|\sum_{i=1}^n d_im^{-1/2}\|_{L^{2q}(\Omega)}\text{ and }\|c\|_{L^2(\Omega)}$$

are sufficiently small then

$$\int_\Omega|Du|^2\zeta^2m\,dx\leq\text{Const}\Big[\int_\Omega u^2(\zeta^2+|D\zeta|^2)\,m\,dx+\int_\Omega f^2\zeta^2m^{-1}\Big]$$

for every C^1-function ζ with compact support in Ω (see Lemma 8.5 in [5]). The assertion then follows from the compactness of $\bar{Q}_{\delta_1}$.

The following result is crucial in the subsequent treatment of the Dirichlet problem.

Theorem 1. *Let u be a solution of* (1) *belonging to $H^{1,2}_{loc}(m,m)$, then the following conditions are equivalent*

(*I*) $M(\delta)$ *is a bounded function on* $(0,\delta_0]$,

(*II*) $\int_Q|Du(x)|^2m(x)r(x)\,dx<\infty$,

(*III*) *There exists* $\lim_{\delta\to0+}M_1(\delta)<\infty$.

Proof. Put

$$v(x)=\begin{cases} u(x)\,(\rho(x)-\delta) & \text{for } x\in Q_\delta\ , \\ 0 & \text{for } x\in Q-Q_\delta, \end{cases}$$

for $0<\delta<\frac{\delta_0}{4}$. It is clear that v is a legitimate test function in (6) and on substitution we obtain

$$\begin{aligned}(10)\quad &\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iuD_ju(\rho-\delta)\,dx+\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iu\circ uD_j\rho dx \\ &+\int_{Q_\delta}\sum_{i=1}^n (d_i+b_i)\,D_iu\circ u(\rho-\delta)\,dx+\int_{Q_\delta}\sum_{i=1}^n d_iu^2D_i\rho dx+\int_{Q_\delta}cu^2(\rho-\delta)\,dx \\ &=\int_{Q_\delta}fu(\rho-\delta)\,dx.\end{aligned}$$

The proof is similar to that of Theorem 1 in [2], but in our situation more care is needed to estimate the resulting integrals in (10).

The proof of "I $\Rightarrow$ II". Let us denote the integrals on the left side of (10) by $J_1,\cdots,J_5$. By virtue of (A) we have

$$(11)\qquad J_1\geq\int_{Q_\delta}|Du|^2(\rho-\delta)m\ dx.$$

To estimate J_2 we set

$$\begin{aligned}J_2&=\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iu\cdot uD_j\rho\Phi\ dx+\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iu\circ uD_j\rho(1-\Phi)\,dx \\ &=J_2'+J_2'',\end{aligned}$$

where Φ is a smooth function on $\bar{Q}$ such that $\Phi=1$ on $Q-Q_{\delta_0/2}$, $\Phi=0$ on Q_{δ_0} and $0\leq\Phi\leq 1$ on $\bar{Q}$.

Since $0<\delta<\delta_0/4$, by Green's formula we have

$$\begin{aligned}J_2'&=\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_i(u^2)\,D_j\rho\Phi dx \\ &=-\frac{1}{2}\int_{\partial Q_\delta}\sum_{i,j=1}^n a_{ij}D_i\rho D_j\rho u^2dS_x-\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^n D_i(a_{ij}D_j\rho\Phi)\,u^2dx.\end{aligned}$$

On the other hand supp $(1-\Phi)\subset Q_{\delta_0/2}$, therefore applying Young's inequality and Lemma 2 to J_2'' we arrive at the following estimate for J_2

$$(12)\qquad |J_2|\leq\frac{1}{2}\int_{\partial Q_\delta}\sum_{i,j=1}^n a_{ij}D_i\rho D_j\rho u^2dS_x+C_1\int_{Q_\delta}u^2m(1+\rho^{-\alpha})\,dx,$$

where a constant $C_1>0$ depends on $||a_{ij}m^{-1}||_{L^\infty}$ and $||D_i a_{ij}m^{-1}\rho^\alpha||_{L^\infty}$. To estimate J_3 we use a decomposition

$$J_3=\int_{Q_{\delta_0}}\sum_{i=1}^n(d_i+b_i)D_iu\cdot u(\rho-\delta)\,dx+\int_{Q_\delta-Q_{\delta_0}}\sum_{i=1}^n(d_i+b_i)D_iu\cdot u(\rho-\delta)\,dx$$
$$=J_3'+J_3''.$$

Since $L^{2\bar{q}}\subset L^{2q}$, using Hölder's inequality, Theorem 3. 1 from [5] and Lemma 2 we get

$$(13)\qquad |J_3'|\leq||\sum_{i=1}^n(b_i+d_i)||_{L^{2q}(Q_{\delta_0})}||Du(\rho-\delta)^{1/2}m^{1/2}||_{L^2(Q_{\delta_0})}||u(\rho-\delta)||_{L^{2^\#}(Q_{\delta_0})}$$
$$\leq C_2\Big[\int_Q f^2m^{-1}\rho^\theta dx+\int_{Q_{\delta_0/2}}u^2m\,dx\Big],$$

where $C_2>0$ depends on $||\sum_{i=1}^n(\mathrm{b}_i+d_i)m^{-1/2}||_{L^{2\bar{q}}}$ and a constant from Lemma 2. To estimate J_3'' we first use the Hölder inequality to obtain

$$|J_3''|\leq||\sum_{i=1}^n(b_i+d_i)m^{-1/2}||_{L^{2\bar{q}}(Q_\delta-Q_{\delta_0})}||(\rho-\delta)^{-\varepsilon}||_{L^{s_1}(Q_\delta-Q_{\delta_0})}$$
$$\times||u(\rho-\delta)^{1/2+\varepsilon/2}||_{L^{2^\#}(Q_\delta-Q_{\delta_0})}||Du(\rho-\delta)^{1/2+\varepsilon/2}m^{1/2}||_{L^2(Q_\delta-Q_{\delta_0})},$$

with $\frac{1}{2\bar{q}}+\frac{1}{\mathrm{s}_1}=\frac{1}{2\mathrm{q}}$, $\frac{1}{2\mathrm{q}}=\frac{1}{2}-\frac{1}{2^\#}$, $0<\varepsilon<\frac{1}{\mathrm{s}_1}$.

Hence be the Sobolev inequality (4) we have

$$||u(\rho-\delta)^{1/2+\varepsilon/2}||_{L^{2^\#}(Q_\delta-Q_{\delta_0})}\leq||u(\rho-\delta)^{1/2+\varepsilon/2}||_{L^{2^\#}(Q_\delta)}$$
$$\leq A\big[||Du(\rho-\delta)^{1/2}m^{1/2}||_{L^2(Q_\delta)}+||u(\rho-\delta)^{\varepsilon/2-1/2}m^{1/2}||_{L^2(Q_\delta)}\big],$$

so that

$$(13)\qquad |J_3''|\leq\delta_0^{-\varepsilon+1/s_1}\int_{Q_\delta}|Du|^2m\rho\,dx+C_3\int_{Q_\delta}u^2m(\rho-\delta)^{\varepsilon-1}dx,$$

where $C_3>0$ depends on the norms of d_i+b_i, A, q and s_1.

Similarly we write

$$(14)\qquad |J_4|\leq\sup_Q|D\rho|\Big[\int_{Q_{\delta_0}}\sum_{i=1}^n|d_i|u^2dx+\int_{Q_\delta-Q_{\delta_0}}\sum_{i=1}^n|d_i|u^2dx\Big]$$
$$=\sup_Q|D\rho|[J_4'+J_4''].$$

It follows then from the Hölder and Sobolev inequalities that

$$|J_4'|\leq ||\sum_{i=1}^{n} d_i m^{-1/2}||_{L^{2q}(Q_{\delta_0})}\circ ||um^{1/2}||_{L^2(Q_{\delta_0})}\circ ||u||_{L^{2^\#}(Q_{\delta_0})}$$

$$\leq A||\sum_{i=1}^{n} d_i m^{-1/2}||_{L^{2q}(Q_{\delta_0})}||um^{1/2}||_{L^2(Q_{\delta_0})}[||um^{1/2}||_{L^2(Q_{\delta_0})}+||Du\circ m^{1/2}||_{L^2(Q_{\delta_0})}].$$

Consequently applying Lemma 2 we arrive at the estimate

$$(15)\qquad |J_4'|\leq C_4[\int_Q f^2 m^{-1}\rho^\theta dx+\int_{Q_{\delta_0/2}} u^2 m dx],$$

where C_4 is a constant depending on the norm of $\sum_{i=1}^{n}|d_i|$, A and a constant from Lemma 2. To estimate J_4'' we first observe that by (3) we have

$$||\sum_{i=1}^{n} d_i||_{L^{2\bar{q}}(Q_\delta - Q_{\delta_0})}\leq \beta_1||\sum_{i=1}^{n} d_i m||_{L^{2\bar{q}}(Q)}<\infty$$

for $\delta\in(0,\delta_0]$. Then using the Hölder inequality, we obtain

$$|J_4''|\leq [\int_{Q_\delta-Q_{\delta_0}}(\sum_{i=1}^{n}|d_i|)^{2\bar{q}}dx]^{1/2\bar{q}}[\int_{Q_\delta-Q_{\delta_0}}(\rho-\delta)^{-\varepsilon_1 s_1}dx]^{1/s_1}$$

$$\times[\int_{Q_\delta-Q_{\delta_0}}u^2(\rho-\delta)^{-\varepsilon_2}dx]^{1/2}[\int_{Q_\delta-Q_{\delta_0}}|u(\rho-\delta)^{\varepsilon_1+\varepsilon_2/2}|^{2^\#}dx]^{1/2^\#}$$

with $\frac{1}{2\bar{q}}+\frac{1}{s_1}=\frac{1}{2q}$, $\frac{1}{2^\#}+\frac{1}{2q}+\frac{1}{2}=1$, $0<\varepsilon<\frac{1}{s_1}$, $\frac{1}{2}<\varepsilon_1+\frac{\varepsilon_2}{2}$ and $0<\varepsilon_2<1$. Let us now apply the Sobolev inequality (4) to obtain

$$(16)\qquad |J_4''|\leq \delta_0^{1/s_1-\varepsilon_1}\int_{Q_\delta}|Du|^2(\rho-\delta)\,dx+C_5\int_{Q_\delta}u^2[(\rho-\delta)^{-\varepsilon_2}+(\rho-\delta)^{2\varepsilon_1+\varepsilon_2-2}]dx,$$

where C_5 is a constant depending on the norm of $\sum_{i=1}^{n} d_i$, A and s_1. Similarly,

$$|J_5|\leq\int_{Q_{\delta_0}}|c|u^2(\rho-\delta)\,dx+\int_{Q_\delta-Q_{\delta_0}}|c|u^2(\rho-\delta)\,dx=J_5'+J_5''.$$

Now by the Sobolev and Hölder inequalities and Lemma 2 we get

$$(17)\qquad |J_5'|\leq C_6[\int_{Q_{\delta_0/2}}u^2 m dx+\int_{Q_{\delta_0}}f^2 m^{-1}\rho^\theta dx],$$

where $C_6>0$ is a constant. On the other hand we have

$$|J_5''|\leq\|c\|_{L^{\bar q}(Q_\delta-Q_{\delta_0})}\|(\rho-\delta)^{-\varepsilon_3}\|_{L^{s_1}(Q_\delta-Q_{\delta_0})}\|u(\rho-\delta)^{1/2+\varepsilon_3/2}\|_{L^{2^\sharp}(Q_\delta-Q_{\delta_0})}$$
$$\times\|u(\rho-\delta)^{1/2+\varepsilon_3/2}\|_{L^{2^\sharp}(Q_\delta-Q_{\delta_0})},$$

with $\dfrac{1}{2\bar q}+\dfrac{1}{2^\sharp}+\dfrac{1}{s_1}+\dfrac{1}{2}=1$, $0<\varepsilon_3<\dfrac{1}{s_1}$, so that

$$|J_5''|\leq\delta^{1/s_1-\varepsilon_3}\int_{Q_\delta}|Du|^2(\rho-\delta)\,m\,dx+C_7\int_{Q_\delta}u^2m(\rho-\delta)^{\varepsilon_3-1}dx, \tag{18}$$

where C_7 depends on $\|c\|_{L^{\bar q}}$, s_1 and A. Finally,

$$\left|\int_{Q_\delta}f\cdot u(\rho-\delta)\,dx\right|\leq\int_{Q_\delta}f^2m^{-1}(\rho-\delta)^\theta dx+\int_{Q_\delta}u^2m(\rho-\delta)^{2-\theta}dx. \tag{19}$$

Inserting the estimates (11) – (19) into (10) we obtain, assuming that δ_0 is sufficiently small, that

$$\int_{Q_\delta}|Du|^2(\rho-\delta)\,m\,dx\leq C_8\Big[\int_Q f^2m^{-1}\rho^\theta dx$$
$$+\int_{Q_\delta}u^2m(\rho-\delta)^{-\mu}dx+\int_{Q_{\delta_0/2}}u^2m\,dx+\sup_{0<\delta<\delta_0}\int_{\partial Q_\delta}u^2dS_x\Big], \tag{20}$$

for all $\delta\in(0,\delta_0/4]$, where $\mu=\max(\alpha,\theta-2,\varepsilon_2,\varepsilon_3,2-\varepsilon_2-2\varepsilon_1)$ and $C_8>0$ depends on δ_0 and constants C_1-C_7. Now note that if (I) holds then we have the estimate (9), therefore the condition (II) follows from the monotone convergence theorem.

Proof of II $\Rightarrow$ III. From the first part of the proof we deduce that

$$\frac{1}{2}\int_{\partial Q_\delta}\sum_{i,j=1}^n a_{ij}D_i\rho D_j\rho u^2dS_x=\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}D_iuD_ju(\rho-\delta)\,dx$$
$$-\frac{1}{2}\int_{Q_\delta}\sum_{i,j=1}^n D_i(a_{ij}D_j\rho)\,u^2dx+\int_{Q_\delta}\sum_{i=1}^n(b_i+d_i)\,D_iu\cdot u(\rho-\delta)\,dx$$
$$+\int_{Q_\delta}\sum_{i=1}^n d_iu^2D_i\rho\,dx+\int_{Q_\delta}cu^2(\rho-\delta)\,dx-\int_{Q_\delta}fu(\rho-\delta)\,dx. \tag{21}$$

It follows from Lemma 1 that

$$\int_{Q_\delta}\frac{u(x)^2m(x)}{(\rho(x)-\delta)^\mu}dx\leq C,\quad 0\leq\mu<1,$$

for $\delta\in(0,\delta_0/2]$, where $C>0$ is independent of δ.

Repeating the argument from "$I\Rightarrow II$" it is easy to show that all integrals are convergent as $\delta\to 0$. Hence "II $\Rightarrow$ III" follows from (8) and the relationship

$$M(\delta)-M_1(\delta)=\int_{\partial Q}u(x_\delta(x))^2\left[\frac{dS_\delta}{dS_0}-1\right]dS_0.$$

Finally "III $\Rightarrow$ I" follows from the proof "II $\Rightarrow$ III.

§4. Traces in $L^2(\partial Q)$ and the Dirichlet Problem

In this section we first establish the existence of a trace of u on ∂Q in $L^2(\partial Q)$, that is, $u(x_\delta)$ converges in $L^2(\partial Q)$ as $\delta\to 0$.

Theorem 2. *Let u be a solution of* (1) *in* $H^{1,2}_{loc}(\mathrm{m},\mathrm{m})$. *If one of the conditions I, II or III holds, then there exists a function ζ belonging to $L^2(\partial Q)$ such that $u(x_\delta)$ converges to ζ in $L^2(\partial Q)$.*

The proof is identical to the proof of Theorem 4 in [2], therefore we only give an outline. It is obvious that there exists a sequence $\delta_\nu\to 0$ as $\nu\to\infty$ and a function $\zeta\in L^2(\partial Q)$ such that

$$\lim_{\nu\to\infty}\int_{\partial Q}u(x_{\delta_\nu}(x))g(x)\,dS_x=\int_{\partial Q}\zeta(x)g(x)\,dS_x$$

for each $g\in L^2(\partial Q)$. Repeating the argument of Theorem 3 in [2] one can easily prove that the above relation holds true if the sequence δ_ν is replaced by δ. Finally to prove that $u(x_\delta)\to\zeta$ in $L^2(\partial Q)$ we show that $\lim_{\delta\to 0}\int_{\partial Q}u(x_\delta(x))^2dS_x=\int_{\partial Q}\zeta(x)^2dS_x$ and the result then follows by uniform convexity of $L^2(\partial Q)$.

Theorem 2 justifies the following approach to the Dirichlet problem.

Let $\phi\in L^2(\partial Q)$. A weak solution u in $H^{1,2}_{loc}(\mathrm{m},\mathrm{m})$ of (1) is a solution of the Dirichlet problem with the boundary condition

$$u(x)=\phi(x)\text{ on }\partial Q \tag{22}$$

if $\lim_{\delta\to 0}\int_{\partial Q}[u(x_\delta(x))-\phi(x)]^2dS_x=0$.

To establish the existence of a solution of the Dirichlet problem (1)-(22) we need some additional assumptions that will be used only in this section.

(a) There exists a constant $\mathrm{a}>-\infty$ such that

$$-\sum_{j=1}^{n} D_j d_j + c \geq a$$

in the sense of distribution.

(b) $$nA_0\Big[\sum_{i=1}^{n} ||b_i m^{-1/2}||_{2q} + \sum_{i=1}^{n} ||d_i m^{-1/2}||_{2q}\Big] + A_0^2 ||c||_q < 1$$

with $A_0 = A \cdot B$, where A and B are constants from the inequalities (4) and (5).

Finally, let us introduce the Hilbert space $\tilde{H}^{1,2}(m, m)$ of all functions u in $H^{1,2}_{loc}(m, m)$ such that

$$||u||^2_{\tilde{H}^{1,2}(m,m)} = \int_Q u(x)^2 m(x)\,dx + \int_Q |Du(x)|^2 m(x) r(x)\,dx < \infty.$$

It is obvious that for all $d < \delta_0$

$$||u||^2_{\tilde{H}^{1,2}(m,m)} \leq \max(1, \beta_1 d)\Big[\int_Q |Du(x)|^2 r(x) m(x)\,dx + \int_Q u(x)^2 m(x)\,dx + \sup_{0<\delta<d} M(\delta)\Big].$$

We can now state the following existence result.

Theorem 3. *Let $\phi \in L^\infty(\partial Q)$ and suppose that $\int_Q |f|^p m^{-1} dx < \infty$ for $p > \frac{n(t-1)}{t-n}$. Then, if the coefficients of L satisfy (a) and (b), the Dirichlet problem (1), (22) has a solution in $H^{1,2}_{loc}(m, m)$.*

Proof. Let $\{\phi_\nu\}$ be a sequence of functions in $C^1(\partial Q)$ converging in $L^2(\partial Q)$ to ϕ. In virtue of assumption (b) and Theorem 4.7 in [5] for every ν there exists a unique solution $u_\nu \in H^{1,2}(m, m)$ of the Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{on} \quad Q, \\ u &= \phi_\nu \quad \text{on} \quad \partial Q. \end{aligned}$$

Since $\phi \in L^\infty(\partial Q)$ we may assume that $\sup_{\nu \geq 1} ||\phi_\nu||_\infty < \infty$. The assumption (b) implies that the maximum principle holds (see Corollary 7.4 in [5]), consequently

(23) $$|u_\nu(x)| \leq \sup_{\nu \geq 1} ||\phi_\nu||_\infty + \int_Q |f|^p m^{-1} dx$$

for all ν. Inspection of the proof of Theorem 1 shows that there exists a constant $C > 0$ such that

$$\int_Q |Du_\nu|^2 mr dx + \int_Q u_\nu^2 m dx + \sup_{0<\delta<d} \int_{\partial Q_\delta} u_\nu^2 dx \tag{24}$$
$$\leq C\left[\int_{\partial Q} \phi_\nu^2 dS_x + \int_{Q_{\delta_0/2}} u_\nu^2 dx + \int_Q f^2 m^{-1} dx\right].$$

It is clear from (23) and (24) that there exists a subsequence $\{u_{\nu_s}\}$ weakly convergent in $\tilde{H}^{1,2}(m, m)$ to a solution of the Dirichlet problem (1)-(22).

§ 5. Case $m \in L^\infty(Q)$

Throughout this section we assume additionally that $m \in L^\infty(Q)$. This assumption allows us to consider the boundary data in $L^2(\partial Q)$. The right Sobolev space in this situation is $H^{1,2}(1, m)$ which is the completion of $C^1(\bar{Q})$ with respect to the norm

$$||u||^2_{H^{1,2}(Q,1,m)} = ||u||^2_{L^2(Q)} + ||Dum^{1/2}||^2_{L^2(Q)}.$$

We briefly denote this space by $H^{1,2}(m)$. We define spaces $H_0^{1,2}(m)$ and $H_{loc}^{1,2}(m)$ in an obvious way.

We commence with the extensions of Theorems 1 and 2.

Theorem 4. *Let u be a solution of* (1) *belonging to $H_{loc}^{1,2}(m)$, then the conditions (I), (II) and (III) are equiva'ent.*

Theorem 5. *Let u be a solution of* (1) *in $H_{loc}^{1,2}(m)$. If one of the conditions (I), (II) or (III) holds, then there exists a function $\zeta \in L^2(\partial Q)$ such that $u(x_\delta)$ converges to ζ in $L^2(\partial Q)$.*

The proofs of these results are essentially the same as of Theorems 1 and 2 and therefore are omitted. We only point out here that the inequality from Lemma 1, which is crucial in the step II$\rightarrow$III, can be stated in the following form: if $u \in H_{loc}^{1,2}(m)$, $0 \leq \mu < 1$ and $0 < \delta_1 \leq \delta_0/2$ then we have

$$\int_{Q_\delta} \frac{u(x)^2}{(\rho(x)-\delta)^\mu} dx \leq K\left[\delta_1^{-\mu} \int_{Q_{\delta_1}} u(x)^2 dx + \delta_1^{1-\mu} \int_{\partial Q_{\delta_1}} u(x)^2 dS_x\right.$$
$$\left. + \delta_1^{1-\mu} \int_{Q_\delta - Q_{\delta_1}} |Du(x)|^2 (\rho(x)-\delta) dx\right]$$

for all $\delta\in(0,\delta_1/2]$, where $K>0$ is a constant independent of δ_1 and δ.

To proceed further let us introduce the equation

$$(1_\lambda)\qquad Lu+\lambda u=f,$$

where λ is a real parameter.

We establish the following energy estimate

Theorem 6. *Let $u\in H^{1,2}_{loc}(m)$ be a solution of the Dirichlet problem $(1_\lambda)-(22)$. Then there exist positive constants d, λ_0 and C independent of u such that*

$$(25)\qquad \int_Q |Du(x)|^2 m(x) r(x)\,dx+\sup_{0<\delta<d} M(\delta)+\lambda\int_{Q_\varrho} u(x)^2 r(x)\,dx \leq [\int_{\partial Q}\phi(x)^2 dS_x+\int_Q f(x)^2 r(x)^\theta m(x)^{-1}dx]$$

for $\lambda\geq\lambda_0$.

Proof. The proof is similar to that of Theorem 5 in [2] (see also Lemma 1 in [1]). An examination of the proof of Theorem 1 shows that we can write the following estimate

$$(26)\qquad \int_{Q_\delta}|Du|^2 m(\rho-\delta)\,dx+\lambda\int_{Q_\delta}u^2(\rho-\delta)\,dx\leq C_1[\int_{Q_\delta}f^2(\rho-\delta)^\theta m^{-1}dx +\int_{\partial Q_\delta}u^2 dS_x+\int_{Q_\delta}u^2(\rho-\delta)^{-\mu}dx],$$

where $\mu=\max(\alpha,\theta-2,\varepsilon_2,\varepsilon_3,2-\varepsilon_2-2\varepsilon_1)$, $C_1>0$ is a positive constant independent of u,λ and $\delta\in(0,\delta_0/2]$, $\varepsilon_1,\varepsilon_2$ and ε_3 are positive constants introduced in Theorem 1. On the other hand, using (21) we easily arrive at the following estimate

$$(27)\qquad \sup_{0<\delta<d}\int_{\partial Q_\delta}u(x)^2 dS_x\leq C_2[\int_Q |Du|^2 m\rho\,dx +\lambda\int_Q u^2\rho\,dx+\int_Q f^2\rho^\theta m^{-1}dx+\int_Q u^2\rho^{-\mu}dx],$$

where $C_2>0$ is a constant independent of u,λ and $\delta\in(0,\delta_0/2]$. Letting $\delta\to 0$ we deduce from (26) that

$$\int_Q|Du|^2 m\rho\,dx+\lambda\int_Q u^2\rho\,dx\leq C_1[\int_{\partial Q}\phi^2 dS_x+\int_Q f^2\rho^\theta m^{-1}dx+\int_Q u^2\rho^{-\mu}dx].$$

Combining this inequality with (27), we obtain

$$(28)\qquad \int_Q |Du|^2 m\rho dx + \lambda \int_Q u^2 \rho dx + \sup_{0<\delta<d} \int_{\partial Q_\delta} u^2 dS_x$$
$$\leq C_3 \Big[\int_{\partial Q} \phi^2 dS_x + \int_Q f^2 \rho^\theta m^{-1} dx + \int_Q u^2 \rho^{-\mu} dx\Big]$$

where $C_3 > 0$ is a constant. Now

$$\int_Q u^2 \rho^{-\mu} dx \leq \frac{d^{1-\mu}}{1-\mu} \sup_{0<\delta\leq d} \int_{\partial Q_\delta} u^2 dS_x + \frac{1}{m_d^{1+\mu}} \int_Q u^2 \rho dx$$

for all $0 \leq \mu < 1$, with $m_d = \inf_{Q_d} \rho(x)$.

Taking λ_0 sufficiently large and d sufficiently small the result follows from (28).

Now we are in a position to state the existence result.

Theorem 7. *Suppose $\phi \in L^2(\partial Q)$. Then there exists a unique solution of the Dirichlet problem $(1_\lambda) - (22)$ for $\lambda \geq \lambda_0$.*

The proof is identical to that of Theorem 6 in [2].

Theorem 8. *Suppose that $\int_Q f^2 m^{-1} dx < \infty$ and let $\phi \in L^2(\partial Q)$. If there is a function $\phi_1 \in H^{1,2}(m)$ such that $\phi_1 = \phi$ on ∂Q in the sense of trace, then a solution $u \in H^{1,2}_{loc}(m)$ of the Dirichlet problem $(1_\lambda) - (22)$ $(\lambda \geq \lambda_0)$ is a solution in $H^{1,2}(m)$ of the same problem.*

This follows from the fact that any solution of $(1_\lambda) - (22)$ in $H^{1,2}(m)$ is also a solution of the same problem in $H^{1,2}_{loc}(m)$ and both problems have a unique solution in respective spaces.

§ 6. Final Remark

One can also establish the existence of a solution of the Dirichlet problem with $\phi \in L^2(\partial Q)$ and with m satisfying (2) and (3) but for the equation

$$(1m, \lambda)\qquad Lu + \lambda m u = f \quad \text{on} \quad Q.$$

Indeed, an examination of the proof of Theorem 6 shows that there exist positive constants λ_0, d and C such that

$$(29)\qquad \int_Q |Du(x)|^2 r(x) m(x)\,dx + \lambda \int_Q u(x)^2 m(x) r(x)\,dx + \sup_{0<\delta<d} M(\delta)$$
$$\leq C\Big[\int_{\partial Q} \phi(x)^2 dS_x + \int_Q f(x)^2 r(x)^\theta m(x)\,dx\Big]$$

for every solution $u \in H^{1,2}_{loc}(m, m)$ of $(1m, \lambda) - (22)$ with $\lambda \geq \lambda_0$.

We can therefore assert the following existence result.

Theorem 9. *Let m satisfy* (2) *and* (3) *and let* $\phi \in L^2(\partial Q)$. *Then there exists a unique solution in* $H^{1,2}_{loc}(m, m)$ *of the problem* $(1m, \lambda)$, (22) *for* $\lambda \geq \lambda_0$.

This follows by a straightforward approximation argument (for details see the proof of Theorem 6 in [2]).

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