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On the Ranks of Homotopy Groups of a Space

By

Kouyemon IRIYE

For a simply connected space X of finite type and a prime p we define two power series

$$P_\pi(X) = \sum(\dim \pi_n(X) \otimes Z_p) \cdot t^n$$

and

$$P_H(X) = \sum(\dim H_n(\Omega X; Z_p)) \cdot t^n.$$

Let $R_\pi(X)$ and $R_H(X)$ (or simply R_π and R_H) be the radiuses of convergence of $P_\pi(X)$ and $P_H(X)$ respectively. Henn [2] proved

Theorem 1. *Let X be a simply connected space of finite type and p be a prime. Then $R_\pi \geqq \min\{R_H, C_p\}$, where C_p is a constant depending only on p and $1 \geqq C_2 \geqq 1/2$, $1 \geqq C_p \geqq 3^{-1/(2p-3)}$ for an odd prime p.*

and conjectured

Conjecture. *If X is a simply connected finite complex then $R_\pi(X) = R_H(X)$.*

In this paper we give a following partial answer to the above.

Theorem 2. $\min\{1, R_\pi\} \leqq R_H$ *for all simply connected spaces of finite type.*

Thus Theorems 1 and 2 imply

Corollary. *Let X be a simply connected space of finite type and p be a prime. If $R_H(X) \leqq C_p$, then $R_\pi(X) = R_H(X)$.*

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Department of Mathematics, Kyoto University, Kyoto 606, Japan.

Remark. (1) Let $R_{\pi_*(X;Z_p)}$ be the radius of convergence of the power series $\sum(\dim(\pi_n(X; Z_p)\otimes Z_p))\cdot t^n$. Using the short exact sequence

$$0\rightarrow\pi_n(X)\otimes Z_p\rightarrow\pi_n(X;Z_p)\rightarrow\pi_{n-1}(X)*Z_p\rightarrow 0$$

for $n\geqq 4$, we see that $R_\pi(X)=R_{\pi_*(X;Z_p)}$. So his statement ([2], Theorem 1) is equivalent to us.

(2) In [2] he proved only that $C_p\geqq 1/2$, but it is not difficult to get the above estimate for an odd prime.

Example. Consider the suspension space of a connected space of finite type. By the Bott-Samelson theorem [1] $H_*(\Omega\Sigma X;Z_p)\cong T(H_*(X;Z_p))$, where T is the tensor algebra functor. If dim $\tilde{H}_*(X;Z_p)>1$, then by Theorem 2 $R_\pi(\Sigma X)\leqq R_H(\Sigma X)<1$. This result implies the existence of an infinite family of integers $\{q_i\}$ and $C>1$ such that dim $\pi_{q_i}(\Sigma X)\otimes Z_p\geqq C^{q_i}$.

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Proof of Theorem 2.

From now on we deal with only the case that $p=2$. Because for an odd prime the following arguments work quite similarly. So $H_*(X)$ stands for $H_*(X;Z_2)$.

For a space X of finite type we define a power series

$$P(X)=\sum(\dim H_n(X))\cdot t^n.$$

For a power series f we denote by $\mathrm{r}(f)$ its radius of convergence. Thus $P_H(X)=P(\Omega X)$ and $R_H(X)=\mathrm{r}(P(\Omega X))$. Theorem 2 is deduced from the following theorem.

Theorem 3. *Let X be a simple space of finite type. Then* $\min\{1, \mathrm{r}(P_\pi(X))\}\leqq\mathrm{r}(P(X))$.

Now we prove Theorem 3. Let X be a simple space of finite type. $\{X_n\}$ denotes the Postonikov decomposition of X. In particular

$$\pi_i(X_n) = \begin{cases} \pi_i(X) & \text{for } i \leqq n \\ 0 & \text{for } i > n, \end{cases}$$

and there is a fibration

(4) $$K(\pi_n(X),\ n) \to X_n \to X_{n-1}.$$

Notice that, since X is simple, the local coefficient system over X_{n-1} associated with the Serre fibration (4) is trivial. Then the Serre spectral sequence associated with (4) induces the following relation among the three power series:

$$P(X_{n-1}) \cdot P(K(\pi_n(X), n)) \geqq P(X_n),$$

where the sign $\geqq$ signifies that each coefficient of the first power series is equal to or larger than the corresponding coefficient to the second. Since $X_0 = *$ and $P(X_n) \equiv P(X) \bmod (t^{n+1})$, the above inequations for $n \geqq 1$ induce the inequation

(5) $$\Pi_{n\geqq 1} P(K(\pi_n(X), n)) \geqq P(X).$$

For a finitely generated abelian group π, $P(K(\pi, n))$ is well-known by Serre.

Theorem 6. ([3]) *For $m \geqq 1$ and $q \geqq 0$,*

$$P(K(Z_{2^m}, q+1)) = \Pi_{h_1 \geqq \cdots \geqq h_q \geqq 0} (1 - t^{2^{h_1} + \cdots + 2^{h_q} + 1})^{-1} \geqq P(K(Z, q+1)).$$

Put

$$m(q) = \dim \pi_q(X) \otimes Z_2$$
$$M(q) = \max\{m(1), \cdots, m(q)\},$$
$$a(n, q) = \#\{h_1 \geqq \cdots \geqq h_q \geqq 0;\ 2^{h_1} + \cdots + 2^{h_q} = n\},$$
$$a(n) = \textstyle\sum_{q\geqq 0} a(n, q)$$

and

$$b(n) = \textstyle\sum_{q\geqq 0} a(n-1, q)\, M(q+1).$$

Then by (5) and Theorem 6 we have

$$\Pi_{n\geqq 1}(1 - t^n)^{-b(n)} = \Pi_{q\geqq 0, n\geqq 1}(1 - t^n)^{-a(n-1, q)M(q+1)}$$
$$\geqq \Pi_{q\geqq 0, n\geqq 1}(1 - t^n)^{-a(n-1, q)m(q+1)}$$

$$=\Pi_{q\geqq 0}\Pi_{h_1\geqq\cdots\geqq h_q\geqq 0}(1-t^{2^{h_1}+\cdots+2^{h_q}+1})^{-m(q+1)}$$
$$\geqq P(X),$$

that is,

$$\Pi_{n\geqq 1}(1-t^n)^{-b(n)}\geqq P(X). \tag{7}$$

Let $0\leqq r<1$. If $\sum_{n\geqq 1}b(n)t^n$ converges for $|t|\leqq r$, then $\Pi_{n\geqq 1}(1-t^n)^{b(n)}$ converges and its limit is nonzero. Thus $\Pi_{n\geqq 1}(1-t^n)^{-b(n)}$ converges. On the other hand we have that $\log \Pi_{n\geqq 1}(1-t^n)^{-b(n)}=\sum_{n\geqq 1}b(n)\{t^n+\cdots\}\geqq\sum_{n\geqq 1}b(n)t^n$. Therefore $\mathrm{r}(P(X))\geqq\mathrm{r}(\Pi_{n\geqq 1}(1-t^n)^{-b(n)})=\min\{1, \mathrm{r}(\sum b(n)t^n)\}$. So it is sufficient to prove that

$$\mathrm{r}(\sum b(n)t^n)\geqq\min\{1, \mathrm{r}P_\pi(X)\}. \tag{8}$$

This is proved by means of the following lemmas. Lemma 9 is directly proved by definition. Lemma 10 is proved by induction on n with Lemma 9, (2) and (3).

Lemma 9. (1) *$a(n, q)=0$ for $n<q$.*
(2) *$a(2n+1, q)=a(2n, q-1)$ and $a(2n+1)=a(2n)$.*
(3) *$a(2n, q)=a(2n-2, q-2)+a(n, q)$ and $a(2n)=a(2n-2)+a(n)$ for $n\geqq 2$.*

Lemma 10. *If we take a real number C $(C>1)$ and an integer N such that $C^N\geqq C^{N-2}+1$, then $a(n)\leqq 2^N\cdot C^n$ for all $n\geqq 0$.*

Now we prove the inequation (8). If we take C and N as in Lemma 10, then

$$\begin{aligned}\mathrm{r}(\sum b(n)t^n)&\geqq\mathrm{r}(\sum a(n-1)M(n)t^n)\\&\geqq\mathrm{r}(\sum 2^N\cdot C^nM(n)t^n)\\&=\mathrm{r}(\sum M(n)t^n)/C\\&=\min\{1, \sum m(n)t^n\}/C\\&=\min\{1, \mathrm{r}P_\pi(X)\}/C.\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{r}(\sum b(n)t^n)&\geqq\sup\ \{\min\ \{1, \mathrm{r}P_\pi(X)\}/C;\ C>1\}\\&=\min\ \{1, \mathrm{r}P_\pi(X)\},\end{aligned}$$

which completes the proof of (8) and of Theorem 3.

Remark. Of course Theorem 3 is valid for odd primes. In this case we can use [4] instead of [3] to prove it.

References

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