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Equivariant Controllable Cutting-Pasting and Cobordism with Vector Fields

By

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Introduction

Throughout this paper G always denotes a finite group of odd order, and manifolds and maps considered are all of class C^∞. $\mathcal{M}_n[G]$ denotes the set of n-dimensional closed G-manifolds. In the set $\mathcal{M}_n[G]$ we will consider the two notions, Reinhart G-cobordism and SKK-equivalence.

For M and $N\in\mathcal{M}_n[G]$, if there is an $(n+1)$-dimensional compact G-manifold L with $\partial L=M+N$, the disjoint union of M and N, then they are called *G-cobordant* and L is called a *G-cobordism* between them. This cobordism relation defines the cobordism group $N_n[G]$ of n-dimensional closed G-manifolds. If a G-cobordism L between M and N admits a nonzero G-vector field which is inward normal on M and outward normal on N, then, following Reinhart [8], M and N are called *Reinhart G-cobordant*, and L a *Reinhart G-cobordism* between them. The set of cobordism classes by this cobordism relation in $\mathcal{M}_n[G]$ forms a semigroup with disjoint union $+$ as its group operation. Denote by $N_n^R[G]$ the Grothendieck group of the semigroup. From the author [4] we obtain a necessary and sufficient condition for M and $N\in\mathcal{M}_n[G]$ to represent the same class in $N_n^R[G]$ in terms of $N_n[G]$ and the Euler characteristics of the fixed point sets of M and N.

Let P, P', Q and Q' be n-dimensional compact G-manifolds with $\partial P=\partial P'$ and $\partial Q=\partial Q'$. Let φ and ψ: $\partial P\to\partial Q$ be G-diffeomorphisms. Then by pasting two G-manifolds along boundary we obtain closed G-manifolds $P\cup_\varphi Q$, $P'\cup_\psi Q'$, etc. Give $\mathcal{M}_n[G]$ the equivalence relation

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$\sim$ generated by relations of the form

$$P\cup_{\varphi}Q+P'\cup_{\phi}Q'\sim P\cup_{\phi}Q+P'\cup_{\varphi}Q'.$$

This relation is called *SKK-equivalence* (*S*chneiden und *K*leben *K*ontrollierbar). The quotient set $\mathscr{M}_n[G]/\sim$ becomes a semigroup with the operation $+$. Denote by $SKK_n[G]$ the Grothendieck group of the semigroup.

In this paper we will establish short exact sequences which contain the groups $N_n[G]$, $N_n^R[G]$ and $SKK_n[G]$. From the exact sequences we will show, under a restriction for the one dimensional components of the fixed point sets, M and N represent the same class in $N_n^R[G]$ if and only if so do they in $SKK_n[G]$.

Heithecker [2, 3] also discusses SKK-equivalence and Reinhart G-cobordism of oriented G-manifolds, G an abelian group of odd order. A modified version of SKK-equivalence is discussed in Prevot [5, 6, 7].

§ 1. Surgery

Given a G-manifold M and a subgroup H of G, M^H denotes the H-fixed point set of M and $M^{=H}$ denotes the union of those components of M^H on which H occurs actually as an isotropy subgroup. If V is a representation of H containing no direct summand of trivial representation, $M^{(H,V)}$ denotes the union of those components of $M^{=H}$ at which the normal representation is isomorphic to V. Suppose $M^{(H,V)}\neq\phi$. Denote by $G_{(H,V)}$ the subgroup of G whose action keeps $M^{(H,V)}$ invariant. We see that $H\subset G_{(H,V)}\subset N(H)$, the normalizer of H in G, and that $G_{(H,V)}$ is determined by (H, V) and independent of M.

Let $\dim M^{(H,V)}>0$ and $\dim M^{(H,V)}+1=p+q$ where p and q are positive integers. Then there is a smooth G-embedding φ: $G\times_H D(V\oplus R^p)\times S(R^q)\rightarrow M$ onto a G-invariant regular submanifold of M, where $D(\ )$, $S(\ )$ and R^k denote the closed unit disc, the unit sphere and the k-dimensional trivial representation, respectively. Let L be a G-manifold obtained from the disjoint union of $M\times[0, 1]$ and $G\times_H D(V\oplus R^p)\times D(R^q)$ by identifying $\operatorname{Im}\varphi\times\{1\}$ with $G\times_H D(V\oplus R^p)\times S(R^q)$. Then, if M is closed, $\partial L=M+N$ where

$$N=(M-\varphi(G\times_H\mathring{D}(V\oplus R^p)\times S(R^q))\cup G\times_H S(V\oplus R^p)\times D(R^q),$$

$\mathring{D}(\ )$ denotes the open unit disc. N is called a *G-manifold obtained from M by G-surgery of type* (H, V, p, q), and L the *trace* of the G-surgery. We see the following.

(1) If $\dim M^{(K,U)} < \dim M^{(H,V)}$, then the G-surgery does not affect $M^{(K,U)}$, i.e., $M^{(K,U)} = N^{(K,U)}$.

(2) If $\dim M^{(H,V)}$ is even, then

$$\chi(N^{(H,V)}) = \chi(M^{(H,V)}) + (-1)^{p+1} 2\chi(G_{(H,V)}/H),$$

where $\chi(\ )$ denotes the Euler characteristic, because $N^{(H,V)}$ is obtained from $M^{(H,V)}$ by deleting $\chi(G_{(H,V)}/H)$ copies of $\mathring{D}(R^p) \times S(R^q)$ and attatching as many copies of $S(R^p) \times D(R^q)$.

(3) If $M^{(H,V)}$ is connected and $p>1$, then $N^{(H,V)}$ is also connected. If $\dim M^{(H,V)}$ is even and greater than 2, we may take p to be odd and greater than 1. Thus, by doing G-surgeries of an appropriate type we may then obtain N such that $N^{(H,V)}$ is connected and $\chi(N^{(H,V)})>0$.

The following lemma is obtained from the existence of *excellent* G-Morse functions (see Field [1]) and a usual connection between surgery and Morse function.

Lemma 1.1. *Let M and $N \in \mathscr{M}_n[G]$ be G-cobordant, and L a G-cobordism between them. Then N is obtained from M by performing a finite series of G-surgeries of type (H_i, V_i, p_i, q_i), $i = 1, 2, \ldots, s$, with trace L.*

§2. Cobordism with Vector Fields

Lemma 2.1 ([3; Satz 1.1], [4; Proposition 1.2], [9; Theorem 4.4]). *Let L be a G-cobordism between closed G-manifolds M and N. Then L admits a nonzero G-vector field which is inward normal on M and outward normal on N, if and only if for any subgroup H of G, every component A of M^H satisfies $\chi(A) = \chi(A \cap M) = \chi(A \cap N)$.*

For a space X and a nonnegative integer k, kX denotes the disjoint union of k copies of X. For a G-manifold M, $M^{H,k}$ denotes the k-dimensional components of M^H. If $M \in \mathscr{M}_n[G]$, $[M]$ denotes the class represented by M in the group $N_n[G]$, $N_n^R[G]$ or $SKK_n[G]$.

Lemma 2.2. *Suppose that $[M] = [N]$ in $N_n[G]$, and that $M^{H,1} = \phi$ and $N^{H,1} = \phi$ for any subgroup H of G. Then there exists a compact G-*

manifold L such that

(i) $\partial L = M_0 + N_0$,

$M_0 = M + \sum_{(H,V)} \alpha_{(H,V)} G \times_H S(V \oplus R^{n-\dim V+1})$,

$N_0 = N + \sum_{(H,V)} \beta_{(H,V)} G \times_H S(V \oplus R^{n-\dim V+1})$,

where $\alpha_{(H,V)}$ *and* $\beta_{(H,V)}$ *are nonnegative integers, and the sums* $\sum_{(H,V)}$ *are taken over pairs* (H, V) *of subgroups* H *of* G *and representations* V *of* H *which contain no direct summand of trivial representation, and*

(ii) *for any subgroup* H *of* G *every component* A *of* L^H *satisfies* $\chi(A) = \chi(A \cap M_0) = \chi(A \cap N_0)$.

From Lemma 2. 1 it follows that the G-manifold L in Lemma 2. 2 admits a nonzero G-vector field which is inward normal on M_0 and outward normal on N_0. Thus we see

Proposition 2. 3. *Suppose that* $[M] = [N]$ *in* $N_n[G]$, *and that* $M^{H,1} = \phi$ *and* $N^{H,1} = \phi$ *for any subgroup* H *of* G. *Then in* $N_n^R[G]$,

$$[M] = [N] + \sum_{(H,V)} \gamma_{(H,V)} [G \times_H S(V \oplus R^{n-\dim V+1})],$$

where $\gamma_{(H,V)}$ *are integers.*

Proof of Lemma 2. 2. (1) From the hypothesis there exists a compact G-manifold L_0 such that (i) $\partial L_0 = M + N$, and (ii) $L_0^{H,2}$ is closed for any subgroup H. We eliminate all the isolated H-fixed points from L_0 as follows. Take in L_0 invariant small open discs $\cup_V \mathring{D}(V)$ with the isolated H-fixed points as their centers. Cut the discs off from L_0, and sew the resulting manifold along the newly arising boundary $\cup_V S(V)$ by antipodal involution. Since G is of odd order, no new fixed points arise by this process, and we obtain a compact G-manifold L_1 such that (i) $\partial L_1 = M + N$, and (ii) for any subgroup H of G, $L_1^{H,0}$ is empty and $L_1^{H,2}$ is closed.

(2) Cut an invariant small open tubular neighborhood of $L_1^{H,2}$ off from L_1. Then the newly arising boundary is a sphere bundle. Sew the resulting manifold along the new boundary by antipodal involution (on the sphere bundle). Since G is of odd order, no new fixed points arise by this process also. Thus we obtain a compact G-manifold L_2 such that (i) $\partial L_2 = M + N$, and (ii) $L_2^{H,0}$ and $L_2^{H,2}$ are empty for any subgroup H of G.

(3) If for a component A of $L_2^{H,1}$, $\partial A\cap M$ or $\partial A\cap N$ is two points, then take a small open disc $\mathring{D}(V\oplus R)$ with a point $\in \mathrm{Int}\, A$ as its center. We cut such discs off from L_2, and then obtain a compact G-manifold L_3 such that

(i) $\partial L_3=M_3+N_3$,

$$M_3=M+\sum\alpha_{(H,V)}G\times_H S(V\oplus R),$$

$$N_3=N+\sum\beta_{(H,V)}G\times_H S(V\oplus R),$$

(ii) for any subgroup H of G, any component of $L_3^{H,1}$ is either a closed curve ($\approx S^1$) or a curve in L_3 which joins a point of M_3 and a point of N_3, and

(iii) $L_3^{H,0}$ and $L_3^{H,2}$ are empty for any subgroup H of G. Thus L_3 is a G-cobordism between M_3 and N_3 such that $\chi(A)=\chi(A\cap M_3)=\chi(A\cap N_3)$ for any component A of $L_3^{H,r}$, $r=0, 1, 2$.

(4) For a positive integer k consider the following assertion:

$P(k)$. *There exists a compact G-manifold L_k such that*

(i) $\partial L_k=M_k+N_k$,

$$M_k=M+\sum_{(H,V)}\alpha_{(H,V)}G\times_H S(V\oplus R^{n-\dim V+1}),$$

$$N_k=N+\sum_{(H,V)}\beta_{(H,V)}G\times_H S(V\oplus R^{n-\dim V+1}),$$

where $\alpha_{(H,V)}$ and $\beta_{(H,V)}$ are nonnegative integers, and

(ii) *for any subgroup H of G and any component A of $L_k^{H,r}$ $(r<k)$,*

$$\chi(A)=\chi(A\cap M_k)=\chi(A\cap N_k).$$

If $k\leqq 3$, the assertion $P(k)$ is already proved by the above arguments. We prove below that $P(k)$ implies $P(k+1)$ for $k\geqq 3$. Since Lemma 2. 2 is equivalent to $P(n+2)$, the lemma is inductively obtained.

For a pair (H, V) suppose that $\dim L_k^{(H,V)}=k$. It is no loss of generality to suppose that $L_k^{(H,V)}$ is connected, since if it is not, we may make it connected by G-surgery of type $(H, V, k, 1)$. Moreover, as noted in §1 this G-surgery does not affect $L_k^{K,r}$ for any subgroup $K\leqq G$ and any $r<k$. Since $M_k^{(H,V)}$ and $N_k^{(H,V)}$ are cobordant with a cobordism $L_k^{(H,V)}$, then $\chi(M_k^{(H,V)})-\chi(N_k^{(H,V)})$ is even. The assertion (ii) of $P(k)$ implies $\chi((M_k^{(H,V)})^K)=\chi((N_k^{(H,V)})^K)$ for any subgroup K with $H\lneqq K\leqq G_{(H,V)}$. From this we see that $\chi(M_k^{(H,V)})-\chi(N_k^{(H,V)})$ is

a multiple of $\chi(G_{(H,V)}/H)$, since H is the principal isotropy subgroup of the $G_{(H,V)}$-manifolds $M_k^{(H,V)}$ and $N_k^{(H,V)}$. Thus we may put

$$\chi(M_k^{(H,V)})-\chi(N_k^{(H,V)})=2m\chi(G_{(H,V)}/H)$$

for some integer m.

(5) Suppose that $k=\dim L_k^{(H,V)}$ is odd. It then follows that

$$\begin{aligned}\chi(L_k^{(H,V)})&=\frac{1}{2}\chi(\partial L_k^{(H,V)})\\&=\frac{1}{2}(\chi(M_k^{(H,V)})+\chi(N_k^{(H,V)}))\\&=\chi(M_k^{(H,V)})-m\chi(G_{(H,V)}/H)\\&=\chi(N_k^{(H,V)})+m\chi(G_{(H,V)}/H).\end{aligned}$$

Take $|m|$ points $x_1, x_2, \ldots, x_{|m|}$ of Int $L_k^{(H,V)}$ whose isotropy subgroups are all H and for which $gx_i\neq x_j$ if $g\in G_{(H,V)}$ and $i\neq j$. Consider a small disc $D_i(V\oplus R^k)$ with x_i as its center, and let

$$L'_k=L_k-\cup_{i=1}^{|m|}G\times_H\mathring{D}_i(V\oplus R^k).$$

We then see that

$$\partial L'_k=M_k+N_k+|m|G\times_H S(V\oplus R^k).$$

If $m\geqq 0$, then let

$$M'_k=M_k, \quad \text{and}$$

$$N'_k=N_k+|m|G\times_H S(V\oplus R^k).$$

If $m<0$, then let

$$M'_k=M_k+|m|G\times_H S(V\oplus R^k), \quad \text{and}$$

$$N'_k=N_k.$$

$L_k'^{(H,V)}$ is then connected and satisfies $\chi(L_k'^{(H,V)})=\chi(M_k'^{(H,V)})=\chi(N_k'^{(H,V)})$. Thus, performing the same as above for all (H, V) with $\dim L_k^{(H,V)}=k$, we obtain a compact G-manifold L_{k+1} as in $P(k+1)$.

(6) Suppose that $k=\dim L_k^{(H,V)}$ is even. Since $\chi(M_k^{(H,V)})=0$ and $\chi(N_k^{(H,V)})=0$, we must then make the Euler characteristic of $L_k^{(H,V)}$ zero (keeping the connectedness of $L_k^{(H,V)}$). The assertion (ii) of $P(k)$ implies $\chi((L_k^{(H,V)})^K)=0$ for any subgroup K with $H\lneqq K\leqq G_{(H,V)}$. From this we see that $\chi(L_k^{(H,V)})$ is a multiple of $\chi(G_{(H,V)}/H)$. Thus let $\chi(L_k^{(H,V)})=m\chi(G_{(H,V)}/H)$, m an integer. We may suppose that m is nonnegative, since if it is not, we may make $\chi(L_k^{(H,V)})$ nonnegative

by the argument (3) in §1. Take m points $x_1, x_2, \ldots, x_m$ of $\mathrm{Int} L_k^{(H,V)}$ whose isotropy subgroups are all H and for which $gx_i \neq x_j$ if $g \in G_{(H,V)}$ and $i \neq j$. Consider a small disc $D_i(V \oplus R^k)$ with x_i as its center, and let

$$L_k' = L_k - \cup_{i=1}^m G \times_H \mathring{D}_i(V \oplus R^k).$$

We then see that

$$\partial L_k' = M_k + N_k + mG \times_H S(V \oplus R^k).$$

Letting $M_k' = M_k$ and $N_k' = N_k + mG \times_H S(V \oplus R^k)$, we see that $L_k'^{(H,V)}$ is connected and $\chi(L_k'^{(H,V)}) = \chi(M_k'^{(H,V)}) = \chi(N_k'^{(H,V)}) = 0$. Thus, performing the same as above for all (H, V) with dim $L_k^{(H,V)} = k$, we obtain a compact G-manifold L_{k+1} as in $P(k+1)$. □

Let $I_n^R[G]$ be the subgroup of $N_n^R[G]$ generated by G-manifolds of the form $G \times_H S(V)$ where H is any subgroup of G, and V is any representation of H with dim $V = n+1$. Let $\bar{N}_n[G]$ and $\bar{N}_n^R[G]$ be the subgroups of $N_n[G]$ and $N_n^R[G]$, respectively, generated by G-manifolds M with $M^{H,1} = \phi$ for any subgroup H of G. Let $\bar{I}_n^R[G] = I_n^R[G] \cap \bar{N}_n^R[G]$. If a representation V of an odd order group contains no direct summand of trivial representation, V has a complex structure, and hence its (real) dimension is even. This implies that the dimensions of M and its fixed point sets are congruent modulo 2. Thus if dim M is even, $M^{H,1}$ is always empty. If G is abelian, the normal bundle of $M^{H,1}$ in M are Reinhart G-cobordant to zero as G-vector bundle (see the author [4; Lemma 5. 1]). Thus if n is even or if G is abelian, then we see that $N_n[G] = \bar{N}_n[G]$, $N_n^R[G] = \bar{N}_n^R[G]$ and $I_n^R[G] = \bar{I}_n^R[G]$.

Theorem 2.4. *There is a short exact sequence*

$$0 \longrightarrow \bar{I}_n^R[G] \xrightarrow{i} \bar{N}_n^R[G] \xrightarrow{j} \bar{N}_n[G] \longrightarrow 0$$

where i is the canonical inclusion, and j is the obvious homomorphism, i.e., the homomorphism sending a Reinhart G-cobordism class $[M]$ to a G-cobordism class $[M]$.

Proof. It is easy that i is monic, j is epic and $j \circ i = 0$. Ker $j \subset$ Im i follows from Proposition 2. 3. □

Proposition 2.5. *If n is odd, then $I_n^R[G]=0$.*

Proof. It is sufficient to prove that $[G\times_H S(U)]=0$ in $I_n^R[G]$ if U is a representation of H with dim $U=n+1$ even. Note that dim U^K is even for any subgroup K of H. Let $RP(U\oplus R)$ be the quotient space $S(U\oplus R)$ by the antipodal involution. It inherits a stucture of an n-dimensional H-manifold. In $RP(U\oplus R)$ take a small disc $D(U)$ with the point $RP(R)$ as its center, and let

$$L=G\times_H RP(U\oplus R)-G\times_H \mathring{D}(U).$$

We then see that $\partial L=G\times_H S(U)$ and that for any subgroup K of G any component A of L^K is diffeomorphic to $RP(U'\oplus R)-\mathring{D}(U')$ (where U' is an even dimensional subspace of U) and A satisfies $\chi(A)=\chi(\partial A)=0$. By Lemma 2.1 L admits a nonzero G-vector field which is inward normal on ∂L. Thus $[G\times_H S(U)]=0$. □

Note that any element of the Grothendieck group of a semigroup S is of the form $s-s'$ where s, $s'\in S$.

Proposition 2.6. *Suppose that $[M]-[N]$ is an element of $I_n^R[G]$. Then $[M]-[N]=0$ in $I_n^R[G]$ if and only if $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any pair (H, V).*

Proof. $[M]-[N]=0$ implies $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ by Lemma 2.1. If n is odd, $[M]-[N]=0$ is clear by the preceding proposition. Suppose that n is even and that $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any (H, V). Note that in this case the dimensions of fixed point sets are all even. It is sufficient to prove that $[M]-[N]=0$ when

$$M=\sum_{(H,V)\in T(G)}\alpha_{(H,V)}G\times_H S(V\oplus R^{n-\dim V+1}),\text{ and}$$

$$N=\sum_{(H,V)\in T(G)}\beta_{(H,V)}G\times_H S(V\oplus R^{n-\dim V+1}).$$

Here $\alpha_{(H,V)}$ and $\beta_{(H,V)}$ are nonnegative integers, and $T(G)$ is a finite set of pairs (H, V) such that if $(H, V)\neq(K, U)$ in $T(G)$, then $G\times_H S(V\oplus R^{n-\dim V+1})$ and $G\times_K S(U\oplus R^{n-\dim U+1})$ are not G-diffeomorphic. Order the pairs in

$$T(G)=\{(H_1, V_1), (H_2, V_2), \dots, (H_a, V_a)\}$$

in such a way that if H_i is conjugate to a subgroup of H_j, then $j\leqq i$.

It follows that

$$(G\times_{H_j}S(V_j\oplus R^{n-\dim V_j+1}))^{(H_i,V_i)}=\begin{cases}G_{(H_i,V_i)}/H_i\times S(R^{n-\dim V_i+1}) & \text{if } i=j\\ \phi & \text{if } i<j.\end{cases}$$

Thus we see that $\chi(M^{(H_i,V_i)})=\sum_{j=1}^{i}\varepsilon_j\alpha_{(H_j,V_j)}$ and $\chi(N^{(H_i,V_i)})=\sum_{j=1}^{i}\varepsilon_j\beta_{(H_j,V_j)}$, where ε_j is an integer, especially $\varepsilon_i=2\chi(G_{(H_i,V_i)}/H_i)\neq 0$. Since $\chi(M^{(H_i,V_i)})=\chi(N^{(H_i,V_i)})$, then $\alpha_{(H_i,V_i)}=\beta_{(H_i,V_i)}$ if $\alpha_{(H_j,V_j)}=\beta_{(H_j,V_j)}$ for any $j<i$. Thus, by induction we see that $\alpha_{(H,V)}=\beta_{(H,V)}$ for any $(H, V)\in T(G)$, or that $[M]=[N]$ in $I_n^R[G]$. □

Corollary 2.7 (cf. [4]). *Suppose that M and N are G-cobordant closed G-manifolds with $M^{H,1}=\phi$ and $N^{H,1}=\phi$ for any subgroup H of G. Then there exists a Reinhart G-cobordism between M and N, if and only if $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any pair (H, V).*

Proof. The "only if" part follows from Lemma 2.1. If $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any (H, V), then $[M]-[N]=0$ in $\bar{I}_n^R[G]$ by Theorem 2.4 and Proposition 2.6. This implies that there exists a Reinhart G-cobordism between M and N. □

§ 3. Controllable Cutting and Pasting

As in §1 and §2 of Heithecker [2] we obtain the following Proposition 3.1, Lemma 3.2 and Lemma 3.3:

Proposition 3.1. *If $[M]=[N]$ in $SKK_n[G]$, then*

(i) $[M]=[N]$ *in* $N_n[G]$, *and*

(ii) $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ *for any pair* (H, V).

Lemma 3.2. *If N is obtained from M by G-surgery of type (H, V, p, q), then in $SKK_n[G]$,*

$$[M]+[G\times_H S(V\oplus R^{p+q})]=[N]+[G\times_H S(V\oplus R^{p+1})\times S(R^q)],$$

where $p+q=n-\dim V+1$.

Lemma 3.3. *In* $SKK_n[G]$,

$$[G\times_H S(V\oplus R^{p+1})\times S(R^q)]=\begin{cases}2[G\times_H S(V\oplus R^{p+q})] & \text{if } q \text{ is odd,}\\ 0 & \text{if } q \text{ is even,}\end{cases}$$

where $p+q=n-\dim V+1$.

The following proposition follows from the preceding two lemmas.

Proposition 3.4. *If N is obtained by performing a finite series of G-sugeries on M of type (H_i, V_i, p_i, q_i), $i=1, 2, \ldots, s$, then in $SKK_n[G]$,*

$$[N]=[M]+\textstyle\sum_{i=1}^{s}(-1)^{q_i}[G\times_{H_i}S(V_i\oplus R^{n-\dim V_i+1})].$$

Let $I_n[G]$ be the subgroup of $SKK_n[G]$ generated by G-manifolds of the form $G\times_H S(V)$. From Proposition 3.4 and Lemma 1.1, if M and $N\in\mathscr{M}_n[G]$ are G-cobordant, then we see that $[M]-[N]\in I_n[G]$. Thus we obtain

Theorem 3.5. *There is a short exact sequence*

$$0\longrightarrow I_n[G]\xrightarrow{\ i\ }SKK_n[G]\xrightarrow{\ j\ }N_n[G]\longrightarrow 0$$

where i and j are the obvious homomorphisms.

Proposition 3.6. *If $[M]=[N]$ in $N_n^R[G]$, then $[M]=[N]$ in $SKK_n[G]$.*

Proof. From the hypothesis there is a Reinhart G-cobordism L between M and N, and by Lemma 2.1 L satisfies $\chi(L^{(H,V)})=\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any pair (H, V). By Lemma 1.1, N is obtained from M by performing a finite series of G-surgeries of type (H_i, V_i, p_i, q_i), $i=1, 2, \ldots, s$, with trace L. Here we may take the subgroups $H_1, H_2, \ldots, H_s$ so that $H_i=H_j$ if H_i and H_j are conjugate. Moreover, we may take the representations $V_1, V_2, \ldots, V_s$ so that $V_i=V_j$ if $H=H_i=H_j$, and if $G\times_H V_i$ and $G\times_H V_j$ are isomorphic as G-vector bundles over G/H. By Proposition 3.4 in $SKK_n[G]$,

$$[N]=[M]+\textstyle\sum_{i=1}^{s}(-1)^{q_i}[G\times_{H_i}S(V_i\oplus R^{n-\dim V_i+1})].$$

Divide the set $I=\{1, 2, \ldots, s\}$ into the disjoint union of subsets, $I=I_1\cup I_2\cup\ldots\cup I_a$, such that for any b $(1\leqq b\leqq a)$, $i, j\in I_b$ if and only if $H_i=H_j$ and $V_i=V_j$. Let $H=H_1$ be a maximal subgroup in

$\{H_1, H_2, \ldots, H_s\}$. Also let $V=V_1$ and $1\in I_1$. Note then that $H=H_i$ and $V=V_i$ for all $i\in I_1$. Note also that $N^{(H,V)}$ is obtained from $M^{(H,V)}$ by performing a finite series of (nonequivariant) surgeries of type $(\{1\}, \{0\}, p_i, q_i)$, $i\in I_1$, in which each type repeats $\chi(G_{(H,V)}/H)$ times. Since $L^{(H,V)}$ is the trace of this series of surgeries, we see

$$\chi(L^{(H,V)})=\chi(M^{(H,V)})+\chi(G_{(H,V)}/H)\textstyle\sum_{i\in I_1}(-1)^{q_i}.$$

Since $\chi(L^{(H,V)})=\chi(M^{(H,V)})$, then $\sum_{i\in I_1}(-1)^{q_i}=0$. Considering inductively the same as above we see that $\sum_{i\in I_b}(-1)^{q_i}=0$ for any b $(1\leqq b\leqq a)$. This implies $[M]=[N]$ in $SKK_n[G]$. □

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{I}^R_n[G] & \xrightarrow{i} & \bar{N}^R_n[G] & \xrightarrow{j} & \bar{N}_n[G] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\
0 & \longrightarrow & I_n[G] & \xrightarrow{i} & SKK_n[G] & \xrightarrow{j} & N_n[G] & \longrightarrow & 0
\end{array}$$

where φ_1, φ_2 and φ_3 are the obvious homomorphisms. Note that φ_1 and φ_2 are well-defined by the preceding proposition.

Proposition 3.7. *$\varphi_2:\bar{N}^R_n[G]\to SKK_n[G]$ is injective.*

Proof. In the above diagram the two rows are exact and φ_3 is injective. Thus it suffices to prove the injectivity of φ_1. Suppose that $[M]-[N]\in\bar{I}^R_n[G]$ and $[M]-[N]=0$ in $I_n[G]$. Proposition 3.1 shows that $\chi(M^{(H,V)})=\chi(N^{(H,V)})$ for any (H, V). By Corollary 2.7 this shows that $[M]-[N]=0$ in $\bar{I}^R_n[G]$. Thus φ_1 is injective. □

Corollary 2.7, Proposition 3.6 and Proposition 3.7 are now summarized as the following theorem.

Theorem 3.8. *Let M and $N\in\mathscr{M}_n[G]$ be such that $M^{H,1}=\phi$ and $N^{H,1}=\phi$ for any subgroup H of G. Then the following* (i), (ii) *and* (iii) *are equivalent:*

(i) $[M]=[N]$ *in* $N^R_n[G]$,

(ii) $[M]=[N]$ *in* $SKK_n[G]$,

(iii) $[M]=[N]$ *in* $N_n[G]$, *and* $\chi(M^{(H,V)})=\chi(N^{(H,V)})$

for any (H, V).

Notes. (1) The restriction $M^{H,1}=\phi$ is caused by the following fact. Given a bounded compact manifold $N\neq\phi$ of dimension $2n-1>2$, then there exists a $2n$-dimensional compact *connected* manifold L which bounds N and has a prescribed Euler characteristic. If $n=1$, however $\chi(L)$ is at most 1.

(2) If n is even or if G is abelian, from what we noted above Theorem 2.4, the restrictions $M^{H,1}=\phi$ and $N^{H,1}=\phi$ in Theorem 3.8 are not needed.

(3) In the case where G is of even order some different matters happen. For example, gluing by antipodal involution yields new fixed points, and the dimensions of fixed point sets are not congruent modulo 2.

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