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Twisted $SU(2)$ Group. An Example of a Non-Commutative Differential Calculus

By

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Abstract

For any number ν in the interval $[-1, 1]$ a C^*-algebra A generated by two elements α and γ satisfying simple (depending on ν) commutation relation is introduced and investigated.

If $\nu=1$ then the algebra coincides with the algebra of all continuous functions on the group $SU(2)$. Therefore one can introduce many notions related to the fact that $SU(2)$ is a Lie group. In particular one can speak about convolution products, Haar measure, differential structure, cotangent boundle, left invariant differential forms, Lie brackets, infinitesimal shifts and Cartan Maurer formulae. One can also consider representations of $SU(2)$.

For $\nu<1$ the algebra A is no longer commutative, however the notions listed above are meaningful. Therefore A can be considered as the algebra of all "*continuous functions*" on a "*pseudospace* $S_\nu U(2)$" and this pseudospace is endowed with a Lie group structure.

The potential applications to the quantum physics are indicated.

§ 0. Introduction

From the point of view of the theory of groups the passage from nonrelativistic to relativistic physics consists in replacing the group of Galilean transformations by the Poincaré group. These groups will be denoted by G and P respectively.

Let g be the Lie algebra of G, $A_1, A_2, \dots, A_{10}$ be a basis in g and C^k_{ij} $(i,j,k=1,2,\dots,10)$ be corresponding structure constants:

$$[A_i, A_j] = \sum_k C^k_{ij} A_k .$$

Then for any $\varepsilon>0$ one can find a basis $B_1, B_2, \dots, B_{10}$ in the Lie algebra p of the group P such that the structure constants C'^k_{ij} $(i, j, k=1, 2, \dots, 10)$ corresponding to this basis

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$$[B_i, B_j] = \sum_k C'^k_{ij} B_k$$

are close to C^k_{ij}:

$$|C'^k_{ij} - C^k_{ij}| < \varepsilon$$

for $i, j, k = 1, 2, \ldots, 10$.

Due to this fact nonrelativistic theory can be considered as a limiting case of the relativistic one and in certain region (small velocities) the relativistic predictions coincide with (are close to) nonrelativistic ones. In other words the correspondence princip e works.

Many theories (especially in elementary particle physics) are based on semisimple Lie groups. For example $SU(2)$ plays the fundamental role in the theory of isotopic spin. It turns out that $SU(2)$ (like any other semisimple Lie group) has the following rigidity property:

Any connected, simply connected Lie group whose structure constants are sufficiently close to the structure constants of $SU(2)$ is isomorphic to $SU(2)$.

This result seems to indicate that any theory based on $SU(2)$ can not be considered as a limiting case of a more general theory based on "*perturbed*" $SU(2)$ *group*.

The content of this paper shows that the above conclusion is not well justified. It turns out that $SU(2)$ can be perturbed. There exists a one parameter family of objects $S_\nu U(2)$ depending continuously on $\nu \in [-1, 1] - \{0\}$ such that for $\nu = 1$ we have $S_\nu U(2) = SU(2)$. However for $\nu < 1$, $S_\nu U(2)$ is not a group in the usual sense. It is a pseudogroup i. e. a locally compact (in fact compact) topological group on which the algebra of "*all continuous functions*" is not commutative.

The concept of quantization entered physics at the beginning of this century. Since 1925 quantization consists in replacing commutative quantities by non-commutative ones. In mathematical language an algebra of continuous functions on a locally compact topological space (e. g. a phase space of a mechanical system with finite degrees of freedom) is replaced by a noncommutative C^*-algebra. It is convenient to consider the latter algebra as the algebra of all "*continuous functions*" on some "*non-commutative locally compact topological space*". In [3] we proposed the name "*pseudospace*" to describe objects of this kind. Consequently a "*pseudogroup*" is a pseudospace endowed with a group structure.

The theory of pseudogroups is now more than 20 years old. After the first papers of G. I. Kac [4] we have the work of M. Takesaki [5] and a series of papers by Jean Marie Schwartz and Michel Enock (see [6] for the list of papers).

The main aim of the theory was to construct a category containing the category of locally compact topological groups and the category of objects dual to them. The (generalized) Pontriagin duality is then a contravariant functor acting within this larger category.

At first the theory was developed in the W^*-algebra framework. In our opinion it was not a natural approach. It means that we neglect the topological structure of pseudogroups concentrating our attention on their measurable structure. As a result we have to start with a very complicated notions (like Haar measure which existence is assumed (not proved)) and axioms which have no direct connections with the postulates of the theory of locally compact groups.

In the program presented in [3] we pointed out that the right approach to the pseudogroup theory is the one based on the C^*-algebra theory.

Recently the C^*-algebra approach to the theory of pseudogroups was used by Jean Michel Vallin [6]. Unfortunately also in his work the existence of Haar measure is postulated.

As far as I know the theory of pseudogroups suffered the lack of interesting examples. Except the one example of finite pseudogroup given by G. I. Kac the only examples supporting very complicated formalism were locally compact groups, objects dual to them and direct products of groups and group duals.

In this paper we present essentially new examples of pseudogroups. The one parameter family of pseudogroups introduced in Section 1 can be regarded as a perturbation of $SU(2)$ group. This fact is very interesting from the point of view of physics. It means that we may try to replace $SU(2)$ group playing an important role in many physical theories by $S_\nu U(2)$ with ν close but not equal to 1.

Having in mind these applications we concentrate our attention on detailed description of the pseudogroups $S_\nu U(2)$ and their representations pushing the general theory to a separate paper [1]. The present paper is mostly devoted to the differential calculus (in the spirit of Alain Connes [7]) that is necessary to introduce infinitesimal generators of representations of $S_\nu U(2)$ (in applications such generators describe important physical quantities related to the group via Noether theorem).

Let us briefly discuss the content of the paper. In the first section we introduce the pseudogroups $S_\nu U(2)$ and investigate their properties. In particular we prove that $S_\nu U(2)$ satisfies the axioms listed in [1]. In Section 2 the basic notions of differential calculus are introduced. The main role is played by a bimodule Γ which elements correspond to differential 1-forms on a Lie group. The external derivative of "smooth functions" is introduced. Section 3 is devoted to higher order differential forms. We derive formulae corresponding to that of Cartan Maurer and find the commutation relations for infinitesimal shifts. Section 4 is of very technical nature and contains the proof of an important Proposition used in Section 3. In Section 5 we use the differential calculus to the representation theory of $S_\nu U(2)$. We describe the irreducible representations of $S_\nu U(2)$. Like in the $SU(2)$ case the irreducible representations are labeled by a non-negative integer or half-integer n. The dimension of representation corresponding to a given n equals to $2n+1$. The tensor product of representations is also discussed. It turns out (this is typical phenomenon for pseudogroups) that the tensor product is no longer commutative. This fact has profound consequences which are not discussed in the present paper. Let us mention two of them.

In the theory of identical particles the operator interchanging particles has no longer the simple form $S(x\otimes y)=y\otimes x$. In particular the operators interchanging particles do not form a representation of the permutation group. Instead we have to deal with representations of an infinite group covering the group of all permutations.

The algebra describing the composed system is no longer the tensor product of algebras associated with the components of the system. In particular observables associated with different parts of the composed system do not commute.

The above remarks mean that introducing pseudogroups to the description of physical system we are forced to abandon many simple principles that we get accustomed to in the usual quantum theory. Nevertheless we belive that despite these unusual features one can formulate the coherent quantum theory with a pseudogroup playing the role of the symmetry group.

At the end of the paper we put two appendices. In the first we discuss the twisted unimodularity condition which for $S_\nu U(2)$ case replaces the condition det $u=1$ satisfied by elements of $SU(2)$. This twisted unimodularity condition was in fact the starting point in the discovery of $S_\nu U(2)$. The second appendix is devoted to the study of the C^*-algebras associated with $S_\nu U(2)$. It turns out that pseudogroups $S_\nu U(2)$ (for $|\nu|<1$) are mutually homeomorphic (but not isomorphic).

§ 1. The Pseudogroup $S_\nu U(2)$

In this section we introduce the basic object investigated in this paper. It is the pseudogroup $S_\nu U(2)$. In general pseudogroups should be considered as "*locally compact topological groups*" on which the "*algebra of continuous functions*" with "*pointwise multiplication*" is not commutative. The particular example considered in this paper is closely related to $SU(2)$ group and can be obtained by a modification of the unimodularity condition that is used in the definition of $SU(2)$. This twisted unimodularity condition leads to the commutation relations in the algebra of "*continuous functions*" collected in the Table 0 below. For details see Appendix A1.

Let ν be a number belonging to $[-1, 1]$. We denote by A the C^*-algebra generated by two elements α, γ satisfying the following

Table 0 The commutation relations in the algebra A

$\alpha^*\alpha+\gamma^*\gamma=I$	$\gamma^*\gamma=\gamma\gamma^*$
$\alpha\alpha^*+\nu^2\gamma^*\gamma=I$	$\alpha\gamma=\nu\gamma\alpha$
	$\alpha\gamma^*=\nu\gamma^*\alpha$

The precise definition of A is the following: Let $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ be the free noncommutative $*$-algebra with unity generated by symbols α and γ. A $*$-representation π of $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ acting on a Hilbert space H is said to be admissible if operators $\alpha'=\pi(\alpha)$ and $\gamma'=\pi(\gamma)$ satisfy the above commutation relations. For any $a\in\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ we set

$$||a||=\sup||\pi(a)|| \tag{1.1}$$

where π runs over the set of all admissible representation of $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$. Clearly $||a||$ is finite for any a belonging to $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ and $||\cdot||$ is a C^*-seminorm. Therefore the set

$$N=\{a\in\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]:||a||=0\} \tag{1.2}$$

is a two-sided ideal in $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ and the seminorm (1.1) produces a C^*-norm on the quotien algebra

$$\mathscr{A}=\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]/N. \tag{1.3}$$

Then A is the completion of $\mathscr{A}$ with respect to this norm. The following result follows easily from the above construction.

Theorem 1.1. *A is a C^*-algebra with unity containing two distinguished elements α, γ satisfying relations of Table 0. The $*$-subalgebra $\mathscr{A}$ generated by α and γ is dense in A. For any two operators α', γ' acting on a Hilbert space H satisfying the relations of Table 0 there exists one and only one representation π of A acting on H such that $\pi(\alpha)=\alpha'$ and $\pi(\gamma)=\gamma'$.*

The more information about the structure of the algebra A and its dependence on ν is given in Appendix A2. In particular it turns out that A is GCR algebra. Therefore it is nuclear and the tensor product of C^*-algebras used in the sequel has unique meaning.

In what follows the $*$-algebra $\mathscr{A}$ generated by α and γ will play a central role. Elements of A should be considered as continuous function on our pseudogroup whereas elements of $\mathscr{A}$ correspond to smooth functions. In other words $\mathscr{A}$ defines a differential structure on the pseudogroup. The following theorem provides us with a necessary information concerning the structure of $\mathscr{A}$.

Theorem 1.2. *Let $\nu\neq 0$. The set of all elements of the form*

$$\alpha^k\gamma^n\gamma^{*m} \quad \textit{and} \quad \alpha^{*k'}\gamma^n\gamma^{*m} \tag{1.4}$$

where k, m, $n=0, 1, 2,\ldots$; $k'=1, 2,\ldots$ *forms a basis in* $\mathscr{A}$: *any element of* $\mathscr{A}$ *can be written in the unique way as a finite linear combination of elements* (1. 4).

Proof. Inserting $k=m=n=0$ we see that I is one of the elements (1.4). Moreover using formulae of Table 0 one can easily check that the product ab, where a is one of the element (1.4) and $b=\alpha, \gamma, \alpha^*, \gamma^*$ is a linear combination of at most two elements (1.4). It shows that the set of all linear combinations of (1.4) coincides with $\mathscr{A}$. To end the proof we have to show that elements (1. 4) are linearly independent.

Assume that $|\nu|<1$. Let H be a separable Hilbert space with an orthonormal basis $\left(\phi_{nk}\colon \begin{matrix} n=0, 1, 2, \ldots \\ k\text{-integer} \end{matrix}\right)$. We introduce operators $\alpha', \gamma' \in B(H)$ such that

$$\alpha'\phi_{nk}=\sqrt{1-\nu^{2n}}\,\phi_{n-1,k}$$
$$\gamma'\phi_{nk}=\nu^n\phi_{n,k+1} \quad .$$

By simple computation one can check that these operators satisfy the relations of Table 0. In virtue of Thm. 1. 1 there exists a representation π of A acting on H such that

$$\pi(\alpha)\phi_{nk}=\sqrt{1-\nu^{2n}}\,\phi_{n-1,k} \tag{1.5}$$
$$\pi(\gamma)\phi_{nk}=\nu^n\phi_{n,k+1} \quad .$$

The case $\nu=\pm 1$ should be treated separately. Let H be a separable Hilbert space with an orthonormal basis (ϕ_{nk}: n, k-integer). For any $t\in[0, 1]$ we consider operators α'_t, γ'_t acting on H such that

$$\alpha'_t\phi_{nk}=\sqrt{1-t^2}\,\phi_{n-1,k}$$
$$\gamma'_t\phi_{nk}=t\nu^n\phi_{n,k+1} \quad .$$

By simple computation one can check that these operators satisfy the relations of Table 0 (with $\nu=\pm 1$). In virtue of Thm 1. 1. there exists a representation π_t of A acting on H such that

$$\pi_t(\alpha)\phi_{nk}=\sqrt{1-t^2}\,\phi_{n-1,k} \tag{1.5'}$$
$$\pi_t(\gamma)\phi_{nk}=t\nu^n\,\phi_{n,k+1} \quad .$$

Let us consider a nontrivial finite linear combination of elements (1.4)

$$c=\sum_{kmn=0} c_{kmn}\alpha^k\gamma^m\gamma^{*n}+\sum_{\substack{mn=0\\k=1}} c'_{kmn}\alpha^{*k}\gamma^m\gamma^{*n}$$

where c_{kmn}, $c'_{k'mn}\in \boldsymbol{C}$ are almost all (but not all) equal to 0. Let s be the smallest integer such that either $c_{kmn}\neq 0$ or $c'_{kmn}\neq 0$ for some k, m, n such that $m+n=s$. Then using (1.5) one obtains for the case $|\nu|<1$

$$\lim_{r\to\infty}\frac{(\psi_{r+k,s-2n}|\pi(c)\psi_{r,0})}{\nu^{sr}}=\begin{cases}c'_{k,s-n,n} & \text{for } k>0\\ c_{-k,s-n,n} & \text{for } k\leq 0\,.\end{cases}$$

Similarly using (1.5′) we get for the case $\nu=\pm 1$

$$\lim_{t\to 0}\frac{(\psi_{k,s-2n}|\pi_t(c)\psi_{0,0})}{t^s}=\begin{cases}c'_{k,s-n,n} & \text{for } k>0\\ c_{-k,s-n,n} & \text{for } k\leq 0\,.\end{cases}$$

Therefore in both cases $c\neq 0$ and the linear independence follows.

Q. E. D.

Theorem 1.3. *Let M be an associative algebra with unity I_m and α_m, γ_m, π^*_m, γ^*_m be elements of M such that*

$$\begin{aligned}&\alpha^*_m\alpha_m+\gamma^*_m\gamma_m=I_m,\ \alpha_m\alpha^*_m+\nu^2\gamma^*_m\gamma_m=I_m\\&\gamma^*_m\gamma_m=\gamma_m\gamma^*_m,\ \alpha_m\gamma_m=\nu\gamma_m\alpha_m,\ \alpha_m\gamma^*_m=\nu\gamma^*_m\alpha_m\\&\gamma_m\alpha^*_m=\nu\alpha^*_m\gamma_m,\ \gamma^*_m\alpha^*_m=\nu\alpha^*_m\gamma^*_m\quad.\end{aligned}\tag{1.6}$$

Assume that $\nu\neq 0$. Then there exists one and only one linear, multiplicative, unital mapping

$$\kappa\colon \mathscr{A}\longrightarrow M$$

such that

$$\kappa(\alpha)=\alpha_m,\ \kappa(\gamma)=\gamma_m,\ \kappa(\alpha^*)=\alpha^*_m,\ \kappa(\gamma^*)=\gamma^*_m\,.\tag{1.7}$$

Proof. The uniqueness of κ is obvious: the algebra $\mathscr{A}$ is generated by α, γ, α^* and γ^*. To prove existence we introduce the ideal $\tilde{N}$ in $\boldsymbol{C}[\![\alpha,\gamma,\alpha^*,\gamma^*]\!]$ generated by the following seven elements: $\alpha^*\alpha+\gamma^*\gamma-I$, $\alpha\alpha^*+\nu^2\gamma^*\gamma-I$, $\gamma^*\gamma-\gamma\gamma^*$, $\alpha\gamma-\nu\gamma\alpha$, $\alpha\gamma^*-\nu\gamma^*\alpha$, $\gamma\alpha^*-\nu\alpha^*\gamma$ and $\gamma^*\alpha^*-\nu\alpha^*\gamma^*$. Since $\boldsymbol{C}[\![\alpha,\gamma,\alpha^*,\gamma^*]\!]$ is free, one can find a linear multiplicative mapping

$$\tilde{\kappa}\colon \boldsymbol{C}[\![\alpha,\gamma,\alpha^*,\gamma^*]\!]\longrightarrow M$$

such that

$$\tilde{\kappa}(\alpha)=\alpha_m,\ \tilde{\kappa}(\gamma)=\gamma_m,\ \tilde{\kappa}(\alpha^*)=\alpha^*_m,\ \tilde{\kappa}(\gamma^*)=\gamma^*_m.$$

It follows immediately from (1.6) that $\tilde{\kappa}$ vanishes on generators of $\tilde{N}$. Therefore $\tilde{\kappa}$ defines a linear multiplicative mapping

$$\kappa\colon \boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]/\tilde{N} \longrightarrow M$$

and the theorem will be proven if we show that

$$\tilde{N}=N \tag{1.8}$$

where N is introduced by (1.2) (cf. (1.3))

Clearly generators of $\tilde{N}$ are mapped to 0 by any admissible $*$-representation of $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$. Therefore $\tilde{N}$ is contained in the ideal N.

To prove the converse inclusion we consider subspace T in $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ of all linear combinations of elements of the form $\alpha^k\gamma^m\gamma^{*n}$ and $\alpha^{*k}\gamma^m\gamma^{*n}$, where k, m, n are nonnegative integers.

One can easily check that any element of $\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]$ is equivalent (modulo $\tilde{N}$) to an element of T. In other words

$$T+\tilde{N}=\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!] . \tag{1.9}$$

On the other hand it follows immediately from Thm. 1.2 that no non-zero element of T is killed by the canonical map

$$\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!] \longrightarrow \mathscr{A}=\boldsymbol{C}[\![\alpha, \gamma, \alpha^*, \gamma^*]\!]/N .$$

It means that

$$T\cap N=\{0\} . \tag{1.10}$$

Now we are able to prove (1.8). We already know that $\tilde{N}$ is contained in N. Let $a\in N$. Then (cf. (1.9) $a=t+n$, where $t\in T$ and $n\in\tilde{N}$. Therefore $t=a-n\in T\cap N$ and according to (1.10) $t=0$. Thus $a=n\in\tilde{N}$ and formula (1.8) is proven. Q. E. D.

In what follows we deal with $N\times N$ matrices (in most cases $N=2$) with entries belonging to a C^*-algebra B (in our case $B=A$ or $B=A\otimes A$). From the formal point of view these matrices can be considered as elements of the algebra $M_N(B)=M_N\otimes B$, where M_N denotes the C^*-algebra of all $N\times N$ matrices with complex entries.

Therefore if

$$u=\begin{pmatrix} u_{11}, \dots, u_{1N} \\ \cdots\cdots\cdots\cdots \\ u_{N1}, \dots, u_{NN} \end{pmatrix}$$

is a $N\times N$ matrix with entries belonging to a C^*-algebra B and φ: $B\rightarrow B'$ is a linear map of B into a C^*-algebra B' then $(\mathrm{id}\otimes\varphi)u$ denotes the $N\times N$ matrix

$$(id\otimes\varphi)u=\begin{pmatrix}\varphi(u_{11}),\dots,\varphi(u_{1N})\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \varphi(u_{N1}),\dots,\varphi(u_{NN})\end{pmatrix}$$

with entries belonging to B'. Clearly "*id*" denotes the identity map of M_N. If

$$u=\begin{pmatrix}u_{11},\dots,u_{1N}\\ \cdots\cdots\cdots\cdots\\ u_{N1},\dots,u_{NN}\end{pmatrix}\qquad v=\begin{pmatrix}v_{11},\dots,v_{1N}\\ \cdots\cdots\cdots\cdots\\ v_{N1},\dots,v_{NN}\end{pmatrix}$$

are $N\times N$ matrices with entries belonging to B and B' resp. then $u \textcircled{\scriptsize T} v$ will denote $N\times N$ matrix

$$w=\begin{pmatrix}w_{11},\dots,w_{1N}\\ \cdots\cdots\cdots\cdots\\ w_{N1},\dots,w_{NN}\end{pmatrix}$$

with entries belonging to $B\otimes B'$ given by the formula

$$w_{ik}=\sum_s u_{is}\otimes v_{sk}\ .$$

In other words $\textcircled{\scriptsize T}$ denotes the usual product of matrices in which the usual product of matrix elements is replaced by tensor product.

Now we are able to formulate our main result showing that A carries a natural matrix pseudogroup structure (cf. [1]).

Theorem 1.4. *Let*

$$u=\begin{pmatrix}\alpha, & -\nu\gamma^*\\ \gamma, & \alpha^*\end{pmatrix}\in M_2\otimes A\ .$$

Then

1° *The* $*$*-algebra* $\mathscr{A}$ *generated by matrix elements of* u *is dense in* A.

2° *There exists a* C^**-algebra homomorphism*

$$\Phi:\ A\longrightarrow A\otimes A$$

such that

$$(id\otimes\Phi)u=u\textcircled{\scriptsize T}u\ . \tag{1.11}$$

3° u *is an invertible element of* $M_2\otimes A$. *If* $\nu\neq 0$ *then there exists a linear antimultiplicative mapping*

$$\kappa: \mathscr{A} \longrightarrow \mathscr{A}$$

such that $\kappa(\kappa(a^*)^*) = a$ *for all* $a \in \mathscr{A}$ *and*

$$(id \otimes \kappa) u = u^{-1} . \tag{1.12}$$

Proof. The first statement is already proven (cf. Thm. 1. 1).

To prove the second statement we notice that (1. 11) is satisfied if and only if

$$\begin{aligned} \Phi(\alpha) &= \alpha \otimes \alpha - \nu \gamma^* \otimes \gamma \\ \Phi(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma . \end{aligned} \tag{1.13}$$

To prove the existence of Φ we assume that $A \subset B(H)$, where H is a Hilbert space and consider operators

$$\begin{aligned} \alpha' &= \alpha \otimes \alpha - \nu \gamma^* \otimes \gamma \\ \gamma' &= \gamma \otimes \alpha + \alpha^* \otimes \gamma \end{aligned}$$

acting on $H \otimes H$. Remembering that α and γ satisfy the commutation relation contained in Table 0, one can easily check by direct computation that α' and γ' also satisfy these relations. (This fact is not accidental; see Appendix A1). Therefore according to the last statement of Thm. 1. 1 there exists a representation

$$\Phi: A \longrightarrow B(H \otimes H)$$

such that equations (1. 13) hold. Clearly the image $\Phi(A)$ is the C^*-subalgebra of $B(H \otimes H)$ generated by α' and γ'. Since these two operators obviously belong to $A \otimes A$, we have $\Phi(A) \subset A \otimes A$. This way the second part of the theorem is proven.

To prove the third statement one checks at first (by direct computation making use of formulae of Table 0) that u is a unitary element of $M_2(A)$. Therefore

$$u^{-1} = u^* = \begin{pmatrix} \alpha^*, & \gamma^* \\ -\nu\gamma, & \alpha \end{pmatrix}$$

Assume that $\nu \neq 0$. We have to show that there exists linear, antimultiplicative mapping $\kappa: \mathscr{A} \to \mathscr{A}$ such that κ composed with the hermition conjugation is an involution and

$$\begin{aligned} \kappa(\alpha) &= \alpha^*, \quad & \kappa(-\nu\gamma^*) &= \gamma^* \\ \kappa(\gamma) &= -\nu\gamma, \quad & \kappa(\alpha^*) &= \alpha \end{aligned} \quad . \tag{1.14}$$

Let $M = \mathscr{A}^{op}$ be the algebra opposite to $\mathscr{A}$ (M is identical with $\mathscr{A}$

as far as linear structure is concerned, whereas the product of two elements in the sense of M coincides with the usual product (i.e. in the sense of $\mathscr{A}$) of the same elements taken in the reverse order) and $\alpha_m=\alpha^*$, $\gamma_m=-\nu\gamma$, $\alpha_m^*=\alpha$ and $\gamma_m^*=-\frac{1}{\nu}\gamma^*$. One can easily check that α_m, γ_m, α_m^* and γ_m^* considered as elements of M satisfy relations (1.6) in Thm. 1.3. Let κ be the linear multiplicative mapping which existence is stated in this theorem. To end the proof we notice that in the considered case relations (1.14) are identical with (1.7) and that any multiplicative mapping into $M=\mathscr{A}^{op}$ is antimultiplicative if it is considered as a mapping into $\mathscr{A}$. Q. E. D.

Remarks: 1. Theorem 1.4 states that for $\nu\neq 0$, (A, u) is a compact matrix pseudogroup in the sense of [1]. It will be denoted by $S_\nu U(2)$ and called twisted $SU(2)$ group for the reasons indicated in Appendix A1 (see also Remark 3 below).

2. Clearly C^*-homomorphism Φ and linear antimultiplicative mapping κ are determined uniquely by conditions (1.11) and (1.12) resp. Moreover the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\Phi\quad} & A\otimes A \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle id\otimes\Phi} \\ A\otimes A & \xrightarrow[\Phi\otimes id]{} & A\otimes A\otimes A \end{array} \tag{1.15}$$

is commutative. The latter can be proved in the following way. At first one checks by direct computation making use of (1.13) that

$$(\Phi\otimes id)\Phi(a) = (id\otimes\Phi)\Phi(a) \tag{1.16}$$

for $a=\alpha$ and γ. Next using the fact that Φ is a $*$-homomorphism one sees that this formula is valid for any $a\in\mathscr{A}$ and finally using the density of $\mathscr{A}$ and the continuity of Φ we obtain (1.16) for all $a\in A$.

3. It follows immediately from Table 0 that for $\nu=1$ the algebra A is commutative. According to the general theory [1] in this case the algebra A can be identified with the algebra $C(G)$ of all continuous functions on G, where

$$G=\{(id\otimes\chi)u:\ \chi \text{ is a character of the algebra } A\}$$

is a compact group of 2×2 matrices. Matrix elements of u are functions of the form $u_{kl}(g)=g_{kl}$ (where $k, l=1, 2$) for

$$g=\begin{pmatrix} g_{11}, & g_{12} \\ g_{21}, & g_{22} \end{pmatrix} \in G .$$

In particular $\alpha(g)=g_{11}$ and $\gamma(g)=g_{21}$. The C^*-homomorphism Φ and the mapping κ are given by formulae

$$(\Phi\ a)(g',g)=a(g'g)$$
$$(\kappa\ a)(g)=a(g^{-1}) \quad .$$

It turns out that

$$G=SU(2) \ . \tag{1.17}$$

Indeed: for any character χ of A we have $\chi(\alpha^*)=\overline{\chi(\alpha)}$, $\chi(\gamma^*)=\overline{\chi(\gamma)}$, $|\chi(\alpha)|^2+|\chi(\gamma)|^2=1$ and the matrix

$$(id\otimes\chi)u=\begin{pmatrix} \chi(\alpha), & -\overline{\chi(\gamma)} \\ \chi(\gamma), & \overline{\chi(\alpha)} \end{pmatrix}$$

is unitary and unimodular. Conversely any $v\in SU(2)$ is of the form

$$v=\begin{pmatrix} \alpha', & -\overline{\gamma'} \\ \gamma', & \overline{\alpha'} \end{pmatrix}$$

where α' and γ' are complex numbers such that $|\alpha'|^2+|\gamma'|^2=1$. Therefore α', γ' obviously satisfy relation of Table 0 (with $\nu=1$) and according to Thm. 1. 1 there exists one dimensional representation (i. e. character) χ of A such that $\chi(\alpha)=\alpha'$ and $\chi(\gamma)=\gamma'$. Then $(id\otimes\chi)u=v$ and formula (1. 17) follows.

It means that the pseudogroup $S_\nu U(2)$ can be regarded as a deformation of $SU(2)$ group.

At the end of this section we introduce some important notions of the general theory of compact matrix pseudogroup and quote some results.

We start with the notion of convolution product. Let χ and ξ be continuous linear functional on A and $a\in A$. Then

$$\chi*a=(id\otimes\chi)\Phi(a) \tag{1.18}$$

$$a*\xi=(\xi\otimes id)\Phi(a) \tag{1.19}$$

$$\xi*\chi=(\xi\otimes\chi)\Phi \quad . \tag{1.20}$$

Clearly $\chi*a$ and $a*\xi$ belong to A; $\xi*\chi$ is a continuous linear functional on A. The commutativity of the diagram (1. 15) implies the associativity of the convolution product. Moreover we have

$$(\xi*\chi)(a) = \chi(a*\xi) = \xi(\chi*a). \qquad (1.21)$$

It turns out that there exists (unique up to a positive factor) positive linear functional on A such that

$$h*a = a*h = h(a)I \qquad (1.22)$$

for any $a\in A$. In the theory of compact groups the above condition expresses the characteristic property of the Haar measure: The invariance under left and right shifts. Also in the pseudogroup case h will be called Haar measure. For $S_\nu U(2)$ with $|\nu|<1$

$$h(a) = \sum_{n=0}^{\infty} \nu^{2n} (\psi_{n0} | \pi(a) \psi_{n0})$$

where π is the representation of A introduced in the proof of Thm. 1.2.

Let (as before) $\mathscr{A}$ denotes the $*$-subalgebra of A generated by α and γ. It follows immediately from (1.13) that

$$\Phi: \mathscr{A} \longrightarrow \mathscr{A} \otimes_{alg} \mathscr{A}. \qquad (1.23)$$

Therefore there exist linear maps

$$r, r': \mathscr{A}\otimes_{alg}\mathscr{A} \longrightarrow \mathscr{A}\otimes_{alg}\mathscr{A}$$

such that

$$r(a\otimes b) = (a\otimes I)\Phi(b)$$
$$r'(a\otimes b) = \Phi(a)(b\otimes I)$$

for any $a, b\in\mathscr{A}$. It follows from the general theory that r and r' are linear bijections. Moreover for any $a\in\mathscr{A}$

$$r'(r^{-1}(I\otimes a)) = I\otimes\kappa(a) \qquad (1.24)$$

where κ is the mapping introduced in Thm. 1.4.

§ 2. First Order Differential Calculus

In the theory of Lie groups the external derivative of a smooth function $a\in\mathscr{C}^\infty(G)$ may be written in the form

$$da = \sum_k (\chi_k * a)\omega_k$$

where (ω_k) is a basis in the space of left invariant differential forms of the first order, χ_k are directional derivatives of the delta-function concentrated at the neutral element of the group G and $*$ denotes the convolution product. We shall use the above formula as a guide-line in our differential calculus. In this calculus the subalgebra $\mathscr{A}$ will play the role of the algebra of smooth functions.

We shall use the convolution product introduced in Section 1. Unfortunately the functional that are important in the differential calculus are defined only on $\mathscr{A}$ and have no continuity property (we meet the same situation in the Lie group case: the directional derivatives of the delta-function are not finite measures).

However due to (1.23) the right hand side of (1.18)-(1.20) are meaningful for any linear functionals χ, ξ defined on $\mathscr{A}$ and any $a\in\mathscr{A}$. In this case $\chi*a$, $a*\xi\in\mathscr{A}$ and $\xi*\chi$ is a linear functional defined on $\mathscr{A}$.

Now we introduce functionals χ_0, χ_1, χ_2 that will play the role of directional derivatives of the delta-function.

Remark. In Sections 2, 3, 4 and 5 $\nu\in[-1, 1]$ and $\nu\neq 0$.

Let M be the set of all 4×4 matrices (with complex entries) having non-zero elements only in the first row and on the diagonal. Clearly M is a subalgebra in M_4 containing unity $I_m=I$. One can easily check that the following elements of M:

$$\alpha_m=\begin{pmatrix} 1, & 0, & 1, & 0 \\ 0, & \nu^{-1}, & 0, & 0 \\ 0, & 0, & \nu^{-2}, & 0 \\ 0, & 0, & 0, & \nu^{-1} \end{pmatrix} \qquad \gamma_m=\begin{pmatrix} 0, & 0, & 0, & -\nu \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}$$

$$\alpha_m^*=\begin{pmatrix} 1, & 0, & -\nu^2, & 0 \\ 0, & \nu, & 0, & 0 \\ 0, & 0, & \nu^2, & 0 \\ 0, & 0, & 0, & \nu \end{pmatrix} \qquad \gamma_m^*=\begin{pmatrix} 0, & -\nu^{-1}, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}$$

satisfy relations (1.6). Therefore (cf. Thm. 1.3) there exists a linear multiplicative unital map

$$f\colon \mathscr{A} \longrightarrow M$$

such that above matrices are equal to $f(\alpha)$, $f(\gamma)$, $f(\alpha^*)$ and $f(\gamma^*)$ resp. Denoting by $e(a)$, $\chi_0(a)$, $\chi_1(a)$, $\chi_2(a)$ matrix elements of $f(a)$ standing in the first row and by $e(a)$, $f_0(a)$, $f_1(a)$, $f_2(a)$ the diagonal elements we introduce linear functional e, χ_i, f_i $(i=0, 1, 2)$ defined on $\mathscr{A}$. We have

$$f(a)=\begin{pmatrix} e(a), & \chi_0(a), & \chi_1(a), & \chi_2(a) \\ 0\ , & f_0(a), & 0\ , & 0 \\ 0\ , & 0\ , & f_1(a), & 0 \\ 0\ , & 0\ , & 0\ , & f_2(a) \end{pmatrix}$$

and

$$e(\alpha)=e(\alpha^*)=1, \qquad e(\gamma)=e(\gamma^*)=0 \tag{2. 1}$$

$$\begin{aligned} &f_0(\alpha)=f_2(\alpha)=\nu^{-1}, && f_1(\alpha)=\nu^{-2} \\ &f_0(\alpha^*)=f_2(\alpha^*)=\nu, && f_1(\alpha^*)=\nu^2 \\ &f_i(\gamma)=f_i(\gamma^*)=0\quad . && (i=0, 1, 2) \end{aligned} \tag{2. 2}$$

Except the following four cases

$$\begin{aligned} &\chi_0(\gamma^*)=-\nu^{-1}, && \chi_1(\alpha)=1 \\ &\chi_1(\alpha^*)=-\nu^2, && \chi_2(\gamma)=-\nu \end{aligned} \tag{2. 3}$$

the all other values of $\chi_i(a)$ (where $i=0, 1, 2;\ a=\alpha, \gamma, \alpha^*, \gamma^*$) are equal to zero.

One can easily check that matrices α_m, γ_m, α_m^* and γ_m^* commute with matrices

$$\begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0 \\ 0, & 0, & \tau, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}$$

where $\tau=(1-\nu^2)\nu^{-2}$. It follows immediately that $f(a)$ commute with the above two matrices for any $a\in\mathscr{A}$ and we obtain

$$f_0(a)=f_2(a) \tag{2. 4}$$

$$f_1(a)=e(a)+\frac{1-\nu^2}{\nu^2}\chi_1(a)\quad . \tag{2. 5}$$

The multiplicativity of f gives rise to the following relations

$$e(ab)=e(a)e(b) \tag{2. 6}$$

$$\chi_i(ab)=\chi_i(a)f_i(b)+e(a)\chi_i(b) \tag{2. 7}$$

$$f_i(ab)=f_i(a)f_i(b) \tag{2. 8}$$

for any a, $b\in\mathscr{A}$ and $i=0, 1, 2$. Using (1. 13) and (2. 1) one can easily compute that $e*a=a$ for $a=\alpha, \gamma, \alpha^*, \gamma^*$. Moreover in virtue of

(2.6) the map $a\to e*a$ is multiplicative:

$$e*ab=(e*a)(e*b). \tag{2.9}$$

Remembering that $\alpha,\gamma,\alpha^*,\gamma^*$ generate the algebra $\mathscr{A}$ we get

$$e*a=a \tag{2.10}$$

for any $a\in\mathscr{A}$. It shows that the functional e plays the role of delta-function concentrated at the neutral element of the (pseudo)group.

Similarly one can prove that

$$(f_0*f_0)(a)=f_1(a) \tag{2.11}$$

for any $a\in\mathscr{A}$: both sides are multiplicative linear functionals on $\mathscr{A}$ and for $a=\alpha,\gamma,\alpha^*,\gamma^*$ the formula follows from easy computation making use of (1.13) and (2.2).

The functional e appears in the following interesting context.

Let m: $\mathscr{A}\otimes_{alg}\mathscr{A}\to\mathscr{A}$ be the multiplication map. This is the linear map such that $m(a\otimes b)=ab$ for any a, b belonging to $\mathscr{A}$. Let κ be the antimultiplicative map introduced in Thm. 1.4. We claim that

$$m(\kappa\otimes id)\Phi(a)=e(a)I \tag{2.12}$$

for any $a\in\mathscr{A}$. Indeed using (1.13) and (1.14) one can easily check that this formula holds for $a=\alpha,\gamma,\alpha^*,\gamma^*$. Moreover if (2.12) is true for $a=b$ and $a=c$, then writting

$$\Phi(b)=\sum_i b_i'\otimes b_i''$$
$$\Phi(c)=\sum_j c_j'\otimes c_j''$$

we have

$$\sum_i \kappa(b_i')b_i''=e(b)I$$
$$\sum_j \kappa(c_j')c_j''=e(c)I$$

and taking into account the antimultiplicativity of κ we get

$$\begin{aligned}m(\kappa\otimes id)\Phi(bc)&=m(\kappa\otimes id)\sum_{ij} b_i'c_j'\otimes b_i''c_j''\\&=\sum_{ij}\kappa(b_i'c_j')b_i''c_j''=\sum_{ij}\kappa(c_j')\kappa(b_i')b_i''c_j''\\&=e(b)\sum_j\kappa(c_j')c_j''=e(b)e(c)I=e(bc)I\ .\end{aligned}$$

It proves (2.12) in full generality. Similarly one can show that

$$m(id\otimes\kappa)\Phi(a)=e(a)I\ . \tag{2.13}$$

Now we are ready to introduce an $\mathscr{A}$-bimodule Γ which in our differential calculus plays the same role as the module of smooth sections of the cotangent bundle in the classical theory of Lie groups. In other words elements of Γ correspond to differential forms of degree one.

Let Γ be the free left module over $\mathscr{A}$ with generators $\omega_0, \omega_1, \omega_2$. It means that any $\omega\in\Gamma$ is of the form

$$\omega=a_0\omega_0+a_1\omega_1+a_2\omega_2 \tag{2. 14}$$

where $a_0, a_1, a_2\in\mathscr{A}$ are uniquely determined and that the left multiplication by an element $a\in\mathscr{A}$ is given by the formula

$$a\omega=(aa_0)\omega_0+(aa_1)\omega_1+(aa_2)\omega_2 \tag{2. 15}$$

(in the following we shall omit the brackets).

For any $\omega=\sum_k a_k\omega_k\in\Gamma$ and any $a\in\mathscr{A}$ we set

$$\omega a=\sum_k a_k(f_k*a)\omega_k\ . \tag{2. 16}$$

Proposition 2. 1. *The left module Γ considered with the right multiplication by elements of $\mathscr{A}$ introduced by* (2. 16) *is an $\mathscr{A}$-bimodule.*

Proof. Obviously the multiplication introduced by (2. 16) is bilinear and the associativity low $(b\omega)a=b(\omega a)$ holds for any $a, b\in\mathscr{A}$ and $\omega\in\Gamma$. Moreover if $a=I$ then $\Phi(a)=I\otimes I$, $f_k*a=f_k(I)I=I$ and $\omega I=\omega$. To end the proof one has to show that

$$\omega(ab)=(\omega a)b$$

for all $a, b\in\mathscr{A}$ and $\omega\in\Gamma$. According to (2. 16) this formula is equivalent to the equation

$$f_k*ab=(f_k*a)(f_k*b) \tag{2. 17}$$

which in turn follows immediately from (2. 8). Q. E. D.

To show how the definition (2. 16) works we shall compute $\omega_0\alpha$. Using (1. 18), (1. 13) and (2. 2) we have

$$\begin{aligned}\omega_0\alpha&=(f_0*\alpha)\omega_0=(id\otimes f_0)\Phi(\alpha)\omega_0\\&=(id\otimes f_0)(\alpha\otimes\alpha-\nu\gamma^*\otimes\gamma)\omega_0\\&=(f_0(\alpha)\alpha-\nu f_0(\gamma)\gamma^*)\omega_0=\frac{1}{\nu}\alpha\omega_0\ .\end{aligned}$$

Similarly one can compute other products $\omega_k a$, where $k=0, 1, 2$ and

$a=\alpha, \gamma, \alpha^*, \gamma^*$. For the readers convenience and for the future references we list the results:

Table 1 Computation rules for $\omega_0, \omega_1, \omega_2$

$\omega_0\alpha=\nu^{-1}\alpha\omega_0,$	$\omega_1\alpha=\nu^{-2}\alpha\omega_1,$	$\omega_2\alpha=\nu^{-1}\alpha\omega_2$
$\omega_0\gamma=\nu^{-1}\gamma\omega_0,$	$\omega_1\gamma=\nu^{-2}\gamma\omega_1,$	$\omega_2\gamma=\nu^{-1}\gamma\omega_2$
$\omega_0\alpha^*=\nu\alpha^*\omega_0,$	$\omega_1\alpha^*=\nu^2\alpha^*\omega_1,$	$\omega_2\alpha^*=\nu\alpha^*\omega_2$
$\omega_0\gamma^*=\nu\gamma^*\omega_0,$	$\omega_1\gamma^*=\nu^2\gamma^*\omega_1,$	$\omega_2\gamma^*=\nu\gamma^*\omega_2$

Now we can introduce the external derivative. For any $a\in\mathscr{A}$ we put

$$da=\sum_k(\chi_k*a)\,\omega_k\ . \tag{2.18}$$

Theorem 2.2. *Formula (2.18) introduces a linear map*

$$d:\ \mathscr{A}\longrightarrow\Gamma.$$

For any a, $b\in\mathscr{A}$ we have

$$d(ab)=(da)b+adb\ . \tag{2.19}$$

Moreover any element $\omega\in\Gamma$ can be written in the form

$$\omega=\sum_{i=1}^{N}a_idb_i$$

where $a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N\in\mathscr{A}$ and $\sum_i a_ib_i=0$.

Proof. Cleary d is linear. The equation (2.19) can be checked in the following way. According to (1.18) and (2.7) we have

$$\begin{aligned}\chi_k*ab&=(id\otimes\chi_k)\Phi(ab)=(id\otimes\chi_k)\Phi(a)\Phi(b)\\&=(id\otimes\chi_k)\Phi(a)\,(id\otimes f_k)\Phi(b)+(id\otimes e)\Phi(a)\,(id\otimes\chi_k)\Phi(b)\\&=(\chi_k*a)\,(f_k*b)+(e*a)\,(\chi_k*b)\qquad\qquad(2.20)\\&=(\chi_k*a)\,(f_k*b)+a(\chi_k*b)\end{aligned}$$

where in the last line we used (2.10). Therefore using (2.16) we obtain

$$\begin{aligned}d(ab)&=\sum_k(\chi_k*ab)\,\omega_k\\&=\sum_k(\chi_k*a)\,(f_k*b)\,\omega_k+a\sum_k(\chi_k*b)\,\omega_k\\&=\sum_k(\chi_k*a)\,\omega_kb+adb=(da)b+adb\ .\end{aligned}$$

To prove the last statement of the theorem we have to compute da for $a=I, \alpha, \gamma, \alpha^*, \gamma^*$. For example, using (1. 13) and (2. 3) we have

$$\begin{aligned}&\Phi(\alpha^*)=\alpha^*\otimes\alpha^*-\nu\gamma\otimes\gamma^*\\&\chi_0*\alpha^*=\chi_0(\alpha^*)\alpha^*-\nu\chi_0(\gamma^*)\gamma=\gamma\\&\chi_1*\alpha^*=\chi_1(\alpha^*)\alpha^*-\nu\chi_1(\gamma^*)\gamma=-\nu^2\alpha^*\\&\chi_2*\alpha^*=\chi_2(\alpha^*)\alpha^*-\nu\chi_2(\gamma^*)\gamma=0\ .\end{aligned}$$

Therefore

$$d\alpha^*=\gamma\omega_0-\nu^2\alpha^*\omega_1\ .$$

Similarly one can check the other formulae listed in the following table

Table 2

$dI=0$	
$d\alpha=\alpha\omega_1+\nu^2\gamma^*\omega_2,$	$d\gamma^*=-\nu^{-1}\alpha\omega_0-\nu^2\gamma^*\omega_1$
$d\gamma=\gamma\omega_1-\nu\alpha^*\omega_2,$	$d\alpha^*=\gamma\omega_0-\nu^2\alpha^*\omega_1$

Now using Table 0 one can check that

Table 3

$$\begin{aligned}\omega_0&=\gamma^*d\alpha^*-\nu\alpha^*d\gamma^*\\\omega_1&=\alpha^*d\alpha+\gamma^*d\gamma\\&=-\gamma d\gamma^*-\nu^{-2}\alpha d\alpha^*\\\omega_2&=\gamma d\alpha-\nu^{-1}\alpha d\gamma\end{aligned}$$

Let $\omega\in\Gamma$. Then $\omega=\sum_k c_k\omega_k$, where $c_0, c_1, c_2\in\mathscr{A}$ and using formulae of Table 3 we obtain

$$\omega=\sum_{i=1}^{4}a_i db_i$$

where $a_i\in\mathscr{A}$ and $b_i=\alpha, \gamma, \alpha^*, \gamma^*$ for $i=1, 2, 3, 4$ resp. If the sum $\sum_i a_i b_i$ (where i runs from 1 to 4) does not vanish then we set $a_5=\sum_i a_i b_i$ and $b_5=-I$. Obviously

$$\omega=\sum_{i=1}^{5}a_i db_i \quad\text{and}\quad \sum_{i=1}^{5}a_i b_i=0\ .$$

This ends the proof of the theorem. Q. E. D.

The following result shows that the pseudogroup $S_\nu U(2)$ is connected.

Theorem 2.3. *If* $a\in\mathscr{A}$ *and* $da=0$ *then* $a=\lambda I$ *for some* $\lambda\in C$.

Proof. We shall use the basis (1.4). For any $m, n=0, 1, 2, \ldots$ and integer k we put

$$a^{kmn}=\begin{cases}\alpha^k\gamma^m\gamma^{*n} & \text{for } k\geq 0\\ \alpha^{*(-k)}\gamma^m\gamma^{*n} & \text{for } k<0\ .\end{cases}$$

It follows immediately from Table 1 that $\omega_1 a^{kmn}=\nu^{2(n-m-k)}a^{kmn}\omega_1$. Using (2.16) we obtain

$$f_1 * a^{kmn}=\nu^{2(n-m-k)}a^{kmn} \tag{2.21}$$

and taking into account (2.5) and (2.10) we get

$$\chi_1 * a^{kmn}=\frac{\nu^2}{1-\nu^2}(\nu^{2(n-m-k)}-1)a^{kmn}\ . \tag{2.22}$$

It is more difficult to compute the convolution products $\chi_0 * a^{kmn}$ and $\chi_2 * a^{kmn}$. In this computation one starts with formulae

$$\chi_0*\alpha=0,\ \chi_0*\alpha^*=\gamma,\ \chi_0*\gamma=0,\ \chi_0*\gamma^*=-\nu^{-1}\alpha$$
$$\chi_2*\alpha=\nu^2\gamma^*,\ \chi_2*\alpha^*=0,\ \chi_2*\gamma=-\nu\alpha^*,\ \chi_2*\gamma^*=0$$

which are essentially contained in Table 2 (cf. Def. (2.18)) and uses the equation $(r=0, 2)$

$$f_r * a^{kmn}=\nu^{n-m-k}a^{kmn}$$

which can be derived in the same way as (2.21). Then using repeatedly the product formula (2.20) one can compute the desired convolution products. We omit the boring details and quote the results:

$\chi_0 * a^{kmn}$ is a linear combination of the following two basis elements: $a^{k+1,m+1,n}$ and $a^{k+1,m,n-1}$. If $n>0$ then the latter element enters into the linear combination with a non-zero coefficient.

$\chi_2 * a^{kmn}$ is a linear combination of the following two basis elements: $a^{k-1,m,n+1}$ and $a^{k-1,m-1,n}$. If $m>0$ then the latter element enters into the linear combination with a non-zero coefficient.

Let $c\in\mathscr{A}$. Then

$$c=\sum_{kmn} c_{kmn}a^{kmn} \tag{2.23}$$

where $c_{kmn}\in \boldsymbol{C}$ and $c_{kmn}=0$ for almost all (kmn). The summation runs over $k=\ldots, -2, -1, 0, 1, 2, \ldots, m, n=0, 1, 2, \ldots$.

Assume that $dc=0$. Then

$$\chi_r * c=0 \tag{2.24}$$

for $r=0, 1, 2$.

Taking into account (2.22) we see that in (2.23) only the terms with $n-m=k$ do not vanish. It means that except the term $k=m=n=0$ in all non-zero terms in (2.23) $n+m$ is strictly positive.

Assume that some coefficients c_{kmn} are different from zero for $n+m>0$. Let s be the smallest strictly positive integer such that for some m, n we have $c_{kmn}\neq 0$ and $s=m+n$. Then either the basis element $a^{k+1,m,n-1}$ enters with a non-zero coefficient into the decomposition of $\chi_0 * c$, or $a^{k+1,m-1,n}$ enters with a non-zero coefficient into the decomposition of $\chi_2 * c$. In both cases we have contradiction with (2.24). It shows that all terms in (2.23) vanish except the one with $k=m=n=0$. Therefore

$$c=c_{000}I \ .$$

Q. E. D.

In the next sections we need a characterisation of those sequences $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$ for which $\sum_i a_i b_i=0$ and $\sum_i a_i db_i=0$.

To formulate this condition we shall use the bijective map

$$r: \mathscr{A}\otimes_{alg}\mathscr{A} \longrightarrow \mathscr{A}\otimes_{alg}\mathscr{A}$$

introduced in Section 1 such that

$$r(a\otimes b) = (a\otimes I)\Phi(b) \tag{2.25}$$

for all $a, b\in\mathscr{A}$. Let us note that for any $z\in\mathscr{A}\otimes_{alg}\mathscr{A}$; $c, c'\in\mathscr{A}$ we have

$$r((c\otimes I)z(I\otimes c')) = (c\otimes I)r(z)\Phi(c') \ . \tag{2.26}$$

Proposition 2.4. *Let* $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N\in\mathscr{A}$. *Then the following three conditions are equivalent*:

I. $\sum_i a_i b_i=0$ *and* $\sum_i a_i db_i=0$

II. $r(\sum_i a_i\otimes b_i) \in\mathscr{A}\otimes_{alg}R$, *where R is the right ideal in $\mathscr{A}$ generated by the following six elements*

$$\alpha^*+\nu^2\alpha-(1+\nu^2)I,\ \gamma^2,\gamma^*\gamma$$
$$\gamma^{*2},\ (\alpha-I)\gamma,\ (\alpha-I)\gamma^* \ . \tag{2.27}$$

III. $\sum_i a_i \otimes b_i$ *can be written as a finite sum of terms of the form* $(c\otimes I)r^{-1}(I\otimes x)(I\otimes c')$, *where* $c,c'\in\mathscr{A}$ *and* x *is one of the element* (2.27).

Proof. At first we have to show that

$$R=\{x\in\mathscr{A}:\ e(x)=0 \text{ and } \chi_r(x)=0 \text{ for } r=0,1,2\}. \tag{2.28}$$

Denote by R' the right hand side of (2.28). Clearly R' is a linear subset of $\mathscr{A}$ and (since functionals e, χ_0,χ_1,χ_2 are linearly independent) dim $\mathscr{A}/R'=4$. Moreover it follows immediately from (2.7) that R' is a right ideal in $\mathscr{A}$.

Using (2.7) and (2.3) one can easily check that functionals e, χ_0,χ_1,χ_2 kill all elements (2.27). It means that these elements belong to R' and we have $R\subset R'$. To prove (2.28) we have to show that

$$\dim \mathscr{A}/R\leq 4 \ . \tag{2.29}$$

For any $a,b\in\mathscr{A}$ we write $a\sim b$ if and only if $a-b$ belongs to R. Since R is a right ideal, both sides of the equivalence relation "$\sim$" may be multiplied from the right by any element of $\mathscr{A}$:

$$(a\sim b)\Rightarrow(ac\sim bc) \tag{2.30}$$

for any $a,b,c\in\mathscr{A}$.

Let us note that

$$\gamma^2\sim\gamma^*\gamma\sim\gamma^{*2}\sim 0$$
$$\alpha\gamma\sim\gamma,\ \alpha\gamma^*\sim\gamma^*.$$

Moreover $\alpha^*\sim-\nu^2\alpha+(1+\nu^2)I$ and in virtue of (2.30)

$$\alpha^*\gamma\sim-\nu^2\alpha\gamma+(1+\nu^2)\gamma\sim\gamma,\ \alpha^*\gamma^*\sim-\nu^2\alpha\gamma^*+(1+\nu^2)\gamma^*\sim\gamma^*$$
$$\alpha^{*2}\sim-\nu^2\alpha\alpha^*+(1+\nu^2)\alpha^*$$
$$=-\nu^2(I-\nu^2\gamma^*\gamma)+(1+\nu^2)\alpha^*\sim-\nu^2 I+(1+\nu^2)\alpha^*$$
$$\alpha^2\sim-\nu^{-2}\alpha^*\alpha+(\nu^{-2}+1)\alpha$$
$$=-\nu^{-2}(I-\gamma^*\gamma)+(\nu^{-2}+1)\alpha\sim-\nu^{-2}I+(\nu^{-2}+1)\alpha \ .$$

Obtained equivalences show that any second order polynomial in $\alpha,\gamma,\alpha^*,\gamma^*$ is equivalent to a linear combination of I,α,γ,γ^*. In virtue of (2.30), any polynomial of total order n in $\alpha,\gamma,\alpha^*,\gamma^*$ is equivalent to a polynomial of order $n-1$. Applying the principle of mathematical induction we conclude that any element of $\mathscr{A}$ is equivalent (modulo R) to a linear combination of the following four elements: $I, \alpha, \gamma, \gamma^*$. This proves inequality (2. 29) and ends the proof of (2. 28).

Taking into account (2. 25) one can easily check that

$$\begin{aligned} ab &= (id\otimes e)\, r(a\otimes b) \\ adb &= \sum_k (id\otimes\chi_k)\, r(a\otimes b)\,\omega_k \ . \end{aligned}$$

Therefore in virtue of (2. 28) the first condition of Prop. 2. 4 is satisfied if and only if

$$r(\sum_i a_i\otimes b_i) \in \mathscr{A}\otimes_{alg} R \ . \tag{2. 31}$$

This way we showed the equivalence of the two first conditions.

Assume now that (2. 31) holds. Then $r(\sum_i a_i\otimes b_i)$ is a finite sum of elements of the form $v\otimes xw$, where $v, w\in\mathscr{A}$ and x is one of the elements (2. 27). Since r is surjective, one can find elements $c_s, c'_s\in\mathscr{A}$ $(s=1, 2, \dots, S)$ such that $r(\sum_s c_s\otimes c'_s) = v\otimes w$. Then we have (cf. (2. 26))

$$\begin{aligned} &\sum_s r((c_s\otimes I)\, r^{-1}(I\otimes x)\,(I\otimes c'_s)) \\ &= \sum_s (c_s\otimes I)\,(I\otimes x)\,\Phi(c'_s) = (I\otimes x)\sum_s (c_s\otimes I)\,\Phi(c'_s) \\ &= (I\otimes x)\, r(\sum_s c_s\otimes c'_s) = v\otimes xw \ . \end{aligned}$$

This way we showed that $r(\sum_i a_i\otimes b_i)$ is a finite sum of elements of the form $r((c\otimes I)\, r^{-1}(I\otimes x)\,(I\otimes c'))$, where $c, c'\in\mathscr{A}$ and x is one of the elements of (2. 27). Now condition III of Prop. 2. 4 is implied by injectivity of r.

Conversely assume that the condition III holds. Then $r(\sum_i a_i\otimes b_i)$ is a finite sum of elements of the form

$$\begin{aligned} r((c\otimes I)\, r^{-1}(I\otimes x)\,(I\otimes c')) &= (c\otimes I)\,(I\otimes x)\,\Phi(c') \\ &= (c\otimes x)\,\Phi(c') \end{aligned} \tag{2. 32}$$

where $c, c'\in\mathscr{A}$ and x is one of the elements (2. 27). In the above computation we used (2. 26). All elements (2. 32) belong to $\mathscr{A}\otimes_{alg} R$. Indeed $x\in R$, $c\otimes x\in\mathscr{A}\otimes_{alg} R$ and $\mathscr{A}\otimes_{alg} R$ is a right ideal of $\mathscr{A}\otimes_{alg}\mathscr{A}$. Therefore $r(\sum_i a_i\otimes b_i)$ belongs to $\mathscr{A}\otimes_{alg} R$ and condition II follows.

Q. E. D.

We shall use the above result to prove the following interesting theorem showing that in a natural way Γ is a $*$-bimodule over $\mathscr{A}$.

Theorem 2.5. *There exists one and only one antilinear involution*

$$*:\ \Gamma \longrightarrow \Gamma \tag{2.33}$$

such that for all $a \in \mathscr{A}$ *and* $\omega \in \Gamma$ *we have*

$$(a\omega)^* = \omega^* a^* \tag{2.34}$$

$$(\omega a)^* = a^* \omega^* \tag{2.35}$$

$$(da)^* = d(a^*). \tag{2.36}$$

Moreover we have

Table 4

$\omega_0^* = \nu\omega_2$,	$\omega_1^* = -\omega_1$,	$\omega_2^* = \frac{1}{\nu}\omega_0$

Proof. Let $\omega \in \Gamma$. Then ω can be written as a finite sum (cf. Thm. 2.2)

$$\omega = \sum_i a_i db_i \tag{2.37}$$

where $a_i, b_i \in \mathscr{A}$ and $\sum_i a_i b_i = 0$. Using the rules (2.34)-(2.36) one easily obtain

$$\omega^* = -\sum_i b_i^* d(a_i^*)$$

and the uniqueness of (2.33) is proven. To prove the existence we have to show that for any $a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N$

$$\begin{pmatrix} \sum_i a_i b_i = 0 \\ \sum_i a_i db_i = 0 \end{pmatrix} \Rightarrow \left(\sum_i b_i^* da_i^* = 0\right). \tag{2.38}$$

Assume that $\sum_i a_i b_i = 0$ and $\sum_i a_i db_i = 0$. Then according to Prop. 2.4, $\sum_i a_i \otimes b_i$ can be written as a sum of terms of the form

$$(c \otimes I) r^{-1} (I \otimes x)(I \otimes c')$$

where $c, c' \in \mathscr{A}$ and x is one of the elements (2.27). Let $r^{-1}(I \otimes x) = \sum_t p_t \otimes q_t$ where $p_t, q_t \in \mathscr{A}$. Then the above expression equals to

$$\sum_t cp_t \otimes q_t c'. \tag{2.39}$$

Therefore $\sum_i b_i^* \otimes a_i^*$ can be written as a sum of terms of the form

$$\sum_t (q_t c')^* \otimes (cp_t)^* = (c'^* \otimes I)(\sum_t q_t^* \otimes p_t^*)(I \otimes c^*).$$

Applying the map r and using (2.26) we see that $r(\sum_i b_i^* \otimes a_i^*)$ can be written as a sum of terms of the form

$$(c'^* \otimes I) r(\sum_t q_t^* \otimes p_t^*) \Phi(c^*) \tag{2.40}$$

where $c, c' \in \mathscr{A}$ and $p_t, q_t \in \mathscr{A}$ are such that $r(\sum_t p_t \otimes q_t) = I \otimes x$, where x is one of the elements (2.27). Now according to (1.24)

$$r(\sum_t q_t^* \otimes p_t^*) = I \otimes \kappa(x)^*$$

and (2.40) equals to

$$(c'^* \otimes \kappa(x)^*) \Phi(c^*). \tag{2.41}$$

The mapping $x \to \kappa(x)^*$ is antilinear, multiplicative and (cf. (1.14)) maps elements $I, \alpha, \gamma, \alpha^*, \gamma^*$ onto $I, \alpha, -\nu\gamma^*, \alpha^*$ and $-\nu^{-1}\gamma$ resp. Therefore this mapping maps elements (2.27) onto elements (2.27) multiplied by a numerical factor. It means that in the formula (2.41) $\kappa(x)^* \in R$. Remembering that $\mathscr{A} \otimes_{alg} R$ is a right ideal in $\mathscr{A} \otimes_{alg} \mathscr{A}$ we see that (2.41) belong to $\mathscr{A} \otimes_{alg} R$. Therefore $r(\sum_i b_i^* \otimes a_i^*) \in \mathscr{A} \otimes_{alg} R$ and using Prop. 2.4 we obtain

$$\sum_i b_i^* da_i^* = 0.$$

This way the implication (2.38) is proved.

Now for any $\omega \in \Gamma$ of the form (2.37) we set

$$\omega^* = -\sum_i b_i^* d(a_i^*). \tag{2.42}$$

The implication (2.38) shows that this definition is correct i.e. the right hand side of the above formula is independent of the particular choice of a_i and b_i in (2.37).

Clearly $*: \Gamma \to \Gamma$ is an antilinear involution. Moreover if ω is given by (2.37) and $a \in \mathscr{A}$ then $a\omega = \sum_i (aa_i) db_i$ and

$$\begin{aligned}(a\omega)^* &= -\sum_i b_i^* d((aa_i)^*) = -\sum_i b_i^* d(a_i^* a^*) \\ &= -\sum_i b_i^* d(a_i^*) a^* - \sum_i b_i^* a_i^* d(a^*) = \omega^* a^*\end{aligned}$$

since $\sum_i b_i^* a_i^* = (\sum_i a_i b_i)^* = 0$. Formula (2. 34) is proved. To prove (2. 35) we use (2. 34) and the fact that $*$ is an involution:

$$(\omega a)^* = ((a^* \omega^*)^*)^* = a^* \omega^*. \tag{2.43}$$

Setting in (2. 37): $a_1 = I$, $a_2 = -a$, $b_1 = a$ and $b_2 = I$ we obtain $\omega = da$ and (2. 42) shows that $(da)^* = d(a^*)$. It proves (2. 36).

Using the expressions for ω_0 and ω_2 given in Table 3 and the definition (2. 42) one easily check that $\omega_0^* = \nu \omega_2$ and $\omega_2^* = \nu^{-1} \omega_0$. To compute ω_1^* one has to use the following expression for ω_1 (cf. Table 3)

$$\omega_1 = \alpha^* d\alpha + \gamma^* d\gamma - IdI.$$

Then using (2. 42) one obtains $\omega_1^* = -\omega_1$. Q. E. D.

§ 3. Higher Order Differential Calculus

In differential geometry second order differential forms on a smooth manifold M are sections of the bundle $\wedge^2 T^*(M)$ which can be constructed starting with the cotangent bundle $T^*(M)$ by taking the tensor product $T^*(M) \otimes_M T^*(M)$ and dividing by the subbundle of symmetric elements. In the algebraic language, denoting by Γ_M the $C^\infty(M)$-bimodule of sections of $T^*(M)$ we construct the bimodules

$$\Gamma_M^{\otimes 2} = \Gamma_M \otimes_{\mathscr{C}^\infty(M)} \Gamma_M$$
$$\Gamma_M^{\wedge 2} = \Gamma_M^{\otimes 2} / S_M^2$$

where S_M^2 denotes the sub-bimodule of $\Gamma_M^{\otimes 2}$ composed of all elements of $\Gamma_M^{\otimes 2}$ which are invariant under the bimodule homomorphism

$$\sigma: \Gamma_M^{\otimes 2} \longrightarrow \Gamma_M^{\otimes 2} \tag{3.1}$$

which maps $\omega \otimes_{\mathscr{C}^\infty(M)} \omega'$ onto $\omega' \otimes_{\mathscr{C}^\infty(M)} \omega$. Then $\Gamma_M^{\wedge 2}$ coincides with the $\mathscr{C}^\infty(M)$-bimodule of second order differential forms on M.

Unfortunately in a noncommutative case (i. e. when $\mathscr{C}^\infty(M)$ is replaced by a noncommutative algebra) the homomorphism (3. 1) can not be introduced and there is no canonical way to distinguish the submodule S^2.

We shall see however that all the main feature of the differential calculus over Lie groups can be reproduced for the pseudogroup $S_\nu U(2)$ if we take as S^2 the sub-bimodule of

$$\Gamma^{\otimes 2} = \Gamma \otimes_{\mathscr{A}} \Gamma$$

generated by six elements of the form

$$\sum_{ki=0}^{2}(\chi_k*\chi_i)(x)\,\omega_k\otimes_{\mathscr{A}}\omega_i \tag{3.2}$$

where x takes values (2.27).

At first we show, how to compute (3.2) for $x=\alpha^*+\nu^2\alpha-(1+\nu^2)I$, $\gamma^2, \gamma^*\gamma, \gamma^{*2}$, $(\alpha-I)\gamma$ and $(\alpha-I)\gamma^*$. Let for example $x=\gamma^2$. Then using the basic rules of differential calculus, Table 2, 1 and 0 we have

$$\begin{aligned} d(\gamma^2) &= (d\gamma)\gamma+\gamma d\gamma \\ &= (\gamma\omega_1-\nu\alpha^*\omega_2)\gamma+\gamma(\gamma\omega_1-\nu\alpha^*\omega_2) \\ &= (\nu^{-2}+1)\gamma^2\omega_1-(1+\nu^2)\alpha^*\gamma\omega_2 \\ &= \frac{1+\nu^2}{\nu^2}(\gamma^2\omega_1-\nu^2\alpha^*\gamma\omega_2). \end{aligned}$$

It means (cf (2.18)) that

$$\sum_{i=0}^{2}(\chi_i*\gamma^2)\,\omega_i=\frac{1+\nu^2}{\nu^2}(\gamma^2\omega_1-\nu^2\alpha^*\gamma\omega_2).$$

Now we have (cf (1.21))

$$\begin{aligned} \sum_{i=0}^{2}(\chi_k*\chi_i)(\gamma^2)\,\omega_i &= \sum_{i=0}^{2}\chi_k(\chi_i*\gamma^2)\,\omega_i \\ &= \frac{1+\nu^2}{\nu^2}(\chi_k(\gamma^2)\,\omega_1-\nu^2\chi_k(\alpha^*\gamma)\,\omega_2). \end{aligned}$$

$\chi_k(\gamma^2)$ and $\chi_k(\alpha^*\gamma)$ can be easily computed due to formulae (2.3) and (2.7). We get

$$\begin{aligned} &\chi_k(\gamma^2)=0 && \text{for } k=0,1,2 \\ &\chi_k(\alpha^*\gamma)=0 && \text{for } k=0,1 \\ &\chi_2(\alpha^*\gamma)=-\nu. \end{aligned}$$

Therefore

$$\sum_{ki=0}^{2}(\chi_k*\chi_i)(\gamma^2)\,\omega_k\otimes_{\mathscr{A}}\omega_i=\nu(1+\nu^2)\,\omega_2\otimes_{\mathscr{A}}\omega_2.$$

In the following the numeric factor will not play any role. Similarly one can compute (3.2) for other x. We list the results of these computations in the following

Proposition 3.1. *The linear span of elements of the form* (3.2), *where x takes values* (2.27) *coincides with the linear span of* $\omega_0\otimes_{\mathscr{A}}\omega_0$, $\omega_1\otimes_{\mathscr{A}}\omega_1$, $\omega_2\otimes_{\mathscr{A}}\omega_2$, $\omega_2\otimes_{\mathscr{A}}\omega_0+\nu^2\omega_0\otimes_{\mathscr{A}}\omega_2$, $\omega_1\otimes_{\mathscr{A}}\omega_0+\nu^4\omega_0\otimes_{\mathscr{A}}\omega_1$, *and* $\omega_2\otimes_{\mathscr{A}}\omega_1+\nu^4\omega_1\otimes_{\mathscr{A}}\omega_2$.

Let us denote by $\zeta_1, \zeta_2, \ldots, \zeta_6$ elements of $\Gamma^{\otimes 2}$ listed in Prop. 3.1 and let

$$S^2 = \{\sum_{r=1}^{6} a_r\zeta_r:\ a_r \in \mathscr{A}\ r=1,2,\ldots,6\}.$$

It follows immediately from Table 1 that S^2 is a sub-bimodule of $\Gamma^{\otimes 2}$.

Now we are going to construct $\mathscr{A}$-bimodules $\Gamma^{\wedge 2}$, $\Gamma^{\wedge 3}, \ldots$ which elements correspond to differential forms of order $2, 3, \ldots$ resp. We introduce these bimodules at once by constructing the graded external algebra $\Gamma^{\wedge}$. Then $\Gamma^{\wedge n}$ will denote the subspace of $\Gamma^{\wedge}$ composed of elements of grade n.

For any natural n we denote by $\Gamma^{\otimes n}$ the $\mathscr{A}$-bimodule $\Gamma \otimes_{\mathscr{A}} \Gamma \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \Gamma$ (n-factors). $\Gamma^{\otimes 0}$ will denote $\mathscr{A}$. Let

$$\Gamma^{\otimes} = \sum_{n=0}^{\infty}{}^{\oplus}\, \Gamma^{\otimes n},$$

Clearly $\Gamma^{\otimes}$ is a graded algebra containing $\mathscr{A}$ as the subalgebra of grade 0 elements and Γ as the subspace of grade 1 elements. We introduce graded $*$-algebra structure in $\Gamma^{\otimes}$ in the following way: on elements of grade 0, $*$ coincides with the hermitian conjugation in $\mathscr{A}$; on elements of grade 1 we use $*$ operation introduced in Section 2 (cf Thm 2.5); for elements of higher order we put

$$(\theta_1 \otimes_{\mathscr{A}} \theta_2 \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta_n)^* = s_n \theta_n^* \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \theta_2^* \otimes_{\mathscr{A}} \theta_1^* \tag{3.3}$$

where $\theta_1, \theta_2, \ldots, \theta_n \in \Gamma$ and s_n is the sign of the permutation $\begin{pmatrix} 1, & 2, & \ldots, & n \\ n, & n-1, & \ldots, & 1 \end{pmatrix}$: $s_n = (-1)^{n(n+1)/2}$. Then

$$(\theta \otimes_{\mathscr{A}} \theta')^* = (-1)^{ki} \theta'^* \otimes_{\mathscr{A}} \theta^*$$

for any homogeneous elements θ and θ' of $\Gamma^{\otimes}$ of grade k and i resp.

Let S be the (two-sided) ideal in $\Gamma^{\otimes}$ generated by S^2. Any element of S can be written as a sum of homogeneous (i.e. of definite grade) elements belonging to S. Therefore the quotient algebra

$$\Gamma^{\wedge} = \Gamma^{\otimes}/S$$

have a natural grading. It is clear that S contains no elements of grade 0 and 1. Therefore in $\Gamma^{\wedge}$, the subalgebra of elements of grade 0 can be identified with $\mathscr{A}$ and the subspace of elements of grade 1 with the bimodule Γ. The multiplication in $\Gamma^\wedge$ will be denoted by $\wedge$ (this sign is usually omitted if one of the factors is of grade 0). The algebra $\Gamma^\wedge$ is generated by $\mathscr{A}$ and grade one elements $\omega_0, \omega_1, \omega_2$.

Taking into account the explicite expressions for $\zeta_1, \zeta_2, \ldots, \zeta_6$ generating S^2 we get the following

Table 5	External product identities
$\omega_0\wedge\omega_0=0$,	$\omega_2\wedge\omega_0=-\nu^2\omega_0\wedge\omega_2$
$\omega_1\wedge\omega_1=0$,	$\omega_1\wedge\omega_0=-\nu^4\omega_0\wedge\omega_1$
$\omega_2\wedge\omega_2=0$,	$\omega_2\wedge\omega_1=-\nu^4\omega_1\wedge\omega_2$

Moreover since the tensor products considered in this section are tensor products over $\mathscr{A}$ we have $\theta a\otimes_{\mathscr{A}}\theta'=\theta\otimes_{\mathscr{A}}a\theta'$ and

$$\theta a\wedge\theta'=\theta\wedge a\theta' \tag{3.4}$$

for any θ, $\theta'\in\Gamma^\wedge$ and $a\in\mathscr{A}$.

One can easily check that S^2 is $*$-invariant. The same holds for S. Due to this fact $\Gamma^\wedge$ has a natural graded $*$-algebra structure. In particular we have

$$(\zeta\wedge\zeta')^*=(-1)^{\partial\zeta\partial\zeta'}\zeta'^*\wedge\zeta^* \tag{3.5}$$

for any homogeneous $\zeta, \zeta'\in\Gamma^\wedge$; $\partial\zeta$ and $\partial\zeta'$ denote grades of ζ and ζ' resp.

Now we can formulate the main theorem of this section.

Theorem 3.2. *There exists one and only one linear map*

$$d\colon \Gamma^\wedge\longrightarrow\Gamma^\wedge \tag{3.6}$$

such that

1° *d rises the grade by one*

2° *On $\mathscr{A}\subset\Gamma^\wedge$, d is given by* (2.18)

3° *d is a graded derivative*:

$$d(\zeta\wedge\zeta')=d\zeta\wedge\zeta'+(-1)^{\partial\zeta}\zeta\wedge d\zeta' \tag{3.7}$$

for all homogeneous elements $\zeta, \zeta'\in\Gamma^\wedge$; $\partial\zeta$ denotes the grade of ζ.

4° $(d\zeta)^*=d(\zeta^*)$ $\qquad \zeta\in\Gamma^\wedge$

5° $d(d\zeta)=0$ $\qquad \zeta\in\Gamma^\wedge$.

The proof of this theorem is based on the following proposition which will be proven in the next section.

Proposition 3.3. *There exists a larger graded $*$-algebra $\tilde{\Gamma}^\wedge$ (with multiplication denoted by $\wedge$ or by $\circ$ if one of the factors is of grade 0) containing $\Gamma^\wedge$ and a grade one element $X \in \tilde{\Gamma}^\wedge$ such that*

$$X \wedge X = 0 \tag{3.9}$$

$$X^* = -X \tag{3.10}$$

and

$$Xa - aX = da \tag{3.11}$$

for any $a \in \mathscr{A}$ (da denotes the derivation introduced in Section 2).

Proof of Thm. 3.2. According to the condition $2°$, d is defined uniquely on elements of grade 0. Any element of grade 1 is a sum of terms of the form adb, where $a, b \in \mathscr{A}$. Using the conditions $3°$ and $5°$ we have $d(adb) = da \wedge db$. Therefore d is defined uniquely on elements of grade 1. Remembering that $\Gamma^\wedge$ is generated by elements of grade 0 and 1 and taking into account the condition $3°$ we see that d (if it exists) is defined uniquely.

To prove the existence we use the larger graded $*$-algebra $\tilde{\Gamma}^\wedge$ and the element $X \in \tilde{\Gamma}^\wedge$ described in Prop. 3.3. For any $\zeta \in \tilde{\Gamma}^\wedge$ we put

$$d\zeta = [X, \zeta]_{\mathrm{grad}}$$

where $[X, \zeta]_{\mathrm{grad}}$ is the graded commutator:

$$[X, \zeta]_{\mathrm{grad}} = \begin{cases} X \wedge \zeta - \zeta \wedge X & \text{if } \zeta \text{ is even} \\ X \wedge \zeta + \zeta \wedge X & \text{if } \zeta \text{ is odd .} \end{cases}$$

We check that the conditions $1°$–$5°$ of Thm. 3.2 are satisfied. Condition $1°$ is obvious (X is of grade 1). Condition $2°$ follows immediately from (3.11). Let $\zeta, \zeta' \in \tilde{\Gamma}^\wedge$ be homogeneous elements of grades $\partial\zeta$ and $\partial\zeta'$ resp. Then

$$\begin{aligned} & d\zeta \wedge \zeta' + (-1)^{\partial\zeta} \zeta \wedge d\zeta' \\ & \quad = (X \wedge \zeta - (-1)^{\partial\zeta} \zeta \wedge X) \wedge \zeta' \\ & \qquad + (-1)^{\partial\zeta} \zeta \wedge (X \wedge \zeta' - (-1)^{\partial\zeta'} \zeta' \wedge X) \\ & \quad = X \wedge \zeta \wedge \zeta' - (-1)^{\partial\zeta + \partial\zeta'} \zeta \wedge \zeta' \wedge \chi = d(\zeta \wedge \zeta') \end{aligned}$$

and the condition $3°$ is satisfied. Let $\zeta \in \tilde{\Gamma}^\wedge$ be a homogeneous element of grade $\partial\zeta$. Then ζ^* is of the same grade and taking into account (3.5) and (3.10) we have

$$(d\zeta)^* = (X\wedge\zeta - (-1)^{\partial\zeta}\zeta\wedge X)^* = (-1)^{\partial\zeta}\zeta^*\wedge X^* - X^*\wedge\zeta^*$$
$$= X\wedge\zeta^* - (-1)^{\partial\zeta}\zeta^*\wedge X = d(\zeta^*).$$

It shows the condition 4°. To prove the condition 5° we compute

$$\begin{aligned}d(d\zeta) &= d(X\wedge\zeta - (-1)^{\partial\zeta}\zeta\wedge X)\\ &= X\wedge(X\wedge\zeta - (-1)^{\partial\zeta}\zeta\wedge X) - (-1)^{\partial\zeta+1}(X\wedge\zeta - (-1)^{\partial\zeta}\zeta\wedge X)\wedge X\\ &= X\wedge X\wedge\zeta - \zeta\wedge X\wedge X\end{aligned}$$

and the condition 5° follows directly from (3.9).

To end the proof we have to show that $d\zeta\in\Gamma^\wedge$ for any $\zeta\in\Gamma^\wedge$. Let us consider the linear set

$$\{\zeta\in\Gamma^\wedge:\ d\zeta\in\Gamma^\wedge\}. \tag{3.12}$$

Condition 3° shows that (3.12) is a subalgebra in $\Gamma^\wedge$. According to the condition 2° this subalgebra contains $\mathscr{A}$ and due to the condition 5° all elements of the form da, where $a\in\mathscr{A}$. Obviously the smallest subalgebra of $\Gamma^\wedge$ containing $\mathscr{A}$ and $d\mathscr{A}$ coincides with $\Gamma^\wedge$. In other words $d\zeta\in\Gamma^\wedge$ for all $\zeta\in\Gamma^\wedge$. Q. E. D.

Later we need the explicite formulae for $d\omega_0$, $d\omega_1$ and $d\omega_2$. If $\theta = \sum_i a_i db_i$ (where $a_i, b_i\in\mathscr{A}$; $i=1, 2,\dots, n$) then using the rules of differential calculus listed in Thm. 3.2 we have

$$d\theta = \sum_i da_i\wedge db_i\ .$$

Using this formula, equations contained in Tables 0, 1, 2, 3 and 5 and the property (3.4) one can compute $d\omega_k$ for $k=0, 1, 2$. E. g.

$$\begin{aligned}d\omega_2 &= d\gamma\wedge d\alpha - \frac{1}{\nu}d\alpha\wedge d\gamma\\ &= (\gamma\omega_1 - \nu\alpha^*\omega_2)\wedge(\alpha\omega_1 + \nu^2\gamma^*\omega_2) - \frac{1}{\nu}(\alpha\omega_1 + \nu^2\gamma^*\omega_2)\wedge(\gamma\omega_1 - \nu\alpha^*\omega_2)\\ &= \nu^2\gamma\omega_1\wedge\gamma^*\omega_2 - \nu\alpha^*\omega_2\wedge\alpha\omega_1 + \alpha\omega_1\wedge\alpha^*\omega_2 - \nu\gamma^*\omega_2\wedge\gamma\omega_1\\ &= \nu^4\gamma^*\gamma\omega_1\wedge\omega_2 - \alpha^*\alpha\omega_2\wedge\omega_1 + \nu^2\alpha\alpha^*\omega_1\wedge\omega_2 - \gamma^*\gamma\omega_2\wedge\omega_1\\ &= \nu^2\omega_1\wedge\omega_2 - \omega_2\wedge\omega_1 = \nu^2(1+\nu^2)\,\omega_1\wedge\omega_2\ .\end{aligned}$$

Similarly one can check two other formulae contained in the following

Table 6	Cartan Maurer formulae
$d\omega_0=\nu^2(1+\nu^2)\,\omega_0\wedge\omega_1$	
$d\omega_1=\quad\nu\quad\omega_0\wedge\omega_2$	
$d\omega_2=\nu^2(1+\nu^2)\,\omega_1\wedge\omega_2$	

Moreover remembering that $d^2=0$ we get

$$\begin{aligned} &d(\omega_0\wedge\omega_1)=0\\ &d(\omega_1\wedge\omega_2)=0\\ &d(\omega_0\wedge\omega_2)=0\ . \end{aligned} \tag{3.13}$$

It follows immediately from Table 5 that any element $\zeta^2\in\Gamma^{\wedge 2}$ is of the form

$$\zeta^2=a_0\omega_1\wedge\omega_2+a_1\omega_2\wedge\omega_0+a_2\omega_0\wedge\omega_1 \tag{3.14}$$

where $a_k\in\mathscr{A}$ $(k=0,1,2)$. Looking more closely to the structure of the ideal S one sees that a_k $(k=0,1,2)$ are uniquely determined by ζ^2. Similarly any element $\zeta^3\in\Gamma^{\wedge 3}$ is of the form

$$\zeta^3=a\omega_0\wedge\omega_1\wedge\omega_2$$

where $a\in\mathscr{A}$ is uniquely determined by ζ^3. Moreover it is clear that $\Gamma^{\wedge n}=0$ for $n>3$. Therefore the de Rham cochain complex for $S_\nu U(2)$ has the following form:

$$0\longrightarrow\mathscr{A}\xrightarrow{d}\Gamma\xrightarrow{d}\Gamma^{\wedge 2}\xrightarrow{d}\Gamma^{\wedge 3}\longrightarrow 0\ .$$

We shall prove in Section 5 that this sequence is exact in Γ and $\Gamma^{\wedge 2}$. It means that the cohomology groups in dimensions 1 and 2 are trivial. The group $H^0(S_\nu U(2))$ is isomorphic to $\boldsymbol{C}$ (cf. Thm. 2.3)). We shall prove in Section 5 that $H^3(S_\nu U(2))$ is isomorphic to $\boldsymbol{C}$. It means that

$$\dim\operatorname{coker}\ (\Gamma^{\wedge 2}\xrightarrow{d}\Gamma^{\wedge 3})=1$$

and there exists unique (up to a complex factor) linear functional

$$\int:\Gamma^{\wedge 3}\longrightarrow\boldsymbol{C}$$

such that for any $\zeta^2\in\Gamma^{\wedge 2}$ we have the Stokes formula

$$\int d\zeta^2=0\ . \tag{3.15}$$

It turns out that for any $a\in\mathscr{A}$

$$\int a\omega_0\wedge\omega_1\wedge\omega_2=h(a) \tag{3.16}$$

where h is the Haar measure (cf. Section 1). Indeed if ζ^2 is given by (3.14) then in virtue of (3.13) and Table 5

$$d\zeta^2=(\chi_0*a_0+\nu^6\chi_1*a_1+\nu^6\chi_2*a_2)\,\omega_0\wedge\omega_1\wedge\omega_2$$

and taking (3.16) as the definition of the integral we have (cf (1.21))

$$\int d\zeta^2=h(\chi_0*a_0+\nu^6\chi_1*a_1+\nu^6\chi_2*a_2)$$
$$=\chi_0(a_0*h)+\nu^6\chi_1(a_1*h)+\nu^6\chi_2(a_2*h)$$

and (3.15) follows (cf. (1.22) and use equation $\chi_k(I)=0$ for $k=0,1,2$)

Remark. $\int$ is not a graded trace in the sense of A. Connes.

For any $a\in\mathscr{A}$ and $k=0,1,2$ we set

$$\nabla_k a=\chi_k*a\,.$$

Then

$$da=\sum_k(\nabla_k a)\,\omega_k\,.$$

It means that in our theory ∇_k play the role of left invariant differential operators of the first order.

Using the basic properties of the external derivative d listed in Thm. 3.2 we have

$$0=d\Big(\sum_k(\nabla_k a)\,\omega_k\Big)=\sum_k d(\nabla_k a)\wedge\omega_k+\sum_k(\nabla_k a)\,d\omega_k$$
$$=\sum_{ik}(\nabla_i\nabla_k a)\,\omega_i\wedge\omega_k+\sum_k(\nabla_k a)\,d\omega_k\,.$$

The last expression can be reduced to the form (3.14) with the help of Tables 5 and 6. This way we obtain

Table 7	Commutation relations for infinitesimal shifts
$\nu\nabla_2\nabla_0-\frac{1}{\nu}\nabla_0\nabla_2=\nabla_1$	
$\nu^2\nabla_1\nabla_0-\frac{1}{\nu^2}\nabla_0\nabla_1=(1+\nu^2)\nabla_0$	
$\nu^2\nabla_2\nabla_1-\frac{1}{\nu^2}\nabla_1\nabla_2=(1+\nu^2)\nabla_2$	

Clearly these relation correspond to the well known formulae

$$\nabla_i\nabla_j-\nabla_j\nabla_i=\sum_k c^k_{ij}\nabla_k$$

for infinitesimal shifts on a Lie group (c^k_{ij} are structure constant of the corresponding Lie algebra).

§ 4. Extended Bimodules

This section is exclusively devoted to the proof of Prop. 3. 3. No notion introduced in this section will be used later. The reader interesting in applications of the differential calculus described in the previous sections may omit Section 4 and pass to Section 5.

To prove Prop. 3. 3 we have to extend $*$-bimodule Γ. Let $\mathscr{A}X$ be a free left $\mathscr{A}$-module with one generator X and

$$\tilde{\Gamma}=\mathscr{A}X\oplus\Gamma. \tag{4.1}$$

Any element $\varpi\in\tilde{\Gamma}$ is of the form

$$\varpi=cX+\omega \tag{4.2}$$

where $c\in\mathscr{A}$ and $\omega\in\Gamma$ are uniquely determined. We introduce right multiplication by elements of $\mathscr{A}$: for any $\varpi\in\tilde{\Gamma}$ of the form (4. 2) and any $a\in\mathscr{A}$ we set

$$\tilde{\omega}a=caX+(cda+\omega a). \tag{4.3}$$

Let us notice that for any $a, b\in\mathscr{A}$ we have $(a\tilde{\omega})b=a(\tilde{\omega}b)$ and

$$\begin{aligned}(\tilde{\omega}a)b&=(caX+(cda+\omega a))b=cabX+cadb+c(da)b+\omega ab\\&=c(ab)X+cd(ab)+\omega(ab)=\tilde{\omega}(ab).\end{aligned}$$

Moreover obviously $\tilde{\omega}I=\tilde{\omega}$. Therefore the left $\mathscr{A}$-module $\tilde{\Gamma}$ endowed with the right multiplication (4. 3) is a bimodule over $\mathscr{A}$.

Inserting in (4. 2): $c=I$, $\omega=0$ and using (4. 3) we obtain $Xa=aX+da$. Therefore

$$da=Xa-aX. \tag{4.4}$$

We know (cf. Thm. 2. 2) that any element of Γ can be written as a sum of terms of the form adb, where $a, b\in\mathscr{A}$. Taking into account (4. 4) we see that any element of $\tilde{\Gamma}$ is a sum of terms of the form aXb, where $a, b\in\mathscr{A}$.

Let Q be a bimodule over $\mathscr{A}$. Then using the above remark and equation $q\otimes_{\mathscr{A}}aXb=qa\otimes_{\mathscr{A}}Xb$ we see that any element of $Q\otimes_{\mathscr{A}}\tilde{\Gamma}$ is a sum of terms of the form $q\otimes_{\mathscr{A}}Xb$, where $q\in Q$ and $b\in\mathscr{A}$. We express this property writing the equation

$$Q\otimes_{\mathscr{A}}\tilde{\Gamma}=Q\otimes_{\mathscr{A}}X\mathscr{A}. \tag{4.5}$$

We introduce $*$-bimodule structure in $\tilde{\Gamma}$: for any $\tilde{\omega}$ of the form (4.2) we set

$$\tilde{\omega}^*=-Xc^*+\omega^*. \tag{4.6}$$

Let us notice that

$$\begin{aligned}(\tilde{\omega}^*)^*&=(-Xc^*+\omega^*)^*=(-c^*X-dc^*+\omega^*)^*\\&=Xc-dc+\omega=cX+\omega=\tilde{\omega}\ .\end{aligned}$$

It shows that the map $*$: $\tilde{\Gamma}\longrightarrow\tilde{\Gamma}$ introduced by (4.6) is an involution. It follows immediately from (4.6) that $(a\tilde{\omega})^*=\tilde{\omega}^*a^*$ for any $a\in\mathscr{A}$ and $\tilde{\omega}\in\tilde{\Gamma}$. Repeating the computation (2.43) we get $(\tilde{\omega}a)^*=a^*\tilde{\omega}^*$. It means that $\tilde{\Gamma}$ is a $*$-bimodule over $\mathscr{A}$.

Inserting in (4.2): $c=I$, $\omega=0$ and using (4.6) we obtain

$$X^*=-X\ . \tag{4.7}$$

It follows immediately from the construction that Γ is a sub-bimodule of $\tilde{\Gamma}$. For any $\tilde{\omega}$ of the form (4.2) we set

$$j(\tilde{\omega})=c.$$

One can easily check that

$$j:\ \tilde{\Gamma}\longrightarrow\mathscr{A} \tag{4.8}$$

is a $*$-bimodule homomorphism and that $\ker j=\Gamma$. Therefore we have the exact sequence

$$0\longrightarrow\Gamma\hookrightarrow\tilde{\Gamma}\xrightarrow{\ j\ }\mathscr{A}\longrightarrow0\ .$$

We repeate for $\tilde{\Gamma}$ the tensor algebra construction that is done in Section 3 for Γ: for any natural n, $\tilde{\Gamma}^{\otimes n}$ will denote the tensor product (over $\mathscr{A}$) of n copies of $\tilde{\Gamma}$. We use the definition (3.3) to endow $\tilde{\Gamma}^{\otimes n}$ with a $*$-bimodule structure. For $n=0$ we set $\tilde{\Gamma}^{\otimes n}=\mathscr{A}$. Since Γ is a sub-bimodule of $\tilde{\Gamma}$ hence $\Gamma^{\otimes n}\subset\tilde{\Gamma}^{\otimes n}$ for all n.

For any natural $k\leq n$ we set

$$j_k=id\otimes_{\mathscr{A}}\dots\otimes_{\mathscr{A}}id\otimes_{\mathscr{A}}j\otimes_{\mathscr{A}}id\otimes_{\mathscr{A}}\dots\otimes_{\mathscr{A}}id$$

where id is the identity map acting on $\tilde{\Gamma}$ and j standing at the k-th place is the $*$-bimodule homomorphism (4.8). More precisely j_k is the $*$-bimodule homomorphism

$$j_k:\ \tilde{\Gamma}^{\otimes n} \longrightarrow \tilde{\Gamma}^{\otimes(n-1)}$$

such that

$$\begin{aligned} &j_k(\tilde{\theta}_1 \otimes_{\mathscr{A}} \tilde{\theta}_2 \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \tilde{\theta}_n) \\ &\quad = \tilde{\theta}_1 \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \tilde{\theta}_{k-2} \otimes_{\mathscr{A}} \tilde{\theta}_{k-1} j(\tilde{\theta}_k) \otimes_{\mathscr{A}} \tilde{\theta}_{k+1} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \tilde{\theta}_n \end{aligned} \tag{4.9}$$

for any $\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n \in \tilde{\Gamma}$. One can easily check that

$$\ker j_k = \tilde{\Gamma}^{\otimes(k-1)} \otimes_{\mathscr{A}} \Gamma \otimes_{\mathscr{A}} \tilde{\Gamma}^{\otimes(n-k)} . \tag{4.10}$$

Let

$$\tilde{\Gamma}^{\otimes} = \sum_{n=0}^{\infty}{}^{\oplus} \tilde{\Gamma}^{\otimes n} .$$

Like in the case considered in Section 3, $\tilde{\Gamma}^{\otimes}$ is a graded $*$-algebra. We denote by $\tilde{S}$ the (two-sided) ideal in $\tilde{\Gamma}^{\otimes}$ generated by single element $X \otimes_{\mathscr{A}} X \in \tilde{\Gamma}^{\otimes 2}$. Obviously $\tilde{S}$ is $*$-invariant. Moreover

$$\tilde{S} = \sum_{n=2}^{\infty}{}^{\oplus} \tilde{S}^n$$

where $\tilde{S}^n$ denotes the set of all elements of $\tilde{S}$ having the grade equal to n. Therefore the quotient

$$\tilde{\Gamma}^{\wedge} = \tilde{\Gamma}^{\otimes} / \tilde{S}$$

is a graded $*$-algebra containing $\mathscr{A}$ as the subalgebra of elements of grade 0 and $\tilde{\Gamma}$ as the $\mathscr{A}$-bimodule of elements of grade one. The multiplication in $\tilde{\Gamma}^{\wedge}$ will be denoted with the same symbol as in $\Gamma^{\wedge}$. We know that $X \otimes_{\mathscr{A}} X \in \tilde{S}$. Therefore

$$X \wedge X = 0 . \tag{4.11}$$

Lemma 4.1. *Let* $p_1, p_2, \ldots, p_s, q_1, q_2, \ldots, q_s \in \mathscr{A}$ *and*

$$r\Big(\sum_t p_t \otimes q_t\Big) = I \otimes z \tag{4.12}$$

where r *is the bijective linear map introduced by* (2.25) *and* z *belongs to the right ideal* R *generated by* (2.27). *Then*

$$\sum_t p_t X \otimes_{\mathscr{A}} X q_t = \sum_{ki=0}^{2} (\chi_k * \chi_i)(z)\, \omega_k \otimes_{\mathscr{A}} \omega_i . \tag{4.13}$$

Proof. In virtue of Prop 2.4

$$\sum_t p_t q_t = 0, \qquad \sum_t p_t d q_t = 0 . \tag{4.14}$$

The second equation means that

$$\sum_t p_t(\chi_i * q_t) = 0 \tag{4.15}$$

for $i=0, 1, 2$. Using (4.4) we have

$$\sum_t p_t X q_t = \sum_t p_t q_t X + \sum_t p_t dq_t$$

and (cf. (4.14))

$$\sum_t p_t X q_t = 0 . \tag{4.16}$$

Taking into account (4.16), (4.4) and (4.15) we compute

$$\begin{aligned}\sum_t p_t X \otimes_{\mathscr{A}} X q_t &= \sum_t p_t X q_t \otimes_{\mathscr{A}} X + \sum_t p_t X \otimes_{\mathscr{A}} dq_t \\ &= \sum_{ti} p_t X(\chi_i * q_t) \otimes_{\mathscr{A}} \omega_i \\ &= \sum_{ti} p_t(\chi_i * q_t) X \otimes_{\mathscr{A}} \omega_i + \sum_{ti} p_t d(\chi_i * q_t) \otimes_{\mathscr{A}} \omega_i \\ &= \sum_{ti} p_t d(\chi_i * q_t) \otimes_{\mathscr{A}} \omega_i .\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_t p_t d(\chi_i * q_t) &= \sum_{tk} p_t(\chi_k * \chi_i * q_t)\omega_k \\ &= \sum_{tk} p_t(id \otimes (\chi_k * \chi_i))\Phi(q_t)\omega_k \\ &= \sum_k (id \otimes (\chi_k * \chi_i))(\sum_t (p_t \otimes I)\Phi(q_t))\omega_k \\ &= \sum_k (id \otimes (\chi_k * \chi_i)) r(\sum_t (p_t \otimes q_t)\omega_k \\ &= \sum_k (\chi_k * \chi_i)(z)\omega_k\end{aligned}$$

and formula (4.13) follows. Q. E. D.

Since r is surjective, for any $z \in R$ one can find p_t, q_t ($t=1, 2, \ldots, s$; s *sufficiently large*) such that the assumption (4.12) is fulfiled. Lemma 4.1 shows that for any $z \in R$

$$\sum (\chi_k * \chi_i)(z)\omega_k \otimes_{\mathscr{A}} \omega_i \in \tilde{S} .$$

In particular setting $z = \alpha^* + \nu^2\alpha - (1+\nu^2)I$, γ^2, $\gamma^*\gamma$, γ^{*2}, $(\alpha - I)\gamma$ and $(\alpha - I)\gamma^*$ we see that all six generators of S^2 belong to $\tilde{S}$. Therefore $S \subset \tilde{S}$ and

$$S \subset \tilde{S} \cap \Gamma^{\otimes}.$$

To complete the proof of Prop. 3.3 it is sufficient to show that

$$\tilde{S} \cap \Gamma^{\otimes} \subset S . \tag{4.17}$$

Indeed if this inclusion holds then the kernel of the composed map

$$\Gamma^{\otimes} \hookrightarrow \tilde{\Gamma}^{\otimes} \longrightarrow \tilde{\Gamma}^{\wedge} \tag{4.18}$$

(where the second arrow denotes the canonical projection) coincides with S and (4. 18) defines the embedding of $\Gamma^{\wedge}=\Gamma^{\otimes}/S$ into $\tilde{\Gamma}^{\wedge}$. Therefore $\tilde{\Gamma}^{\wedge}$ can be considered as a larger graded $*$-algebra containing $\Gamma^{\wedge}$ and equation (4. 11), (4. 7) and (4. 4) show that $X\in\tilde{\Gamma}$ satisfies the conditions listed in Prop. 3. 3.

Relation (4. 17) means that for any $n=0, 1, 2, 3, \ldots$

$$\tilde{S}^{n}\cap\Gamma^{\otimes n}\subset S^{n} \tag{4. 19}$$

where $S^{n}(\tilde{S}^{n}, \Gamma^{\otimes n})$ denotes the $\mathscr{A}$-bimodule of elements of grade n belonging to S ($\tilde{S}$, $\Gamma^{\otimes}$ resp.). For $n=0, 1$ this relation obviously holds (both sides equal 0).

We consider the case $n=2$. Any element $\zeta^{2}\in\tilde{S}^{2}$ is of the form

$$\zeta^{2}=\sum_{i} a_{i}X\otimes_{\mathscr{A}}Xb_{i}$$

where $a_{i}, b_{i}\in\mathscr{A}$. If $\zeta^{2}\in\Gamma^{\otimes}$ then (cf. (4. 10) and (4. 9))

$$j_{2}(\zeta^{2})=\sum_{i} a_{i}Xb_{i}=0 .$$

Taking into account the definition (4. 3) we obtain $\sum_{i} a_{i}b_{i}=0$ and $\sum_{i} a_{i}db_{i}=0$. Using Prop. 2. 4 and repeating the argumentation used in Section 2 immediately after implication (2. 38) we see that $\sum_{i} a_{i}\otimes b_{i}$ can be written as a sum of terms of the form (cf. (2. 39))

$$\sum_{t} cp_{t}\otimes q_{t}c'$$

where $c, c'\in\mathscr{A}$ and $p_{t}, q_{t}\in\mathscr{A}$ are such that

$$r(\sum_{t} p_{t}\otimes q_{t})=I\otimes z$$

where z is one of the elements (2. 27). Therefore ζ^{2} is a sum of terms of the form

$$c\sum_{t} p_{t}X\otimes_{\mathscr{A}}Xq_{t}c'.$$

Using Lemma 4. 1 we see that ζ^{2} is a sum of terms of the form

$$c[\sum(\chi_{k}*\chi_{i})(z)\omega_{k}\otimes_{\mathscr{A}}\omega_{i}]c'$$

where c, $c'\in\mathscr{A}$ and z is one of the element (2. 27). According to Prop. 3. 1 elements in square brackets generate S^{2}. Therefore $\zeta^{2}\in S^{2}$. This way we showed that

$$\tilde{S}^{2}\cap\ker j_{2}\subset S^{2}. \tag{4. 20}$$

In particular

$$\tilde{S}^2 \cap \Gamma^{\otimes 2} \subset S^2 . \tag{4.21}$$

Let $n>2$. Clearly $\zeta^n \in \tilde{S}^n$ if and only if ζ^n is a sum of terms of the form

$$\zeta^s \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} \zeta^r \tag{4.22}$$

where $\zeta^s \in \tilde{\Gamma}^{\otimes s}$, $\zeta^r \in \tilde{\Gamma}^{\otimes r}$, $s, r=0, 1, 2, \ldots, n-2$ and $s+r=n-2$. Separating terms with $s=n-2$ we see that

$$\tilde{S}^n = \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \tilde{\Gamma}$$

and using (4.5) we obtain

$$\tilde{S}^n = \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-1} \otimes_{\mathscr{A}} X \mathscr{A} \ . \tag{4.23}$$

Inserting $n-1$ instead of n we obtain

$$\tilde{S}^{n-1} = \tilde{\Gamma}^{\otimes (n-3)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-2} \otimes_{\mathscr{A}} X \mathscr{A} \ .$$

Inserting this expression into (4.23) we get

$$\begin{aligned} \tilde{S}^n = \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{\Gamma}^{\otimes (n-3)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} \otimes_{\mathscr{A}} X \mathscr{A} \\ + \tilde{S}^{n-2} \otimes_{\mathscr{A}} X \mathscr{A} \otimes_{\mathscr{A}} X \mathscr{A} . \end{aligned}$$

Now, using simple formula

$$Xa \otimes_{\mathscr{A}} Xb = X \otimes_{\mathscr{A}} aXb = X \otimes_{\mathscr{A}} Xab - X \otimes_{\mathscr{A}} (da) b \tag{4.24}$$

we see that

$$\tilde{\Gamma}^{\otimes (n-3)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} \otimes_{\mathscr{A}} X \mathscr{A} \subset \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma$$

and

$$\tilde{S}^{n-2} \otimes_{\mathscr{A}} X \mathscr{A} \otimes_{\mathscr{A}} X \mathscr{A} \subset \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma .$$

Therefore

$$\tilde{S}^n \subset \tilde{\Gamma}^{\otimes (n-2)} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X \mathscr{A} + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma \tag{4.25}$$

Let $q \in \tilde{S}^n$. Then

$$q = \sum_i \zeta_i \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X b_i + q' \tag{4.26}$$

where $\zeta_i \in \tilde{\Gamma}^{\otimes (n-2)}$, $b_i \in \mathscr{A}$, $q' \in \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma$.

In order to have compact notation we put $X = \omega_{-1}$ and for any sequence $r = (r_1, r_2, \ldots r_{n-2})$ of elements of the set $\{-1, 0, 1, 2\}$ we put

$$\omega_r = \omega_{r_1} \otimes_{\mathscr{A}} \omega_{r_2} \otimes_{\mathscr{A}} \ldots \otimes_{\mathscr{A}} \omega_{r_{n-2}} .$$

Then, using Table 1 and the formula $aX = Xa - da$ one can write any element $\zeta \in \tilde{\Gamma}^{\otimes (n-2)}$ in the form

$$\zeta=\sum_r \omega_r a_r$$

where $a_r \in \mathscr{A}$ and the summation runs over all possible sequences of length $n-2$. In particular all ζ_i in (4.26) can be written in this way. Therefore

$$\begin{aligned} q &= \sum_{ri} \omega_r a_{ri} \otimes_{\mathscr{A}} X \otimes_{\mathscr{A}} X b_i + q' \\ &= \sum_r \omega_r \otimes_{\mathscr{A}} (\sum_i a_{ri} X \otimes_{\mathscr{A}} X b_i) + q'. \end{aligned} \tag{4.27}$$

Assume that $j_n(q)=0$, where j_n is the last of the mappings (4.9). Remembering that $q' \in \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma$ we see that $j_n(q')=0$. Therefore using (4.27) we obtain

$$\sum_r \omega_r \otimes_{\mathscr{A}} (\sum_i a_{ri} X b_i) = 0 \ .$$

It means that for any r

$$j_2(\sum_i a_{ri} X \otimes_{\mathscr{A}} X b_i) = \sum_i a_{ri} X b_i = 0$$

and in virtue of (4.20) we get

$$\sum_i a_{ri} X \otimes_{\mathscr{A}} X b_i \in S^2 \ .$$

It shows that the first term in (4.27) belongs to $\tilde{\Gamma}^{\otimes(n-2)} \otimes_{\mathscr{A}} S^2$ and

$$q \in \tilde{\Gamma}^{\otimes(n-2)} \otimes_{\mathscr{A}} S^2 + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma .$$

This way we showed that

$$\tilde{S}^n \cap \ker j_n \subset \tilde{\Gamma}^{\otimes(n-2)} \otimes_{\mathscr{A}} S^2 + \tilde{S}^{n-1} \otimes_{\mathscr{A}} \Gamma \ . \tag{4.28}$$

Let $m>0$. Using (4.5) we obtain

$$\tilde{\Gamma}^{\otimes m} = \tilde{\Gamma}^{\otimes(m-1)} \otimes_{\mathscr{A}} X \mathscr{A} .$$

Therefore for $m>1$ we have

$$\tilde{\Gamma}^{\otimes m} = \tilde{\Gamma}^{\otimes(m-2)} \otimes_{\mathscr{A}} X \mathscr{A} \otimes_{\mathscr{A}} X \mathscr{A} .$$

Now using (4. 24) we see that

$$\tilde{\Gamma}^{\otimes m} = \tilde{S}^m + \tilde{\Gamma}^{\otimes(m-1)} \otimes_{\mathscr{A}} \Gamma . \tag{4.29}$$

We shall prove that for $k=1, 2, \ldots$

$$\tilde{\Gamma}^{\otimes k} = \tilde{S}^k + \tilde{\Gamma} \otimes_{\mathscr{A}} \Gamma^{\otimes(k-1)} . \tag{4.30}$$

For $k=1$ the formula is evident. For $k=2$ it coincides with (4.29) with m=2. Assume that (4.30) holds for $k=m-1$. Then using (4.29) we have

$$\begin{aligned} \tilde{\Gamma}^{\otimes m} &= \tilde{S}^m + (\tilde{S}^{m-1} + \tilde{\Gamma} \otimes_{\mathscr{A}} \Gamma^{\otimes(m-2)}) \otimes_{\mathscr{A}} \Gamma \\ &= \tilde{S}^m + \tilde{S}^{m-1} \otimes_{\mathscr{A}} \Gamma + \tilde{\Gamma} \otimes_{\mathscr{A}} \Gamma^{\otimes(m-1)} \end{aligned}$$

and taking into account the obvious inclusion $\tilde{S}^{m-1}\otimes_{\mathscr{A}}\Gamma\subset\tilde{S}_m$ we obtain (4.30) for $k=m$. This way (4.30) is proven in full generality.

Let us put $k=n-2$ in (4.30) and insert the obtained expression for $\tilde{\Gamma}^{\otimes(n-2)}$ into (4.28):

$$\tilde{S}^n\cap\ker j_n\subset\tilde{S}^{n-2}\otimes_{\mathscr{A}}S^2+\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-3)}\otimes_{\mathscr{A}}S^2+\tilde{S}^{n-1}\otimes_{\mathscr{A}}\Gamma.$$

Clearly

$$\tilde{S}^{n-2}\otimes_{\mathscr{A}}S^2\subset\tilde{S}^{n-1}\otimes_{\mathscr{A}}\Gamma \text{ and } \Gamma^{\otimes(n-3)}\otimes_{\mathscr{A}}S^2\subset S^{n-1}.$$

Therefore

$$\tilde{S}^n\cap\ker j_n\subset\tilde{\Gamma}\otimes_{\mathscr{A}}S^{n-1}+\tilde{S}^{n-1}\otimes_{\mathscr{A}}\Gamma\ . \tag{4.31}$$

for $n=3, 4, 5, \dots.$

We shall prove that

$$\tilde{S}^k\cap(\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(k-1)})\subset X\otimes_{\mathscr{A}}S^{k-1}+S^k \tag{4.32}$$

for $k=2, 3, 4, \dots$

For $k=2$ this formula follows from (4.20). Let $n>2$. Assume that (4.32) holds for $k=n-1$. Let

$$\zeta^n\in\tilde{S}^n\cap(\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-1)}).$$

Then $j_n(\zeta^n)=0$ and using (4.31) we see that

$$\zeta^n=X\otimes_{\mathscr{A}}\xi_{-1}+\sum_i\omega_i\otimes_{\mathscr{A}}\xi_i+\sum_i\xi'_i\otimes_{\mathscr{A}}\omega_i \tag{4.33}$$

where $\xi_{-1}, \xi_0, \xi_1, \xi_2\in S^{n-1}$ and $\xi'_0, \xi'_1, \xi'_2\in\tilde{S}^{n-1}$. The first and the second term in (4.33) belong to $\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-1)}$. Therefore $\sum_i\xi'_i\otimes_{\mathscr{A}}\omega_i\in\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-1)}$. It means that ξ'_i belong to $\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-2)}$. Now using (4.32) with $k=n-1$ we obtain $\xi'_i\in X\otimes_{\mathscr{A}}S^{n-2}+S^{n-1}$ and the last term in (4.33) belongs to $X\otimes_{\mathscr{A}}S^{n-2}\otimes_{\mathscr{A}}\Gamma+S^{n-1}\otimes_{\mathscr{A}}\Gamma\subset X\otimes_{\mathscr{A}}S^{n-1}+S^n$. On the other hand the first and the second terms belong to $X\otimes_{\mathscr{A}}S^{n-1}$ and S^n resp. Therefore

$$\zeta^n\in X\otimes_{\mathscr{A}}S^{n-1}+S^n$$

and (4.32) holds for $k=n$. This way we proved (4.32) in full generality.

Inserting in (4.32) $k=n$ and remembering that $\Gamma^{\otimes n}$ is contained in $\tilde{\Gamma}\otimes_{\mathscr{A}}\Gamma^{\otimes(n-1)}$ we get

$$\tilde{S}^n\cap\Gamma^{\otimes n}\subset X\otimes_{\mathscr{A}}S^{n-1}+S^n\ . \tag{4.34}$$

We know that $S^n\subset\Gamma^{\otimes n}$ and $(X\otimes_{\mathscr{A}}S^{n-1})\cap\Gamma^{\otimes n}=0$. Therefore the inclusion (4.34) means that

$$\tilde{S}^n \cap \Gamma^{\otimes n} \subset S^n$$

and (4. 19) holds in full generality. This ends the proof of Prop. 3. 3.

§ 5. Finite Dimensional Representations of the Pseudogroup $S_\nu U(2)$

In this section we apply the differential calculus built in Sections 2 and 3 to the investigation of representations of $S_\nu U(2)$. We shall assume that these representations are finite dimensional. This assumption is not restrictive. According to the general theory [1] any unitary representation of a compact matrix pseudogroup is a direct sum of finite dimensional irreducible representations.

At the beginning we remind the notion of representation. Let V be a finite dimensional vector space. We say that v is a representation of $S_\nu U(2)$ acting on V if v is an invertible element of the algebra $B(V)\otimes A$ such that

$$(id\otimes\Phi)v = v \mathbin{\bigcirc\!\!\!\!\top} v \tag{5. 1}$$

where $\mathbin{\bigcirc\!\!\!\!\top}$ denotes the bilinear multiplication defined on elements of $B(V)\otimes A$ with values in $B(V)\otimes A\otimes A$ such that

$$(m_1\otimes a)\mathbin{\bigcirc\!\!\!\!\top}(m_2\otimes b) = m_1 m_2\otimes a\otimes b$$

for any $m_1, m_2 \in B(V)$, $a, b\in A$. If $V=\boldsymbol{C}^N$ then $B(V)=M_N$ and the product $\mathbin{\bigcirc\!\!\!\!\top}$ introduced here coincides with the one considered in Section 1. In particular 2×2 matrix u considered in Thm 1. 4 is a representation of $S_\nu U(2)$ acting on $\boldsymbol{C}^2$. This representation is called fundamental.

In the group representation theory we consider equivalent representations, invariant subspaces, subrepresentations and irreducible representations. All these notions based on the concept of intertwining operator are meaningful in the representation theory of pseudogroup (cf. [1]).

Let v and w be representations of $S_\nu U(2)$ acting on V and W resp. A linear mapping S: $V\to W$ intertwines v with w if

$$(S\otimes I)v = w(S\otimes I). \tag{5. 2}$$

Representations v and w are equivalent if there exists an invertible S intertwining v with w. A subspace V' of V is invariant under v if there exists a representation v' acting on V' such that the embedding

$V' \to V$ intertwines v' with v. The representation v' is then a subrepresentation of v. It is uniquely determined. v is irreducible if there is no non-trivial invariant subspace. If v and v' are representations acting on V and V' resp., then a representation w acting on $V \oplus V'$ is a direct sum of v and v' if the canonical embeddings $V \to V \oplus V'$ and $V' \to V \oplus V'$ intertwine v and v' resp. with w. A representation v is called unitary if the space V on which it acts is a Hilbert space and if v is a unitary element of $B(V) \otimes A$. It is known that any finite dimensional representation is equivalent to a unitary representation.

Let V and V' be a finite dimensional vector spaces. We consider bilinear multiplication $\circledT$ defined on elements of $B(V) \otimes A$ and $B(V') \otimes A$ with values in $B(V \otimes V') \otimes A$ such that

$$(m \otimes a) \circledT (n \otimes b) = m \otimes n \otimes ab \tag{5.3}$$

for any $m \in B(V)$, $n \in B(V')$ and $a, b \in A$. One can easily check that if v and v' are representations of $S_\nu U(2)$ acting on V and V' resp. then $v \circledT v'$ is a representation of $S_\nu U(2)$ acting on $V \otimes V'$. It is called tensor product of representations v and v'. Let us notice that in the pseudogroup case the tensor product is not commutative. More precisely the flip map $V \otimes V' \to V' \otimes V$ interchanging V and V' does not in general intertwine $v \circledT v'$ with $v' \circledT v$.

In the theory of Lie groups the matrix elements of finite dimensional representations are smooth functions. Due to this fact we can use the differential calculus in the representation theory. Similar fact holds for pseudogroups (see [1]). If v is a representation of $S_\nu U(2)$ acting on a finite dimensional vector space V then v and v^{-1} belong to $B(V) \otimes \mathscr{A}$. Therefore for any linear functional χ on $\mathscr{A}$ we may introduce an operator

$$(id \otimes \chi) v \in B(V).$$

In particular we set

$$A_k = (id \otimes \chi_k) v \tag{5.4}$$

for $k=0, 1, 2$ (see Section 2 for the definition of χ_0, χ_1, χ_2). Operators A_0, A_1, A_2 introduced by (5.4) will be called infinitesimal generators of v.

Example 5.1. *If u is the representation of $S_\nu U(2)$ considered in Thm.*

1.4 *then using formulae* (2.3) *we compute infinitesimal generators of* u:

$$A_0=(id\otimes\chi_0)u=\begin{pmatrix}\chi_0(\alpha), & -\nu\chi_0(\gamma^*)\\ \chi_0(\gamma), & \chi_0(\alpha^*)\end{pmatrix}=\begin{pmatrix}0, & 1\\ 0, & 0\end{pmatrix}$$

$$A_1=(id\otimes\chi_1)u=\begin{pmatrix}\chi_1(\alpha), & -\nu\chi_1(\gamma^*)\\ \chi_1(\gamma), & \chi_1(\alpha^*)\end{pmatrix}=\begin{pmatrix}1, & 0\\ 0, & -\nu^2\end{pmatrix}$$

$$A_2=(id\otimes\chi_2)u=\begin{pmatrix}\chi_2(\alpha), & -\nu\chi_2(\gamma^*)\\ \chi_2(\gamma), & \chi_2(\alpha^*)\end{pmatrix}=\begin{pmatrix}0, & 0\\ -\nu, & 0\end{pmatrix}.$$

Let v be a representation of $S_\nu U(2)$ acting on a finite dimensional vector space V. Then v can be written in the form

$$v=\sum_i m_i\otimes v_i$$

where $m_i\in B(V)$ and $v_i\in\mathscr{A}$. Condition (5.1) means that

$$\sum_i m_i\otimes\Phi(v_i)=\sum_{i,j} m_im_j\otimes v_i\otimes v_j\ .$$

Applying to the both sides the map $id\otimes id\otimes\chi$, where χ is a linear functional on $\mathscr{A}$ we obtain

$$\begin{aligned}\sum_i m_i\otimes(\chi*v_i)&=\sum_{i,j} m_im_j\otimes v_i\chi(v_j)\\ &=v[(id\otimes\chi)v\otimes I]\ .\end{aligned}\tag{5.5}$$

If $\chi=e$ then (cf. (2.10)) $\chi*v_i=v_i$, the left hand side of the above equation equals to v and remembering that v is invertible we get

$$(id\otimes e)v=I\ .\tag{5.6}$$

Inserting in (5.5) χ_k instead of χ, multiplying both sides (from the right) by $I\otimes\omega_k$ and summing over k we get

$$(id\otimes d)v=v(\sum_k A_k\otimes\omega_k)\tag{5.7}$$

where A_0, A_1, A_2 are infinitesimal generators of v.

Proposition 5.2. *Let* v, v' *be representations of* $S_\nu U(2)$ *acting on finite dimensional vector spaces* V *and* V' *resp.,* A_0, A_1, A_2, *and* A_0', A_1', A_2' *be infinitesimal generators of* v *and* v' *resp. and* $S\in B(V, V')$. *Then the following conditions are equivalent*:

I. S *intertwines* v *with* v'

II. $SA_k=A_k'S$ *for* $k=0, 1, 2.$

Proof. If $(S\otimes I)v=v'(S\otimes I)$ then applying to both sides the map

$id\otimes\chi_k$ $(k=0, 1, 2)$ we obtain $S(id\otimes\chi_k)v=(id\otimes\chi_k)v'S$. Therefore I implies II.

To prove the converse we have to compute $(id\otimes d)(v^{-1})$ where v^{-1} is the inverse of v. Using relation $dI=0$ we get

$$\begin{aligned} &v^{-1}(id\otimes d)v+(id\otimes d)(v^{-1})v \\ &=(id\otimes d)(v^{-1}v)=0 \end{aligned}$$

and using (5.7) we obtain

$$(id\otimes d)(v^{-1})=-(\sum_k A_k\otimes\omega_k)v^{-1}.$$

Now, using the rules of differential calculus, formula (5.7) with v replaced by v' and the above equation we have

$$\begin{aligned} &(id\otimes d)(v'(S\otimes I)v^{-1}) \\ &=(id\otimes d)(v')(S\otimes I)v^{-1}+v'(S\otimes I)(id\otimes d)(v^{-1}) \\ &=v'(\sum_k A'_kS\otimes\omega_k)v^{-1}-v'(\sum_k SA_k\otimes\omega_k)v^{-1} \\ &=v'(\sum_k(A'_kS-SA_k)\otimes\omega_k)v^{-1}. \end{aligned}$$

If condition II is satisfied then

$$(id\otimes d)(v'(S\otimes I)v^{-1})=0$$

and using Thm. 2.3 we see that

$$v'(S\otimes I)v^{-1}=S'\otimes I$$

where $S'\in B(V, V')$. Therefore

$$v'(S\otimes I)=(S'\otimes I)v\ . \tag{5.8}$$

Applying to the both sides of the above equation the map $id\otimes e$, using (5.6) and the same formula for v' we get $S'=S$ and (5.8) shows that condition I holds. Q. E. D.

If $V=V'$ and $A_k=A'_k$ for $k=0, 1, 2$, then setting $S=I$ (the identity acting on V) we satisfy condition II. Therefore $S=I$ intertwines v with v' and $v=v'$. This way we get

Corollary 5.3. *Any representation of $S_\nu U(2)$ is uniquely determined by its infinitesimal generators.*

Now we shall show that the infinitesimal generators A_0, A_1, A_2 introduced by (5.4) satisfy the same commutation relations as infinitesimal shifts (cf. Table 7 in Section 3).

We apply $(id\otimes d)$ to the both sides of (5.7). Remembering that $d^2=0$ we get

$$(id\otimes d)\,(v\sum_{k=0}^{2} A_k\otimes\omega_k)=0\ .$$

Therefore

$$(id\otimes d)\,(v)\wedge(\sum_{k=0}^{2} A_k\otimes\omega_k)+v\sum_{k=0}^{2} A_k\otimes d\omega_k=0$$

where $\wedge$ denotes the usual product in $B(V)$ tensored with the $\wedge$ product in $\Gamma^\wedge$. Using once more (5.7) we get

$$\sum_{ki=0}^{2} A_iA_k\otimes\omega_i\wedge\omega_k+\sum_{k=0}^{2} A_k\otimes d\omega_k=0\ . \tag{5.9}$$

Proceeding in the same way as at the end of Section 3 we obtain

Table 8	The commutation relations for infinitesimal generators
$\nu A_2A_0-\frac{1}{\nu}A_0A_2=A_1$	
$\nu^2A_1A_0-\frac{1}{\nu^2}A_0A_1=(1+\nu^2)A_0$	
$\nu^2A_2A_1-\frac{1}{\nu^2}A_1A_2=(1+\nu^2)A_2$	

Assume now that V is a Hilbert space and that v is a unitary representation of $S_\nu U(2)$ acting on V. Then $v^*v=I\otimes I$ and

$$\begin{aligned}0&=(id\otimes d)\,(v^*v)=(id\otimes d)\,(v^*)\,v+v^*(id\otimes d)\,(v)\\&=[v^*(id\otimes d)\,v]^*+v^*(id\otimes d)\,v\end{aligned}$$

where in the last line we used (2.36). The star $*$ standing just after the square bracket denotes the hermitian conjugation in $B(V)$ tensored with the involution (2.33) in Γ. Taking into account (5.7) we get

$$(\sum_{k=0}^{2} A_k\otimes\omega_k)^*=-\sum_{k=0}^{2} A_k\otimes\omega_k$$

which means (cf. Table 4) that

Table 9	The selfadjointness relation for infinitesimal generators
$-\nu A_0^*=A_2,$	$A_1^*=A_1$

Let V be a finite dimensional vector space. We say that (A_0, A_1, A_2) is an infinitesimal representation of $S_\nu U(2)$ acting on V if $A_0, A_1, A_2 \in B(V)$ and if the relations of Table 8 hold. An infinitesimal representation (A_0, A_1, A_2) is called selfadjoint if the relations of Table 9 are satisfied. According to the results presented above the investigation of (global) representations of $S_\nu U(2)$ can be reduced to study of infinitesimal representations.

Theorem 5.4. *Let (A_0, A_1, A_2) be an infinitesimal representation of $S_\nu U(2)$ acting on a finite dimensional vector space V. We assume that this representation is irreducible (i.e. there is no non-trivial subspace of V invariant under A_k where $k=0,1,2$).*

Then the eigenvalues of A_1 are real and denoting by λ_{max} the maximal one we have the following possibilities:

I. $\lambda_{max} = -\dfrac{\nu^2}{1-\nu^2}$.

Then dim $V=1$ *and there exists $c \in \mathbf{C}$ such that*

$$A_0 = c\frac{\nu}{1-\nu^2} I \tag{5.10}$$

$$A_1 = -\frac{\nu^2}{1-\nu^2} I \tag{5.11}$$

$$A_2 = \frac{1}{c}\,\frac{\nu^2}{1-\nu^2} I \ . \tag{5.12}$$

II. $\lambda_{max} = \dfrac{\nu^2}{1-\nu^2}(\nu^{-4n}-1)$

where n is a nonnegative integer or half-integer: $n=0, \frac{1}{2}, 1, 1\frac{1}{2}, \ldots$. Then dim $V=2n+1$ *and there exists a basis*

$$(\Xi_{-n}, \Xi_{-n+1}, \ldots, \Xi_n) \tag{5.13}$$

in V such that for $k=-n, -n+1, \ldots, n$ we have

$$A_0 \Xi_k = -c_{k+1} \Xi_{k+1} \tag{5.14}$$

$$A_1 \Xi_k = \frac{\nu^2}{1-\nu^2}(\nu^{-4k}-1)\,\Xi_k \tag{5.15}$$

$$A_2 \Xi_k = \nu c_k \Xi_{k-1} \tag{5.16}$$

where

$$c_k = \frac{\nu}{1-\nu^2}[(\nu^{-2k}-\nu^{2n})(\nu^{-2n}-\nu^{-2(k-1)})]^{1/2}. \tag{5.17}$$

If V is a Hilbert space and the infinitesimal representation (A_0, A_1, A_2) is selfadjoint then the case I *cannot occur and in the case* II *the basis* (5.13) *is orthonormal (more precisely $(\Xi_k|\Xi_{k'})=c\delta_{kk'}$ where c is a positive constant).*

Remark. If $\nu=\pm 1$ then the case I cannot occur and in the case II the expression for λ_{max} and the right hand sides of (5.15) and (5.17) should be replaced by the suitable limits: $\lambda_{max}=2n$, $A_1\Xi_k=2k\Xi_k$, $c_k=\nu[(n+k)(n-k+1)]^{1/2}$.

Proof. For $\nu=1$ the relations contained in Table 8 coincides with the commutation relations in the Lie algebra $su(2)$. One can check that for $\nu=-1$ the relations of Table 8 are identical with the commutation relations in $su(1, 1)$. Since these two cases are covered by many textbooks in the following we assume that $|\nu|<1$.

Let

$$B=I+\frac{1-\nu^2}{\nu^2}A_1 \tag{5.18}$$

$$C=I+\frac{1-\nu^2}{(1+\nu^2)\nu^3}(\nu^4A_2A_0-A_0A_2) . \tag{5.19}$$

Then using Table 8 one can check that operators B and C commute and

$$BA_0=\nu^{-4}A_0B, \qquad CA_0=\nu^{-2}A_0C \tag{5.20}$$

$$BA_2=\nu^4A_2B, \qquad CA_2=\nu^2A_2C . \tag{5.21}$$

Moreover

$$A_0A_2=\frac{\nu^3}{(1-\nu^2)^2}[I+\nu^2B-(1+\nu^2)C] \tag{5.22}$$

$$A_2A_0=\frac{\nu}{(1-\nu^2)^2}[\nu^2I+B-(1+\nu^2)C]. \tag{5.23}$$

Let $V'=(\ker B)\cap(\ker C)$. It follows immediately from (5.20) and (5.21) and the first equation of Table 8 that V' is invariant under A_0, A_2, A_1. Therefore either $V'=V$ or $V'=\{0\}$. Assume that $V'=V$. Then $B=C=0$ and (cf. (5.18)) formula (5.11) follows. Moreover in virtue of (5.22) and (5.23) operators A_0 and A_2 commute and using the irreducibility we get dim $V=1$ and all operators on V are multiple of identity: $A_0=c_0I$ and $A_2=c_2I$. In virtue of (5.22) $c_0c_2=\nu^3(1-\nu^2)^{-2}$ and setting $c=\nu^{-1}(1-\nu^2)c_0$ we obtain (5.10) and (5.12).

Now we consider case $V'=\{0\}$.

Assume that ker $A_0=\{0\}$. Then using (5.20) we see that the common spectrum of the pair of commuting operators (B, C) is invariant under the map $(b, c)\rightarrow(\nu^{-4}b, \nu^{-2}c)$. Clearly the orbit of any point $(b, c)\neq(0, 0)$ is infinite. The case $(b, c)=(0, 0)$ is excluded since $V'=\{0\}$. Therefore we obtain contradiction with the assumption that dim V is finite. It proves that ker A_0 is non-trivial.

First of the equations (5.20) shows that ker A_0 is B-invariant. Let Ξ_{max} be an eigenvector of B belonging to ker A_0:

$$\begin{aligned} A_0\Xi_{max}&=0 \\ B\Xi_{max}&=b\Xi_{max} \end{aligned} \tag{5.24}$$

where $b\in\boldsymbol{C}$. Then using (5.23) we obtain

$$C\Xi_{max}=c\Xi_{max}$$

where

$$c=\frac{b+\nu^2}{1+\nu^2}\ . \tag{5.25}$$

Using k-times equations (5.21) we obtain

$$BA_2^k\Xi_{max}=\nu^{4k}bA_2^k\Xi_{max} \tag{5.26}$$

$$CA_2^k\Xi_{max}=\nu^{2k}cA_2^k\Xi_{max} \tag{5.27}$$

for $k=0, 1, 2, \ldots$. Assume that $A_2^k\Xi_{max}\neq 0$ for all k. Then we have an infinite sequence of vectors $(A_2^k\Xi_{max})_{k=0,1,2,\cdots}$ being common eigenvectors of (B, C) corresponding to different pairs of eigenvalues $(\nu^{4k}b, \nu^{2k}c)$ which cannot happen for finite-dimensional V. Therefore $A_2^k\Xi_{max}=0$ for some k.

Let N be the smallest integer such that $A_2^{N+1}\Xi_{max}=0$. Then $A_2^N\Xi_{max}\neq 0$ and applying both sides of (5.22) to $A_2^N\Xi_{max}$ and using (5.26) and (5.27) we get

$$1+\nu^{4N+2}b=(1+\nu^2)\nu^{2N}c\ . \tag{5.28}$$

Solving (5.25) and (5.28) with respect to b and c we obtain

$$b=\nu^{-2N}, \quad c=\frac{\nu^{-2N}+\nu^2}{1+\nu^2}\ .$$

Let $n=\frac{N}{2}$. We set

$$\Xi_n=\Xi_{max} \tag{5.29}$$

and for $k=n,\ n-1,\ n-2,\ldots,\ -n+1$

$$\Xi_{k-1}=\frac{1}{\nu c_k}A_2\Xi_k \tag{5.30}$$

where c_k is given by (5.17). This way we introduce subsequently vectors $\Xi_n, \Xi_{n-1}, \ldots, \Xi_{-n+1}, \Xi_{-n}\in V$. Clearly Ξ_k is proportional to $A_2^{n-k}\Xi_{max}$: $\Xi_k=c_k' A_2^{n-k}\Xi_{max}$ with $c_k'\neq 0$.

Using (5.26) and (5.27) with k replaced by $n-k$ and formulae for b and c we obtain

$$\begin{aligned} B\Xi_k&=\nu^{-4k}\Xi_k \\ C\Xi_k&=\frac{\nu^{-2n}+\nu^{2n+2}}{1+\nu^2}\nu^{-2k}\Xi_k\ . \end{aligned} \tag{5.31}$$

Now formula (5.15) follows directly from (5.18).

For $k=-n$: $c_k=0$ and the right hand side of (5.16) vanishes. On the other hand Ξ_{-n} is proportional to $A_2^{2n}\Xi_{max}$ and $A_2\Xi_{-n}$ vanishes (because $A_2A_2^{2n}\Xi_{max}=A_2^{N+1}\Xi_{max}=0$). Therefore (5.16) holds for $k=-n$. For other values of k it holds in virtue of (5.30).

For $k=n$: $c_{k+1}=0$ and both sides of (5.14) vanish (cf. (5.24) and (5.29)). To prove (5.14) for other values of k we insert $k+1$ instead of k in (5.30) and compute using (5.22) and (5.31) with k replaced by $k+1$:

$$\begin{aligned} A_0\Xi_k&=\frac{1}{\nu c_{k+1}}A_0A_2\Xi_{k+1}=\frac{\nu^2}{(1-\nu^2)^2c_{k+1}}[I+\nu^2B-(1+\nu^2)C]\Xi_{k+1} \\ &=\frac{\nu^2}{(1-\nu^2)^2c_{k+1}}[1+\nu^{-4k-2}-\nu^{-2n-2k-2}-\nu^{2n-2k}]\Xi_{k+1} \\ &=-\frac{c_{k+1}^2}{c_{k+1}}\Xi_{k+1} \end{aligned}$$

and (5.14) follows.

It follows from (5.14)-(5.16) that the subspace of V spanned by vectors (5.13) is invariant under A_0, A_1, A_2; therefore in virtue of irreducibility this subspace coincides with V. In other words (5.13) is a basis in V and $\dim V=2n+1$.

Assume that V is a Hilbert space and that $A_1^*=A_1$ and $A_2^*=-\nu A_0$. If A_0 and A_2 are given by (5.10) and (5.12) then the last equality in the previous sentence leads to contradiction $\bar{c}c=-1$. If A_0, A_1, A_2 are given by (5. 14)-(5. 16) then vectors (5. 13) are mutually orthogonal (as eigenvectors of selfadjoint operator A_1 corresponding to different eigenvalues) and

$$(\Xi_{k-1}|\Xi_{k-1})=(\nu c_k)^{-1}(\Xi_{k-1}|A_2\Xi_k)=(\nu c_k)^{-1}(A_2^*\Xi_{k-1}|\Xi_k)$$
$$=-c_k^{-1}(A_0\Xi_{k-1}|\Xi_k)=(\Xi_k|\Xi_k)$$

for $k=n, n-1, \ldots, -n+1$. Q. E. D.

Remark 5. 5. One can easily check that operators A_0, A_1, A_2 defined by (5. 10)-(5. 12) satisfy the relations of Table 8. However they are not infinitesimal generators of any representation of $S_\nu U(2)$. Indeed since any representation is equivalent to a unitary one, the infinitesimal generators always satisfy the selfadjointness relations, which are not fulfiled in the considered case.

Remark 5. 6. If n is nonnegative integer or half-integer and V is a $(2n+1)$-dimensional vector space with a basis (5. 13) then elementary computations show that operators A_0, A_1, A_2 introduced by (5. 14)-(5. 16) satisfy the relations of Table 8. In other words (A_0, A_1, A_2) is an infinitesimal representation of $S_\nu U(2)$. This representation will be denoted by d_n^{inf}. One can easily check that d_n^{inf} is irreducible (V contains no non-trivial subspace invariant under A_k $k=0, 1, 2$). If V is the Hilbert space and the basis (5. 13) is orthonormal then the selfadjointness relation hold.

Remark 5. 7. Let $d^{inf}=(A_0, A_1, A_2)$ be an infinitesimal representation of $S_\nu U(2)$ acting on a vector space W. Assume that W contains a non-zero vector Ξ_{max} such that $A_0\Xi_{max}=0$ and $A_1\Xi_{max}=\lambda_{max}\Xi_{max}$, where $\lambda_{max}\in\boldsymbol{C}$. Then repeating the reasoning started with formulae (5. 24) we see that

$$\lambda_{max}=\frac{\nu^2}{1-\nu^2}(\nu^{-4n}-1)$$

for some nonnegative integer or half-integer n and d^{inf} contains a subrepresentation equivalent to d_n^{inf}.

Theorem 5. 8. *Any infinitesimal representation d_n^{inf} (where n is a non-*

negative integer or half-integer corresponds to a (global) representation of $S_\nu U(2)$.

Proof. Let n be a nonnegative integer or half-integer and

$$T=\{-n, -n+1, \ldots, n\}.$$

For $k\in T$ we put

$$x_k=\alpha^{n+k}\gamma^{*n-k}.$$

It follows immediately from (1.13) that

$$\Phi(x_k)=\sum_{i\in T} x_i\otimes w_{ik} \tag{5.32}$$

where $w_{ik}\in\mathscr{A}$ $(i, k\in T)$. According to Thm. 1.2 elements (x_k) are linearly independent. Therefore elements w_{ik} are uniquely determined.

Taking into account the commutativity of the diagram (1.15) we have

$$\begin{aligned}\sum_{i\in T} x_i\otimes\Phi(w_{ik}) &= (id\otimes\Phi)\,\Phi(x_k)\\ &= (\Phi\otimes id)\,\Phi(x_k)=\sum_{s\in T}\Phi(x_s)\otimes w_{sk}\\ &= \sum_{is\in T} x_i\otimes w_{is}\otimes w_{sk}.\end{aligned}$$

Therefore

$$\Phi(w_{ik})=\sum_{s\in T} w_{is}\otimes w_{sk}\ . \tag{5.33}$$

Let w be the $(2n+1)\times(2n+1)$ matrix with matrix elements equal to w_{ik}: $w=(w_{ik})_{ik\in T}$. Then $w\in M_{2n+1}\otimes\mathscr{A}$ and (5.33) means that

$$(id\otimes\Phi)\,w=w\oplus w\ . \tag{5.34}$$

We shall prove that w is invertible. Indeed using (5.32) and (2.10) we have

$$\sum_i x_i e(w_{ik})=(id\otimes e)\,\Phi(x_k)=x_k\ .$$

Therefore $e(w_{ik})=\delta_{ik}$ where δ_{ik} is the Kronecker symbol. Using (2.12) we get

$$\begin{aligned}\sum_s \kappa(w_{is})\,w_{sk} &= m(\kappa\otimes id)\sum_s w_{is}\otimes w_{sk}\\ &= m(\kappa\otimes id)\,\Phi(w_{ik})=e(w_{ik})\,I=\delta_{ik}I\ .\end{aligned}$$

Similarly using (2.13) we obtain

$$\sum_s w_{is}\kappa(w_{sk})=\delta_{ik}I\ .$$

These relations show that $(\mathrm{id}\otimes\kappa)w$ is inverse of w and (5.33) shows that w is a (global) representation of $S_\nu U(2)$.

Let d^{inf} be the infinitesimal representation corresponding to w: $d^{inf}=(A_0, A_1, A_2)$, where (cf. (5.4)) A_r are $(2n+1)\times(2n+1)$ matrices

$$A_r=(\chi_r(w_{ik}))_{i,k\in T} \quad . \tag{5.35}$$

In virtue of (5.32)

$$\sum_i x_i\chi_r(w_{ik})=(id\otimes\chi_r)\Phi(x_k)=\chi_r*x_k \quad . \tag{5.36}$$

It follows from Table 2 that $\chi_0*\alpha=0$. Using (2.20) we obtain $\chi_0*x_n=\chi_0*\alpha^{2n}=0$ and the above equality shows that $\chi_0(w_{in})=0$. It means that the last (i.e. corresponding to $k=n$) column of A_0 vanishes. Therefore denoting by Ξ_{max} the vector in $\boldsymbol{C}^{2n+1}$ having all except the last component equal to zero we have

$$A_0\Xi_{max}=0 \quad .$$

Moreover using formulae (5.35), (5.36) and (2.22) we see that A_1 is a diagonal matrix with elements $\nu^2(1-\nu^2)^{-1}[\nu^{-4k}-1]$ on the diagonal. In particular for $k=n$ we get

$$A_1\Xi_{max}=\nu^2(1-\nu^2)^{-1}[\nu^{-4n}-1]\Xi_{max} \quad .$$

Now using Remark 5.7 we see that d^{inf} contain a subrepresentation equivalent to d_n^{inf}. Since the dimensions of the two representations are equal to $2n+1$ we conclude that d^{inf} is equivalent to d_n^{inf}.

Q. E. D.

The representation constructed in the above proof will be denoted by d_n. Clearly any irreducible representation of $S_\nu U(2)$ is equivalent to d_n for some n. Any finite-dimensional representation of $S_\nu U(2)$ is equivalent to a direct sum of representations d_n.

Let $\Gamma_c^\wedge$ be a subalgebra in $\Gamma^\wedge$ (see Section 3) generated by I and $\omega_0, \omega_1, \omega_2$. Clearly $\Gamma_c^\wedge$ is a graded $*$-algebra and due to Cartan-Maurer formulae (Table 6) $\Gamma_c^\wedge$ is closed under the external derivative d. Therefore denoting by $\Gamma_c^{\wedge r}$ the subspace of all elements of grade r we have the following cochain complex:

$$0\longrightarrow\Gamma_c^{\wedge 0}\xrightarrow{d}\Gamma_c^{\wedge 1}\xrightarrow{d}\Gamma_c^{\wedge 2}\xrightarrow{d}\Gamma_c^{\wedge 3}\longrightarrow 0 \quad . \tag{5.37}$$

According to (3.13) d: $\Gamma_c^{\wedge 2}\rightarrow\Gamma_c^{\wedge 3}$ vanishes. Moreover equation $dI=0$ shows that d: $\Gamma_c^{\wedge 0}\rightarrow\Gamma_c^{\wedge 1}$ vanishes. On the other hand in virtue of Cartan-Maurer formulae (Table 6) d: $\Gamma_c^{\wedge 1}\rightarrow\Gamma_c^{\wedge 2}$ is a linear isomorphism. Therefore denoting by H^0, H^1, H^2, H^3 the cohomology groups of (5.37) we obtain

$$H^0=\boldsymbol{C}, H^1=0, H^2=0, H^3=\boldsymbol{C}. \tag{5.38}$$

Let v be a representation of $S_\nu U(2)$ acting on a vector space V, A_0, A_1 and A_2 be infinitesimal generators of v. We consider the tensor product $\Gamma_V^\wedge=V\otimes\Gamma_c^\wedge$. Clearly $\Gamma_V^\wedge$ is a graded vector space: the subspace of all elements of grade r coincides with $\Gamma_V^{\wedge r}=V\otimes\Gamma_c^{\wedge r}$.

For any $x\in V$ and $\zeta\in\Gamma_c^\wedge$ we put

$$d_v(x\otimes\zeta)=\sum_{k=0}^{2}A_kx\otimes\omega_k\wedge\zeta+x\otimes d\zeta\ . \tag{5.39}$$

This formula defines a linear map

$$d_v\colon\ \Gamma_V^\wedge\longrightarrow\Gamma_V^\wedge\ .$$

Clearly d_v increases the grade of any homogeneous element by one

$$0\longrightarrow\Gamma_V^{\wedge 0}\xrightarrow{d_v}\Gamma_V^{\wedge 1}\xrightarrow{d_v}\Gamma_V^{\wedge 2}\xrightarrow{d_v}\Gamma_V^{\wedge 3}\xrightarrow{d_v}0\ . \tag{5.40}$$

Using (5.39), (3.7) and remembering that $d^2\zeta=0$ we obtain

$$d_v^2(x\otimes\zeta)=\sum_{ik=0}^{2}A_iA_kx\otimes\omega_i\wedge\omega_k\wedge\zeta+\sum_{k=0}^{2}A_kx\otimes d\omega_k\wedge\zeta$$

and formula (5.9) shows that $d_v^2=0$. It means that (5.40) is a cochain complex. We shall prove

Theorem 5.9. *Let q be the multiplicity of one dimensional trivial representation in v. Then denoting by $H_v^0, H_v^1, H_v^2, H_v^3$ the cohomology groups of* (5.40) *we have*

$$H_v^0=\boldsymbol{C}^q,\ H_v^1=0,\ H_v^2=0,\ H_v^3=\boldsymbol{C}^q. \tag{5.41}$$

Proof. It is known that any representation of $S_\nu U(2)$ can be decomposed into a direct sum of irreducible representations. Therefore one may assume that v is irreducible.

Let v be irreducible trivial representation: $v=I\otimes I\in B(V)\otimes\mathscr{A}$, where dim $V=1$. Then $A_r=0$ $(r=0,1,2)$ and complex (5.40) coincides with (5.37). In this case $q=1$ and (5.41) follows from (5.38).

Assume that v is a non-trivial representation of $S_\nu U(2)$. Then the infinitesimal generators of v are given by formulae (5.14)-(5.16) with some strictly positive n (the case $n=0$ corresponds to the trivial representation). In this case $q=0$ and we have to show that the sequence (5.40) is exact.

Let us notice that if n is half-integer then (cf. (5.15)) A_1 is invertible. If n is integer then Ξ_0 is the only basis vector killed by A_1. Since $A_0\Xi_0=-c_1\Xi_1\neq 0$, we see that the intersection of kernels of A_0, A_1, A_2 contains only zero. Thus

$$\ker\ (d:\ \Gamma_V^{\wedge 0} \longrightarrow \Gamma_V^{\wedge 1})=0$$

and sequence (5.40) is exact at $\Gamma_V^{\wedge 0}$.

Using formulae (5.39), (3.13) and Table 5 we get

$$\begin{aligned} &d_v(x_0\otimes\omega_1\wedge\omega_2+x_1\otimes\omega_2\wedge\omega_0+x_2\otimes\omega_0\wedge\omega_1) \\ &\quad=(A_0x_0+\nu^6A_1x_1+\nu^6A_2x_2)\otimes\omega_0\wedge\omega_1\wedge\omega_2 \ . \end{aligned}$$

In virtue of (5.14)-(5.16) any vector of basis (5.13) belongs to the range of at least one generator A_r $(r=0, 1, 2)$. The above formula shows that the mapping

$$d_v:\ \Gamma_V^{\wedge 2}\longrightarrow\Gamma_V^{\wedge 3}$$

is surjective and sequence (5.40) is exact at $\Gamma_V^{\wedge 3}$.

Let

$$x=\sum_{k=0}^{2} x_k\otimes\omega_k$$

be an element of $\Gamma_V^{\wedge 1}=V\otimes\Gamma_c^{\wedge 1}$. Assume that $d_vx=0$. It means that

$$\nu A_2x_0-\nu^{-1}A_0x_2-x_1=0 \tag{5.42}$$

$$\nu^2A_1x_0-\nu^{-2}A_0x_1-(1+\nu^2)x_0=0 \tag{5.43}$$

$$\nu^2A_2x_1-\nu^{-2}A_1x_2-(1+\nu^2)x_2=0 \ . \tag{5.44}$$

If n is half-integer then A_1 is invertible and

$$x_1=A_1y \tag{5.45}$$

where $y=A_1^{-1}x_1$. If n is integer then in virtue of (5.15) the $(+1)$-component of $\nu^2A_1x_0-(1+\nu^2)x_0$ vanishes (the term "*(s)-component of a vector $z\in V$*" means z^s if $z=\sum_k z^k\Xi_k$) and using (5.43) and (5.14) we see that the (0)-component of x_1 vanishes. In this case $x_1\in$ Range A_1 and again we have (5.45) with some $y\in V$. Let

$$x'=x-d_vy \ . \tag{5.46}$$

Then $d_vx'=0$ and writing

$$x'=\sum_k x'_k\otimes\omega_k$$

we see that x'_0, x'_1, x'_2 satisfy relations (5.42)-(5.44). However now $x'_1=x_1-A_1y=0$ and these relations take the following form

$$\nu A_2x'_0-\nu^{-1}A_0x'_2=0 \tag{5.47}$$

$$(\nu^2A_1-(1+\nu^2)I)x'_0=0 \tag{5.48}$$

$$(\nu^{-2}A_1+(1+\nu^2)I)x'_2=0\ . \tag{5.49}$$

If n is half-integer then operators $\nu^2A_1-(1+\nu^2)I$ and $\nu^{-2}A_1+(1+\nu^2)I$ are invertible (cf. (5.15)) and we obtain $x'_0=x'_2=0$. Therefore $x'=0$ and

$$x=d_vy\ .$$

Therefore in this case the sequence (5.40) is exact at $\Gamma_V^{\wedge 1}$.

If n is integer then equations (5.48) and (5.49) show that $x'_0=\lambda_0\Xi_1$ and $x'_2=\lambda_2\Xi_{-1}$, where $\lambda_0, \lambda_2\in\mathbf{C}$ (cf. (5.15)). Now (5.47) implies equality

$$\nu^2c_1\lambda_0+\frac{c_0}{\nu}\lambda_2=0\ . \tag{5.50}$$

Using (5.17) one can easily check that $c_0^2=\nu^2c_1^2$. Therefore (5.50) is equivalent to

$$\frac{\lambda_0}{c_1}+\frac{\lambda_2}{\nu c_0}=0\ .$$

Let

$$y'=-\frac{\lambda_0}{c_1}\Xi_0=\frac{\lambda_2}{\nu c_0}\Xi_0\ .$$

Then $A_0y'=\lambda_0\Xi_1=x'_0$, $A_1y'=0$, $A_2y'=\lambda_2\Xi_{-1}=x'_2$. It means that $x'=d_vy'$ and using (5.46) we get

$$x=d_v(y+y').$$

Therefore also in this case the sequence (5.40) is exact at $\Gamma_V^{\wedge 1}$.

Exactness at $\Gamma_V^{\wedge 2}$ follows now easily from simple dimension computations which we left to the reader. Q. E. D.

We are going to apply this theorem to the regular representation. It is known that any irreducible representation enters into regular representation with the multiplicity equal to the dimension of the representation. In particular the multiplicity of the trivial representation equals to 1. In this case the sequence (5.40) coincides with the de Rham sequence

$$0 \longrightarrow \mathscr{A} \xrightarrow{d} \Gamma \xrightarrow{d} \Gamma^{\wedge 2} \xrightarrow{d} \Gamma^{\wedge 3} \longrightarrow 0$$

and we obtain

Corollary 5.10. *The de Rham cohomology groups of $S_\nu U(2)$ equal* $\boldsymbol{C}$ *in dimensions* 0 *and* 3 *and equal* 0 *in dimensions* 1 *and* 2.

The end of this section is devoted to some remarks concerning the tensor product of representations.

Let v and w be representations of $S_\nu U(2)$ acting on finite-dimensional vector spaces V and W resp. If

$$v = \sum_i m_i \otimes a_i$$
$$w = \sum_j n_j \otimes b_j$$

where $m_i \in B(V)$, $n_j \in B(W)$, a_i, $b_j \in \mathscr{A}$, then the tensor product of representations v and w is a representation of $S_\nu U(2)$ acting on $V \otimes W$ introduced by (cf. (5.3))

$$v \mathbin{\text{⊤}\!\!\!\!\bigcirc} w = \sum_{ij} m_i \otimes n_j \otimes a_i b_j$$

and denoting by $A_r^{v \mathbin{\text{⊤}\!\!\!\!\bigcirc} w}$ the infinitesimal generators of $v \mathbin{\text{⊤}\!\!\!\!\bigcirc} w$ we have

$$A_r^{v \mathbin{\text{⊤}\!\!\!\!\bigcirc} w} = \sum_{ij} \chi_r(a_i b_j) m_i \otimes n_j .$$

Using (2.7) we get

$$A_r^{v \mathbin{\text{⊤}\!\!\!\!\bigcirc} w} = A_r^v \otimes B_r^w + I \otimes A_r^w \tag{5.51}$$

where $r = 0, 1, 2,$

$$A_r^v = \sum_i \chi_r(a_i) m_i$$
$$A_r^w = \sum_j \chi_r(b_j) n_j$$

are infinitesimal generators of v and w resp. and

$$B_r^w = \sum_j f_r(b_j) n_j = (id \otimes f_r) w.$$

In virtue of (2.4), (2.5) and (5.6)

$$B_0^w = B_2^w$$
$$B_1^w = I + \frac{1 - \nu^2}{\nu^2} A_1^w . \tag{5.52}$$

Moreover using (2.11) and (5.1) we obtain

$$(B_0^w)^2 = B_1^w.$$

It turns out that for any representation w operator B_0^w has only positive eigenvalues. This fact can be checked by direct computations for irreducible w (then w is equivalent to d_n with some n and one can use formulae obtained in the proof of Thm. 5.4) and clearly holds for any w (since w can be decomposed into a direct sum of irreducible representations). Therefore

$$B_0^w = B_2^w = \left(I + \frac{1-\nu^2}{\nu^2} A_1^w\right)^{1/2}. \qquad (5.53)$$

Formulae (5.51)-(5.53) express generators of tensor product of two representations in terms of generators of these representations. They correspond to the simple formula

$$A_r^{v \oplus w} = A_r^v \otimes I + I \otimes A_r^w$$

known in the Lie group representation theory. The asymmetry between first and the second factor in formulae (5.51) reflects the noncommutativity of the tensor product mentioned at the beginning of this section.

Using formulae (5.14)-(5.16), (5.51)-(5.53) and Remark 5.7 one can prove the following

Theorem 5.11. *Let n, m be non-negative integer or half-integer. Then the tensor product $d_n \textcircled{\scriptsize T} d_m$ is equivalent to the direct sum*

$$d_{|n-m|} \oplus d_{|n-m|+1} \oplus \dots \oplus d_{n+m}.$$

Theorems 5.4 and 5.11 show that the representation theory for $S_\nu U(2)$ is similar to that of $SU(2)$.

Appendices

A1. The Twisted Unimodularity Condition

Let K be a two-dimensional Hilbert space. We denote by $\Lambda^2 K$ and $S^2 K$ the subspaces of $K \otimes K$ composed of antisymmetric and symmetric elements resp. Then

$$K \otimes K = \Lambda^2 K \oplus S^2 K$$

and dim $\Lambda^2 K=1$.

The determinant of an operator $u\in B(K)$ can be introduced in the following way: One has to consider $u\otimes u$ acting on $K\otimes K$. Then $\Lambda^2 K$ is an eigenspace of $u\otimes u$ and the corresponding eigenvalue coincides with det u. Therefore denoting by ξ a nonzero vector belonging to $\Lambda^2 K$ we have

$$(u\otimes u)\xi=(\det u)\xi.$$

In particular unimodular operators u are distinguished by the condition

$$(u\otimes u)\xi=\xi \;. \tag{A1. 1}$$

Replacing ξ by another non-zero vector $\xi'\in K\otimes K$ we obtain the following "*twisted unimodularity condition*"

$$(u\otimes u)\xi'=\xi' \;. \tag{A1. 2}$$

The vector ξ' can not be arbitrary. In order to have non-trivial unitary solutions of (A1. 2), ξ' should have the following property: There exists a number complex number τ such that

$$(x\otimes\xi'\,|\,\xi'\otimes y)=\tau(x\,|y) \tag{A1. 3}$$

for any x, $y\in K$. One can easily verify that (A1. 3) is satisfied if and only if ξ' is of the form

$$\xi'=k(e_1\otimes e_2-\nu e_2\otimes e_1) \tag{A1. 4}$$

where (e_1, e_2) is an orthonormal basis in K, ν is a real number in the interval $[-1, 1]$, $k\in \boldsymbol{C}$ and $|k|=(1+\nu^2)^{-1/2}\|\xi'\|$. Then $\tau=-\nu(1+\nu^2)^{-1}\|\xi'\|^2$. If $\nu=1$ then $\xi'\in\Lambda^2 K$ and (A1. 2) coincides with the usual (non-twisted) condition (A1. 1).

Even if (A1. 3) is satisfied, equation (A1. 2) has no interesting solutions if we are restricted to unitaries u belonging to $B(K)$. In order to find non-trivial unitary u satisfying (A1. 2) one has to consider elements of $B(K)\otimes A$, where A is a C^*-algebra which need not be commutative. In this case (A1. 2) should be rewritten in the following more precise way

$$(u \,\textcircled{\scriptsize T}\, u)(\xi'\otimes I)=\xi'\otimes I$$

where Ⓣ denotes the tensor product elements of $B(K)$ combined with the usual product in A (cf. def. (5. 3)) and I is the unity of A.

In the following theorem we consider $\boldsymbol{C}^2$ instead of K assuming that the basis entering to the formula (A1.4) coincides with the canonical basis in $\mathbb{C}^2$. Then $B(K)\otimes A$ can be identified with the algebra $M_2(A)$ of all 2×2 matrices with entries belonging to A.

Theorem A1.1. *Let $\nu\in[-1, 1]$ and $\xi_\nu=e_1\otimes e_2-\nu e_2\otimes e_1$, where (e_1, e_2) is the canonical basis in $\mathbb{C}^2$. Then for any 2×2 matrix u with entries belonging to a C^*-algebra A the following two conditions are equivalent:*

I. *u is unitary and*

$$(u \textcircled{\scriptsize T} u)(\xi_\nu\otimes I)=\xi_\nu\otimes I \ . \tag{A1.5}$$

II. *u is of the form*

$$u=\begin{pmatrix}\alpha, & -\nu\gamma^* \\ \gamma, & \alpha^*\end{pmatrix} \tag{A1.6}$$

where α, $\gamma\in A$ and (cf. Table 0)

$$\begin{matrix} & \gamma^*\gamma=\gamma\gamma^* \\ \alpha^*\alpha+\gamma^*\gamma=I & \\ & \alpha\gamma=\nu\gamma\alpha \\ \alpha\alpha^*+\nu^2\gamma^*\gamma=I & \\ & \alpha\gamma^*=\nu\gamma^*\alpha \ . \end{matrix} \tag{A1.7}$$

Proof. I$\Rightarrow$II

Let

$$u=\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta\in A$. We use the unitarity to simplify (A1.5). Multiplying both sides of (A1.5) by $u^* \textcircled{\scriptsize T} I_2$ (where I_2 denotes the unity of $M_2(A)$) we get

$$(I_2 \textcircled{\scriptsize T} u)(\xi_\nu\otimes I)=(u^* \textcircled{\scriptsize T} I_2)(\xi_\nu\otimes I) \ . \tag{A1.8}$$

We rewrite this equation in the matrix form using the basis in $\mathbb{C}^2\otimes\mathbb{C}^2$ composed of the elements $e_1\otimes e_1$, $e_1\otimes e_2$, $e_2\otimes e_1$ and $e_2\otimes e_2$. In this basis $I_2 \textcircled{\scriptsize T} u$ and $u^* \textcircled{\scriptsize T} I_2$ are represented by the matrices:

$$\begin{pmatrix}\alpha, & \beta, & 0, & 0 \\ \gamma, & \delta, & 0, & 0 \\ 0, & 0, & \alpha, & \beta \\ 0, & 0, & \gamma, & \delta\end{pmatrix} \qquad \begin{pmatrix}\alpha^*, & 0, & \gamma^*, & 0 \\ 0, & \alpha^*, & 0, & \gamma^* \\ \beta^*, & 0, & \delta^*, & 0 \\ 0, & \beta^*, & 0, & \delta^*\end{pmatrix}$$

whereas $\xi_\nu\otimes I$ is represented by the column

$$\begin{pmatrix} 0 \\ I \\ -\nu I \\ 0 \end{pmatrix}.$$

Therefore we have

$$\begin{pmatrix} \alpha, & \beta, & 0, & 0 \\ \gamma, & \delta, & 0, & 0 \\ 0, & 0, & \alpha, & \beta \\ 0, & 0, & \gamma, & \delta \end{pmatrix} \begin{pmatrix} 0 \\ I \\ -\nu I \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha^*, & 0, & \gamma^*, & 0 \\ 0, & \alpha^*, & 0, & \gamma^* \\ \beta^*, & 0, & \delta^*, & 0 \\ 0, & \beta^*, & 0, & \delta^* \end{pmatrix} \begin{pmatrix} 0 \\ I \\ -\nu I \\ 0 \end{pmatrix}. \tag{A1.9}$$

This relation is satisfied if and only if $\beta=-\nu\gamma^*$ and $\delta=\alpha^*$. It proves (A1. 6). To end this part of the proof we notice that (A 1. 6) is unitary if and only if equations (A1. 7) hold.

II$\Rightarrow$I

Assume that u is given by (A1. 6), where α, $\gamma\in A$ satisfy (A1. 7). Then (see the remark at the end of the first part of the proof) u is unitary and (A1. 5) is implied by (A1. 8). The latter is equivalent to (A1. 9) with β and δ replaced by $-\nu\gamma^*$ and α^* resp. and evidently holds. Q. E. D.

Assume that a unitary $u\in M_2(A)$ satisfies the twisted unimodularity condition (A1. 5). Let $v=u\oplus u\in M_2(A\otimes A)$ (see page 1. 7 for the definition of $\oplus$). Then v is unitary and using the obvious relation

$$(u\oplus u)\,\textcircled{\scriptsize T}\,(u\oplus u) = (u\,\textcircled{\scriptsize T}\,u)\oplus(u\,\textcircled{\scriptsize T}\,u)$$

we see that v also satisfies the twisted unimodularity condition with the same ν. This explains why elements α' and γ' considered in the proof of Thm. 1. 4 satisfy relations of Table 0.

A2. Structure of the Algebra A

We shall use the notions and results of [2].

For any Hilbert space H we denote by $D_\nu(H)$ the set of all pairs of operators (α,γ) acting on H and satisfying relations of Table 0. Then D_ν is a measurable domain and for any H, $D_\nu(H)$ is a closed subset of $B(H)^2$. Therefore D_ν is a compact domain.

One can easily verify that the algebra A introduced in Section 1 coincides with the algebra $\mathscr{C}(D_\nu)$ of all continuous operator functions defined on D_ν.

At first we consider case $|\nu|<1$.

It is not difficult to prove the following

Lemma A2.1. *Let $|\nu|<1$. Then for any Hilbert space H and any (α, γ) belonging to $D_\nu(H)$ we have*

$$Sp\ \alpha^*\alpha \subset \{0, 1-\nu^2, 1-\nu^4, 1-\nu^6, \dots, 1\}$$
$$Sp\ \gamma^*\gamma \subset \{1, \nu^2, \nu^4, \nu^6, \dots, 0\}.$$

Let f, g be continuous functions on $[0, 1]$ such that:

$$f(t) = \begin{cases} 1 & \text{for } t \geq 1-\nu^2 \\ 0 & \text{for } t=0 \end{cases}$$

$$g(t) = \begin{cases} 0 & \text{for } t \leq \nu^2 \\ 1 & \text{for } t=1 \end{cases}.$$

For any Hilbert space H and any $(\alpha, \gamma) \in D_\nu(H)$ we set

$$T_H^1(\alpha, \gamma) = \alpha f(\alpha^*\alpha)$$
$$T_H^2(\alpha, \gamma) = \gamma g(\gamma^*\gamma).$$

Obviously $T^1, T^2 \in \mathscr{C}(D_\nu)$. Moreover using Lemma A2.1 one can easily check that $(T_H^1(\alpha, \gamma), T_H^2(\alpha, \gamma)) \in D_0(H)$ for any $(\alpha, \gamma) \in D_\nu(H)$. Therefore $T=(T^1, T^2)$ is a morphism from D_ν into D_0:

$$T\colon D_\nu \longrightarrow D_0. \qquad \text{(A2.1)}$$

It turns out that this morphism is invertible. The inverse $\tilde{T}=(\tilde{T}^1, \tilde{T}^2)$ is given by the formulae

$$\begin{aligned}\tilde{T}_H^1(\alpha, \gamma) &= \sum_{n=1}^{\infty} \sqrt{1-\nu^{2n}}(\alpha^{*(n-1)}\alpha^n - \alpha^{*n}\alpha^{n+1}) \\ &= \sum_{n=0}^{\infty} \frac{(1-\nu^2)\nu^{2n}}{\sqrt{1-\nu^{2n+2}}+\sqrt{1-\nu^{2n}}}\alpha^{*n}\alpha^{n+1} \qquad \text{(A2.2)}\end{aligned}$$

$$\tilde{T}_H^2(\alpha, \gamma) = \sum_{n=0}^{\infty} \nu^n \alpha^{*n}\gamma\alpha^n. \qquad \text{(A2.3)}$$

Clearly $\tilde{T}^1, \tilde{T}^2 \in \mathscr{C}(D_0)$ (the series (A2.2) and (A2.3) are uniformly converging). Therefore (A2.1) is a homeomorphism and using [2] we obtain

Theorem A2.2. *For any* $\nu\in[-1, 1]$ *the algebra* A *is isomorphic to* $\mathscr{C}(D_0)$.

Remark. This result does not mean that the pseudogroups $S_\nu U(2)$ are isomorphic for all ν in the interval $[-1, 1]$. It only means that the underlying pseudospaces are homeomorphic.

For any Hilbert space H, S^1_H will denote the set of all unitaries acting on H. Clearly S^1 is a compact domain and the algebra $\mathscr{C}(S^1)$ coincides with the algebra of all continuous functions on the unit circle.

Let us notice that any pair of the form (α, γ), where α is unitary and $\gamma=0$ satisfies the relation of Table 0. Therefore we have morphism (injection)

$$R: S^1 \longrightarrow D_\nu \tag{A2.4}$$

such that $R(U)=(U, 0)$. By inverse image (A2.4) defines C^*-algebra homomorphism (surjection):

$$R_*: A=C(D_\nu)\longrightarrow \mathscr{C}(S^1) .$$

The kernel of this homomorphism consists of all continuous operator functions a defined on D_ν such that $a_H(U, 0)=0$ for any Hilbert space H and any unitary U acting on H. One can prove that this kernel is isomorphic to the tensor product $C\otimes\mathscr{C}(S^1)$, where C denotes the algebra of all compact operators acting on a separable Hilbert space (at first one has to show that the representation π constructed in the proof of Thm. 1.2 is faithful). Therefore we have

Theorem A2.3. *The algebra* A *is a GCR algebra.*

Remark. We considered only the case $\nu\in[-1, 1]$. However for $\nu=1$ the algebra A is commutative, whereas for $\nu=-1$ all irreducible representations of A are two and one-dimensional. Therefore in these cases A is a CCR algebra.

References

[1] Woronowicz, S. L., Compact matrix pseudogroups, in preparation.

[2] Kruszynski, P. and Woronowicz, S. L., A noncommutative Gelfand Naimark theorem, *Journal of Operator Theory*, 8 (1982), 361-389.

[3] Woronowicz, S. L., Pseudospaces, pseudogroups and Pontriagin duality, *Proceedings of the International Conference on Mathematics and Physics Lausanne 1979*, Lecture Notes in Physics, **116** (1980).

[4] Kac, G. I., Ring-groups and the principle of duality. I, II, *Trudy Moskov. Mat. Obsc.*, **12** (1963), 259-301; **13** (1965), 84-113.

[5] Takesaki, M., Duality and von Neumann algebras, *Lecture notes, Fall 1970, Tulane University*, New Orleans, Louisiana.

[6] Vallin, J. M., C^*-algebre de Hopf et C^*-algebre de Kac, *Proceedings of London Mathematical Society* (3), **50** (1985), 131-174.

[7] Connes, A., Non commutative differential geometry, Chapter I. The Chern character in K homology, *IHES/M/82/53*, Chapter II. De Rham homology and non commutative algebra *IHES/M/83/19*.