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Fermion Ito's Formula II: The Gauge Process in Fermion Fock Space

By

David APPLEBAUM*

Abstract

The stochastic calculus constructed in [2] for fermion Brownian motion is augmented through the inclusion of stochastic integration with respect to the gauge process. The solutions of certain non-commutative stochastic differential equations are used to construct dilations of contraction semigroups on a Hilbert space $\mathfrak{h}_0$ and of uniformly continuous, completely positive semigroups on $\boldsymbol{B}(\mathfrak{h}_0)$. Finally we construct a fermion analogue of the classical Poisson process and investigate some of its properties.

§ 0. Introduction

In the recent paper [2], the present author together with R. L. Hudson developed a stochastic calculus on fermion Fock space over $L^2(\boldsymbol{R}^+)$ in which the Brownian motion process of classical stochastic calculus was replaced by fermion Brownian motion of variance 1 [4] i.e. the process formed from pairs $(A_t, A_t^\dagger)$ of annihilation and creation operators smeared by the indicator function of the interval $[0, t)$ which satisfy the anticommutation relation

$$\{A_s, A_t^\dagger\} = \min\{s, t\} I \tag{0.1}$$

for $s, t \in \boldsymbol{R}^+$. An Itô formula for products of stochastic integrals was obtained wherein the "Itô correction" term arose from the relation

$$dA\ dA^\dagger = dt \tag{0.2}$$

between stochastic differentials.

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*Dipartimento di Matematica, II Università degli Studi di Roma, Via Orazio Raimondo 00173 (La Romanina), Roma, Italy.
Current address: Department of Mathematics, University Park, Nottingham, NG7 2RD, England.

With regard to its sister theory previously developed in [10] by R. L. Hudson and K. R. Parthasarathy for boson Fock space, the scheme of [2] contained two major defects
(i) The structure of the algebra of stochastic integrals involved unnatural parity assumptions.
(ii) There was no fermion analogue of the gauge process which in [10], to some extent, took over the role of the Poisson process from the classical theory.

(i) was rectified by the author in [3] thus bringing the theory into line with the general format proposed by L. Accardi and K. R. Parthasarathy in [1]. The aim of the present paper is to rectify (ii).

We recall that in [10], the gauge process was defined through its action on the total set of exponential vectors $\{\psi(f), f\in L^2(R^+)\}$ in boson Fock space by

$$\Lambda_t\psi(f)=\frac{d}{d\varepsilon}\psi(e^{\varepsilon\chi[0,t]}f)\,|_{\varepsilon=0}\,. \tag{0.3}$$

Together with the boson Brownian motion process of variance 1 formed from pairs of annihilation and creation operators $(B_t, B_t^\dagger)$ smeared as above, a stochastic calculus was obtained with the "Itô correction" rules

$$\begin{aligned} dB\ dB^\dagger&=dt, & d\Lambda\ d\Lambda&=d\Lambda \\ d\Lambda\ dB^\dagger&=dB^\dagger, & dB\ d\Lambda&=dB\,. \end{aligned} \tag{0.4}$$

Furthermore, the classical Poisson Process $\Pi^l=(\Pi^l(t), t\geq 0)$ of intensity $l>0$ was realized in boson Fock space as

$$\Pi^l(t)=\Lambda_t+\sqrt{l}\,(B_t+B_t^\dagger)+lt \tag{0.5}$$

where we note that the process formed by $B_t+B_t^\dagger$ is itself a realization of classical Brownian motion.

The organization of the present paper is as follows; after collecting together some useful algebraic facts in § 1 we proceed to define the fermion gauge operator in § 2 and examine its relevant properties. As in [2] our strategy is to use n-particle vectors in place of exponential vectors to define our operators. In § 4 and § 5 we develop our stochastic calculus and find that the extended version of (0.2) is precisely (0.4) with the boson processes therein replaced by their fermion analogues.

In § 6 we prove the existence of solutions to a certain class of stochastic differential equations and investigate those conditions under which the solutions are unitary. Such equations are a source of cocycles for perturbing the group of shift operators on fermion Fock space over $L^2(\mathbb{R})$. We use the perturbed group to construct a dilation scheme for contraction semigroups on Hilbert space in the sense of [17] and of quantum dynamical semigroups [14] on the algebra of bounded operators on Hilbert space in the sense of [7].

Finally we investigate the fermion analogue of (0. 5) which defines an object which, we propose, deserves to be called a fermion Poisson process of intensity l.

We employ the following notation:

If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are inner product spaces, we denote by $\mathfrak{h}_1 \underline{\otimes} \mathfrak{h}_2$ their algebraic tensor product and $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ it's Hilbert space completion.

Densely defined maps between Hilbert spaces are mutually adjoint if each is contained in the adjoint of the other. Pairs of such maps will be denoted $(T, T^{\dagger})$.

Whenever a proposition contains the symbol $T^{\ddagger}$ it should be read for both T and $T^{\dagger}$.

Let V be a vector space and

$$\underset{\sim}{c} = (c_1, \ldots, c_n) \in \overset{n}{\underset{j=1}{\times}} V \quad \text{where } n \in \mathbb{N}$$

for $1 \leq j \leq n$, define the maps

$$r_j \colon \overset{n}{\underset{j=1}{\times}} V \to \overset{n-1}{\underset{j=1}{\times}} V$$

by

$$r_j(\underset{\sim}{c}) = (c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n).$$

We write

$$\sum_{k=1}^{n-1}{}^{(k)} c_k \equiv \sum_{\iota=1}^{n-1} (r_j(\underset{\sim}{c}))_{\iota}.$$

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§ 1. Preliminaries (cf. [2], [9])

A Hilbert space $\mathscr{H}$ is said to be $\boldsymbol{Z}_2$*-graded* if it may be written $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ where $\mathscr{H}_+$ and $\mathscr{H}_-$ are called the *even* and *odd* subspaces (respectively). Suppose that T is a densely defined operator on $\mathscr{H}$ whose domain $\mathscr{E}$ is an internal direct sum $\mathscr{E}_+\oplus\mathscr{E}_-$ where $\mathscr{E}_+$ and $\mathscr{E}_-$ are dense in $\mathscr{H}_+$ and $\mathscr{H}_-$ (respectively). $\mathscr{E}$ is then said to be *densely graded.* We say T is *even* if $T\mathscr{E}_\pm\subseteq\mathscr{H}_\pm$ and *odd* if $T\mathscr{E}_\pm\subseteq\mathscr{H}_\mp$.

Let ρ denote the parity automorphism of $B(\mathscr{H})$ defined by linear extension of

$$\rho(T)=T \text{ if } T \text{ is even}$$
$$\rho(T)=-T \text{ if } T \text{ is odd}$$

ρ is implemented by a self-adjoint, unitary operator θ on $\mathscr{H}$ (called the *parity operator*) which acts as I on $\mathscr{H}_+$ and $-I$ on $\mathscr{H}_-$.

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be $\boldsymbol{Z}_2$-graded Hilbert spaces and $\mathscr{H}$ be their Hilbert space tensor product. $\mathscr{H}$ is $\boldsymbol{Z}_2$-graded by the prescription

$$\mathscr{H}_+=(\mathscr{H}_{1+}\otimes\mathscr{H}_{2+})\oplus(\mathscr{H}_{1-}\otimes\mathscr{H}_{2-})$$
$$\mathscr{H}_-=(\mathscr{H}_{1+}\otimes\mathscr{H}_{2-})\oplus(\mathscr{H}_{1-}\otimes\mathscr{H}_{2+}).$$

It is easy to see that the parity automorphism of $\boldsymbol{B}(\mathscr{H})$ is $\rho=\rho_1\otimes\rho_2$ and that ρ is unitarily implemented by $\theta_1\otimes\theta_2$ where $\rho_i(\cdot)=\theta_i(\cdot)\theta_i$ is the parity automorphism of $\boldsymbol{B}(\mathscr{H}_i)$, $(i=1,2)$.

Let $\psi_i\in\mathscr{H}_i$ and $T_i\in\boldsymbol{B}(\mathscr{H}_i)$ $(i=1,2)$ with ψ_1 and T_2 of definite parity.

The *Chevalley product* $T_1\hat{\otimes}T_2$ is defined by continuous linear extension of

$$(T_1\hat{\otimes}T_2)(\psi_1\otimes\psi_2)=(-1)^{\delta(T_2)\varepsilon(\psi_1)}T_1\psi_1\otimes T_2\psi_2 \tag{1.1}$$

where $\delta(T_2)=\text{sgn}\,\rho_2(T_2)$ and $\varepsilon(\psi_1)=\text{sgn}\,\theta_1\psi_1$. (1.1) extends by linearity to the case of arbitrary $T_2\in\boldsymbol{B}(\mathscr{H}_2)$.

For $S_i, T_i\in\boldsymbol{B}(\mathscr{H}_i)$ $(i=1,2)$ with S_2 and T_1 of definite parity we have

$$(S_1\hat{\otimes}T_1)(S_2\hat{\otimes}T_2)=(-1)^{\delta(S_2)\delta(T_1)}S_1S_2\hat{\otimes}T_1T_2 \tag{1.2}$$

which again extends by linearity to the case of arbitrary S_2 and T_1.

If $S\in\boldsymbol{B}(\mathscr{H}_1)$ its *ampliation* to $\mathscr{H}$ is the operator $S\hat{\otimes}I\in\boldsymbol{B}(\mathscr{H})$. If

T is an operator on $\mathscr{H}_1$ with densely graded domain $\mathscr{E}_1$ its *algebraic ampliation* to $\mathscr{H}$ is the operator $T\underline{\hat{\otimes}}I$ with domain $\mathscr{E}_1\underline{\hat{\otimes}}\mathscr{H}_2$ where the symbol $\underline{\hat{\otimes}}$ is manipulated in the same way as $\hat{\otimes}$ (with due regard for limitations of domain).

For $\mathfrak{h}$ a complex, separable Hilbert space let $\mathscr{C}(\mathfrak{h})$ denote the *C.A.R. algebra* over $\mathfrak{h}$. $\mathscr{C}(\mathfrak{h})$ is a C^*-algebra with identity generated by $\{a(f), f\in\mathfrak{h}\}$ satisfying

$$\begin{aligned}\{a(f),\ a(g)\} &= 0\\ \{a(f),\ a^\dagger(g)\} &= \langle f,g\rangle I\end{aligned} \tag{1.3}$$

for each $f,g\in\mathfrak{h}$.

From (1.3) we deduce (see e.g. [6])

$$||a(f)|| = ||f|| = ||a^\dagger(f)|| \tag{1.4}$$

for each $f\in\mathfrak{h}$.

Let ω be the (vacuum) gauge invariant quasi-free state on $\mathscr{C}(\mathfrak{h})$ whose two point functions are given by

$$\begin{aligned}\omega(a^\dagger(f)a(g)) &= 0\\ \omega(a(g)a^\dagger(f)) &= \langle f,g\rangle\end{aligned} \tag{1.5}$$

for $f,g\in\mathfrak{h}$.

The G.N.S. representation of $(\mathscr{C}(\mathfrak{h}),\omega)$ lives on fermion Fock space $\Gamma(\mathfrak{h})$. For each $f\in\mathfrak{h}$, $a(f)$ and $a^\dagger(f)$ are realized as annihilation and creation operators respectively on $\Gamma(\mathfrak{h})$, these being bounded and mutually adjoint. ω acts as expectation with respect to the vacuum vector ψ_0 in $\Gamma(\mathfrak{h})$ which is characterized by the relation

$$a(f)\psi_0 = 0 \quad \text{for all} \quad f\in\mathfrak{h}. \tag{1.6}$$

Let $\mathscr{S}$ be a dense subspace of $\mathfrak{h}$ and let

$$f = (f_1,\ldots,f_n)\in \mathop{\times}_{j=1}^{n}\mathscr{S} \quad \text{for} \quad n\in N.$$

We write

$$f^j = (f_1,\ldots\overset{j}{\wedge}\ldots,f_n)$$

and

$$f^{jk}=(f_1 \dots \overset{j}{\wedge} \dots \overset{k}{\wedge} \dots f_n)$$

where $1 \leq j, k \leq n \ (j \neq k)$.

For $T \in \boldsymbol{B}(\mathfrak{h})$ we write

$$Tf=(Tf_1, \dots, Tf_n).$$

The set $\mathscr{I}=\{\psi_n(f), f \in \overset{n}{\underset{j=1}{\times}} \mathscr{S}, n \in \boldsymbol{N} \cup \{0\}\}$ is total in $\Gamma(\mathfrak{h})$ where each

$$\psi_n(f)=a^{\dagger}(f_n) \dots a^{\dagger}(f_1)\Omega.$$

By (1.4) we have

$$\|\psi_n(f)\| \leq \prod_{j=1}^{n} \|f_j\|. \tag{1.7}$$

From now on, $\mathscr{E}$ will denote the linear span of $\mathscr{I}$. For each $f \in \overset{n}{\underset{j=1}{\times}} \mathscr{S}, n \in \boldsymbol{N}, g \in \mathfrak{h}$ we have, by (1.3) and (1.6)

$$a(g)\psi_n(f)=\sum_{j=1}^{n}(-1)^{n-j}\langle g, f_j\rangle \psi_{n-1}(f^j). \tag{1.8}$$

$\Gamma(\mathfrak{h})$ is $\boldsymbol{Z}_2$-graded as the direct sum of even and odd antisymmetric tensor powers with respect to which, the annihilation and creation operators are odd. Let $\rho_F(\cdot)=\theta_F(\cdot)\theta_F$ denote the parity automorphism of $\Gamma(\mathfrak{h})$. We have

$$\theta_F \psi_n(f)=(-1)^n \psi_n(f). \tag{1.9}$$

for all $f \in \overset{n}{\underset{j=1}{\times}} \mathscr{S}, n \in \boldsymbol{N}$.

Let $\mathfrak{h}=\mathfrak{h}_1 \oplus \mathfrak{h}_2$, then we may make the canonical identification $\Gamma(\mathfrak{h})=\Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2)$ for which $\psi_0=\psi_0^1 \otimes \psi_0^2$ (where $\psi^{(j)}$ is the vacuum vector in $\Gamma(\mathfrak{h}_j)\ (j=1,2)$) and

$$a(g)=a(g_1)\hat{\otimes} I+I \hat{\otimes} a(g_2)$$

for $g=(g_1, g_2) \in \mathfrak{h}$ [9].

Hence for $T=0 \oplus T_2 \in \boldsymbol{B}(\mathfrak{h})$ we may identify $a(Tg)$ with $I \hat{\otimes} a(T_2 g_2)$.

Let $\mathfrak{h}_0$ be a complex separable $\boldsymbol{Z}_2$-graded Hilbert space with parity automorphism $\rho_0(\cdot)=\theta_0(\cdot)\theta_0$. We write

$$\begin{aligned}\mathscr{H} &= \mathfrak{h}_0 \otimes \Gamma(\mathfrak{h}) \\ &= \mathfrak{h}_0 \otimes \Gamma(\mathfrak{h}_1) \otimes \Gamma(\mathfrak{h}_2) \\ &= \mathscr{H}_1 \otimes \mathscr{H}_2\end{aligned}$$

where $\mathscr{H}_1 = \mathfrak{h}_0 \otimes \Gamma(\mathfrak{h}_1)$ and $\mathscr{H}_2 = \Gamma(\mathfrak{h}_2)$. $\tilde{\mathscr{E}}_1$ will denote the dense subspace $\mathfrak{h}_0 \underline{\oplus} \mathscr{E}_1$ in $\mathscr{H}_1$ (where $\mathscr{E}_1$ is the linear span of n-particle vectors in $\Gamma(\mathfrak{h}_1)$). We denote by $\rho(\cdot) = \theta(\cdot)\theta$ the parity automorphism of $\boldsymbol{B}(\mathscr{H})$. For arbitrary $X \in \boldsymbol{B}(\mathscr{H}_1)$ and odd $Y \in \boldsymbol{B}(\mathscr{H}_2)$ we deduce from (1.2) the formula

$$(I \hat{\otimes} Y)(X \hat{\otimes} I) = \rho((X \hat{\otimes} I)) I \hat{\otimes} Y. \tag{1.10}$$

$\mathscr{E}_1$ is $\boldsymbol{Z}_2$-graded by the prescription $(\mathscr{E}_1)_\pm = \mathscr{E}_1 \cap \Gamma(\mathfrak{h})_\pm$.

We will make frequent use of the relation

$$\begin{aligned} \theta \cdot \theta_0 \otimes I &= (\theta_0 \otimes \theta_F)(\theta_0 \otimes I) \\ &= (\theta_0 \hat{\otimes} \theta_F)(\theta_0 \hat{\otimes} I) \\ &= I \hat{\otimes} \theta_F . \end{aligned} \tag{1.11}$$

§ 2. The Gauge Operator

Let $T \in \boldsymbol{B}(\mathfrak{h})$. We define an operator $\lambda(T)$ on the dense domain $\mathscr{E}$ by linear extension of the prescription

$$\begin{aligned} &\lambda(T)\psi_0 = 0 \\ &\lambda(T)\psi_n(f) = \frac{d}{d\varepsilon}\psi_n(e^{\varepsilon T}f)\,|_{\varepsilon=0} \end{aligned} \tag{2.1}$$

where differentiation is in the strong sense. $\lambda(T)$ is called the *gauge operator*.

It is well defined for suppose that, for some $m \in \boldsymbol{N}$, we can find

$$c_j \in \boldsymbol{C},\ n_j \in \boldsymbol{N},\ f_{(j)} \in \mathop{\times}_{k=1}^{n_j} \mathscr{S} \quad (1 \le j \le m)$$

for which

$$\sum_{j=1}^m c_j \psi_{n_j}(f_{(j)}) = 0$$

then for each $\psi_p(g) \in \mathscr{S}$ we have

$$\langle \sum_{j=1}^m c_j \psi_{n_j}(e^{\varepsilon T} f_{(j)}),\ \psi_p(g) \rangle = \langle \sum_{j=1}^m c_j \psi_{n_j}(f_{(j)}), \psi_p(e^{\varepsilon T^\dagger} g) \rangle = 0$$

whence

$$\sum_{j=1}^n c_j \psi_{n_j}(e^{\varepsilon T} f_{(j)}) = 0$$

so, by linearity,

$$\lambda(T)\sum_{j=1}^{n} c_j\psi_{n_j}(f_{(j)}) = 0\,,$$

as required.

Lemma 2.1. *The folloing relations hold on the dense set $\mathscr{E}$ for all $g\in\mathfrak{h}$ and $T, U\in\boldsymbol{B}(\mathfrak{h})$ with $[T, U]=0$.*

(*i*) $[a(g), \lambda(T)] = a(T^{\dagger}g)$ (2.2)

(*ii*) $[a^{\dagger}(g), \lambda(T)] = -a^{\dagger}(Tg)$ (2.3)

(*iii*) $[\lambda(T), \lambda(U)] = 0$ (2.4)

(*iv*) $[\lambda(T), \theta_F] = 0$. (2.5)

The proof is by straightforward computation.

From (iv) we see that $\lambda(T)$ is an even operator.

Let A and B be operators in $\mathscr{H}_1$ with domain $\mathscr{E}_1$. We identify them with their algebraic ampliations in $\mathscr{H}$. For $S, T\in\boldsymbol{B}(\mathfrak{h}_2), e\in\mathfrak{h}_2$ we similarly identify $\lambda(S)$ and $\lambda(T)$ with their algebraic ampliations in $\mathscr{H}$ and $a(e)$ with its ampliation in $\mathscr{H}$.

Lemma 2.2. *For $u, v\in\mathfrak{h}_0$, $f\in\times_{j=1}^{n}\mathscr{S}$, $g\in\times_{k=1}^{m}\mathscr{S}$, $m, n\in\boldsymbol{N}$ we have*

(*i*) $$\langle Au\otimes\psi_n(f), B\lambda(T)v\otimes\psi_m(g)\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\langle f_j, Tg_k\rangle\langle Au\otimes\psi_{n-1}(f^{j}), Bv\otimes\psi_{m-1}(g^{k})\rangle \tag{2.6}$$

(*ii*) $$\langle Aa(e)u\otimes\psi_n(f), B\lambda(T)v\otimes\psi_m(g)\rangle = \sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{k=1}^{m}(-1)^{n-j+\iota+m}\langle f_j, e\rangle\langle f_\iota, Tg_k\rangle\langle A\theta_0 u\otimes\psi_{n-2}(f^{j\iota}), Bv\otimes\psi_{m-1}(f^{k})\rangle \tag{2.7}$$

(*iii*) $$\langle a^{\dagger}(e)Au\otimes\psi_n(f), B\lambda(T)v\otimes\psi_m(g)\rangle = \sum_{k=1}^{m}(-1)^{m-k}\langle T^{\dagger}e, g_k\rangle\langle Au\otimes\psi_n(f), \rho(B)\theta_0 v\otimes\psi_{m-1}(g^{k})\rangle + \sum_{k=1}^{m}\sum_{\iota=1}^{m-1}{}^{(k)}\sum_{j=1}^{n}(-1)^{m-k+\iota+j}\langle e, g_k\rangle\langle Tf_j, g_\iota\rangle\langle Au\otimes\psi_{n-1}(f^{j}), \rho(B)\theta_0 v\otimes\psi_{m-2}(g^{k\iota})\rangle \tag{2.8}$$

(*iv*) $$\langle A\lambda(T)u\otimes\psi_n(f), B\lambda(U)v\otimes\psi_m(g)\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\langle Tf_j, Ug_k\rangle\langle Au\otimes\psi_{n-1}(f^{j}), Bv\otimes\psi_{m-1}(g^{k})\rangle$$

$$-\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{p=1}^{m-1}{}^{(k)}(-1)^{i+k+\iota+p}\langle f_\iota, Ug_k\rangle\langle Tf_j, g_p\rangle$$
$$\langle Au\otimes\psi_{n-2}(f^{j\iota}), Bv\otimes\psi_{m-}(g^{kp})\rangle. \tag{2.9}$$

Proof. (i) By repeated use of (2.3) we obtain

$$\langle Au\otimes\psi_n(f), B\lambda(T)v\otimes\psi_m(g)\rangle$$
$$=\langle Au\otimes\psi_n(f), Bv\otimes\sum_{k=1}^{m}a^\dagger(g_m)\dots a^\dagger(Tg_k)\dots a^\dagger(g_1)\psi_0\rangle$$
$$=\sum_{k=1}^{m}(-1)^{m-k}\langle Au\otimes\psi_n(f), Bv\otimes a^\dagger(Tg_k)\psi_{m-1}(g^k)\rangle \quad \text{by (1.3)}$$
$$=\sum_{k=1}^{m}(-1)^{m-k}\langle a(Tg_k)Au\otimes\psi_n(f), \theta B\theta\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
by (1.1) and (1.6)
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{m+n-j-k}\langle f_j, Tg_k\rangle\langle\theta A\theta\theta_0 u\otimes\psi_{n-1}(f^j), \theta B\theta\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
by (1.3) and (1.8).

The result follows from applying (1.11), (1.9) and the unitarity of θ.

(ii) From (1.8) we obtain

$$\langle Aa(e)u\otimes\psi_n(f), B(T)v\otimes\psi_m(g)\rangle=$$
$$\sum_{j=1}^{n}(-1)^{n-j}\langle f_j, e\rangle\langle A\theta_0 u\otimes\psi_{n-1}(f^j), B\lambda(T)v\otimes\psi_m(g)\rangle$$

and the result follows from applying (2.6) to this expression.

(iii) By (1.10) and (2.2) we have

$$\langle a^\dagger(e)Au\otimes\psi_n(f), B\lambda(T)v\otimes\psi_m(g)\rangle$$
$$=\langle Au\otimes\psi_n(f), \rho(B)a(T^\dagger e)v\otimes\psi_m(g)\rangle$$
$$+\langle Au\otimes\psi_n(f), \rho(B)\lambda(T)a(h)v\otimes\psi_m(g)\rangle$$
$$=\sum_{k=1}^{m}(-1)^{m-k}\langle T^\dagger e, g_k\rangle\langle Au\otimes\psi_n(f), \rho(B)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle\rho(B)a(e)v\otimes\psi_m(g), A\lambda(T)u\otimes\psi_n(f)\rangle^-$$

by (1.8) and (2.5). The result follows by applying (2.7) to this expression.

(iv) Repeated application of (2.3) yields

$$\langle A\lambda(T)u\otimes\phi_n(f), B\lambda(U)v\otimes\phi_m(g)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{n+m-j-k}\langle Au\otimes a^\dagger(Tf_j)\phi_{n-1}(f^j), Bv\otimes a^\dagger(Ug_k)\phi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{n+m-j-k}\langle\rho(A)\theta_0 u\otimes\phi_{n-1}(f^j) a(Tf_j)a^\dagger(Ug_k)\rho(B)\theta_0 v \otimes\phi_{m-1}(g^k)\rangle \quad \text{by (1.10)}$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{n+m-j-k}\langle Tf_j, Ug_k\rangle\langle\rho(A)\theta_0 u\otimes\phi_{n-1}(f^j), \rho(B)\theta_0 v \otimes\phi_{m-1}(g^k)\rangle$$
$$-\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{n+m-j-k}\langle\rho(A)\theta_0 u\otimes\phi_{n-1}(f^j), a^\dagger(Ug_k)a(Tf_j)\rho(B)\theta_0 v \otimes\phi_{m-1}(g^k)\rangle \quad \text{by (1.3)}$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\langle Tf_j, Ug_k\rangle\langle Au\otimes\phi_{n-1}(f^j), Bv\otimes\phi_{m-1}(g^k)\rangle$$
$$-\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{n+m-j-k}\langle Au\otimes a(Ug_k)\phi_{n-1}(f^j), Bv\otimes a(Tf_j)\phi_{m-1}(g^k)\rangle$$

by (1.11), (1.9) and (1.10). The result follows from applying (1.8) twice in the second term. □

§ 3. Adapted Processes—The Gauge Process

From now on, we will take $\mathfrak{h}=\boldsymbol{L}^2(\boldsymbol{R}^+)$, $\mathfrak{h}_1=\boldsymbol{L}^2([0,t))$ and $\mathfrak{h}_2=\boldsymbol{L}^2([t,\infty))$. Indeed, we will find it convenient to use the notation $\mathfrak{h}_1=\mathfrak{h}_t$, $\mathfrak{h}_2=\mathfrak{h}^t$, $\phi_0^{(1)}=\phi_{0,t}\phi_0^{(2)}=\phi_0^t$, $\tilde{\mathscr{E}}_1=\tilde{\mathscr{E}}_t$, $\mathscr{H}_1=\mathscr{H}_t$ and $\mathscr{H}_2=\mathscr{H}^t\cdot\mathscr{S}$ will consists of locally bounded functions. We recall the following definitions from [2].

An *adapted process* is a family $F=(F(t), t\in\boldsymbol{R}^+)$ of operators in $\mathscr{H}$ such that

a) For each $t\in\boldsymbol{R}^+, F(t)$ is the algebraic ampliation to $\tilde{\mathscr{E}}_t\underline{\otimes}\mathscr{H}^t$ of an operator in $\mathscr{H}_t$ with domain $\tilde{\mathscr{E}}_t$.

b) A family of operators $F^\dagger=(F^\dagger(t)\in\boldsymbol{R}^+)$ exists satisfying the conditions of a) with each $F^\dagger(t)$ adjoint to $F(t)$. $F^\dagger$ is called the *adjoint process*. It is clearly adapted.

We denote by $\mathscr{A}$ the complex vector space of all adapted processes in $\mathscr{H}$.

$F\in\mathscr{A}$ is said to be *simple* if there exists an increasing sequence $(t_r)_{r\in N\cup\{0\}}$ in $\boldsymbol{R}^+$ with $t_0=0$, $\lim_{r\to\infty} t_r=\infty$ and $F=\sum_{r=0}^{\infty} F_r\chi_{[t_r,t_{r+1})}$ where each $F_r=F(t_r)$, *continuous* if for arbitrary $u\in h_0, f\in \times_{j=1}^{n} \mathscr{S}, n\in N\cup\{0\}$, the maps from $\boldsymbol{R}^+$ to $\mathscr{H}$ given by $t\to F(t)u\otimes\psi_n(f)$ are strongly continuous and *locally square integrable* if each of these maps are strongly measurable and F satisfies

$$\int_0^t ||F(s)u\otimes\psi_n(f)||^2 ds<\infty \quad \text{for each} \quad t\in\boldsymbol{R}^+.$$

We denote by $\mathscr{A}_0, \mathscr{A}_c$ and $\mathscr{L}^2_{loc}$ the subspaces of $\mathscr{A}$ of simple, continuous and locally square integrable processes, respectively.

We have $\mathscr{A}_0, \mathscr{A}_c\subset\mathscr{L}^2_{loc}$.

We say that $F\in\mathscr{A}$ has the property α (e.g. unitarity, even parity) whenever each $F(t)$ is the algebraic ampliation of an operator in $\mathscr{H}_t$ with domain $\tilde{\mathscr{E}}_t$ possessing the property α.

The *annihilation* and *creation processes* are the mutually adjoint, odd continuous processes defined by

$$A_t=I\hat{\otimes}a(\chi_{[0,t)}),\ A_t^\dagger=I\hat{\otimes}a^\dagger(\chi_{[0,t)}) \quad \text{for } t\in\boldsymbol{R}^+.$$

We define the *gauge process* $(\Lambda_t, t\in\boldsymbol{R}^+)$ by the prescription

$$\Lambda_t=I\hat{\otimes}\lambda(\chi_{[0,t)}) \tag{3.1}$$

for each $t\in\boldsymbol{R}^+$, where the indicator function $\chi_{[0,t)}$ acts by multiplication on $L^2(\boldsymbol{R}^+)$.

By (2.5), it is clearly an even process. It is continuous for given $u\in\mathfrak{h}_0, f\in\times_{j=1}^{n}\mathscr{S}, n\in N, s,t\in\boldsymbol{R}^+$ with $s\leq t$ we have by (2.9)

$$\begin{aligned}
&||(\Lambda_t-\Lambda_s)u\otimes\psi_n(f)||^2\\
&=\langle\lambda(\chi_{(s,t)})u\otimes\psi_n(f), \lambda(\chi_{(s,t)})u\otimes\psi_n(f)\rangle\\
&=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_s^t\overline{f_j(\tau)}f_k(\tau)d\tau\langle u\otimes\psi_{n-1}(f^j), v\otimes\psi_{n-1}(f^k)\rangle\\
&\quad-\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{p=1}^{n-1}{}^{(k)}(-1)^{j+k+\iota+p}\Big(\int_s^t\overline{f_\iota(\tau)}f_k(\tau)d\tau\Big)\Big(\int_s^t\overline{f_j(\tau)}f_p(\tau)d\tau\Big)\\
&\quad\langle u\otimes\psi_{n-2}(f^\iota), u\otimes\psi_{n-2}(f^p)\rangle\\
&\to 0 \quad \text{as} \quad s\to t.
\end{aligned}$$

For each $t\in\boldsymbol{R}^+$, we define the *time t vacuum conditional expectation*

$\boldsymbol{E}_{t]}: \boldsymbol{B}(\mathscr{H}) \to \boldsymbol{B}(\mathscr{H}_t) \hat{\otimes} I$ by continuous, linear extension of the prescription

$$\boldsymbol{E}_{t]}(X \otimes Y) = X \hat{\otimes} I \langle \psi_0^t, Y\psi_0^t \rangle \tag{3.2}$$

where $X \in \boldsymbol{B}(\mathscr{H}_t), Y \in \boldsymbol{B}(\mathscr{H}^t)$. $\boldsymbol{E}_{t]}$ extends in the obvious way to densely defined operators on $\mathscr{H}$ whose domain includes ψ_0.

We say that $F \in \mathscr{A}$ is a *martingale* if for each $s, t \in \boldsymbol{R}^+$ with $s \leq t$

$$\boldsymbol{E}_{s]}(F(t)) = F(s). \tag{3.3}$$

We show that the gauge process is a martingale. For all $\phi_s, \chi_s \in \mathscr{H}_s$, we have

$$\begin{aligned}
&\langle \phi_s \otimes \psi_0^s, \Lambda_t \chi_s \otimes \psi_0^s \rangle \\
&= \langle \phi_s \otimes \psi_0^s, (\Lambda_s + \lambda(\chi_{[s,t)}))\chi_s \otimes \psi_0^s \rangle \\
&= \langle \phi_s \otimes \psi_0^s, \Lambda_s \chi_s \otimes \psi_0^s \rangle + \langle \phi_s \otimes \psi_0^s, \lambda(\chi_{(s,t)})\chi_s \otimes \psi_0^s \rangle \\
&= \langle \phi_s, \Lambda_s \chi_s \rangle + \langle \phi_s, \chi_s \rangle \langle \psi_0^s, \lambda(\chi_{(s,t)})\psi_0^s \rangle \\
&= \langle \phi_s, \Lambda_s \chi_s \rangle
\end{aligned}$$

whence $\boldsymbol{E}_{s]}(\Lambda_t) = \Lambda_s$, as required.

It follows that the martingale representation theorem of [15] is not valid in this case.

We will find a use in §6 for the following estimate. Let L be the ampliation in $\mathscr{H}$ of a bounded operator in $\mathfrak{h}_0, u \in \mathfrak{h}_0, f \in \times_{j=1}^n \mathfrak{h}, n \in \boldsymbol{N}$, $t > 0$, then

$$\begin{aligned}
&||L\Lambda_t u \otimes \psi_n(f)||^2 \\
&= ||\sum_{j=1}^n (-1)^{n-j} Lu \otimes a^\dagger(\chi_{[0,t)} f_j)\psi_{n-1}(f^j)||^2 && \text{by (2.3)} \\
&\leq ||L||^2 ||u||^2 n ||\sum_{j=1}^n a^\dagger(\chi_{[0,t)} f_j)\psi_{n-1}(f^j)||^2 \\
&\leq ||L||^2 ||u||^2 n \sum_{j=1}^n (||\chi_{[0,t)} f_j||^2 ||\psi_{n-1}(f^j)||^2) \\
&\leq n^2 t ||L||^2 ||u||^2 ||f_1||^2 \ldots ||f_n||^2 && (3.4)
\end{aligned}$$

by (1.4) and (1.7).

§4. Stochastic Integrals of Simple Processes

Let $E, F, G, H \in \mathscr{A}_0$ with

$$E=\sum_{r=0}^{\infty}E_r\chi_{[t_r,t_{r+1})},\qquad F=\sum_{r=0}^{\infty}F_r\chi_{[t_r,t_{r+1})}$$

$$G=\sum_{r=0}^{\infty}G_r\chi_{[t_r,t_{r+1})},\qquad H=\sum_{r=0}^{\infty}H_r\chi_{[t_r,t_{r+1})} \tag{4.1}$$

where $0<t_0<\dots<t_r\xrightarrow[r]{}\infty$.

We define the *stochastic integral* of (E,F,G,H) to be the family of operators $M=(M(t),t\geq 0)$ defined inductively by the prescription

$$M(t)=M(t_r)+E_r(A_t-A_{t_r})+(A_t^\dagger-A_{t_r}^\dagger)F_r+G_r(A_t-A_{t_r})$$
$$+H_r(t-t_r) \tag{4.2}$$

whenever $t_r<t\leq t_{r+1}$, where $M(0)$ is the ampliation in $\mathscr{H}$ of an element of $B(\mathfrak{h}_0)$.

We write

$$M(t)=M(0)+\int_0^t(EdA+dA^\dagger F+GdA+Hdt) \tag{4.3}$$

whenever (4.2) holds. Equivalently, we will use the differential notation

$$dM=EdA+dA^\dagger F+GdA+Hdt\ . \tag{4.4}$$

We note that by (1.10) we have the formal relation

$$dA^\dagger F=\rho(F)dA^\dagger. \tag{4.5}$$

By formal adjunction in (4.2) we see that M is adapted, the adjoint process being given by

$$M^\dagger(t)=M^\dagger(0)+\int_0^t(E^\dagger dA+dA^\dagger G^\dagger+F^\dagger dA+H^\dagger dt). \tag{4.6}$$

Clearly M depends linearly on (E,F,G,H).

Theorem 4.1. *Let* $E,F,G,H\in\mathscr{A}_0$ *and* M *be their stochastic integral. For each* $u,v\in\mathfrak{h}_0, f\in\times_{j=1}^{n}\mathscr{S}, g\in\times_{k=1}^{m}\mathscr{S}, n,m\in N, t\in\mathbb{R}^+$ *we have*

$$\langle u\otimes\psi_n(f),(M(t)-M(0))v\otimes\psi_m(g)\rangle$$
$$=\int_0^t\{\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(\tau)}\langle u\otimes\psi_{n-1}(f^j),E(\tau)v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)$$
$$+\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(\tau)}\langle\theta_0u\otimes\psi_{n-1}(f^j),F(\tau)v\otimes\psi_m(g)\rangle$$
$$+\sum_{k=1}^{m}\langle u\otimes\psi_n(f),G(\tau)\theta_0v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)$$
$$+\langle u\otimes\psi_n(f),H(\tau)v\otimes\psi_m(g)\rangle d\tau. \tag{4.7}$$

Furthermore $M(t)=M(0)$ *for all* $t\in\boldsymbol{R}^{+}$ *if and only if each* $E(t)=F(t)=G(t)=H(t)\equiv 0$.

Proof. By Theorem 4.1 of [2] we have

$$\{\langle u\otimes\psi_n(f),(M(t)-M(0)-\int_0^t Ed\Lambda)v\otimes\psi_m(g)\rangle$$
$$=\int_0^t\{\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(\tau)}\langle\theta_0 u\otimes\psi_{n-1}(f^j),F(\tau)v\otimes\psi_m(g)\rangle$$
$$+\sum_{k=1}^{m}(-1)^{m-k}\langle u\otimes\psi_n(f),G(\tau)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)$$
$$+\langle u\otimes\psi_n(f),H(\tau)v\otimes\psi_m(g)\rangle\}\,d\tau \tag{4.8}$$

so it is sufficient to establish

$$\langle u\otimes\psi_n(f),\int_0^t Ed\Lambda v\otimes\psi_m(g)\rangle=$$
$$\int_0^t\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(\tau)}\langle u\otimes\psi_{n-1}(f^j),E(\tau)v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)\,d\tau \tag{4.9}$$

whence (4.7) follows from adding (4.8) to (4.9).

Without loss of generality, we take

$$E=\sum_{j=0}^{\infty}E_r\chi_{[t_r,t_{r+1})}\ \text{with}\ 0=t_0<\ldots<t_r\xrightarrow{r}\infty$$

and establish (4.9) by induction. (4.9) clearly holds for $t=0$, assume that it holds for $t=t_r$, then for $t_r<t\leq t_{r+1}$, by (4.2) we have

$$\langle u\otimes\psi_n(f),\int_{t_r}^t Ed\Lambda v\otimes\psi_m(g)\rangle$$
$$=\langle u\otimes\psi_n(f),E_r(\Lambda_t-\Lambda_{t_r})v\otimes\psi_m(g)\rangle$$
$$=\langle u\otimes\psi_n(f),E_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle \qquad \text{by (3.1)}$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\langle f_j,\chi_{[t_r,t)}g_k\rangle\langle u\otimes\psi_{n-1}(f^j),E_r v\otimes\psi_{m-1}(g^k)\rangle \quad \text{by (2.6)}$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_{t_r}^t\overline{f_j(\tau)}g_k(\tau)\,d\tau\langle u\otimes\psi_{n-1}(f^j),E_r v\otimes\psi_{m-1}(g^k)\rangle.$$

For $\tau\in(t_r,t)$ we have $E_r=E(\tau)$ and the result follows from additivity of the Lebesgue integral.

Now suppose that $M(t)=M(0)$ for all $t\in\boldsymbol{R}^{+}$. Applying the argument on p. 479 of [2] in (4.7) we deduce that $F=G=H\equiv 0$. Thus we conclude that for all $t\in\mathbb{R}^+$

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\langle u\otimes\psi_{n-1}(f^j),E(t)v\otimes\psi_{m-1}(g^k)\rangle g_k(t)=0$$

and since the set of $\psi_n(f)$ corresponding to a choice of $f\in\times_{j=1}^{n}\mathscr{S}$ such that each $f_j(t)\neq 0$ for all $t\in\mathbb{R}^+$ remains total in $\Gamma(L^2(\mathbb{R}^+))$ we conclude that $E\equiv 0$. □

The following theorem is essentially the fermion Itô formula for simple processes and plays a central role in the further development of our theory.

Theorem 4.2. *Let* $E,F,G,H,E',F',G',H'\in\mathscr{A}_0$,

$$M(t)=\int_0^t Ed\Lambda+dA^\dagger F+GdA+Hdt$$

$$M'(t)=\int_0^t E'd\Lambda+dA^\dagger F'+G'dA+D'dt\,.$$

For arbitrary $u,v\in\mathfrak{h}_0, f\in\times_{j=1}^{n}\mathscr{S}, g\in\times_{k=1}^{m}\mathscr{S}, n,m\in\mathbb{N}$, *the mapping*

$$t\to\langle M(t)u\otimes\psi_n(f),M'(t)v\otimes\psi_m(g)\rangle$$

from $\mathbb{R}^+\to\mathbb{C}$ *is absolutely continuous with derivative*

$$\begin{aligned}
&\frac{d}{dt}\langle M(t)u\otimes\psi_n(f),M'(t)v\otimes\psi_m(g)\rangle\\
&=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}[\langle M(t)u\otimes\psi_{n-1}(f^j),E'(t)v\otimes\psi_{m-1}(g^k)\rangle\\
&\quad+\langle E(t)u\otimes\psi_{n-1}(f^j),M'(t)v\otimes\psi_{m-1}(g^k)\rangle\\
&\quad+\langle E(t)u\otimes\psi_{n-1}(f^j),E'(t)v\otimes\psi_{m-1}(g^k)\rangle]g_k(t)\\
&\quad+\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}[\langle G(t)\theta_0u\otimes\psi_{n-1}(f^j),M'(t)v\otimes\psi_m(g)\rangle\\
&\quad+\langle\rho(M(t)\theta_0u\otimes\psi_{n-1}(f^j),F'(t)v\otimes\psi_m(g)\rangle\\
&\quad+\langle\rho(E(t))\theta_0u\otimes\psi_{n-1}(f^j),F'(t)v\otimes\psi_m(g)\rangle]\\
&\quad+\sum_{k=1}^{m}(-1)^{m-k}[\langle F(t)u\otimes\psi_n(f),\rho(M'(t))\theta_0v\otimes\psi_{m-1}(g^k)\rangle\\
&\quad+\langle F(t)u\otimes\psi_n(f),\rho(E'(t))\theta_0v\otimes\psi_{m-1}(g^k)\rangle\\
&\quad+\langle M(t)u\otimes\psi_n(f),G'(t)\theta_0v\otimes\psi_{m-1}(g^k)\rangle]g_k(t)\\
&\quad+\langle H(t)u\otimes\psi_n(f),M'(t)v\otimes\psi_m(g)\rangle\\
&\quad+\langle M(t)u\otimes\psi_n(f),H'(t)v\otimes\psi_m(g)\rangle\\
&\quad+\langle F(t)u\otimes\psi_n(f),F'(t)v\otimes\psi_m(g)\rangle\,.
\end{aligned}\tag{4.10}$$

Proof. By Theorem 4.2 of [2] and proposition 4 of [4] we have

$$\frac{d}{dt}\langle (M(t)-\int_0^t Ed\Lambda)u\otimes\psi_n(f),\ (M'(t)-\int_0^t E'd\Lambda)v\otimes\psi_m(g)\rangle$$
$$=\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}[\langle G(t)\theta_0 u\otimes\psi_{m-1}(f^j),\ (M'(t)-\int_0^t E'd\Lambda)$$
$$v\otimes\psi_m(g)\rangle+\langle\rho(M(t)-\int_0^t Ed\Lambda)\theta_0 u\otimes\psi_{m-1}(f^j),F'(t)v\otimes\psi_m(g)\rangle]$$
$$+\sum_{k=1}^{m}(-1)^{m-k}[\langle F(t)u\otimes\psi_n(f),\rho(M'(t)-\int_0^t E'd\Lambda)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle (M(t)-\int_0^t Ed\Lambda)u\otimes\psi_n(f),G'(t)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle]g_k(t)$$
$$+\langle H(t)u\otimes\psi_n(f),\ (M'(t)-\int_0^t E'd\Lambda)v\otimes\psi_m(g)\rangle$$
$$+\langle (M(t)-\int_0^t Ed\Lambda)u\otimes\psi_n(f),H'(t)v\otimes\psi_m(g)\rangle$$
$$+\langle F(t)u\otimes\psi_n(f),F'(t)v\otimes\psi_m(g)\rangle. \tag{4.11}$$

Without loss of generality, we take E,F,G,H as in (4.1) and assume that E',F',G',H' have the same intervals of constancy. Thus we may write $M(t)$ and $M'(t)$ in the form (4.2).

We establish (4.1) by induction. It clearly is valid for $t=0$ and we assume that it holds for $t=t_r$.

Now suppose that $t_r<t\leq t_{r+1}$. Given (4.11) it is sufficient to compute

$$\frac{d}{dt}\{\langle\int_{t_r}^t Ed\Lambda u\otimes\psi_n(f),\ (M'(t)-\int_{t_r}^t E'd\Lambda)v\otimes\psi_m(g)\rangle$$
$$+\langle (M(t)-\int_{t_r}^t Ed\Lambda)u\otimes\psi_n(f),\int_{t_r}^t E'd\Lambda v\otimes\psi_m(g)\rangle$$
$$+\langle\int_{t_r}^t Ed\Lambda u\otimes\psi_n(f),\int_{t_r}^t E'd\Lambda v\otimes\psi_m(g)\rangle\}$$
$$=\frac{d}{dt}\{\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f),\ (M'(t_r)+a^\dagger(\chi_{[t_r,t)})F'_r$$
$$+G'_r a(\chi_{[t_r,t)})+H'_r(t-t_r))v\otimes\psi_m(g)\rangle$$
$$+\langle (M(t_r)+a^\dagger(\chi_{[t_r,t)})F_r+G_r a(\chi_{[t_r,t)})+H_r(t-t_r))u\otimes\psi_n(f),$$
$$E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle+\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f),E'_r v\otimes\psi_m(g)\rangle\} \tag{4.12}$$

by (4.2).

We compute each of the terms in (4.12) separately

$$\frac{d}{dt}\langle M(t_r)u\otimes\psi_n(f), E_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$=\frac{d}{dt}\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_{t_r}^{t}\overline{f_j(\tau)}g_k(\tau)\,d\tau\langle M(t_r)u\otimes\psi_{n-1}(f^j),$$
$$E_r'v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}g_k(t)\langle M(t_r)u\otimes\psi_{n-1}(f^j), E_r'v\otimes\psi_{m-1}(g^k)\rangle \tag{4.13}$$

by (2.6).

A similar argument yields

$$\frac{d}{dt}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), M'(t_r)v\otimes\psi_m(g)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}g_k(t)\langle E_ru\otimes\psi_{n-1}(f^j), M'(t_r)v\otimes\psi_{m-1}(g^k)\rangle. \tag{4.14}$$

Again, by (2.6) we have

$$\frac{d}{dt}\langle H_r(t-t_r)u\otimes\psi_n(f), E_r'\lambda(\chi_{[t_r,t)})v\otimes\psi_n(g)\rangle$$
$$=\frac{d}{dt}\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_{t_r}^{t}\overline{f_j(\tau)}g_k(\tau)\,d\tau\langle H_r(t-t_r)u\otimes\psi_{n-1}(f^j),$$
$$E_r'v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_{t_r}^{t}\overline{f_j(\tau)}g_k(\tau)\,d\tau\langle H_ru\otimes\psi_{n-1}(f^j), E_r'v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}g_k(t)\langle H_r(t-t_r)u\otimes\psi_{n-1}(f^j), E_r'v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\langle H_ru\otimes\psi_n(f), E_r'\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$+\sum_{j=1}^{n}(-1)^{j+k}\overline{f_j(t)}\langle H_r(t-t_r)u\otimes\psi_{n-1}(f^j), E_r'v\otimes\psi_{m-1}(g^k)\rangle g_k(t). \tag{4.15}$$

Similarly

$$\frac{d}{dt}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), H_r'(t-t_r)v\otimes\psi_m(g)\rangle$$
$$=\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), H_r'v\otimes\psi_m(g)\rangle+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}$$
$$\langle E_ru\otimes\psi_{n-1}(f^j), H_r'(t-t_r)v\otimes\psi_{m-1}(g^k)\rangle. \tag{4.16}$$

Using (2.7) we obtain

$$\frac{d}{dt}\langle G_r a(\chi_{[t_r,t)})u\otimes\psi_n(f), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$=\frac{d}{dt}\sum_{j=1}^{n}\sum_{\iota=1}^{n-j}{}^{(j)}\sum_{k=1}^{m}(-1)^{n-j+\iota+k}\left(\int_{t_r}^{t}\overline{f_j(\tau)}d\tau\right)\left(\int_{t_r}^{t}\overline{f_\iota(\tau)}g_k(\tau)d\tau\right)$$
$$\langle G_r\theta_0 u\otimes\psi_{n-2}(f^{j\iota}), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{k=1}^{m}(-1)^{n-j+\iota+k}\Big[\overline{f_j(t)}\int_{t_r}^{t}\overline{f_\iota(\tau)}g_k(\tau)d\tau$$
$$+\left(\int_{t_r}^{t}\overline{f_j(\tau)}d\tau\right)\overline{f_\iota(t)}g_k(t)\Big]\langle G_r\theta_0 u\otimes\psi_{n-2}(f^{j\iota}), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}\langle G_r\theta_0 u\otimes\psi_{n-1}(f^j), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\langle G_r a(\chi_{[t_r,t)})u\otimes\psi_{n-1}(f^j),$$
$$E'_r v\otimes\psi_{m-1}(g^k)\rangle g_k(t) \tag{4.17}$$

by (2.6) and Appendix (i).

Similarly

$$\frac{d}{dt}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), G'_r a(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$=\sum_{k=1}^{n}(-1)^{m-k}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), G'_r\theta_0 v\otimes\psi_{m-1}(g^k)\rangle g_k(t)$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\langle E_r u\otimes\psi_{n-1}(f^j), G'_r a(\chi_{[t_r,t)})$$
$$v\otimes\psi_{m-1}(g^k)\rangle g_k(t). \tag{4.18}$$

By (2.8) we deduce

$$\frac{d}{dt}\langle a^\dagger(\chi_{[t_r,t)})F_r u\otimes\psi_n(f), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$=\frac{d}{dt}\sum_{k=1}^{m}(-1)^{m-k}\int_{t_r}^{t}g_k(\tau)d\tau\langle F_r u\otimes\psi_n(f), \rho(E'_r)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\frac{d}{dt}\sum_{k=1}^{m}\sum_{\iota=1}^{m-1}{}^{(k)}\sum_{j=1}^{n}(-1)^{m-k+\iota+j}\left(\int_{t_r}^{t}g_k(\tau)d\tau\right)\left(\int_{t_r}^{t}\overline{f_j(\tau)}g_\iota(\tau)d\tau\right)$$
$$\langle F_r u\otimes\psi_{n-1}(f^j), \rho(E'_r)\theta_0 v\otimes\psi_{m-2}(g^{k\iota})\rangle$$
$$=\sum_{k=1}^{m}(-1)^{m-k}\langle F_r u\otimes\psi_n(f), \rho(E'_r)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle g_k(t)$$
$$+\sum_{k=1}^{m}(-1)^{m-k}\langle F_r u\otimes\psi_n(f), \rho(E_r)\lambda(\chi_{[t_r,t)})\theta_0 v\otimes\psi_{m-1}(g^k)\rangle g_k(t)$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\langle a^\dagger(\chi_{[t_r,t)})F_r u\otimes\psi_{n-1}(f^j),$$
$$E'_r v\otimes\psi_{m-1}(g^k)\rangle g_k(t) \tag{4.19}$$

where we have used the same arguments as in the previous computation.

Similarly, we obtain

$$\frac{d}{dt}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), a^\dagger(\chi_{[t_r,t)})F'_r v\otimes\psi_m(g)\rangle$$
$$=\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}\langle\rho(E_r)\theta_0 u\otimes\psi_{n-1}(f^j), F'_r v\otimes\psi_m(g)\rangle$$
$$+\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}\langle\rho(E_r)\lambda(\chi_{[t_r,t)})\theta_0 u\otimes\psi_{n-1}(f^j), F'_r v\otimes\psi_m(g)\rangle$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\langle E_r u\otimes\psi_{n-1}(f^j), a^\dagger(\chi_{[t_r,t)})F'_r v\otimes\psi_{m-1}(g^k)\rangle g_k(t). \tag{4.20}$$

Finally, using (2.9) we have

$$\frac{d}{dt}\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$=\frac{d}{dt}\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\int_{t_r}^{t}\overline{f_j(\tau)}g_k(\tau)\,d\tau\langle E_r u\otimes\psi_{n-1}(f^j), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$-\frac{d}{dt}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{p=1}^{m-1}{}^{(k)}(-1)^{i+k+\iota+p}\left(\int_{t_r}^{t}\overline{f_\iota(\tau)}g_k(\tau)\,d\tau\right)$$
$$\left(\int_{t_r}^{t}\overline{f_j(\tau)}g_p(\tau)\,d\tau\right)\langle E_r u\otimes\psi_{n-2}(f^{j\iota}), E'_r v\otimes\psi_{m-2}(g^{kp})\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\{\langle E_r u\otimes\psi_{n-1}(f^j), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_{n-1}(f^j), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle E_r u\otimes\psi_{n-1}(f^j), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_{n-1}(g^k)\rangle\}g_k(t). \tag{4.21}$$

by Appendix (ii).

Collecting together the terms in (4.13) to (4.21), we find that the right hand side of (4.12) is equal to

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\{\langle E_r u\otimes\psi_{n-1}(f^j), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle M(t)u\otimes\psi_{n-1}(f^j), E'_r v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle E_r u\otimes\psi_{n-1}(f^j), M'(t)v\otimes\psi_{m-1}(g^k)\rangle\}g_k(t)$$
$$+\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}\{\langle G_r\theta_0 u\otimes\psi_{n-1}(f^j), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$+\langle\rho(E_r)\lambda(\chi_{[t_r,t)})\theta_0 u\otimes\psi_{n-1}(f^j), F'_r v\otimes\psi_m(g)\rangle$$

$$+\langle\rho(E_r)\theta_0 u\otimes\psi_{n-1}(f^j), F'_r v\otimes\psi_m(g)\rangle$$
$$+\sum_{k=1}^{m}(-1)^{m-k}\{\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), G'_r\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle F_r u\otimes\psi_n(f), \rho(E'_r)\lambda(\chi_{[t_r,t)})\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$+\langle F_r u\otimes\psi_n(f), \rho(E'_r)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle\}g_k(t)$$
$$+\langle H_r u\otimes\psi_n(f), E'_r\lambda(\chi_{[t_r,t)})v\otimes\psi_m(g)\rangle$$
$$+\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_n(f), H'_r v\otimes\psi_m(g)\rangle. \tag{4.22}$$

Adding (4.2) to (4.11), where the latter equation is cut down to the range (t_r, t), we obtain (4.10) as required. □

§5. Extension to Square Integrable Processes

In (4.10) we take $E'=E, F'=F, G'=G, H'=H, v=u, m=n$, and $g=f$ to obtain

$$\frac{d}{dt}||M(t)u\otimes\psi_n(f)||^2$$
$$=2\mathrm{Re}\{\langle\sum_{j=1}^{n}(-1)^j f_j(t)M(t)u\otimes\psi_{n-1}(f^j), \sum_{k=1}^{n}(-1)^k f_k(t)$$
$$E(t)u\otimes\psi_{n-1}(f^k)\rangle+\langle\sum_{j=1}^{n}(-1)^{n-j}f_j(t)\rho(M(t))\theta_0 u\otimes\psi_{n-1}(f^j),$$
$$F(t)u\otimes\psi_n(f)\rangle+\langle\sum_{j=1}^{n}(-1)^{n-j}f_j(t)\rho(E(t))\theta_0 u\otimes\psi_{n-1}(f^j),$$
$$F(t)u\otimes\psi_n(f)\rangle+\sum_{j=1}^{n}(-1)^{n-j}\langle G(t)\theta_0 u\otimes\psi_{n-1}(f^j),$$
$$\overline{f_j(t)}M(t)v\otimes\psi_n(f)\rangle+\langle H(t)u\otimes\psi_n(f), M(t)v\otimes\psi_n(f)\rangle\}$$
$$+||\sum_{j=1}^{n}(-1)^j f_j(t)E(t)u\otimes\psi_{n-1}(f^j)||^2+||F(t)u\otimes\psi_n(f)||^2$$
$$\leq\sum_{j=1}^{n}\{|f_j(t)|^2[2n||M(t)u\otimes\psi_{n-1}(f^j)||^2+3n||E(t)u\otimes\psi_{n-1}(f^j)||^2$$
$$+||M(t)u\otimes\psi_n(f)||^2]+||G_t\theta_0 u\otimes\psi_{n-1}(f^j)||^2\}$$
$$+3||F(t)u\otimes\psi_n(f)||^2+||M(t)u\otimes\psi_n(f)||^2$$
$$+||H(t)u\otimes\psi_n(f)||^2 \tag{5.1}$$

where we have used the inequalities $2\mathrm{Re}\langle\phi_1,\phi_2\rangle\leq||\phi_1||^2+||\phi_2||^2$ and $||\sum_{j=1}^{n}\phi_j||^2\leq n\sum_{j=1}^{n}||\phi_j||^2$ together with the unitarity of θ.

Let $E, F, G, H\in\mathscr{L}^2_{loc}$. By Proposition 3.1 of [2] there exist $(E_p)_{p\in N}$,

$(F_p)_{p\in N}$, $(G_p)_{p\in N}$, $(H_p)_{p\in N}$ such that each $E_p, F_p, G_p, H_p \in \mathscr{A}_0$ ($p\in N$) and for arbitrary $u\in\mathfrak{h}_0$, $f\in \times_{j=1}^{n}\mathscr{S}$, $n\in N$, $\alpha_j, \beta_j, \gamma, \delta\in\mathscr{S}$, $1\leq j\leq n$, $t\in \mathbb{R}^+$,

$$\int_0^t \{\sum_{j=1}^n[\alpha_j(\tau)\|(E^\sharp(\tau)-E_p^\sharp(\tau))u\otimes\psi_{n-1}(f^j)\|^2 + \beta_j(\tau)\|G^\sharp(\tau)-G_p^\sharp(\tau))\theta_0 u\otimes\psi_{n-1}(f^j)\|^2] + \gamma(\tau)\|(F^\sharp(\tau)-F_p^\sharp(\tau))u\otimes\psi_n(f)\|^2 + \delta(\tau)\|(H^\sharp(\tau)-H_p^\sharp(\tau))u\otimes\psi_n(f)\|^2\}\,d\tau \xrightarrow[n]{} 0. \quad (5.2)$$

Let $M_p(t)=\int_0^t E_p dA + dA^\dagger F_p + G_p dA + H_p dt$.

Theorem 5.1. *For arbitrary* $u\in\mathfrak{h}_0, f\in\times_{j=1}^{n}\mathscr{S}, n\in N$ *and* $T>0$, $(M_p(t)u\otimes\psi_n(f))_{p\in N}$ *converges uniformly in* $\mathscr{H}$ *for* $t\in[0,T]$ *to a limit independent of the choice of* $(E_p)_{p\in N}$, $(F_p)_{p\in N}$, $(G_p)_{p\in N}$, $(H_p)_{p\in N}$ *satisfying* (5.2).

Proof. The argument is very similar to that of the proof of theorem 5.1 in [2], indeed replacing M by M_p-M_q in (5.1) and using the integrating factor $\exp\{-t-\int_0^t\sum_{j=1}^n|f_j(\tau)|^2 d\tau\}$ yields

$$\begin{aligned}&\|(M_p(t)-M_q(t))u\otimes\psi_n(f)\|^2\\ &\leq \exp\{\int_0^T(\sum_{j=1}^n|f_j(s)|^2+1)ds\}\int_0^T[\sum_{j=1}^n\{|f_j(\tau)|^2\cdot\\ &\quad[2n\|(M_p(\tau)-M_q(\tau))u\otimes\psi_{n-1}(f^j)\|^2+3n\|(E_p(\tau)-E_q(\tau))\\ &\quad u\otimes\psi_{n-1}(f^j)\|^2]+\|(G_p(\tau)-G_q(\tau))\theta_0 u\otimes\psi_{n-1}(f^j)\|^2\}\\ &\quad+3\|(F_p(\tau)-F_q(\tau))u\otimes\psi_n(f)\|^2+\|(H_p(\tau)-H_q(\tau))\\ &\quad u\otimes\psi_n(f)\|^2]d\tau.\end{aligned} \quad (5.3)$$

Putting $n=0$ in (5.3), the sequence $(M_p(t)u\otimes\psi_0)_{p\in N}$ is uniformly convergent for $t\in[0,T]$ by an identical argument to that of [2].

Making the inductive hypothesis that each $(M_p(t)u\otimes\psi_{n-1}(f^j))_{p\in N}$ is uniformly convergent, hence uniformly Cauchy for $t\in[0,T]$, the result follows from applying (5.2) in (5.3) where we take each $\alpha_j(t)=3n|f_j(t)|^2$, $\beta_j(t)=\chi_{[0,T]}(t)$, $\gamma(t)=3$, $\delta(t)=\chi_{[0,T]}(t)$, $1\leq j\leq n$, $t\in[0,T]$.

The analogous result for $(M_p^\dagger(t)u\otimes\psi_n(f))_{p\in N}$ is proved identically.

A similar inductive argument establishes the required independence of the convergence. □

We define the operator $M(t)$ on $\mathscr{E}$ by the prescription

$$M(t)u\otimes\psi_n(f)=\lim_{p\to\infty}M_p(t)u\otimes\psi_n(f)$$

for all $t\in\boldsymbol{R}^+$. We denote also by $M(t)$ its extension as an algebraic ampliation to $\mathscr{E}_t\underline{\otimes}\mathscr{H}^t$. Clearly $M(t)$ is an adapted process, we call it the *stochastic integral* of the square integrable processes E, F, G and H and write

$$M(t)=\int_0^t EdA+dA^\dagger F+GdA+Hdt \tag{5.4}$$

for $t\in\boldsymbol{R}^+$, the adjoint process being given by

$$M^\dagger(t)=\int_0^t E^\dagger dA+dA^\dagger G^\dagger+F^\dagger dA+H^\dagger dt\,.$$

By identical arguments to those used in [2], we see that M is a continuous process, furthermore the maps $t\to M(t)u\otimes\psi_n(f)$ are bounded on finite intervals whence we may pass to the limit of simple approximation in Theorems 4.1 and 4.2, hence establishing the validity of these for $E, F, G, H\in\mathscr{L}^2_{loc}$.

Let $\mathfrak{M}$ denote the set of all stochastic integrals of square integrable processes which satisfy (5.4) with each $M(t), E(t), F(t), G(t), H(t)\in\boldsymbol{B}(\mathscr{H})$ for $t\in\boldsymbol{R}^+$ and

$$\sup_{0\le s\le t}\max\{\|M(s)\|, \|E(s)\|, \|F(s)\|, \|G(s)\|, \|H(s)\|\}<\infty.$$

Theorem 6.1 below demonstrates that $\mathfrak{M}$ is by no means empty.

Theorem 5.2 (*Fermion Itô formula in bounded form*). *$\mathfrak{M}$ is a $\boldsymbol{Z}_2$-graded *-algebra under pointwise operator multiplication and the involution $M\to M^\dagger$.*

Furthermore for $M_1, M_2\in\mathfrak{M}$ with

$$dM_i=E_idA+dA^\dagger F_i+G_idA+H_idt \qquad (i=1,2)$$

we have

$$d(M_1M_2)=dM_1\cdot M_2+M_1\cdot dM_2+dM_1\cdot dM_2 \tag{5.5}$$

where

$$dM_1\cdot M_2=E_1M_2dA+dA^\dagger F_1M_2+G_1\rho(M_2)dA+H_1M_2dt \tag{5.6}$$

$$M_1\cdot dM_2=M_1E_2dA+dA^\dagger\rho(M_1)F_2+M_1G_2dA+M_1H_2dt \tag{5.7}$$

and dM_1dM_2 *is evaluated by bilinear extension of the multiplication rules*

	$d\Lambda$	$dA^\dagger$	dA	dt
$d\Lambda$	$d\Lambda$	$dA^\dagger$	0	0
dA	dA	dt	0	0
$dA^\dagger$	0	0	0	0
dt	0	0	0	0

Proof. The structure of $\mathfrak{M}$ ensures that the right hand sides of (5.6) and (5.7) are well defined. The result follows from putting $M=M_1^\dagger$, $M'=M_2$ in (4.10) and then applying (4.7). □

§ 6. Stochastic Evolutions

Let $L_j \in \mathscr{L}^2_{loc}$ $(j=1,2,3,4)$ be such that each $L_j(t)$ is the ampliation in $\mathscr{H}$ of a bounded operator $\tilde{L}_j(t)$ in $\mathfrak{h}_0$ such that $\{\tilde{L}_j(t)u, t\in R^+\}$ is bounded for each $u\in\mathfrak{h}_0$.

Hence, by the principle of uniform boundedness, there exists $C_j>0$ such that

$$||\tilde{L}_j(t)|| \leq C_j \text{ for each } t\in R^+ \ (j=1,2,3,4).$$

Let $C=\max\{C_j, j=1,2,3,4\}$.

Our aim in this section is to establish the existence of a unique solution to the stochastic differential equation

$$\begin{aligned} &dU=U(L_1d\Lambda+dA^\dagger L_2+L_3dA+L_4dt)\\ &U(0)=I \end{aligned} \tag{6.1}$$

and to establish conditions on the L_j's under which each $U(t)$ is a unitary operator in $\mathscr{H}$.

We establish the existence of a process $U=(U(t), t\in R^+)$ satisfying (6.1) as the limit of the sequence of stochastic integrals defined inductively by

$$U_0(t)=I$$

$$U_p(t)=I+\int_0^t [U_{p-1}L_1d\Lambda+dA^\dagger\rho(U_{p-1})L_2+U_{p-1}L_3dA+U_{p-1}L_4]dt \tag{6.2}$$

for all $t \in \boldsymbol{R}^+$.

U_p is well defined for all p, by the same arguments as used in [2], with adjoint process given by

$$U_0^\dagger(t) = I$$
$$U_p^\dagger(t) = I + \int_0^t [L_1^\dagger U_{p-1}^\dagger dA + dA^\dagger L_3^\dagger U_{p-1}^\dagger + L_2^\dagger \rho(U_{p-1}^\dagger) dA + L_4^\dagger U_{p-1}^\dagger] dt\,. \tag{6.3}$$

The following estimate is crude, but as we shall see, effective.

Theorem 6.1. *For arbitrary* $p > 0, u \in \mathfrak{h}_0, f \in \times_{j=1}^{n} \mathcal{S}, n \in N$ *and* $t > 0$ *we have*

$$\begin{aligned}&||(U_p(t) - U_{p-1}(t))u \otimes \psi_n(f)||^2 \\ &\leq \frac{1}{p!} \exp\{t + \int_0^t \sum_{j=1}^n |f_j(\tau)|^2 d\tau\} (t+n)^p C^{2p} 6^{n+p} (n+1)^{2n} \\ &\quad ||u||^2 \exp \sum_{j=1}^n ||f_j||^2 .\end{aligned} \tag{6.4}$$

Proof. This follows very closely the argument of [2], whence we argue by induction on n and on p. By (5.1) we have

$$\begin{aligned}&\frac{d}{dt} ||(U_p(t) - U_{p-1}(t))u \otimes \psi_n(f)||^2 \\ &\leq \sum_{j=1}^n \{|f_j(t)|^2 [2n ||(U_p(t) - U_{p-1}(t))u \otimes \psi_{n-1}(f^j)||^2 \\ &+ 3n ||(U_{p-1}(t) - U_{p-2}(t)) L_1(t) u \otimes \psi_{n-1}(f^j)||^2 \\ &+ ||(U_p(t) - U_{p-1}(t))u \otimes \psi_n(f)||^2] \\ &+ ||(U_{p-1}(t) - U_{p-2}(t)) L_3(t) \theta_0 u \otimes \psi_{n-1}(f^j)||^2\} \\ &+ 3 ||\rho(U_{p-1}(t) - U_{p-2}(t)) L_2(t) u \otimes \psi_n(f)||^2 \\ &+ ||(U_p(t) - U_{p-1}(t))u \otimes \psi_n(f)||^2 \\ &+ ||(U_{p-1}(t) - U_{p-2}(t)) L_4(t) u \otimes \psi_n(f)||^2 .\end{aligned}$$

Using the integrating factor $\exp\{-t - \int_0^t \sum_{j=1}^n |f_j(\tau)|^2 d\tau\}$, and the unitarity of θ we obtain upon integration

$$\begin{aligned}&||(U_p(t) - U_{p-1}(t))u \otimes \psi_n(f)||^2 \\ &\leq \int_0^t \exp\{t - \tau + \int_\tau^t \sum_{j=1}^n |f_j(s)|^2 ds\}\end{aligned}$$

$$\times\Big[\sum_{j=1}^{n}|f_j(\tau)|^2\{2n\|(U_p(\tau)-U_{p-1}(\tau))u\otimes\psi_{n-1}(f^j)\|^2$$
$$+3n\|(U_{p-1}(\tau)-U_{p-2}(\tau))L_1(\tau)u\otimes\psi_{n-1}(f^j)\|^2\}$$
$$+\sum_{j=1}^{n}\|(U_{p-1}(\tau)-U_{p-2}(\tau))L_3(\tau)\theta_0 u\otimes\psi_{n-1}(f^j)\|^2$$
$$+3\|(U_{p-1}(\tau)-U_{p-2}(\tau))\theta L_2(\tau)u\otimes\psi_n(f)\|^2$$
$$+\|(U_{p-1}(\tau)-U_{p-2}(\tau))L_4(\tau)u\otimes\psi_n(f)\|^2\Big]d\tau\,. \qquad (6.5)$$

When $n=0$, we obtain from (6.5), for all $p>0$

$$\|(U_p(t)-U_{p-1}(t))u\otimes\psi_0\|^2$$
$$\leq\int_0^t e^{t-\tau}\{3\|(U_{p-1}(\tau)-U_{p-2}(\tau))\theta L_2(\tau)u\otimes\psi_0\|^2$$
$$+\|(U_{p-1}(\tau)-U_{p-2}(\tau))L_4(\tau)u\otimes\psi_0\|^2\}\,d\tau$$
$$\leq\frac{1}{p!}e^t t^p 6^p C^{2p}\|u\|^2$$

by iteration.

Now, making the inductive hypothesis that

$$\|(U_p(t)-U_{p-1}(t))u\otimes\psi_{n-1}(f^j)\|^2$$
$$\leq\frac{1}{p!}\exp\{t+\int_0^t\sum_{k=1}^{n-1}{}^{(j)}|f_k(\tau)|^2d\tau\}\,(t+n-1)^pC^{2p}6^{n+p-1}$$
$$\times n^{2n-2}\|u\|^2\exp\sum_{k=1}^{n-1}{}^{(j)}\|f_k\|^2$$

we find that the first term on the right of (6.5) is bounded above by

$$2n\frac{1}{p!}6^{n+p-1}C^{2p}n^{2n-2}\sum_{j=1}^{n}\exp(t+\int_0^t\sum_{k=1}^{n-1}{}^{(j)}|f_k(s)|^2ds\}$$
$$\times\int_0^t\exp(\int_\tau^t|f_j(s)|^2ds)\,|f_j(\tau)|^2(\tau+n-1)^pd\tau\|u\|^2\exp\sum_{k=1}^{n-1}{}^{(j)}\|f_k\|^2. \qquad (6.6)$$

Now, integrating by parts

$$\int_0^t\exp(\int_\tau^t|f_j(s)|^2ds)\,|f_j(\tau)|^2(\tau+n-1)^pd\tau$$
$$=\int_0^t\exp(\int_\tau^t|f_j(s)|^2ds)\,p(\tau+n-1)^{p-1}d\tau+\exp(\int_0^t|f_j(s)|^2ds)\,(n-1)^p$$
$$\leq\exp(\int_0^t|f_j(s)|^2ds)\,(t+n-1)^p$$

whence (6.6) is bounded above by

$$2n^2 \exp\{t+\int_0^t \sum_{j=1}^n |f_j(s)|^2 ds\} \frac{1}{p!} 6^{n+p-1} C^{2p}$$
$$\times n^{2n-2}(t+n-1)^p ||u||^2 \exp \sum_{j=1}^n ||f_j||^2 . \tag{6.7}$$

Similarly the second term on the right hand side of (6.5) is bounded above by

$$3n \frac{1}{(p-1)!} C^{2p-2} 6^{n+p-2} n^{2n-2} \sum_{j=1}^n \exp\{t+\int_0^t \sum_{k=1}^{n-1}{}^{(j)} |f_k(s)|^2 ds\}$$
$$\times \int_0^t \exp(\int_\tau^t |f_j(s)|^2 ds) |f_j(\tau)|^2 (\tau+n-1)^{p-1} ||L_1(\tau) u||^2 d\tau$$
$$\times \exp \sum_{k=1}^{n-1}{}^{(j)} ||f_k||^2$$
$$\leq 3n^2 \exp\{t+\int_0^t \sum_{j=1}^n |f_j(s)|^2 ds\} \frac{1}{(p-1)!} 6^{n+p-2} C^{2p} n^{2n-2}$$
$$\times (t+n-1)^{p-1} ||u||^2 \exp \sum_{j=1}^n ||f_j||^2 \tag{6.8}$$

and the third term on the right hand side of (6.5) is bounded above by

$$\frac{1}{(p-1)!} C^{2p-2} 6^{n+p-2} n^{2n-2} \sum_{j=1}^n \exp\{t+\int_0^t \sum_{k=1}^{n-1}{}^{(j)} |f_k(s)|^2 ds\}$$
$$\times \int_0^t \exp(\int_\tau^t |f_j(s)|^2 ds) (\tau+n-1)^{p-1} ||L_3(\tau) \theta_0 u||^2 d\tau \exp \sum_{k=1}^{n-1}{}^{(j)} ||f_k||^2$$
$$\leq n \exp\{t+\int_0^t \sum_{j=1}^n |f_j(s)|^2 ds\} \frac{1}{p!} 6^{n+p-2} C^{2p} n^{2n-2}$$
$$\times (t+n-1)^p ||u||^2 \exp \sum_{j=1}^n ||f_j||^2 . \tag{6.9}$$

Adding together (6.7), (6.8) and (6.9) we find that the first three terms on the right hand side of (6.5) are bounded above by

$$\exp\{t+\int_0^t \sum_{j=1}^n |f_j(s)|^2 ds\} 6^{n+p-2} C^{2p} n^{2n-2}$$
$$\times \left\{12n^2 \frac{(t+n-1)^p}{p!} + n \frac{(t+n-1)^p}{p!}\right.$$
$$\left. + 3n^2 \frac{(t+n-1)^{p-1}}{(p-1)!}\right\} ||u||^2 \exp \sum_{j=1}^n ||f_j||^2$$
$$\leq \exp\{t+\int_0^t \sum_{j=1}^n |f_j(s)|^2 ds\} 6^{n+p-2} C^{2p} (12n^2+n)$$
$$\times n^{2n-2} \frac{1}{p!} (t+n)^p ||u||^2 \exp \sum_{j=1}^n ||f_j||^2 \tag{6.10}$$

where we have used the binomial theorem to deduce that

$$\frac{1}{p!}[(t+n-1)^{p}+p(t+n-1)^{p-1}]\leq\frac{1}{p!}(t+n)^{p}.$$

So substituting this into (6.5) we have

$$\begin{aligned}&||(U_p(t)-U_p(t)u\otimes\psi_n(f)||\\&\leq\exp\{t+\int_0^t\sum_{j=1}^n|f_j(s)|^2ds\}\,6^{n+p-2}C^{2p}(12n^2+n)n^{2n-2}\\&\times\frac{1}{p!}(t+n)^p||u||^2\exp\sum_{j=1}^n||f_j||^2+\int_0^t\exp\{t-\tau\\&+\int_\tau^t\sum_{j=1}^n|f_j(s)|^2ds\}[3||(U_{p-1}(\tau)-U_{p-2}(\tau))\theta L_2(\tau)u\otimes\psi_n(f)||^2\\&+||(U_{p-1}(\tau)-U_{p-2}(\tau))L_4(\tau)u\otimes\psi_n(f)||^2]d\tau\ .\end{aligned}\tag{6.11}$$

We proceed now to establish our result by induction on p.

When $p=1$, for all $n>0$ we have

$$\begin{aligned}&||(U_1(t)-I)u\otimes\psi_n(f)||^2\\&=||\int_0^t(L_1dA+dA^{\dagger}L_2+L_3dA+L_4dt)u\otimes\psi_n(f)||^2\\&\qquad\leq4C^2(||A_tu\otimes\psi_n(f)||^2+||A_t^{\dagger}u\otimes\psi_n(f)||^2\\&\qquad+||A_tu\otimes\psi_n(f)||^2+||tu\otimes\psi_n(f)||^2)\\&\qquad\leq4C^2(n^2t+2t+t^2)||u||^2\prod_{j=1}^n||f_j||^2\\&\qquad\leq\exp\{t+\int_0^t\sum_{j=1}^n|f_j(s)|^2ds\}\,(n+1)^{2n}(t+n)C^26^n\\&\qquad\times||u||^2\exp\sum_{j=1}^n||f_j||^2\end{aligned}$$

as required, where we have used (3.4), (1.4) and (1.7).

Now making the inductive hypothesis in (6.11) that

$$\begin{aligned}&||(U_{p-1}(t)-U_{p-2}(t))u\otimes\psi_n(f)||^2\\&\leq\frac{1}{(p-1)!}\exp\{t+\int_0^t\sum_{j=1}^n|f_j(\tau)|^2d\tau\}\,(t+n)^{p-1}C^{2p-2}\\&\times6^{n+p-1}(n+1)^{2n}||u||^2\exp\sum_{j=1}^n||f_j||^2\end{aligned}$$

our result follows from the observation that

$$\begin{aligned}&6^{n+p-2}(12n^2+n)n^{2n-2}+4.\,6^{n+p-1}(n+1)^{2n}\\&\leq6^{n+p-2}(n+1)^{2n-2}[12n^2+n+24(n+1)^2]\\&\leq6^{n+p}(n+1)^{2n}.\end{aligned}$$

A similar argument establishes (6. 4) for the case where $U_p(t)$ is replaced by $U_p^\dagger(t)$ $(p>0)$. □

From (6. 4) we obtain

$$\sum_{p=1}^{\infty}\|(U_p^\sharp(t)-U_{p-1}^\sharp(t))u\otimes\psi_n(f)\|^2<\infty$$

whence

$$U(t)u\otimes\psi_n(f)=u\otimes\psi_n(f)+\sum_{p=1}^{\infty}(U_p(t)-U_{p-1}(t))u\otimes\psi_n(f) \tag{6. 12}$$

exists and defines an adapted, square integrable process.

To see that the limit indeed satisfies (6. 1), we use (6. 10), the Schwartz inequality and the uniformity of the convergence in (6. 12) on finite intervals of $\boldsymbol{R}^+$ to show that, for each $t\in\boldsymbol{R}^+$.

$$\begin{aligned}\lim_{p\to\infty}\|(\int_0^t[(U-U_{p-1})L_1dA+dA^\dagger\rho(U-U_{p-1})L_2\\ +(U-U_{p-1})L_3dA+(U-U_{p-1})L_4dt])u\otimes\psi_n(f)\|\\ =0.\end{aligned}$$

Theorem 6. 2. *The solution* U *of* (6. 1) *is unique.*

The proof is similar enough to the proof of theorem 6. 2 of [2] to make repetition unnecessary here. Note however that our more general estimate (5. 1) frees us from the requirement of [2] that U be of definite parity.

We will now investigate the conditions under which U defines a unitary process.

Let us, first of all, assume that U is indeed unitary whence it is bounded and U, $UL_j(j=1,2,3,4)$ are uniformly bounded on finite intervals, whence by Theorem 5. 2,

$$\begin{aligned}0&=d(U^\dagger U)\\ &=dU^\dagger.U+U^\dagger.dU+dU^\dagger dU\\ &=L_1^\dagger dA+L_2^\dagger dA+dA^\dagger L_3^\dagger+L_4^\dagger dt\\ &\quad+L_1dA+dA^\dagger L_2+L_3dA+L_4dt\\ &\quad+L_1^\dagger L_1dA+L_2^\dagger\rho_0(L_1)dA+dA^\dagger\rho_0(L_1^\dagger)L_2\\ &\quad+L_2^\dagger L_2dt\end{aligned}$$

where we have used the relation $U^\dagger U=I$ and (1. 10) together with the odd and even parities of dA and $d\Lambda$ respectively.

Using the independence of the stochastic differentials established in Theorem 5.1 we find the conditions

$$\begin{aligned} &L_1+L_1^\dagger+L_1^\dagger L_1=0\\ &L_3+L_2^\dagger+L_2^\dagger\rho_0(L_1)=0\\ &L_4+L_4^\dagger+L_2^\dagger L_2=0 \end{aligned} \tag{6.13}$$

whence we may take

$$\begin{aligned} &L_1=W-I\\ &L_2=-\rho_0(W)L^\dagger\\ &L_3=L\\ &L_4=iH-\frac{1}{2}LL^\dagger \end{aligned} \tag{6.14}$$

where L, W and H are arbitrary, unitary and self-adjoint processes, respectively.

Theorem 6.3. *A necessary and sufficient condition for the process U to be unitary is that (L_1, L_2, L_3, L_4) be of the form (6.14).*

Proof. We need to show the conditions (6.14) are sufficient.

Let $u, v\in\mathfrak{h}_0, f\in \times_{j=1}^{n}\mathscr{S}, g\in\times_{k=1}^{m}\mathscr{S}, n, m\in N$. From (4.10) we have

$$\frac{d}{dt}\langle U^\dagger(t)u\otimes\psi_n(f), U^\dagger(t)v\otimes\psi_m(g)\rangle=0$$

which together with the initial condition $U^\dagger(0)=I$ ensures that $U^\dagger$ is isometric.

To show that U is an isometry, we again use (4.10) and after some simplification, we obtain

$$\begin{aligned} &\frac{d}{dt}\langle U(t)u\otimes\psi_n(f), U(t)v\otimes\psi_m(g)\rangle\\ &=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}[\langle U(t)W(t)u\otimes\psi_{n-1}(f^j), U(t)W(t)v\\ &\quad\otimes\psi_{m-1}(g^k)\rangle-\langle U(t)u\otimes\psi_{n-1}(f^j), U(t)v\otimes\psi_{m-1}(g^k)\rangle]g_k(t)\\ &\quad+\sum_{j=1}^{n}(-1)^{n-j}\overline{f_j(t)}[\langle U(t)L(t)\theta_0u\otimes\psi_{n-1}(f^j), U(t)v\otimes\psi_m(g)\rangle\\ &\quad-\langle\rho(U(t))\rho_0(W(t))\theta_0u\otimes\psi_{n-1}(f^j), \rho(U(t))\rho_0(W(t))L^\dagger(t)\\ &\qquad v\otimes\psi_m(g)\rangle] \end{aligned}$$

$$+\sum_{k=1}^{m}(-1)^{m-k}[\langle U(t)u\otimes\psi_n(f),U(t)L(t)\theta_0 v\otimes\psi_{m-1}(g^k)\rangle$$
$$-\langle\rho(U(t))\rho_0(W(t))L^\dagger(t)u\otimes\psi_n(f),\rho(U(t))\rho_0(W(t))\theta_0 v\otimes\psi_{m-1}(g^k)\rangle]g_k(t)$$
$$+\langle U(t)(iH(t)-\frac{1}{2}L(t)L^\dagger(t))u\otimes\psi_n(f),U(t)v\otimes\psi_m(g)\rangle$$
$$+\langle U(t)u\otimes\psi_n(f),U(t)(iH(t)-\frac{1}{2}L(t)L^\dagger(t))v\otimes\psi_m(g)\rangle$$
$$+\langle\rho(U(t))\rho_0(W(t))L^\dagger(t)u\otimes\psi_n(f),\rho(U(t))\rho_0(W(t))L^\dagger(t)v\otimes\psi_m(g)\rangle.$$

We define bounded operators $K_{m,n}(f,g;t)$ on $\mathfrak{h}_0$ by the prescription

$$\langle u,K_{m,n}(f,g;t)v\rangle=\langle U(t)u\otimes\psi_n(f),U(t)v\otimes\psi_m(g)\rangle$$

and using the unitarity of θ and (1.11) we see that these satisfy the (weak sense) ordinary differential equations

$$\frac{d}{dt}K_{m,n}(f,g;t)$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}[W^\dagger(t)K_{m-1,n-1}(f^j,g^k;t)W(t)-K_{m-1,n-1}(f^j,g^k;t)]g_k(t)$$
$$+\sum_{j=1}^{n}(-1)^j\overline{f_j(t)}[(-1)^nK_{m,n-1}(f^j,g;t)L(t)+(-1)^mL(t)W^\dagger(t)K_{m,n-1}(f^j,g;t)W(t)]$$
$$+\sum_{k=1}^{m}(-1)^k[(-1)^mL^\dagger(t)K_{m-1,n}(f,g^k;t)+(-1)^nW^\dagger(t)K_{m-1,n}(f,g^k;t)W(t)L^\dagger(t)]g_k(t)$$
$$+i[K_{m,n}(f,g;t),H(t)]-\frac{1}{2}\{L(t)L^\dagger(t),K_{m,n}(f,g;t)\}$$
$$+(-1)^{m+n}L(t)\theta_0W^\dagger(t)K_{m,n}(f,g;t)W(t)\theta_0L^\dagger(t) \tag{6.15}$$

together with the initial value

$$K_{m,n}(f,g;0)=\langle\psi_n(f),\psi_m(g)\rangle I=\delta_{m,n}D_n(f,g)I$$

where $D_n(f,g)$ is the determinant of the $n\times n$ matrix whose (i,j)th entry is $\langle f_i,g_j\rangle$ (which follows from $U(0)=I$).

We prove by induction on $N=n+m$ that $K_{m,n}(f,g;t)=\delta_{m,n}D_n(f,g)I$.

When $N=0, n=m=0$ and all except the last three terms in (6.15) vanish. Since θ and $W^\dagger(t)$ are unitary, it is easy to check that

$K_{0,0}(t)=K_{0,0}(0)=I$ is the unique solution to the given initial value problem.

We now consider the case $N=1$ and take $n=0$ and $m=1$. Here (6.15) reduces to

$$\frac{d}{dt}K_{1,0}(0,g;t)=g(t)\,(L^{\dagger}(t)K_{0,0}(t)-W^{\dagger}(t)K_{0,0}(t)W(t)L^{\dagger}(t))$$
$$+i[K_{1,0}(0,g;t),H(t)]-\frac{1}{2}\{L(t)L^{\dagger}(t),K_{1,0}(0,g;t)\}$$
$$-L(t)\theta_0W^{\dagger}(t)K_{1,0}(0,g;t)W^{\dagger}(t)\theta_0L^{\dagger}(t)$$

and using $K_{0,0}(t)=I$, we see that the unique solution to the given initial value problem is $K_{1,0}(t)=K_{1,0}(0)=\delta_{1,0}I=0$.

A similar argument holds for the case $n=1$ and $m=0$.

We now make the inductive hypothesis for $N-2$ and $N-1$ so that in particular we require that

$$K_{m-1,n-1}(f^j,g^k;t)=\delta_{m-1,n-1}D_{n-1}(f^j,g^k)I$$
$$K_{m,n-1}(f^j,g;t)=\delta_{m,n-1}D_{n-1}(f^j,g)I$$

and

$$K_{m-1,n}(f,g^k;t)=\delta_{m-1,n}D_{n-1}(f,g^k)I\ .$$

It is necessary to make the inductive step in the three cases $m=n, m=n-1$ and $m-1=n$.

When $m=n$, (6.15) yields

$$\frac{d}{dt}K_{n,n}(f,g;t)$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}(-1)^{j+k}\overline{f_j(t)}\,[W^{\dagger}(t)D_{n-1}(f^j,g^k)W(t)-D_{n-1}(f^j,g^k)I]g_k(t)$$
$$+i[K_{n,n}(f,g;t),H(t)]-\frac{1}{2}\{L(t)L^{\dagger}(t),K_{n,n}(f,g;t)\}$$
$$+L(t)\theta_0W^{\dagger}(t)K_{n,n}(f,g;t)W(t)\theta_0L^{\dagger}(t).$$

The first term vanishes by unitarity of $W(t)$ and $K_{n,n}(f,g;t)=D_n(f,g)I$ is the unique solution of the resulting initial value problem by the same reasoning as used in the case $N=0$.

Similarly, the cases when $m=n-1$ and $m-1=n$ follow by the same argument as that used for $N=1$.

Whence $U(t)$ is isometric, as required. □

§ 7. Applications to the Construction of Dilations

Throughout this section we will take U to be the unitary solution of the stochastic differential equation

$$dU=U((W-I)d\Lambda-dA^{\dagger}\rho_0(W)L^{\dagger}+LdA+\left(iH-\frac{1}{2}LL^{\dagger}\right)dt) \tag{7.1}$$

with L, H and W constant processes in $\boldsymbol{B}(\mathfrak{h}_0)$. Since U is a stochastic integral we have $U\in\mathscr{A}_c$, furthermore it is unitary, therefore bounded whence the map $t\to U(t)$ is strongly continuous from $\boldsymbol{R}^+$ to $\boldsymbol{B}(\mathscr{H})$.

For T a contraction on $\boldsymbol{L}^2(\boldsymbol{R})$, its second quantisation $\Gamma(T)$ on $\Gamma(\boldsymbol{L}^2(\boldsymbol{R}))$ is defined by

$$\begin{aligned}&\Gamma(T)\psi_0=\psi_0\\&\Gamma(T)\psi_n(f)=\psi_n(Tf)\\&\qquad\text{for } f\in\mathop{\times}_{j=1}^{n}L^2(\boldsymbol{R}),\ n\in\boldsymbol{N}.\end{aligned} \tag{7.2}$$

In particular, $\Gamma(T)$ is itself a contraction and we have $\Gamma(T^{\dagger})=\Gamma(T)^{\dagger}$ and $\Gamma(TV)=\Gamma(T)\Gamma(V)$ where V is another contraction on $L^2(\boldsymbol{R})$.

Let $\{S_t, t\in\boldsymbol{R}\}$ be the strongly continuous unitary group of shift operators on $\boldsymbol{L}^2(\boldsymbol{R})$ where for each $f\in\boldsymbol{L}^2(\boldsymbol{R})$

$$(S_tf)(s)=f(t-s)\quad s,t\in\boldsymbol{R}. \tag{7.3}$$

It is not difficult to verify that $\{\Gamma(S_t), t\in\boldsymbol{R}\}$ is a strongly continuous unitary group on $\Gamma(\boldsymbol{L}^2(\boldsymbol{R}))$.

Let $\hat{\mathscr{H}}=\mathfrak{h}_0\otimes\Gamma(\boldsymbol{L}^2(\boldsymbol{R}))$. We write $\{\gamma(S_t), t\in\boldsymbol{R}^+\}$ for the strongly continuous unitary group on $\hat{\mathscr{H}}$ given by

$$\gamma(S_t)=I\hat{\otimes}\Gamma(S_t),\quad t\in\boldsymbol{R}.$$

We will use the same notation to denote these operators acting on the subspace $\mathscr{H}$ of $\hat{\mathscr{H}}$ (where they are still a semigroup).

Theorem 7.1 (*c.f. Theorem* 7.1 *of* [11]). *For arbitrary* $s, t\in\boldsymbol{R}^+$

$$U(t)=\gamma(S_s^{\dagger})U^{\dagger}(s)U(s+t)\gamma(S_s). \tag{7.4}$$

Proof. For $t\in\boldsymbol{R}^+$ write

$$V(t)=\gamma(S_s^\dagger)U^\dagger(s)U(s+t)\gamma(S_s).$$

By analogous arguments to those used in [11] we see that $V=(V(t), t\in\mathbb{R}^+)$ is adapted and it inherits continuity from U. Thus we may consider the stochastic differential equation

$$dM=V\Big((W-I)dA-dA^\dagger\rho_0(W)L^\dagger+LdA+\Big(iH-\frac{1}{2}L^\dagger L\Big)dt\Big).$$

By Theorem 4.1, for each $u,v\in\mathfrak{h}_0, f\in\times_{j=1}^{n}\mathcal{S}, g\in\times_{k=1}^{m}\mathcal{S}, t\in\mathbb{R}^+$ we have

$$\begin{aligned}
&\langle u\otimes\psi_n(f),(M(t)-M(0))v\otimes\psi_m(g)\rangle\\
&=\int_0^t\{\sum_{j=1}^n\sum_{k=1}^m(-1)^{j+k}\overline{f_j(\tau)}\langle u\otimes\psi_{n-1}(f^j),V(\tau)(W-1)\\
&\qquad\qquad v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)\\
&+\sum_{j=1}^n(-1)^{n-j}\langle\overline{f_j(\tau)}<\theta_0u\otimes\psi_{n-1}(f^j),\rho(V(\tau))\rho_0(W)Lv\otimes\psi_m(g)\rangle\\
&+\sum_{k=1}^m(-1)^{m-k}\langle u\otimes\psi_n(f),V(\tau)L\theta_0v\otimes\psi_{m-1}(g^k)\rangle g_k(\tau)\\
&+\langle u\otimes\psi_n(f),V(\tau)\Big(iH-\frac{1}{2}LL^\dagger\Big)v\otimes\psi_m(g)\rangle\}\,d\tau\\
&=\int_0^t\{\sum_{j=1}^n\sum_{k=1}^m(-1)^{j+k}\overline{f_j(\tau)}\langle u\otimes\psi_{n-1}(S_sf^j),U^\dagger(s)U(s+\tau)(W-I)\\
&\qquad\qquad v\otimes\psi_{m-1}(S_sg^k)\rangle g_k(\tau)\\
&+\sum_{j=1}^n(-1)^{n-j}\overline{f_j(\tau)}\langle\theta_0u\otimes\psi_{n-1}(S_sf^j),\rho(U^\dagger(s))\rho(U(s+\tau))\rho_0(W)L^\dagger\\
&\qquad\qquad v\otimes\psi_m(S_sg)\rangle\\
&+\sum_{k=1}^m(-1)^{m-k}\langle u\otimes\psi_n(S_sf),U^\dagger(s)U(s+\tau)L\theta_0v\otimes\psi_{m-1}(S_sg^k)\rangle g_k(\tau)\\
&+\langle u\otimes\psi_n(S_sf),U^\dagger(s)U(s+\tau)\Big(iH-\frac{1}{2}LL^\dagger\Big)v\otimes\psi_m(S_sg)\rangle\}\,d\tau
\end{aligned}\tag{7.5}$$

where we have used (7.2) and the even parity of $\Gamma(S_s)$.

Replacing each $f_j(\tau)$ with $S_sf_j(\tau+s)$ $(i\le j\le n)$, each $g_k(\tau)$ with $S_sg_k(\tau+s)$ $(1\le k\le m)$ and making the substitution $\tau\to\tau-s$ in (7.5) we obtain

$$\begin{aligned}
&\langle u\otimes\psi_n(f),(M(t)-M(0))v\otimes\psi_m(g)\rangle\\
&=\int_s^{t+s}\{\sum_{j=1}^n\sum_{k=1}^m(-1)^{j+k}\overline{S_sf_j(\tau)}\langle u\otimes\psi_{n-1}(S_sf^j),U^\dagger(s)U(\tau)(W-I)\\
&\qquad\qquad v\otimes\psi_{m-1}(S_sg^k)\rangle S_sg_k(\tau)
\end{aligned}$$

$$+\sum_{j=1}^{n}(-1)^{n-j}\overline{S_s f_j(\tau)}\langle\theta_0 u\otimes\phi_{n-1}(S_s f^j),\,\rho(U^\dagger(s))\rho(U(\tau))\rho_0(W)L^\dagger$$
$$v\otimes\phi_m(S_s g)\rangle$$
$$+\sum_{k=1}^{m}(-1)^{m-k}\langle u\otimes\phi_n(S_s f),\,U^\dagger(s)U(\tau)L\theta_0 v\otimes\phi_{m-1}(S_s g^k)\rangle S_s g_k(\tau)$$
$$+\langle u\otimes\phi_n(S_s f),\,U^\dagger(s)U(\tau)\left(iH-\frac{1}{2}LL^\dagger\right)v\otimes\phi_m(S_s g)\rangle\}\,d\tau$$
$$=\langle u\otimes\phi_n(S_s f),\,(U^\dagger(s)U(s+t)-I)v\otimes\phi_m(S_s g)\rangle$$
$$=\langle\gamma(S_s)u\otimes\phi_n(f),\,(U^\dagger(s)U(s+t)-I)\gamma(S_s)v\otimes\phi_m(g)\rangle$$
$$=\langle u\otimes\phi_n(f),\,(V(t)-I)v\otimes\phi_m(g)\rangle$$

where we have used Theorem 4.1 and the isometry of $\gamma(S_s)$. Whence $M(t)=V(t)$ and so V satisfies (6.1) from which we conclude by Theorem 6.2 that $V=U$, as required. □

We define a family of operators on $\mathscr{H}$, $\{\hat{P}_t, t\in\boldsymbol{R}\}$ by the prescription

$$\hat{P}_t=\begin{cases}U(t)\gamma(S_t) & \text{if } t\geq 0\\ \gamma(S_t)U^\dagger(-t) & \text{if } t<0\,.\end{cases}\tag{7.6}$$

Corollary 7.2. *$\{\hat{P}_t, t\in\boldsymbol{R}\}$ is a strongly continuous unitary group on $\mathscr{H}$* (c.f. [8], [16]).

Proof. The group property is a trivial consequence of the cocycle condition (7.4). Strong continuity follows by an $\varepsilon/2$ argument from the strong continuity of the maps $t\to\gamma(S_t)$ and $t\to U_t$. □

Let $\{P_t, t\in\boldsymbol{R}^+\}$ denote the uniformly continuous contraction semi-group on $\mathfrak{h}_0$ with infinitesimal generator $iH-\frac{1}{2}LL^\dagger$ and let r denote the injective isometry from $\mathfrak{h}_0$ into $\mathscr{H}$ given by

$$r(u)=u\otimes\phi_0\quad\text{for each } u\in\mathfrak{h}_0.$$

Theorem 7.3. *$(\mathscr{H}, \{\hat{P}_t, t\in\boldsymbol{R}\})$ is a dilation of $(\mathfrak{h}_0, \{P_t, t\in\boldsymbol{R}^+\})$ in the sense that the following diagram commutes for all $t\in\boldsymbol{R}^+$*

$$\begin{array}{ccc}\mathfrak{h}_0 & \xrightarrow{P_t} & \mathfrak{h}_0\\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r^\dagger}\\ \mathscr{H} & \xrightarrow{\hat{P}_t} & \mathscr{H}\end{array}$$

(c.f. [13], [18]).

The proof imitates that of theorem 7.1 of [2].

Let $\hat{T}_t$ denote the strongly continuous group of automorphisms of $\boldsymbol{B}(\mathscr{H})$ defined by

$$\hat{T}_t(X) = \hat{P}_t X \hat{P}_t^\dagger \quad \text{for} \quad X \in \boldsymbol{B}(\mathscr{H}).$$

In particular let $X \in \boldsymbol{B}(\mathfrak{h}_0)$ and j be the canonical injection of $\boldsymbol{B}(\mathfrak{h}_0)$ into $\boldsymbol{B}(\mathscr{H})$ given by

$$j(X) = X \hat{\otimes} I .$$

Let $\{T_t, t \in \boldsymbol{R}^+\}$ be the norm continuous, identity preserving, completely positive semigroup of operators on the Banach space $\boldsymbol{B}(\mathfrak{h}_0)$ with infinitesimal generator $\mathscr{L}$[14] given by

$$\mathscr{L}(X) = L\rho_0(X)L^\dagger - \frac{1}{2}\{LL^\dagger, X\} + i[H, X] .$$

Theorem 7.4. *$(\boldsymbol{B}(\mathscr{H}), j^{-1}\circ \boldsymbol{E}_0, \{T_t, t \in \boldsymbol{R}\})$ is a dilation of $(\boldsymbol{B}(\mathfrak{h}_0), \{T_t, t \in \boldsymbol{R}^+\})$ in the sense that the following diagram commutes for all $t \in \boldsymbol{R}^+$*

$$\begin{array}{ccc} \boldsymbol{B}(\mathfrak{h}_0) & \xrightarrow{\;T_t\;} & \boldsymbol{B}(\mathfrak{h}_0) \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle j^{-1}\circ E_0} \\ \boldsymbol{B}(\mathscr{H}) & \xrightarrow{\;\hat{T}_t\;} & \boldsymbol{B}(\mathscr{H}) \end{array}$$

where j^{-1} is the left inverse of j (c.f. [7], [13]).

The proof is again a slight variation on that of Theorem 7.1 in [2].

We remark that neither of the dilations constructed in Theorems 7.3 and 7.4 is unique since both of the semigroups are independent of the choice of coefficient $W-I$ of the gauge differential in (7.1). Furthermore these semigroups may also be dilated using the boson stochastic calculus of [10].

§ 8. Fermion Poisson Processes

Let $\mathfrak{A}$ be the $\boldsymbol{Z}_2$-graded vector space comprising stochastic integrals of the form

$$dM = F_1 dA + dA^\dagger F_2 + F_3 dA + F_4 dt$$

where, for each $t \in \boldsymbol{R}^+$, the restriction of $F_i(t)$ to $\mathfrak{h}_0$ is an even operator $(i=1,2,3,4)$.

We denote by η the automorphism of period 2 of $\mathfrak{A}$ given by

$$\eta(X)=X \quad \text{when } X \text{ is even}$$
$$\eta(X)=(\theta_0\hat{\otimes}I)X \quad \text{when } X \text{ is odd} .$$

(We remark that the restriction of η to the $\boldsymbol{Z}_2$-graded *-algebra $\mathscr{A}\cap\mathfrak{M}$ is a *-automorphism of period 2.)

In (7.1) take $L=\sqrt{l}(W-I)\theta_0$ and $H=-\frac{i}{2}l(W-W^\dagger)$, for $l\in\boldsymbol{R}^+$ so that we obtain

$$dU=U(W-I)d\eta(\Pi^l) \tag{8.1}$$

where $\Pi^l=(\Pi^l(t),t\in\boldsymbol{R}^+)$ is the solution of

$$d\Pi^l=d\Lambda+\sqrt{l}(dA+dA^\dagger)+ldt \quad \text{with } \Pi^l(0)=0 . \tag{8.2}$$

Clearly, by Theorem 5.2, $(d\Pi^l)^2=d\Pi^l$.

Furthermore substituting these values of L and H into (7.7) we find that (8.1) yields a cocycle for the dilation of the semigroup on $\boldsymbol{B}(\mathfrak{h}_0)$ with infinitesimal generator

$$\mathscr{L}(X)=l(WXW^\dagger-X) . \tag{8.3}$$

Equations (8.1) to (8.3) indicate a striking resemblance between the process Π^l and the realisation of the classical Poisson process in boson Fock space over $\boldsymbol{L}^2(\boldsymbol{R}^+)$ [10] whose application to the dilation of semigroups of the form (8.3) was discussed in [3].

Π^l is clearly not a classical Poisson process, for although it is self adjoint, it does not commute with itself at different times.

We investigate this process more closely in the spirit of [10] by taking $\mathfrak{h}_0=\boldsymbol{C}$ so that $\mathscr{H}=\boldsymbol{L}^2(\boldsymbol{R}^+)$ and $\theta_0=I$. Let α be a measurable, locally bounded function on $\boldsymbol{R}^+$.

By Theorem 6.3 we may take W in (8.1) to be the process given by

$W(t)=e^{i\alpha(t)}$, where we note that only in the case where α is constant may we dilate the semigroup (8.3).

Now applying Theorem 4.1 in (8.1) with $n=m=0$ we obtain

$$\langle\psi_0, (U(t)-I)\psi_0\rangle = l\int_0^t U(\tau)\,(e^{i\alpha(\tau)}-1)\,d\tau$$

which upon iteration yields

$$\langle\psi_0, U(t)\psi_0\rangle = \exp\{l\int_0^t (e^{i\alpha(s)}-1)\,ds\}\ . \tag{8.4}$$

The right hand side of (8.4) is recognizable as the expectation of $\exp(i\int_0^t \alpha dX_l)$ where X_l is a classical Poisson process of intensity l (c.f.[10]). We feel that this justifies our calling Π^l a *fermion Poisson process of intensity l.*

Remark that the integrated form of (8.2) yields the following "central limit theorem"

$$l^{-\frac{1}{2}}(\Pi_l(t)-lt) = \psi(t) + l^{-\frac{1}{2}}\Lambda(t) \tag{8.5}$$

where $\psi(t) = A_t + A_t^\dagger$ is the Clifford process of [5] which plays the role of an anticommuting analogue of the classical Brownian motion process.

Note

Since this paper was written, [18] has appeared in which the main results of §5 and §6 are obtained in a much more elegant and economical manner. Furthermore, in the light of these new results, it becomes immediately apparent why classical and fermionic Poisson processes share so many of the same properties.

Appendix

We give the proofs of the following formulae which were used to establish Theorem 4.2

(i) $$\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{k=1}^{m}(-1)^{n-j+\iota+k}\Big(\int_{t_r}^{t}\overline{f_j(\tau)}\,d\tau\Big)\overline{f_\iota(t)}\,g_k(t)$$
$$\times\langle G_r\theta_0 u\otimes\psi_{n-2}(f^{j\iota}),\,E_r'v\otimes\psi_{m-1}(g^k)\rangle$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}\,g_k(t)\langle G_r a(\chi_{[t_r,t)})u\otimes\psi_{n-1}(f^j),\,E_r'v\otimes\psi_{m-1}(g^k)\rangle$$

$$\text{(ii)}\quad -\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\iota=1}^{n-1}{}^{(j)}\sum_{p=1}^{m-1}{}^{(k)}(-1)^{j+k+\iota+p}\left(\int_{t_r}^{t}\overline{f_\iota(\tau)}g_k(\tau)\,d\tau\right)$$

$$\times\overline{f_j(t)}g_p(t)\langle E_r u\otimes\psi_{n-2}(f^{j\iota}),\,E'_r v\otimes\psi_{m-2}(g^{kp})\rangle$$

$$=\sum_{j=1}^{n}\sum_{k=1}^{m}(-1)^{j+k}\overline{f_j(t)}g_k(t)\langle E_r\lambda(\chi_{[t_r,t)})u\otimes\psi_{n-1}(f^j),$$

$$E'_r v\otimes\psi_{m-1}(g^k)\rangle.$$

Proof of (i). It is sufficient to establish the following for $n\geq 2$

$$\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}(-1)^{n+j-\iota}\left(\int_{t_r}^{t}f_j(\tau)\,d\tau\right)f_\iota(t)\psi_{n-2}(f^{j\iota})$$

$$=\sum_{j=1}^{n}(-1)^{j}f_j(t)\,a(\chi_{[t_r,t)})\psi_{n-1}(f^j). \tag{A. 1}$$

We prove (A. 1) by induction, noting that when $n=2$, the left hand side of (A. 1) becomes

$$\left(\int_{t_r}^{t}f_1(\tau)\,d\tau\right)f_2(t)\psi_0-\left(\int_{t_r}^{t}f_2(\tau)\,d\tau\right)f_1(t)\psi_0$$

$$=f_2(t)\,a(\chi_{[t_r,t)})a^\dagger(f_1)\psi_0-f_1(t)\,a(\chi_{[t_r,t)})\,a^\dagger(f_2)\psi_0$$

as required.

More generally we find that

$$\sum_{j=1}^{n+1}\sum_{\iota=1}^{n}{}^{(j)}(-1)^{n+1+j+\iota}\left(\int_{t_r}^{t}f_j(\tau)\,d\tau\right)f_\iota(t)\psi_{n-1}(f^{j\iota})$$

$$=\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}(-1)^{n-1-j+\iota}\left(\int_{t_r}^{t}f_j(\tau)\,d\tau\right)f_\iota(t)\psi_{n-1}(f^{j\iota})$$

$$+\sum_{\iota=1}^{n}(-1)^{\iota}\left(\int_{t_r}^{t}f_{n+1}(\tau)\,d\tau\right)f_\iota(t)\psi_{n-1}(f^{\iota})$$

$$+\sum_{j=1}^{n}(-1)^{j+1}\left(\int_{t_r}^{t}f_j(\tau)\,d\tau\right)f_{n+1}(t)\psi_{n-1}(f^{j}). \tag{A. 2}$$

Now

$$\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}(-1)^{n+1-j+\iota}\left(\int_{t_r}^{t}f_j(\tau)\,d\tau\right)f_\iota(t)\psi_{n-1}(f^{j\iota})$$

$$=a^\dagger(f_{n+1})\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}(-1)^{n+1-j+\iota}\left(\int f_j(\tau)\,d\tau\right)f_\iota(t)\psi_{n-2}(f^{j\iota})$$

$$=\sum_{j=1}^{n}(-1)^{j+1}f_j(t)\,a^\dagger(f_{n+1})\,a(\chi_{[t_r,t)})\psi_{n-1}(f^j)$$

$$=\sum_{j=1}^{n}(-1)^{j+1}f_j(t)\int_{t_r}^{t} f_{n+1}(\tau)\,d\tau\psi_{n-1}(f^j)$$

$$+\sum_{j=1}^{n}(-1)^{j}f_j(t)a(\chi_{[t_r,t)})\psi_n(f^j) \qquad (A.3)$$

by the inductive hypothesis and (1.3).

We also have by (1.8)

$$\sum_{j=1}^{n}(-1)^{j+1}\left(\int_{t_r}^{t} f_j(\tau)\,d\tau\right)f_{n+1}(t)\psi_{n-1}(f^j)$$

$$=(-1)^{n+1}f_{n+1}(t)a(\chi_{[0,t)})\psi_n(f^{n+1}) \qquad (A.4)$$

and the result follows upon substituting (A.3) and (A.4) back into (A.2).

Proof of (ii). By a slight generalization of the proof of (A.1) we obtain for each $1\leq k\leq m$

$$\sum_{j=1}^{n}\sum_{\iota=1}^{n-1}{}^{(j)}(-1)^{j+\iota}\left(\int_{t_r}^{t} f_\iota(\tau)g_k(\tau)\,d\tau\right)f_j(t)\psi_{n-2}(f^{j\iota})$$

$$=\sum_{j=1}^{n}(-1)^{n+j-1}f_j(t)a(\chi_{[t_r,t)}g_k)\psi_{n-1}(f^j). \qquad (A.5)$$

A straightforward inductive argument establishes the formula

$$\sum_{k=1}^{m}\sum_{p=1}^{m-1}{}^{(k)}(-1)^{k+p}g_p(t)a^{\dagger}(\chi_{[t_r,t)}g_k)\psi_{m-2}(g^{kp})$$

$$=-\sum_{k=1}^{m}\sum_{p=1}^{m-1}{}^{(k)}(-1)^{k+p}g_k(t)a^{\dagger}(\chi_{[t_r,t)}g_p)\psi_{m-2}(g^k) \qquad (A.6)$$

and the result follows from (A.5), (A.6), (1,11), (1.9) and (2.3).

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