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Conditions for Well-posedness in Gevrey Classes of the Cauchy Problems for Fuchsian Hyperbolic Operators II

By

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Detected to Professor Shoji IRIE on his sixtieth birthday

Introduction

In this article we shall present a sufficient condition for well-posedness in Gevrey classes of some Fuchsian hyperbolic Cauchy problems. Namely we show that we can determine a function space in which the Cauchy problem for a given Fuchsian hyperbolic operator is well-posed.

In the case that the initial surface is non-characteristic, there are many results.

The results independent of the lower order terms were obtained by Ohya [12], Leray-Ohya [8], Steinberg [13], Ivrii [5], Trepreau [15], Bronstein [2], Kajitani [7] and Nishitani [11], which show that the multiplicity of the characteristic roots determines the well-posed class.

On the other hand, it is an interesting problem to study how the lower order terms have an effect on the well-posed class. Ivrii showed the following in [6].

(I) Let $P=\partial'_t-t^{2l}\partial_x^2+at^s\partial_x$, where l and s are non-negative integers and a is a non-zero constant. When $0\leqq s<l-1$, the Cauchy problem for P is $\gamma_{\mathrm{loc}}^{(\kappa)}$-well-posed if and only if $1\leqq\kappa<(2l-s)/(l-s-1)$.

(II) Let $P=\partial_t^2-x^{2\mu}\partial_x^2+ax^{\nu}\partial_x$, where μ and ν are non-negative integers and a is a non-zero constant. When $0\leqq\nu<\mu$, the Cauchy problem for P is $\gamma_{\mathrm{loc}}^{(\kappa)}$-well-posed if and only if $1\leqq\kappa<(2\mu-\nu)/(\mu-\nu)$.

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These examples are extended for more general operators by Igari [3], Uryu [17] and Tahara [14] concerning (I) and Uryu-Itoh [18] and Itoh [4] concerning (II).

Furthermore we propose the following operator.

(III) $P=\partial_t^2-t^{2\ell}x^{2\mu}\partial_x^2+at^s x^\nu\partial_x$, where ℓ, μ, s and ν are non-negative integers and a is a non-zero constant.

In this paper we consider the Cauchy problem for the operators which are the most general extension of (III), noting that Fuchsian partial differetial operators introduced by Baouendi-Goulaouic [1] are the natural extension of non-characteristic operators.

§1. Main Result and Remarks

Let $(x, t)\in \boldsymbol{R}^n\times[0, T]$ and $(D_x, D_t)=(D_{x_1}, \cdots, D_{x_n}, D_t)=(-\sqrt{-1}\partial/\partial x_1, \cdots, -\sqrt{-1}\partial/\partial x_n, -\sqrt{-1}\partial/\partial t)$. Let us denote by (ξ, τ) the dual variable of (x, t).

Now we shall define the Gevrey classes.

Definition 1.1. $(\gamma_{\mathrm{loc}}^{(\kappa)}, \gamma^{(\kappa)}; \kappa\geqq 1)$ $f(x)\in\gamma_{\mathrm{loc}}^{(\kappa)}$ implies that $f(x)\in C^\infty(\boldsymbol{R}^n)$ nad for any compact set $K\subset\boldsymbol{R}^n$, there exist constants c, $R>0$ such that

$$(1.1) \qquad |D_x^\alpha f(x)|\leqq cR^{|\alpha|}|\alpha|!^\kappa, \ x\in K, \qquad \text{for any } \alpha .$$

$f(x)\in\gamma^{(\kappa)}$ implies that $f(x)\in C^\infty(\boldsymbol{R}^n)$ and (1.1) holds for any $x\in\boldsymbol{R}^n$.

Next we shall define Fuchsian partial differential operators according to Baouendi-Goulaouic [1].

Let

$$\begin{aligned} L &= L(x, t, D_x, D_t) \\ &= t^k D_t^m+L_1(x, t, D_x)t^{k-1}D_t^{m-1}+\cdots+L_k(x, t, D_x)D_t^{m-k} \\ &\quad +L_{k+1}(x, t, D_x)D_t^{m-k-1}+\cdots+L_m(x, t, D_x) . \end{aligned}$$

Then L is said to be of Fuchsian type with weight $m-k$ with respect to t when it has the following properties:

(A-1) $k\in\boldsymbol{Z}$, $0\leqq k\leqq m$,

(A-2) $\operatorname{ord} L_j(x, t, D_x)\leqq j$,

(A-3) $\operatorname{ord} L_j(x, 0, D_x) = 0$ for $1\leqq j\leqq k$.

From (A-3), we can set $L_j(x, 0, D_x)=a_j(x)$ for $1\leqq j\leqq k$.

A characteristic polynomial associated with L is

$$(1.2) \qquad \mathcal{C}(\lambda, x) = \lambda(\lambda-1)\cdots(\lambda-m+1)+\sqrt{-1}a_1(x)\lambda(\lambda-1)\cdots(\lambda-m+2) + \cdots + \sqrt{-1}^k a_k(x)\lambda(\lambda-1)\cdots(\lambda-m+k+1)\,.$$

It's roots, called characteristic exponents, are denoted by $0, 1, \cdots, m-k-1$, $\lambda_1(x), \cdots, \lambda_k(x)$.

(A–4) there exists a constant $c>0$ such that

$$|(\lambda-\lambda_1(x))\cdots(\lambda-\lambda_k(x))| \geqq c/\lambda(\lambda-1)\cdots(\lambda-m+k+1) \quad \text{for} \quad \lambda\in\mathbb{Z},\ \lambda\geqq m-k\,.$$

In this paper we deal with the following Fuchsian partial differential operator. Let

$$t^{m-k}L=\tilde{L}(x, t, D_x, D_t)=\tilde{L}_0(x, t, D_x, D_t)+\tilde{L}_1(x, t, D_x, D_t),$$

where

$$(1.3) \qquad \tilde{L}_0(x, t, D_x, D_t) = t^m D_t^m + \sum_{\substack{|\alpha|+j=m \\ j\leqq m-1}} t^{|\alpha|\ell+j}\sigma(x)^{|\alpha|\mu} a_{\alpha,j}(x, t) D_x^\alpha D_t^j$$

and

$$(1.4) \qquad \tilde{L}_1(x, t, D_x, D_t) = \sum_{|\alpha|+j\leqq m-1} t^{s(\alpha,j)+j}\sigma(x)^{\nu(\alpha,j)} a_{\alpha,j}(x, t) D_x^\alpha D_t^j\,.$$

We assume the following conditions on $\tilde{L}$.

(A–5) λ-roots of $\lambda^m+\sum_{\substack{|\alpha|+j=m \\ j\leqq m-1}} a_{\alpha,j}(x, t)\xi^\alpha\lambda^j=0$ are real and distinct.

(A–6) $a_{\alpha,j}(x, t)\in\mathcal{B}([0, T], \gamma^{(\kappa)})$.

(A–7) $\sigma(x)\in\gamma^{(\kappa)}$ and is a real-valued function.

(A–8) ℓ is a positive rational number and μ, $s(\alpha, j)$ and $\nu(\alpha, j)$ are integers such that $\mu\geqq 1$, $s(\alpha, j)\geqq 0$ and $\nu(\alpha, j)\geqq 0$.

We define ρ as follows:

$$(1.5) \qquad \rho = \max_{|\alpha|+j\leqq m-1} \{(m-j-s(\alpha, j)/\ell)/(m-j-|\alpha|)\,,\ (m-j-\nu(\alpha, j)/\mu)/(m-j-|\alpha|),\ 1\}\,.$$

Then we have

Theorem 1.1. *Under* (A–1)~(A–8), *if* $1\leqq\kappa<\rho/(\rho-1)$, *the Cauchy problem for* L:

$$(1.6) \qquad \begin{cases} Lu(x, t) = f(x, t) & \text{in } \mathbb{R}^n\times(0, T] \\ D_t^i u(x, t)|_{t=0} = u^i(x)\,, \quad 0\leqq i\leqq m-k-1 & \text{on } \mathbb{R}^n \end{cases}$$

is $\gamma_{\text{loc}}^{(\kappa)}$*-well-posed, i.e. for any* $f(x, t)\in\mathcal{B}([0, T], \gamma_{\text{loc}}^{(\kappa)})$ *and any* $u^i(x)\in\gamma_{\text{loc}}^{(\kappa)}$, $0\leqq i\leqq m-k-1$, *there exists a unique solution* $u(x,t)\in\mathcal{B}([0, T], \gamma_{\text{loc}}^{(\kappa)})$ *of* (1.6).

Remark 1.1. From the definition of ρ, we may only consider the case that $s(\alpha, j)\leqq|\alpha|\ell$ and $\nu(\alpha, j)\leqq|\alpha|\mu$.

Remark 1.2. From (A–3), $s(\alpha, j)>0$ if $|\alpha|>0$.

Remark 1.3. In the case that $k=0$, $\sigma(x)$ is a polynomial and $a_{\alpha,j}(x, t)\in \mathscr{B}([0, T], \gamma^{(1)})$, Ivrii showed in [6] that if (1.6) is locally $\gamma^{(\kappa)}_{\mathrm{loc}}$-well-posed, then $1\leqq\kappa\leqq\rho/(\rho-1)$.

§2. Proof of Theorem 1.1

In this section we shall reduce Theorem 1.1 to Theorem 2.1.

Definition 2.1. We say that $f(x)\in H^{\infty}(\mathbf{R}^n)$ belongs to $\Gamma^{(\kappa)}$ if there exist constants c, $R>0$ such that

(2.1) $\|D_x^{\alpha}f(x)\|\leqq cR^{|\alpha|}|\alpha|!^{\kappa}$ for any α,

where $\|\circ\|$ denotes L^2-norm with respect to x.

Theorem 2.1. *Under* (A–1)$\sim$(A–8), *if* $1\leqq\kappa<\rho/(\rho-1)$, *then the assertions* (1°) *and* (2°) *hold.*

(1°) (1.6) *is* $\Gamma^{(\kappa)}$*-well-posed.*

(2°) *If* $\operatorname{supp} u^i(x)\subset K$, $0\leqq i\leqq m-k-1$ *and* $\operatorname{supp} f(x, t)\subset C_{\ell}(K)$ *for any compact set* $K\subset\mathbf{R}^n$, *then* $\operatorname{supp} u(x, t)\subset C_{\ell}(K)$. *Here*

$$C_{\ell}(K) = \{(x, t)\in\mathbf{R}^n\times[0, T];\ \min_{y\in K}|x-y|\leqq\lambda_{\max}|t|^{\ell}/\ell\}\ ,$$

where $\lambda_{\max} = \max\limits_{1\leqq j\leqq m}\ \sup\limits_{(x,t)\in\mathbf{R}^n\times[0,T],|\xi|=1}|\sigma(x)^{\mu}\lambda_j(x, t, \xi)|$ *and* $\lambda_j(x, t, \xi)$ *are* λ-*roots in* (A–5).

Lemma 2.1. *Theorem* 1.1 *follows from Theorem* 2.1.

Proof. (I; the case that $\kappa>1$) First we shall show the existence of a solution of (1.6). Let $\{\phi_p(x)\}$ be a partition of unity. Namely $\phi_p(x)$ are compactly supported $\gamma^{(\kappa)}$-functions satisfying the following three conditions: (i) $0\leqq\phi_p(x)\leqq1$, (ii) $\sum\phi_p(x)$ is locally finite and (iii) $\sum\phi_p(x)\equiv1$ on $\mathbf{R}^n$. For any $u^i(x)\in\gamma^{(\kappa)}_{\mathrm{loc}}$, $0\leqq i\leqq m-k-1$ and any $f(x, t)\in\mathscr{B}([0, T], \gamma^{(\kappa)}_{\mathrm{loc}})$, we set $u_p^i(x)=\phi_p(x)u^i(x)\in\Gamma^{(\kappa)}$ and $f_p(x, t)=\phi_p(x)f(x, t)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$. Then from (1°) in Theorem 2.1, there exists a unique solution $u_p(x, t)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ of the Cauchy problem:

$$\begin{cases} Lu_p(x, t) = f_p(x, t) \\ D_t^i u_p(x, t)|_{t=0} = u_p^i(x)\ , \qquad 0\leqq i\leqq m-k-1\ . \end{cases}$$

We note that $\Gamma^{(\kappa)}\subset\gamma^{(\kappa)}$ by Sobolev's lemma. Therefore $u_p(x, t)\in\mathcal{B}([0, T], \gamma^{(\kappa)})$. Furthermore since the summation $\sum u_p(x, t)$ is locally finite, then $u(x, t)=\sum u_p(x, t)$ belongs to $\mathcal{B}([0, T], \gamma^{(\kappa)}_{\mathrm{loc}})$ and is a solution of (1.6).

Next we shall show the uniqueness of solutions. For any $(x_0, t_0)\in\mathbb{R}^n\times(0, T]$, we set

$$D_0(x_0, t_0) = \{(x, t)\in\mathbb{R}^n\times[0, T];\ |x-x_0|<\lambda_{\max}(t_0^{\ell}-t^{\ell})/\ell\} \quad \text{and}$$
$$K = D_0(x_0, t_0)\cap\{(x, 0);\ x\in\mathbb{R}^n\}.$$

Let $\phi(x)$ be a compactly supported $\gamma^{(\kappa)}$-function such that $\phi(x)=1$ on K. Let us assume that $u(x, t)\in\mathcal{B}([0, T], \gamma^{(\kappa)}_{\mathrm{loc}})$ satisfies the following equation:

$$\begin{cases} Lu(x, t) = 0 & \text{in } \mathbb{R}^n\times(0, T] \\ D_t^i u(x, t)|_{t=0} = 0, \quad 0\leqq i\leqq m-k-1 & \text{on } \mathbb{R}^n. \end{cases}$$

Since $L(\phi u)=\phi Lu+[L, \phi]u=[L, \phi]u\equiv\tilde{f}(x, t)$ and L is a differential operator, we get that $\operatorname{supp}\tilde{f}(x, t)\subset C_\ell(K^c)$. Here $[\cdot, \cdot]$ is the commutator. Therefore from (2°) in Theorem 2.1, we find that $\operatorname{supp}\phi u\subset C_\ell(K^c)$. Then $u\equiv0$ on $D_0(x_0, t_0)$. Hence $u(x_0, t_0)=0$.

(II; the case that $\kappa=1$) In (I), we have already showed that if $1<\kappa<\rho/(\rho-1)$, there exists a unique solution $u(x, t)\in\mathcal{B}([0, T], \gamma^{(\kappa)}_{\mathrm{loc}})$ of (1.6). Therefore it is sufficient to show the analyticity of the solution. If we refer to the method of Mizohata [9] and § 5 in this paper, we can easily see this fact. Q.E.D.

We shall prove Theorem 2.1 by the method of successive approximations. Therefore we decompose $\tilde{L}$ as follows and consider the following scheme.

$$\tilde{L} = Q_0(x, t, D_x, D_t)+Q_1(x, t, D_x, D_t). \tag{2.2}$$

For α, j such that $s(\alpha, j)=|\alpha|\ell$ and $\nu(\alpha, j)=|\alpha|\mu$, we set

$$Q_0(x, t, D_x, D_t) = \tilde{L}_0(x, t, D_x, D_t) + \sum_{|\alpha|+j\leqq m-1} t^{s(\alpha,j)+j}\sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x, t)D_x^\alpha D_t^j \tag{2.3}$$

and for α, j such that $s(\alpha, j)<|\alpha|\ell$ or $\nu(\alpha, j)<|\alpha|\mu$, we set

$$Q_1(x, t, D_x, D_t) = \sum_{\substack{|\alpha|+j\leqq m-1\\ |\alpha|\neq0}} t^{s(\alpha,j)+j}\sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x, t)D_x^\alpha D_t^j. \tag{2.4}$$

$$\begin{cases} Q_0u_0(x, t) = t^{m-k}f(x, t) & \text{in } \mathbb{R}^n\times(0, T] \\ D_t^i u_0(x, t)|_{t=0} = u^i(x), \quad 0\leqq i\leqq m-k-1 & \text{on } \mathbb{R}^n \end{cases} \tag{2.5$_0$}$$

and for $j\geqq1$

$$(2.5)_j \quad \begin{cases} Q_0 u_j(x, t) = -Q_1 u_{j-1}(x, t) & \text{in } \mathbf{R}^n \times (0, T] \\ D_t^i u_j(x, t)|_{t=0} = 0, \quad 0 \leqq i \leqq m-k-1 & \text{on } \mathbf{R}^n. \end{cases}$$

The following proposition will be proved in §3.

Proposition 2.1. *Under* (A–1)~(A–8), (1°) *and* (2°) *hold.*

(1°) *The Cauchy problem for* Q_0:

$$(2.6) \quad \begin{cases} Q_0 v(x, t) = t^{m-k} f(x, t) & \text{in } \mathbf{R}^n \times (0, T] \\ D_t^i v(x, t)|_{t=0} = v^i(x), \quad 0 \leqq i \leqq m-k-1 & \text{on } \mathbf{R}^n \end{cases}$$

is H^∞*-well-posed.*

(2°) *If* $\operatorname{supp} v^i(x) \subset K$, $0 \leqq i \leqq m-k-1$ *and* $\operatorname{supp} f(x, t) \subset C_l(K)$ *for any compact set* $K \subset \mathbf{R}^n$, *then* $\operatorname{supp} u(x, t) \subset C_l(K)$.

Corollary 2.1. *When* $\rho=1$, (1.6) *is* C^∞*-well-posed.*

If we note that Q_1 is a differential operator and $\Gamma^{(\kappa)} \subset H^\infty$ and use Proposition 2.1, then we find that $u_j(x, t) \in \mathcal{B}([0, T], H^\infty)$ for any $j \geqq 0$. Therefore our aim is to show the formal solution

$$(2.7) \quad u(x, t) = \sum_{j=0}^{\infty} u_j(x, t)$$

converges in $\mathcal{B}([0, T], \Gamma^{(\kappa)})$.

Our plan is as follows. In §4, we shall get an energy inequality for Q_0. In §5, we shall estimate derivatives of a solution of the Cauchy problem:

$$(2.8) \quad \begin{cases} Q_0 v(x, t) = g(x, t) \\ D_t^i v(x, t)|_{t=0} = 0, \quad 0 \leqq i \leqq m-k-1, \end{cases}$$

where $g(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ such that for any sufficiently large fixed integer s, $D_t^i g(x, t)|_{t=0}=0$, $0 \leqq i \leqq s-1$. And in §6, we shall obtain an estimate of $Q_1 v(x, t)$. Using the consequence in §5 and §6, we shall prove Theorem 2.1 in §7.

§3. Proof of Proposition 2.1

Let us note that

$$\tilde{L}_0(x, t, \xi, \tau) = \prod_{j=1}^{m} (t\tau - t^l \sigma(x)^\mu \lambda_j(x, t, \xi)),$$

where $\lambda_j(x, t, \xi)$ are λ-roots in (A–5). And modifying $\lambda_j(x, t, \xi)$ near $\xi=0$, we may assume that if $i \neq j$, there exists a constant $\delta > 0$ such that $|(\lambda_i - \lambda_j)(x, t, \xi)| \geq \delta \langle \xi \rangle$, where $\lambda_j(x, t, \xi) \in \mathcal{B}([0, T], S^1)$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Here for real k, S^k is the symbol class of classical pseudo-differential operators.

We shall define the modules W_k, $0 \leq k \leq m-1$, over the ring of pseudo-differential operators in x of order zero.

Let $\partial_j = tD_t - t^l \sigma(x)^\mu \lambda_j(x, t, D_x)$ and $\Pi_m = \partial_1 \cdots \partial_m$. Let W_{m-1} be the module generated by the monomial operators $\Pi_m / \partial_i = \partial_1 \cdots \partial_{i-1} \partial_{i+1} \cdots \partial_m$ of order $m-1$ and let W_{m-2} be the module generated by the operators $\Pi_m / \partial_i \partial_j$, $i \neq j$, of order $m-2$ and so on.

Lemma 3.1. *For any i,j, there exist pseudo-differential operators A_{ij}, B_{ij} and $C_{ij} \in \mathcal{B}([0, T], S^0)$ such that*

$$(3.1) \qquad [\partial_i, \partial_j] = A_{ij}\partial_i + B_{ij}\partial_j + C_{ij} ,$$

where $[\cdot, \cdot]$ is the commutator.

Proof. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then by the product formula of pseudo-differential operators, we get

$$\begin{aligned} \sigma_0([\partial_i, \partial_j]) &= \partial_\tau (t\tau - t^l \sigma(x)^\mu \lambda_i) D_t (t\tau - t^l \sigma(x)^\mu \lambda_j) \\ &\quad - \partial_\tau (t\tau - t^l \sigma(x)^\mu \lambda_j) D_t (t\tau - t^l \sigma(x)^\mu \lambda_i) \\ &\quad + \sum_{k=1}^{n} \{ \partial_{\xi_k} (t\tau - t^l \sigma(x)^\mu \lambda_i) D_{x_k} (t\tau - t^l \sigma(x)^\mu \lambda_j) \\ &\quad - \partial_{\xi_k} (t\tau - t\, \sigma^l(x)^\mu \lambda_j) D_{x_k} (t\tau - t^l \sigma(x)^\mu \lambda_i) \} \\ &= t^l \sigma(x)^\mu D_{ij}(x, t, \xi) , \quad \text{where} \quad D_{ij} \in \mathcal{B}([0, T], S^1) . \end{aligned}$$

If we set $A_{ij} = D_{ij}/(\lambda_j - \lambda_i)$ and $B_{ij} = D_{ij}/(\lambda_i - \lambda_j)$, then A_{ij}, $\mathrm{B}_{ij} \in \mathcal{B}([0, T], S^0)$ and $A_{ij}(x, t, \xi)\,(t\tau - t^l \sigma(x)^\mu \lambda_i) + B_{ij}(x, t, \xi)\,(t\tau - t^l \sigma(x)^\mu \lambda_j) = t^l \sigma(x)^\mu D_{ij}(x,t,\xi)$. Q.E.D.

Lemma 3.2. *For any monomial $\omega_k^\alpha \in W_k$, $0 \leq k \leq m-1$, there exist ∂_i and $\omega_{k+1}^\beta \in W_{k+1}$ such that*

$$(3.2) \qquad \partial_i \omega_k^\alpha = \omega_{k+1}^\beta + \sum_{j=1}^{k+1} \sum_{\gamma} C_{\gamma j} \omega_{k+1-j}^\gamma ,$$

where $C_{\gamma j} \in \mathcal{B}([0, T], S^0)$.

Proof. For any $\omega_k^\alpha = \partial_{j_1} \cdots \partial_{j_k}$, $1 \leq j_1 < \cdots < j_k \leq m$, there exists some $i \notin \{j_1, \cdots, j_k\}$ with $1 \leq i \leq m$. Hence if we use Lemma 3.1, we easily obtain (3.2). Q.E.D.

Lemma 3.3. *Let*

$$\Psi(t)=\sum_{k=0}^{m-1}\sum_{\alpha}\|\omega_k^\alpha u\|\,,$$

then there exists a constant $c_1>0$ *such that*

$$t\frac{d}{dt}\Psi(t)\leqq c_1\{\|\Pi_m u\|+\Psi(t)\}\,, \tag{3.3}$$

for $u(x,t)\in\mathcal{B}([0,T],H^\infty)$.

Proof. From Lemma 3.2 and Lemma A.2 in Appendix, we get that for any k with $0\leqq k\leqq m-1$,

$$\begin{aligned}t\frac{d}{dt}\|\omega_k^\alpha u\|^2&=2\mathrm{Re}(\sqrt{-1}t^l\sigma(x)^\mu\lambda_i\omega_k^\alpha u+\omega_{k+1}^\beta u+\sum_{j=1}^{k+1}\sum_{\gamma}C_{\gamma j}\omega_{k+1-j}^\gamma u,\ \omega_k^\alpha u)\\&\leqq c_2(\|\omega_k^\alpha u\|+\|\omega_{k+1}^\beta u\|+\sum_{j=1}^{k+1}\sum_{\gamma}\|\omega_{k+1-j}^\gamma u\|)\|\omega_k^\alpha u\|\,.\end{aligned}$$

Therefore we obtain (3.3). Q.E.D.

Lemma 3.4. *Let* $\Pi_s=\partial_{i_1}\cdots\partial_{i_s}$, $1\leqq i_1<\cdots<i_s\leqq m$. *Then* $\sigma(\Pi_s)$, *the symbol of* Π_s, *is expressed in the form*:

$$\sigma(\Pi_s)=\prod_{j=1}^{s}(t\tau-t^l\sigma(x)^\mu\lambda_{i_j})+R_{s-1}+\cdots+R_0\,, \tag{3.4}$$

where $R_{s-j}=\sum_{p+q=s-j}t^{pl+q}\sigma(x)^{p\mu}b_{pj}(x,t,\xi)\tau^q$ *for some* $b_{pj}\in\mathcal{B}([0,T],S^p)$.

Proof. We carry out the proof by induction on s. When $s=1$, (3.4) is trivial. Suppose (3.4) holds for s. Since $\Pi_{s+1}=\Pi_s\partial_{i_{s+1}}$,

$$\sigma(\Pi_{s+1})=\sigma(\Pi_s)\,(t\tau-t^l\sigma(x)^\mu\lambda_{i_{s+1}})+\sum_{|\alpha|\neq 0}\partial_{\xi,\tau}^\alpha\sigma(\Pi_s)D_{x,t}^\alpha(t\tau-t^l\sigma(x)^\mu\lambda_{i_{s+1}})\,.$$

Substituting the right hand side of (3.4) for $\sigma(\Pi_s)$, we have (3.4) with $s+1$. Q.E.D.

Lemma 3.5. *There exist* $A_j(x,t,\xi)\in\mathcal{B}([0,T],S^0)$ *such that for* $i'+j'=m-k$, $1\leqq k\leqq m$,

$$\begin{aligned}&t^{i'l+j'}\sigma(x)^{i'\mu}b_{i'j'}(x,t,\xi)\tau^{j'}\\&\quad=\sum_{j=k}^{m}A_j(x,t,\xi)\prod_{i\neq j,i\geqq k}(t\tau-t^l\sigma(x)^\mu\lambda_i(x,t,\xi))\,,\end{aligned} \tag{3.5}$$

where $b_{ij}\in\mathcal{B}([0,T],S^i)$.

Proof. Substituting $t^l\sigma(x)^\mu\lambda_j(x,t,\xi)$ for $t\tau$, then we obtain

$$t^{(m-k)\ell}\sigma(x)^{(m-k)\mu}K_j(x, t, \xi) = A_j(x, t, \xi)t^{(m-k)\ell}\sigma(x)^{(m-k)\mu}\prod_{i\neq j, i\geqq k}(\lambda_j-\lambda_i),$$

where $K_j(x, t, \xi)\in\mathscr{B}([0, T], S^{m-k})$. Therefore if we set $A_j(x, t, \xi)=K_j(x, t, \xi)\times\{\sum_{i\neq j, i\geqq k}(\lambda_j-\lambda_i)\}^{-1}$, (3.5) is realized. Q.E.D.

Corollary 3.1. *There exist pseudo-differential operators* $C_k(x, t, D_x)\in\mathscr{B}([0, T], S^0)$ *such that*

$$Q_0-\Pi_m = \sum_{k=0}^{m-1}\sum_{\alpha}C_k(x, t, D_x)\omega_k^{\alpha}. \tag{3.6}$$

Proof. From (3.4) with $s=m$,

$$\sigma(Q_0-\Pi_m) = \sum_{j=1}^{m}\sum_{p+q=m-j}t^{p\ell+q}\sigma(x)^{p\mu}b_{pj}(x, t, \xi)\tau^q,$$

where $b_{pj}(x, t, \xi)\in\mathscr{B}([0, T], S^p)$. Using Lemma 3.5, the principal symbol of $Q_0-\Pi_m$ is

$$\sum_{j=1}^{m}A_j(x, t, \xi)\prod_{i\neq j}(t\tau-t^{\ell}\sigma(x)^{\mu}\lambda_i(x, t, \xi)),$$

where $A_j(x, t, \xi)\in\mathscr{B}([0, T], S^0)$. Applying (3.4) for $s=m-1$,

$$\sigma(Q_0-\Pi_m-\sum_{j=1}^{m}A_j\prod_{i\neq j}\partial_i) = \sum_{j=1}^{m-1}\sum_{p+q=m-1-j}t^{p\ell+q}\sigma(x)^{p\mu}\tilde{b}_{pj}(x, t, \xi)\tau^q,$$

where $\tilde{b}_{pj}(x, t, \xi)\in\mathscr{B}([0, T], S^p)$. Repeating these steps, (3.6) is verified.

Q.E.D.

Lemma 3.6. *There exists a constant* $c_3>0$ *such that*

$$t\frac{d}{dt}\Psi(t)\leqq c_3\{||Q_0u||+\Psi(t)\}. \tag{3.7}$$

Proof. Using Lemma 3.3 and Corollary 3.1, we obtain that

$$\begin{aligned} t\frac{d}{dt}\Psi(t)&\leqq c_1\{||\Pi_m u||+\Psi(t)\} \\ &\leqq c_1\{||Q_0u||+||(Q_0-\Pi_m)u||+\Psi(t)\}\leqq c_3\{||Q_0u||+\Psi(t)\}. \end{aligned}$$

Q.E.D.

For a sufficiently large integer N, we put

$$u_N(x, t) = u(x, t)-\sum_{j=0}^{m+N}\frac{t^j}{j!}\partial_t^j u(x, 0).$$

Then $u_N(x, t)$ satisfies the equation:

$$Q_0 u_N(x, t) = f(x, t) - Q_0(\sum_{j=0}^{m+N} t^j \partial_t^j u(x, 0)) \equiv f_N(x, t) .$$

Here we note that from (A–4), for any $i \geqq 0$, $D_t^i u(x, 0)$ is represented by $f(x, t)$ and $u^i(x)$, $0 \leqq i \leqq m-k-1$ (cf. Baouendi-Goulaouic [1]).

Lemma 3.7. *For sufficiently large N, the following energy estimate holds.*

$$\|u(\cdot, t)\|_s \leqq \text{const.} \{\sum_{j=0}^{m+N} \frac{t^j}{j!} \|\partial_t^j u(\cdot, 0)\|_s + t^N \int_0^t \|D_\tau^{N+1} f_N(\cdot, \tau)\|_s d\tau\} , \tag{3.8}$$

where $\|\cdot\|_s$ denotes H^s-norm with respect to x.

Proof. If we redefine $\Psi(t)$ replacing $u(x, t)$ by $u_N(x, t)$, then from Lemma 3.6,

$$\frac{d}{dt}(t^{-c_3}\Psi(t)) \leqq c_3 t^{-c_3-1} \|f_N(\cdot, t)\| .$$

We can choose N such that $t^{-c_3}\Psi(t)|_{t=0} = 0$. Then

$$\Psi(t) \leqq c_3 t^{c_3} \int_0^t \tau^{-c_3-1} \|f_N(\cdot, \tau)\| d\tau .$$

On the other hand, since $D_t^i f_N(x, 0) = 0$ for $0 \leqq i \leqq N$,

$$f_N(x, t) = \frac{1}{N!} \int_0^t (t-\tau)^N \partial_\tau^{N+1} f_N(x, \tau) d\tau .$$

Thus

$$\|u_N(\cdot, t)\| \leqq \text{const.}\, t^N \int_0^t \|D_\tau^{N+1} f_N(\cdot, \tau)\| d\tau .$$

Similarly we get that for real s,

$$\|u_N(\cdot, t)\|_s \leqq \text{const.}\, t^N \int_0^t \|D_\tau^{N+1} f_N(\cdot, \tau)\|_s d\tau .$$

Therefore we can obtain the desired estimate. Q.E.D.

Proof of Proposition 2.1. For any i with $m-k \leqq i \leqq m-1$, we calculate $D_t^i v(x, 0)$ and let them $v^i(x)$, $m-k \leqq i \leqq m-1$. Next we define the δ-translation $Q_0^\delta(x, t, D_x, D_t)$ of Q_0 by

$$Q_0^\delta(x, t, D_x, D_t) = Q_0(x, t+\delta, D_x, D_t) \quad \text{for} \quad 0 \leqq \delta \leqq 1 . \tag{3.9}$$

Now we consider the following non-characteristic Cauchy problem:

$$\begin{cases} Q_0^\delta v_\delta(x, t) = t^{m-k} f(x, t) & \text{in} \quad \mathbb{R}^n \times (0, T] \\ D_t^i v_\delta(x, t)|_{t=0} = v^i(x) , \quad 0 \leqq i \leqq m-1 & \text{on} \quad \mathbb{R}^n . \end{cases} \tag{3.10}$$

For $\delta>0$, (3.10) is H^∞-well-posed (cf. Uryu [16]). Further from Lemma 3.7, the following energy estimate holds uniformly in δ:

$$||v_\delta(\cdot, t)||_s \leqq \text{const.} \{\sum_{j=0}^{m+N} \frac{t^j}{j!} ||\partial_t^j v_\delta(\cdot, 0)||_s + t^N \int_0^t ||D_\cdot^{N+1} f_N(\cdot, \tau)||_s d\tau\} .$$

Therefore there exists a subsequence $\{v_{\delta_j}\}$ which converges weakly in $\mathscr{B}([0, T], H^s)$ as $\delta_j \to 0$. This limit function v is a unique solution of (2.6). Hence (1°) has proved.

In order to prove (2°), we note the following fact. For $\delta>0$, initial surface $\{t=0\}$ is non-characteristic with respect to Q_0^δ and Q_0^δ is invariant under the Holmgren transformation:

$$\begin{cases} x' = x \\ t' = t + |x|^2 . \end{cases}$$

Thus by the well-known method (for example, see Mizohata [10]), we find that the domain of dependence is finite, i.e. for any $(x_0, t_0) \in \mathbb{R}^n \times (0, T]$, if $f(x, t) \equiv 0$ in D_δ and $v^i(x) \equiv 0$ on $D_\delta \cap \{(x, 0); x \in \mathbb{R}^n\}$, then $v_\delta(x, t) \equiv 0$ in D_δ, where $D_\delta = \{(x, t) \in \mathbb{R}^n \times [0, T]; |x - x_0| < \lambda_{\max} \{(t_0+\delta)^l - (t+\delta)^l\}/l\}$.

Then the following fact holds for limit function $v(x, t)$. If $f(x, t) \equiv 0$ in $\tilde{D}$ and $v^i(x) \equiv 0$ on $\tilde{D} \cap \{(x, 0); x \in \mathbb{R}^n\}$, then $v(x, t) \equiv 0$ in $\tilde{D}$, where $\tilde{D} = \bigcap_{\delta>0} D_\delta$. Since we can easily see that $\tilde{D} = D_0$, (2°) is verified.

This completes the proof. Q.E.D.

§4. Energy Inequality for Q_0

The aim of this section is to show the following lemma.

Lemma 4.1. *Let*

$$\Psi_r(t) = \sum_{k=0}^{m-1} \sum_\alpha ||\Lambda^r \omega_k^\alpha u|| ,$$

where Λ is the pseudo-differential operator with symbol $\langle\xi\rangle$. Then there exist constants c_4, $\hat{R}>0$ such that

(4.1)
$$t\frac{d}{dt}\Psi_r(t) \leqq c_4 \{||\Lambda^r Q_0 u|| + \Psi_r(t) + t^l \sum_{j=1}^{r^*} \hat{R}^{j-1}(j-1)!^\kappa \binom{r^*}{j} \Psi_{r+1-j}(t) + \sum_{j=1}^{r^*-1} \hat{R}^j j!^\kappa \binom{r^*}{j} \Psi_{r-j}(t) + \hat{R}^r r!^\kappa \Psi_0(t)\} .$$

Proof. For $r>0$, operating Λ^r on both sides of (3.2), we get that

$$\partial_i \Lambda^r \omega_k^\alpha u = [\partial_i, \Lambda^r]\omega_k^\alpha u + \Lambda^r \omega_{k+1}^\beta u + \sum_{j=1}^{k+1} \sum_\gamma (C_{\gamma j} \Lambda^r \omega_{k+1-j}^\gamma u + [\Lambda^r, C_{\gamma j}]\omega_{k+1-j}^\gamma u) .$$

Similar to the proof of Lemma 3.3, we have that for any k with $0 \leqq k \leqq m-1$,

$$t\frac{d}{dt} ||\Lambda^r \omega_k^\alpha u|| \leqq c_5 \{||\Lambda^r \omega_k^\alpha u|| + ||[\Lambda^r, \partial_i]\omega_k^\alpha u|| + ||\Lambda^r \omega_{k+1}^\beta u|| + \sum_{j=1}^{k+1} \sum_\gamma (||\Lambda^r \omega_{k+1-j}^\gamma u|| + ||[\Lambda^r, C_{\gamma j}]\omega_{k+1-j}^\gamma u||)\} .$$

It follows from Lemma A.3 in Appendix that

$$||[\Lambda^r, \partial_i]\omega_k^\alpha u|| \leqq t^\iota \sum_{j=1}^{r^*} \hat{c}\hat{R}^{j-1}(j-1)!^\kappa \binom{r^*}{j} ||\Lambda^{r+1-j}\omega_k^\alpha u|| + t^\iota \hat{c}\hat{R}^r r!^\kappa ||\omega_k^\alpha u||$$

and

$$||[\Lambda^r, C_{\gamma i}]\omega_{k+1-i}^\gamma u|| \leqq \sum_{j=1}^{r^*-1} \hat{c}\hat{R}^j j!^\kappa \binom{r^*}{j} ||\Lambda^{r-j}\omega_{k+1-i}^\gamma u|| + \hat{c}\hat{R}^r r!^\kappa ||\omega_{k+1-i}^\gamma u|| .$$

Therefore we obtain that

$$t\frac{d}{dt}\Psi_r(t) \leqq c_6 \{||\Lambda^r \Pi_m u|| + \Psi_r(t) + t^\iota \sum_{j=1}^{r^*} \hat{R}^{j-1}(j-1)!^\kappa \binom{r^*}{j} \Psi_{r+1-j}(t) + \sum_{j=1}^{r^*-1} \hat{R}^j j!^\kappa \binom{r^*}{j} \Psi_{r-j}(t) + \hat{R}^r r!^\kappa \Psi_0(t)\} .$$

If we use Corollary 3.1 and refer to the proof of Lemma 3.6, then we get (4.1). Q.E.D.

Here $r! = \Gamma(r+1)$ and r^* is the lowest integer greater than or equal to r, where $\Gamma(\cdot)$ is the gamma function.

§5. Estimate of $\Lambda^r v(x, t)$

We assume the existence of solutions of the following Cauchy problem:

$$\begin{cases} Q_0 v(x, t) = g(x, t) \\ D_t^i v(x, t)|_{t=0} = 0 , \qquad 0 \leqq i \leqq m-k-1 , \end{cases}$$

where $g(x, t) \in \mathscr{B}([0, T], \Gamma^{(\kappa)})$ such that for any sufficiently large fixed integer s, $D_t^i g(x, t)|_{t=0} = 0$, $0 \leqq i \leqq s-1$.

Therefore we may assume that for any $r \geqq 0$, there exist constants $c, R, M > 0$ such that

(5.1) $$||\Lambda^r g(x, t)|| \leqq c R^r r!^\kappa t^s \exp (M r^* t^\iota) .$$

For simplification we use the notation

$$w_r(s, t, R) = R^r r!^\kappa t^s \exp(Mr^* t^\ell).$$

Lemma 5.1. *For any $r \geqq 0$, there exists a constant $A' > 0$ such that for sufficiently large R, M, s,*

$$\Psi_r(t) \leqq cA' s^{-1} w_r(s, t, R). \tag{5.2}$$

Proof. We carry out the proof by induction on r.

When $r=0$, it follows from Lemma 3.6 and (5.1) that

$$t\frac{d}{dt}\Psi_0(t) \leqq c_3\{cw_0(s, t, R) + \Psi_0(t)\}.$$

From this inequality,

$$\frac{d}{dt}(t^{-c_3}\Psi_0(t)) \leqq cc_3 t^{-c_3-1} w_0(s, t, R).$$

If we note that s is sufficiently large,

$$\Psi_0(t) \leqq t^{c_3}\int_0^t cc_3\tau^{s-c_3-1}d\tau = cc_3 t^{c_3}(s-c_3)^{-1}t^{s-c_3} \leqq cA' s^{-1} w_0(s, t, R),$$

if we choose A' such that $A' \geqq 2c_3$.

We assume (5.2) is valid for any r such that $0 \leqq r \leqq n$. Let us show that (5.2) is valid for $n < r \leqq n+1$. It follows from Lemma 4.1 that

$$\frac{d}{dt}\{t^{-c_4}\exp(-c_4 r^* t^\ell/\ell)\Psi_r(t)\} \leqq c_4 t^{-c_4-1}\exp(-c_4 r^* t^\ell/\ell)\{\|A^r Q_0 v\|$$
$$+ t^\ell \sum_{j=2}^{r^*} \hat{R}^{j-1}(j-1)!^\kappa \binom{r^*}{j}\Psi_{r+1-j}(t) + \sum_{j=1}^{r^*-1}\hat{R}^j j!^\kappa\binom{r^*}{j}\Psi_{r-j}(t) + \hat{R}^r r!^\kappa \Psi_0(t)\}.$$

Hence we get that

$$\Psi_r(t) \leqq c_4 t^{c_4}\exp(c_4 r^* t^\ell/\ell)\int_0^t \tau^{-c_4-1}\exp(-c_4 r^*\tau^\ell/\ell)\{\|A^r Q_0 v\|$$
$$+ \tau^\ell \sum_{j=2}^{r^*}\hat{R}^{j-1}(j-1)!^\kappa\binom{r^*}{j}\Psi_{r+1-j}(\tau) + \sum_{j=1}^{r^*-1}\hat{R}^j j!^\kappa\binom{r^*}{j}\Psi_{r-j}(\tau) + \hat{R}^r r!^\kappa\Psi_0(\tau)\}d\tau$$
$$\leqq c_4 t^{c_4}\exp(c_4 r^* t^\ell/\ell)\int_0^t \tau^{-c_4-1}\exp(-c_4 r^*\tau^\ell/\ell)\{cw_r(s, \tau, R)$$
$$+ \tau^\ell \sum_{j=2}^{r^*}\hat{R}^{j-1}(j-1)!^\kappa\binom{r^*}{j} cA' s^{-1} w_{r+1-j}(s, \tau, R)$$
$$+ \sum_{j=1}^{r^*-1}\hat{R}^j j!^\kappa\binom{r^*}{j} cA' s^{-1} w_{r-j}(s, \tau, R) + \hat{R}^r r!^\kappa cA' s^{-1} w_0(s, \tau, R)\}d\tau$$
$$\leqq c_4 t^{c_4}\exp(c_4 r^* t^\ell/\ell)\int_0^t \tau^{-c_4-1}\exp(-c_4 r^*\tau^\ell/\ell)$$

$$\times\{cw_r(s,\tau,R)+\tau^{\ell}\sum_{j=2}^{r^*}(\hat{R}/R)^{j-1}\binom{r}{j-1}^{-\kappa}\binom{r^*}{j}cA's^{-1}w_r(s,\tau,R)$$

$$+\sum_{j=1}^{r^*-1}(\hat{R}/R)^j\binom{r}{j}^{-\kappa}\binom{r^*}{j}cA's^{-1}w_r(s,\tau,R)+(\hat{R}/R)^r cA's^{-1}w_r(s,\tau,R)\}\,d\tau\,.$$

Let $R\geqq 2\hat{R}$, then

$$\Psi_r(t)\leqq c_7 t^{c_4}\exp(c_4 r^* t^{\ell}/\ell)\int_0^t \tau^{-c_4-1}\exp(-c_4 r^*\tau^{\ell}/\ell)$$

$$\times\{cw_r(s,\tau,R)+\tau^{\ell}r^*cA's^{-1}w_r(s,\tau,R)+cA's^{-1}w_r(s,\tau,R)\}\,d\tau$$

$$\leqq cc_7 t^{c_4}\exp(c_4 r^* t^{\ell}/\ell)R^r r!^{\kappa}\exp\{(M-c_4/\ell)r^* t^{\ell}\}\int_0^t \tau^{s-c_4-1}d\tau$$

$$+r^*cA's^{-1}c_7 t^{c_4}\exp(c_4 r^* t^{\ell}/\ell)R^r r!^{\kappa}t^{s-c_4}\int_0^t \tau^{\ell-1}\exp\{(M-c_4/\ell)r^*\tau^{\ell}\}\,d\tau$$

$$+cA's^{-1}c_7 t^{c_4}\exp(c_4 r^* t^{\ell}/\ell)R^r r!^{\kappa}\exp\{(M-c_5/\ell)r^* t^{\ell}\}\int_0^t \tau^{s-c_4-1}d\tau$$

$$\leqq cA's^{-1}w_r(s,t,R)\,,$$

if we choose A' such that $A'\geqq 3c_4^{-1}c_7$ and note that s and M are sufficiently large. Q.E.D.

Lemma 5.2. *Let*

$$\Phi_r(t)=\sum_{i+j\leqq m-1}t^{i\ell+j}\|\Lambda^r\{\sigma(x)^{i\mu}A^i D_t^j v\}\|,$$

then

$$\Phi_r(t)\leqq c_9\{\sum_{j=0}^{r^*-1}\hat{R}^j j!^{\kappa}\binom{r^*}{j}\Psi_{r-j}(t)+\hat{R}^r r!^{\kappa}\Psi_0(t)\}\,.$$

Proof. From Lemma 3.4 and Lemma 3.5, we get that

$$t^{i\ell+j}\|\Lambda^r\{\sigma(x)^{i\mu}A^i D_t^j v\}\|=\|\Lambda^r\{\sum_{k=0}^{i+j}\sum_{\alpha}A_k(x,t,D_x)\omega_k^{\alpha}v\}\|$$

$$\leqq c_9\sum_{k=0}^{i+j}\sum_{\alpha}(\|\Lambda^r\omega_k^{\alpha}v\|+\|[\Lambda^r,A_k]\omega_k^{\alpha}v\|)\,.$$

Using Lemma A.3 in Appendix, we have

$$\|[\Lambda^r,A_k]\omega_k^{\alpha}v\|\leqq\sum_{j=1}^{r^*-1}\hat{c}\hat{R}^j j!^{\kappa}\binom{r^*}{j}\|\Lambda^{r-j}\omega_k^{\alpha}v\|+\hat{c}\hat{R}^r r!^{\kappa}\|\omega_k^{\alpha}v\|\,.$$

Thus we can obtain the desired inequality. Q.E.D.

Corollary 5.1. *For any $r\geqq 0$, there exists a constant $\tilde{A}>0$ such that for sufficiently large R, M, s,*

(5.3) $$\Phi_r(t)\leqq c\tilde{A}s^{-1}w_r(s,t,R)\,.$$

Proof. Applying Lemma 5.1 to Lemma 5.2, we find that

$$\Phi_r(t)\leqq c_8\{\sum_{j=0}^{r^*-1}\hat{R}^j j!^\kappa\binom{r^*}{j}cA's^{-1}w_{r-j}(s,\ t,\ R)+\hat{R}^r r!^\kappa cA's^{-1}w_0(s,\ t,\ R)\}$$

$$\leqq c_8\{\sum_{j=0}^{r^*-1}(\hat{R}/R)^j\binom{r}{j}^{-\kappa}\binom{r^*}{j}cA's^{-1}w_r(s,\ t,\ R)+(\hat{R}/R)^r cA's^{-1}w_r(s,\ t,\ R)\}$$

$$\leqq c\tilde{A}s^{-1}w_r(s,\ t,\ R)\,,$$

if we make $R\geqq 2\hat{R}$ and choose $\tilde{A}$ such that $\tilde{A}\geqq 3c_8A'$. Q.E.D.

Lemma 5.3. *For any $r\geqq 0$ and $i+j\leqq m-1$, there exists a constant $A>0$ such that for sufficiently large R, M, s,*

$$t^{i\ell+j}||\Lambda^r\{\sigma(x)^{i\mu}\Lambda^i D_t^j v\}||\leqq cAs^{-(m-i-j)}w_r(s,\ t,\ R)\,. \tag{5.4}$$

Proof. It follows from Corollary 5.1 that

$$||\Lambda^r\{\sigma(x)^{i\mu}\Lambda^i D_t^{j+(m-i-j-1)}v\}||\leqq c\tilde{A}s^{-1}w_r(s-i\ell-m+i+1,\ t,\ R)\,.$$

Hence we get that if we put $q=m-i-j-1$,

$$||\Lambda^r\{\sigma(x)^{i\mu}\Lambda^i D_t^j v\}||\leqq\int_0^t\cdots\int_0^{\tau_2}||\Lambda^r\{\sigma(x)^{i\mu}\Lambda^i D_\tau^{j+q}v\}||d\tau_1\cdots d\tau_q$$

$$\leqq c\tilde{A}s^{-1}R^r r!^\kappa\exp(Mr^*t^\ell)\int_0^t\cdots\int_0^{\tau_2}\tau_1^{s-i\ell-j-q}d\tau_1\cdots d\tau_q$$

$$\leqq c(2^q\tilde{A})s^{-(q+1)}w_r(s-i\ell-j,\ t,\ R)\,.$$

If we set $A=2^q\tilde{A}$, we get (5.4). Q.E.D.

Lemma 5.4. *For any $r\geqq 0$ and i,j such that $i+j=0,\cdots,m-1$,*

$$t^{i\ell+j}||\sigma(x)^{i\mu}\Lambda^{r+i}D_t^j v||\leqq c_{10}cAw_{r+i}(s,\ t,\ R) \tag{5.5}$$
$$\times\sum_{k=0}^{i}s^{-(m-i-j+k)}\{(r+i)\cdots(r+k+1)\}^{-\kappa}\{(r+k)\cdots(r+1)\}^{1-\kappa}\,.$$

Proof. We carry out the proof by induction on i. When $i=0$, (5.5) is trivial from (5.4). Using (5.4) and Lemma A.3 in Appendix and noting $\mu\geqq 1$, we obtain that

$$t^{i\ell+j}||\sigma(x)^{i\mu}\Lambda^{r+i}D_t^j v||$$

$$\leqq t^{i\ell+j}||\Lambda^r\{\sigma(x)^{i\mu}\Lambda^i D_t^j v\}||+t^{i\ell+j}||[\Lambda^r,\ \sigma(x)^{i\mu}]\Lambda^i D_t^j v||$$

$$\leqq cAs^{-(m-i-j)}w_r(s,\ t,\ R)+\sum_{k=1}^{i-1}\hat{c}\hat{R}^k k!^\kappa\binom{r^*}{k}t^{(i-k)\ell+j}||\sigma(x)^{(i-k)\mu}\Lambda^{r+i-k}D_t^j v||$$

$$+\sum_{k=i}^{r^*-1}\hat{c}\hat{R}^k k!^\kappa\binom{r^*}{k}t^j||\Lambda^{r+i-k}D_t^j v||+\hat{c}\hat{R}^r r!^\kappa t^j||\Lambda^i D_t^j v||$$

$$\leqq cAs^{-(m-i-j)}w_r(s, t, R)+\sum_{k=1}^{i-1}\hat{c}\hat{R}^k k!^\kappa\binom{r^*}{k}cAw_{r+i-k}(s, t, R)$$

$$\times\sum_{k'=0}^{i-k}s^{-(m-i-j+k+k')}\{(r+i-k)\cdots(r+k'+1)\}^{-\kappa}\{(r+k')\cdots(r+1)\}^{1-\kappa}$$

$$+\sum_{k=i}^{r^*-1}\hat{c}\hat{R}^k k!^\kappa\binom{r^*}{k}cAs^{-(m-j)}w_{r+i-k}(s, t, R)+\hat{c}\hat{R}^r r!^\kappa cAs^{-(m-j)}w_i(s, t, R)$$

$$\leqq cAs^{-(m-i-j)}\{(r+i)\cdots(r+1)\}^{-\kappa}w_{r+i}(s, t, R)+c_{11}cAw_{r+i}(s, t, R)$$

$$\times\sum_{k=1}^{i-1}\sum_{k'=0}^{i-k}s^{-(m-i-j+k+k')}\{(r+i)\cdots(r+k+k'+1)\}^{-\kappa}\{(r+k+k')\cdots(r+1)\}^{1-\kappa}$$

$$+cAs^{-(m-j)}w_{r+i}(s, t, R)\sum_{k=i}^{r^*-1}\hat{c}(\hat{R}/R)^k\binom{r+i}{k}^{1-\kappa}$$

$$+\hat{c}(\hat{R}/R)^r\binom{r+i}{i}^{-\kappa}cAs^{-(m-j)}w_{r+i}(s, t, R)$$

$$\leqq cAs^{-(m-i-j)}\{(r+i)\cdots(r+1)\}^{-\kappa}w_{r+i}(s, t, R)$$

$$+c_{11}cAw_{r+i}(s, t, R)\sum_{k=1}^{i}s^{-(m-i-j+k)}\{(r+i)\cdots(r+k+1)\}^{-\kappa}\{(r+k)\cdots(r+1)\}^{1-\kappa}$$

$$+\hat{c}cAs^{-(m-j)}\{(r+i)\cdots(r+1)\}^{1-\kappa}w_{r+i}(s, t, R)\sum_{k=i}^{r^*-1}(1/2)^k k^{i(\kappa-1)}\binom{r}{k-i}^{1-\kappa}$$

$$+\hat{c}cAs^{-(m-j)}\{(r+i)\cdots(r+1)\}^{-\kappa}w_{r+i}(s, t, R)$$

$$\leqq c_{10}cAw_{r+i}(s, t, R)\sum_{k=0}^{i}s^{-(m-i-j+k)}\{(r+i)\cdots(r+k+1)\}^{-\kappa}\{(r+k)\cdots(r+1)\}^{1-\kappa}$$

Q.E.D.

§6. Estimate of $A^rQ_1v(x, t)$

Lemma 6.1. *If* $\sigma(x)\in\mathscr{B}(R^n)$ *and* $0\leqq\nu<\mu$, *then*

$$\|\sigma(x)^\nu u\|\leqq\|u\|^{1-\nu/\mu}\|\sigma(x)^\mu u\|^{\nu/\mu}. \tag{6.1}$$

Proof. By Holder's inequality,

$$\|\sigma(x)^\nu u\|^2=\int|\sigma(x)^\nu u|^2dx=\int|u|^{2(1-\nu/\mu)}|\sigma(x)^\mu u|^{2\nu/\mu}dx$$
$$\leqq\left(\int|u|^2dx\right)^{1-\nu/\mu}\left(\int|\sigma(x)^\mu u|^2dx\right)^{\nu/\mu}$$
$$=\|u\|^{2(1-\nu/\mu)}\|\sigma(x)^\mu u\|^{2\nu/\mu}.$$

Q.E.D.

Lemma 6.2. *Let*

$$\rho_\theta(\alpha, j)=\begin{cases}\nu(\alpha, j)/|\alpha|\mu & \text{if } \ell\nu(\alpha, j)<\mu s(\alpha, j)\\ s(\alpha, j)/(|\alpha|\ell+\theta) & \text{if } \ell\nu(\alpha, j)\geqq\mu s(\alpha, j)\end{cases} \tag{6.2}$$

with respect to $0<\theta\leqq1$, *then for any* $r\geqq0$, (6.3) $$t^{s(\alpha,j)+j}\|\sigma(x)^{\nu(\alpha,j)}\Lambda^{r+|\alpha|}D_t^j v\| \leqq c_{12}cAw_{r+|\alpha|}(s+\varepsilon_1, t, R)\sum_{k=0}^{|\alpha|} s^{-\{m-j-(|\alpha|-k)\rho_\theta(\alpha,j)\}} \times \{(r+|\alpha|)\cdots(r+k+1)\}^{-\kappa\rho_\theta(\alpha,j)}\{(r+k)\cdots(r+1)\}^{-(\kappa-1)\rho_\theta(\alpha,j)},$$

where $\varepsilon_1=\min\{s(\alpha,j)-\ell\nu(\alpha,j)/\mu,\ s\theta/(|\alpha|\ell+\theta)\}>0$.

Proof. First we consider the case that $\ell\nu(\alpha,j)<\mu s(\alpha,j)$. If we use Lemma 5.4 and Lemma 6.1, we get

$$\begin{aligned}
&t^{s(\alpha,j)+j}\|\sigma(x)^{\nu(\alpha,j)}\Lambda^{r+|\alpha|}D_t^j v\| \\
&\leqq \{t^{j(1-\rho_\theta(\alpha,j))}\|\Lambda^{r+|\alpha|}D_t^j v\|^{1-\rho_\theta(\alpha,j)}\} \\
&\quad\times\{t^{s(\alpha,j)+j\rho_\theta(\alpha,j)}\|\sigma(x)^{\nu(\alpha,j)\rho_\theta(\alpha,j)^{-1}}\Lambda^{r+|\alpha|}D_t^j v\|^{\rho_\theta(\alpha,j)}\} \\
&\leqq c_{13}t^{s(\alpha,j)-\ell\nu(\alpha,j)/\mu}\{t^j\|\Lambda^{r+|\alpha|}D_t^j v\|\}^{1-\rho_\theta(\alpha,j)} \\
&\quad\times\{t^{|\alpha|\ell+j}\|\sigma(x)^{|\alpha|\mu}\Lambda^{r+|\alpha|}D_t^j v\|\}^{\rho_\theta(\alpha,j)} \\
&\leqq c_{14}cAw_{r+|\alpha|}(s+s(\alpha,j)-\ell\nu(\alpha,j)/\mu, t, R)s^{-(m-j)(1-\rho_\theta(\alpha,j))} \\
&\quad\times\{\sum_{k=0}^{|\alpha|} s^{-(m-|\alpha|-j+k)}\{(r+|\alpha|)\cdots(r+k+1)\}^{-\kappa}\{(r+k)\cdots(r+1)\}^{1-\kappa}\}^{\rho_\theta(\alpha,j)} \\
&\leqq c_{14}cAw_{r+|\alpha|}(s+s(\alpha,j)-\ell\nu(\alpha,j)/\mu, t, R)\sum_{k=0}^{|\alpha|} s^{-\{m-j-(|\alpha|-\kappa)\rho_\theta(\alpha,j)\}} \\
&\quad\times\{(r+|\alpha|)\cdots(r+k+1)\}^{-\kappa\rho_\theta(\alpha,j)}\{(r+k)\cdots(r+1)\}^{-(\kappa-1)\rho_\theta(\alpha,j)}.
\end{aligned}$$

Next in the case that $\ell\nu(\alpha,j)\geqq\mu s(\alpha,j)$, we have

$$\begin{aligned}
&t^{s(\alpha,j)+j}\|\sigma(x)^{\nu(\alpha,j)}\Lambda^{r+|\alpha|}D_t^j v\| \\
&\leqq \{t^{j(1-\rho_\theta(\alpha,j))}\|\Lambda^{r+|\alpha|}D_t^j v\|^{1-\rho_\theta(\alpha,j)}\} \\
&\quad\times\{t^{s(\alpha,j)+j\rho_\theta(\alpha,j)}\|\sigma(x)^{\nu(\alpha,j)\rho_\theta(\alpha,j)^{-1}}\Lambda^{r+|\alpha|}D_t^j v\|^{\rho_\theta(\alpha,j)}\} \\
&\leqq c_{15}t^{s\theta/(|\alpha|\ell+\theta)}\{t^j\|\Lambda^{r+|\alpha|}D_t^j v\|\}^{1-\rho_\theta(\alpha,j)} \\
&\quad\times\{t^{|\alpha|\ell+j}\|\sigma(x)^{|\alpha|\mu}\Lambda^{r+|\alpha|}D_t^j v\|\}^{\rho_\theta(\alpha,j)} \\
&\leqq c_{16}cAw_{r+|\alpha|}(s+s\theta/(|\alpha|\ell+\theta), t), R)\sum_{k=0}^{|\alpha|} s^{-\{m-j-(|\alpha|-k)\rho_\theta(\alpha,j)\}} \\
&\quad\times\{(r+|\alpha|)\cdots(r+k+1)\}^{-\rho\kappa_\theta(\alpha,j)}\{(r+k)\cdots(r+1)\}^{-(\kappa-1)\rho_\theta(\alpha,j)}.
\end{aligned}$$

Setting $\varepsilon_1=\min\{s(\alpha,j)-\ell\nu(\alpha,j)/\mu,\ s\theta/(|\alpha|\ell+\theta)\}$, then we obtain (6.3).

Q.E.D.

We note that for any α, j such that $|\alpha|\neq 0$,

(6.4) $\nu(\alpha,j)=0$ or there exists a non-negative integer $p(\alpha,j)$ such that $p(\alpha,j)\times\mu<\nu(\alpha,j)\leqq(p(\alpha,j)+1)\mu$. And there exists a non-negative integer $q(\alpha,j)$ such that $q(\alpha,j)\ell<s(\alpha,j)\leqq(q(\alpha,j)+1)\ell$.

Lemma 6.3. *For any* $r \geqq 0$ *and* $|\alpha|>0$,

$$(6.5)\quad t^{s(\alpha,j)+j}\|[\Lambda^r, \sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x,t)D_x^\alpha]D_t^j v\| \leqq c_{17}cAw_{r+|\alpha|}(s+\varepsilon_2, t, R)\sum_{k=0}^{h(\alpha,j)+1} s^{-(m-j-h(\alpha,j)-1+k)} \times\{(r+|\alpha|)\cdots(r+|\alpha|-h(\alpha,j)+k)\}^{-\kappa} \times\{(r+|\alpha|-h(\alpha,j)+k-1)\cdots(r+|\alpha|-h(\alpha,j))\}^{-(\kappa-1)},$$

where $h(\alpha,j)=\begin{cases} p(\alpha,j) & \text{if } l\nu(\alpha,j)<\mu s(\alpha,j) \\ q(\alpha,j) & \text{if } l\nu(\alpha,j)\geqq \mu s(\alpha,j)\end{cases}$ *and* $\varepsilon_2=s(\alpha,j)-lh(\alpha,j)>0.$

Proof. First we consider the case that $l\nu(\alpha,j)<\mu s(\alpha,j)$. Since

$$\sigma([\Lambda^r, \sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x,t)D_x^\alpha]) = \sum_{k=1}^{r^*+|\alpha|-1}\sum_{|\beta|=k}\frac{1}{\beta!}\partial_\xi^\beta\langle\xi\rangle^r D_x^\beta\{\sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x,t)\}\xi^\alpha+r(x,t,\xi),$$

if we note that

$$\nu(\alpha,j)-k=(p(\alpha,j)+1-k)\mu+(\nu(\alpha,j)-p(\alpha,j)\mu-1)+(k-1)(\mu-1),$$

then we obtain that

$$\begin{aligned} I(\alpha,j)&\equiv t^{s(\alpha,j)+j}\|[\Lambda^r, \sigma(x)^{\nu(\alpha,j)}a_{\alpha,j}(x,t)D_x^\alpha]D_t^j v\| \\ &\leqq \sum_{k=1}^{p(\alpha,j)+1} t^{s(\alpha,j)-(p(\alpha,j)+1-k)l}\binom{r^*}{k}t^{(p(\alpha,j)+1-k)l+j} \\ &\quad\times\|\sigma(x)^{(p(\alpha,j)+1-k)\mu}\Lambda^{r+|\alpha|-k}D_t^j v\| \\ &\quad+\hat{c}t^{s(\alpha,j)}\sum_{k=p(\alpha,j)+2}^{|\alpha|}\binom{r^*}{k}t^j\|\Lambda^{r+|\alpha|-k}D_t^j v\| \\ &\quad+\hat{c}t^{s(\alpha,j)}\sum_{k=|\alpha|+1}^{r^*}\hat{R}^{k-|\alpha|}(k-|\alpha|)!^\kappa\binom{r^*}{k}t^j\|\Lambda^{r+|\alpha|-k}D_t^j v\| \\ &\quad+\hat{c}t^{s(\alpha,j)}\sum_{k=r^*+1}^{r^*+|\alpha|-1}\hat{R}^{k-|\alpha|}(k-|\alpha|)!^\kappa t^j\|\Lambda^{r+|\alpha|-k}D_t^j v\| \\ &\quad+\hat{c}t^{s(\alpha,j)}\hat{R}^r r!^\kappa t^j\|D_t^j v\|. \end{aligned}$$

Using Lemma 5.4 and noting Remark 1.2,

$$\begin{aligned} I(\alpha,j)&\leqq c_{18}t^{\varepsilon_2}\Big[\sum_{k=1}^{p(\alpha,j)+1}\sum_{k'=0}^{p(\alpha,j)+1-k}\binom{r^*}{k}cAs^{-(m-j-p(\alpha,j)-1+k+k')} \\ &\quad\times\{(r+|\alpha|-k)\cdots(r+|\alpha|-p(\alpha,j)+k')\}^{-\kappa} \\ &\quad\times\{(r+|\alpha|-p(\alpha,j)+k'-1)\cdots(r+|\alpha|-p(\alpha,j))\}^{1-\kappa}w_{r+|\alpha|-k}(s,t,R) \\ &\quad+\sum_{k=p(\alpha,j)+2}^{|\alpha|}\binom{r^*}{k}cAs^{-(m-j)}w_{r+|\alpha|-k}(s,t,R) \end{aligned}$$

$$+\sum_{k=|\alpha|+1}^{r^*} \hat{R}^{k-|\alpha|}(k-|\alpha|)!^{\kappa}\binom{r^*}{k} cAs^{-(m-j)}w_{r+|\alpha|-k}(s,t,R)$$

$$+\sum_{k=r^*+1}^{r^*+|\alpha|-1} \hat{R}^{k-|\alpha|}(k-|\alpha|)!^{\kappa} cAs^{-(m-j)}w_{r+|\alpha|-k}(s,t,R)$$

$$+\hat{R}^r r!^{\kappa} cAs^{-(m-j)}w_0(s,t,R)]$$

$$\leqq c_{18}cAw_{r+|\alpha|}(s+\varepsilon_2,t,R)[\sum_{k=1}^{p(\alpha,j)+1}\sum_{k'=0}^{p(\alpha,j)+1-k} s^{-(m-j-p(\alpha,j)-1+k+k')}$$

$$\times\{(r+|\alpha|)\cdots(r+|\alpha|-p(\alpha,j)+k+k')\}^{-\kappa}$$

$$\times\{(r+|\alpha|-p(\alpha,j)+k+k'-1)\cdots(r+|\alpha|-p(\alpha,j))\}^{1-\kappa}$$

$$+s^{-(m-j)}\sum_{k=p(\alpha,j)+2}^{|\alpha|}\{(r+|\alpha|)\cdots(r+|\alpha|-k+1)\}^{1-\kappa}$$

$$+s^{-(m-j)}\sum_{k=|\alpha|+1}^{r^*}(\hat{R}/R)^{k-|\alpha|}\binom{r}{k-|\alpha|}^{-\kappa}\binom{r^*}{k}\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa}$$

$$+s^{-(m-j)}\sum_{k=r^*+1}^{r^*+|\alpha|-1}(\hat{R}/R)^{k-|\alpha|}\binom{r}{k-|\alpha|}^{-\kappa}\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa}$$

$$+s^{-(m-j)}(\hat{R}/R)^r\{(r+|\alpha|)\cdots(r+1)\}^{-\kappa}]$$

$$\leqq c_{17}cAw_{r+|\alpha|}(s+\varepsilon_2,t,R)\sum_{k=0}^{p(\alpha,j)+1} s^{-(m-j-p(\alpha,j)-1+k)}$$

$$\times\{(r+|\alpha|)\cdots(r+|\alpha|-p(\alpha,j)+k)\}^{-\kappa}$$

$$\times\{(r+|\alpha|-p(\alpha,j)+k-1)\cdots(r+|\alpha|-p(\alpha,j))\}^{1-\kappa}.$$

The calculation of the case that $l\nu(\alpha,j)\geqq\mu s(\alpha,j)$ is quite similar to the first case. Q.E.D.

From $\Lambda^r Q_1=[\Lambda^r, Q_1]+Q_1\Lambda^r$, Lemma 6.2 and Lemma 6.3, we obtain

Lemma 6.4.

$$\|\Lambda^r Q_1 v\|\leqq\tilde{c}cA\sum_{\substack{|\alpha|+j\leqq m-1\\ |\alpha|\neq 0}} K_j^\alpha(s,r)w_{r+|\alpha|}(s+\varepsilon,t,R), \tag{6.6}$$

where $\tilde{c}>0$, $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}>0$ *and*

$$K_j^\alpha(s,r)=\sum_{k=0}^{|\alpha|} s^{-\{m-j-(|\alpha|-k)\rho_\theta(\alpha,j)\}}$$

$$\times\{(r+|\alpha|)\cdots(r+k+1)\}^{-\kappa\rho_\theta(\alpha,j)}\{(r+k)\cdots(r+1)\}^{-(\kappa-1)\rho_\theta(\alpha,j)}$$

$$+\sum_{k=0}^{h(\alpha,j)+1} s^{-(m-j-h(\alpha,j)-1+k)}$$

$$\times\{(r+|\alpha|)\cdots(r+|\alpha|-h(\alpha,j)+k)\}^{-\kappa}$$

$$\times\{(r+|\alpha|-h(\alpha,j)+k-1)\cdots(r+|\alpha|-h(\alpha,j))\}^{-(\kappa-1)}.$$

§7. Proof of Theorem 2.1

In order to prove Theorem 2.1, we prepare several lemmas.

Lemma 7.1. *For any $f(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$ and $u^i(x)\in\Gamma^{(\kappa)}$, $0\leqq i\leqq m-k-1$, there exists a unique solution $u(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$ of the equation*:

$$(7.1)\qquad \begin{cases} Q_0u(x,t)=t^{m-k}f(x,t) \\ D_t^iu(x,t)|_{t=0}=u^i(x), \qquad 0\leqq i\leqq m-k-1\,. \end{cases}$$

And especially, if $u^i(x)\equiv 0$, $0\leqq i\leqq m-k-1$ and $D_t^if(x,t)|_{t=0}=0$, $0\leqq i\leqq \hat{s}-1$, then we obtain that $D_t^iu(x,t)|_{t=0}=0$, $0\leqq i\leqq m-k-1+\hat{s}$, where $\hat{s}$ is a positive integer.

Proof. It follows from Proposition 2.1 that there exists a unique solution $u(x,t)\in\mathcal{B}([0,T],H^\infty)$ of (7.1). Therefore let us show that $u(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$.

From (A–4), we note that we can calculate the derivatives of $u(x,t)$ at $t=0$ and each derivatives belongs to $\Gamma^{(\kappa)}$.

For any fixed integer $s\geqq 1$, let

$$u_s(x,t)=u(x,t)-\sum_{j=0}^{s-1}\frac{t^j}{j!}\,\partial_t^ju(x,0)\,,$$

then $u_s(x,t)$ satisfies the equation

$$Q_0u_s(x,t)=f(x,t)-Q_0\Big(\sum_{j=0}^{s-1}\frac{t^j}{j!}\,\partial_t^ju(x,0)\Big)\equiv f_s(x,t)\,.$$

Thus we get that $f_s(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$ such that $D_t^if_s(x,t)|_{t=0}=0$, $0\leqq i\leqq s-1$. From the consequence of §5, it is easily seen that $u_s(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$. Hence $u(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$.

The second assertion is clear from (A–4). Q.E.D.

Lemma 7.2. *Let $u_j(x,t)$ be the solution of $(2.5)_j$, then $u_j(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$ for $j\geqq 0$. Moreover there exists an integer $\tilde{s}\geqq 1$ such that for $j\geqq 1$, $D_t^iu_j(x,t)|_{t=0}=0$, $0\leqq i\leqq m-k-1+\tilde{s}(j-1)$.*

Proof. It follows from the first assertion of Lemma 7.1 that $u_0(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$. If we remember (2.2)∼(2.4), then we find that

$$-Q_1u_0(x,t)=t^{m-k}f_1(x,t)$$

such that $f_1(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$. Using Lemma 7.1 once more, we can get that $u_1(x,t)\in\mathcal{B}([0,T],\Gamma^{(\kappa)})$. Therefore repeating these steps, we have $u_j(x,t)\in$

$\mathcal{B}([0, T], \Gamma^{(\kappa)})$ for $j \geqq 0$.

Let us consider the second assertion. From $(2.5)_1$, $D_t^i u_1(x, t)|_{t=0}=0$, $0 \leqq i \leqq m-k-1$. Put

$$\tilde{s} = \min_{\substack{|\alpha|+j \leqq m-1 \\ |\alpha| \neq 0}} \{s(\alpha, j)\} \geqq 1 .$$

Thus from (2.4) and the second assertion of Lemma 7.1, we obtain that $D_t^i u_2(x, t)|_{t=0}=0$, $0 \leqq i \leqq m-k-1+\tilde{s}$. Similarly we conclude the second assertion of Lemma 7.2. Q.E.D.

From Lemma 7.2, for any fixed integer $s \geqq 1$, there exists $N=N(s) \in \mathbb{N}$ such that for any $j \geqq N-1$, $D_t^i u_j(x, t)|_{t=0}=0$, $0 \leqq i \leqq s-1$.

Therefore we may assume that for any $r \geqq 0$, there exist positive constants c and R such that

$$(7.2) \qquad ||\Lambda^r Q_1 u_{N-1}|| \leqq c w_r(s, t, R) .$$

Lemma 7.3. *Under (7.2), if* $1 \leqq \kappa < \rho/(\rho-1)$, *there exist constants* $\tilde{A}, B, q > 0$ *which are independent of r such that*

$$(7.3) \qquad ||\Lambda^r u_{N+n}|| \leqq c\tilde{A}B^n n^{-qn} w_r(s, t, 2^\kappa R)$$

for $n=0, 1, 2, \cdots$.

Proof. From (7.2) and Lemma 5.3, we get that

$$||\Lambda^r u_N|| \leqq cAs^{-m} w_r(s, t, R) .$$

It follows from Lemma 6.4 that

$$||\Lambda^r Q_1 u_N|| \leqq \tilde{c}cA \sum_{\substack{|\alpha_1|+j_1 \leqq m-1 \\ |\alpha_1| \neq 0}} K_{j_1}^{\alpha_1}(s, r) w_{r+|\alpha_1|}(s+\varepsilon, t, R) .$$

If we use Lemma 5.3, we have that

$$||\Lambda^r u_{N+1}|| \leqq \tilde{c}cA^2 \sum_{\substack{|\alpha_1|+j_1 \leqq m-1 \\ |\alpha_1| \neq 0}} (s+\varepsilon)^{-m} K_{j_1}^{\alpha_1}(s, r) w_{r+|\alpha_1|}(s+\varepsilon, t, R) .$$

Applying Lemma 6.4 again, we obtain that

$$||\Lambda^r Q_1 u_{N+1}|| \leqq \tilde{c}^2 cA^2 \sum_{\substack{|\alpha_1|+j_1 \leqq m-1 \\ |\alpha_1| \neq 0}} \sum_{\substack{|\alpha_2|+j_2 \leqq m-1 \\ |\alpha_2| \neq 0}} K_{j_1}^{\alpha_1}(s, r) K_{j_2}^{\alpha_2}(s+\varepsilon, r+|\alpha_1|)$$
$$\times w_{r+|\alpha_1|+|\alpha_2|}(s+2\varepsilon, t, R) .$$

Using Lemma 5.3 again, we get that

$$||A^r u_{N+2}|| \leqq \tilde{c}^2 c A^3 \sum \sum (s+2\varepsilon)^{-m} K^{\alpha_1}_{j_1}(s, r) K^{\alpha_1, \alpha_2}_{j_1, j_2}(s, r) w_{r+|\alpha_1|+|\alpha_2|}(s+2\varepsilon, t, R),$$

where $K^{\alpha_1, \alpha_2}_{j_1, j_2}(s, r) = K^{\alpha_2}_{j_2}(s+\varepsilon, r+|\alpha_1|)$.

Setting

$$K^{\alpha_1, \cdots, \alpha_i}_{j_1, \cdots, j_i}(s, r) = K^{\alpha_i}_{j_i}(s+(i-1)\varepsilon, r+|\alpha_1|+\cdots+|\alpha_{i-1}|),$$

inductively we obtain that for any $n \geqq 0$,

$$||A^r u_{N+n}|| \leqq cA(\tilde{c}A)^n \sum \cdots \sum K^{\alpha_1}_{j_1}(s, r) \cdots K^{\alpha_1, \cdots, \alpha_n}_{j_1, \cdots, j_n}(s, r) \times w_{r+|\alpha_1|+\cdots+|\alpha_n|}(s+n\varepsilon, t, R).$$

By the way,

$$\begin{aligned} &K^{\alpha_1}_{j_1}(s, r) \cdots K^{\alpha_1, \cdots, \alpha_n}_{j_1, \cdots, j_n}(s, r) \\ &\quad = \sum \cdots \sum s^{-a_1}(r+1)^{-b_1^1} \cdots (r+|\alpha_1|)^{-b_1^{|\alpha_1|}} \\ &\qquad \times (s+\varepsilon)^{-a_2}(r+|\alpha_1|+1)^{-b_2^1} \cdots (r+|\alpha_1|+|\alpha_2|)^{-b_2^{|\alpha_2|}} \cdots \\ &\quad \times (s+(n-1)\varepsilon)^{-a_n}(r+|\alpha_1|+\cdots+|\alpha_{n-1}|+1)^{-b_n^1} \cdots (r+|\alpha_1|+\cdots+|\alpha_n|)^{-b_n^{|\alpha_n|}}, \end{aligned}$$

where

$$a_d \in \{m-j_d-(|\alpha_d|-k_d)\rho_\theta(\alpha_d, j_d),\ m-j_d-h(\alpha_d, j_d)-1+k_d\}$$

and

$$b_d^{d'} \in \{\kappa\rho_\theta(\alpha_d, j_d), (\kappa-1)\rho_\theta(\alpha_d, j_d), \kappa, \kappa-1, 0\}.$$

We note the following.

(7.4) If $a_d = m-j_d-(|\alpha_d|-k_d)\rho_\theta(\alpha_d, j_d)$, then $b_d^1, \cdots, b_d^{k_d} = (\kappa-1)\rho_\theta(\alpha_d, j_d)$ and $b_k^{k_d+1}, \cdots, b_d^{|\alpha_d|} = \kappa\rho_\theta(\alpha_d, j_d)$.

(7.5) If $a_d = m-j_d-h(\alpha_d, j_d)-1+k_d$, then $b_d^1, \cdots, b_d^{|\alpha_d|-h(\alpha_d, j_d)-1} = 0$, $b_d^{|\alpha_d|-h(\alpha_d, j_d)}, \cdots, b_d^{|\alpha_d|-h(\alpha_d, j_d)+k_d-1} = \kappa-1$, and $b_d^{|\alpha_d|-h(\alpha_d, j_d)+k_d}, \cdots, b_d^{|\alpha_d|} = \kappa$.

Let $s \geqq \varepsilon$ and $a = \min\{a_d\}$ and if we use Lemma A.4 in Appendix, then we have that

$$\begin{aligned} s^{-a_1} \cdots (s+(n-1)\varepsilon)^{-a_n} &\leqq \varepsilon^{-a_1} \cdots (n\varepsilon)^{-a_n} \\ &= \varepsilon^{-(a_1+\cdots+a_n)} 1^{-a_1} \cdots n^{-a_n} \leqq \varepsilon^{-an} A_1 R_1^n n^{-(a_1+\cdots+a_n)}. \end{aligned}$$

Let $r=0$ and using Lemma A.4 again,

$$\begin{aligned} &(r+1)^{-b_1^1} \cdots (r+|\alpha_1|)^{-b_1^{|\alpha_1|}} \times \cdots \times (r+|\alpha_1|+\cdots+|\alpha_{n-1}|+1)^{-b_n^1} \cdots \\ &\quad \times (r+|\alpha_1|+\cdots+|\alpha_n|)^{-b_n^{|\alpha_n|}} \\ &\leqq A_1 R_1^n (|\alpha_1|+\cdots+|\alpha_n|)^{-(b_1^1+\cdots+b_n^{|\alpha_n|})} \leqq A_1 R_1^n n^{-(b_1^1+\cdots+b_n^{|\alpha_n|})}. \end{aligned}$$

Further we estimate $w_{r+|\alpha_1|+\cdots+|\alpha_n|}(s+n\varepsilon, t, R)$ as follows:

$$R^{r+|\alpha_1|+\cdots+|\alpha_n|} \leqq R^r R^{(m-1)n},$$

by Lemma A.5 in Appendix,

$$\begin{aligned}(r+|\alpha_1|+\cdots+|\alpha_n|)!^\kappa &\leqq 2^{(r+|\alpha_1|+\cdots+|\alpha_n|)\kappa} r!^\kappa (|\alpha_1|+\cdots+|\alpha_n|)!^\kappa \\ &\leqq 2^{\kappa r} 2^{(m-1)\kappa n} r!^\kappa A_2 R_2^n n^{(|\alpha_1|+\cdots+|\alpha_n|)\kappa},\end{aligned}$$

$$t^{s+\varepsilon_n} \leqq t^s T^{\varepsilon n}$$

and

$$\exp\{M(r^*+|\alpha_1|+\cdots+|\alpha_n|)t^l\} \leqq \exp(Mr^*t^l)\exp\{M(m-1)T^l n\}.$$

Hence we find that

$$\begin{aligned}||A^r u_{N+n}|| &\leqq cAA_1^2A_2\{\tilde{c}AR_1^2R_2\varepsilon^{-a}R^{m-1}T^\varepsilon 2^{(m-1)\kappa}\exp(M(m-1)T)\}^n w_r(s, t, 2^\kappa R) \\ &\quad\times\sum\cdots\sum n^{(|\alpha_1|+\cdots+|\alpha_n|)\kappa-(a_1+\cdots+a_n)-(b_1^1+\cdots+b_n^{|\alpha_n|})}.\end{aligned}$$

Let i be the number of $\{m-j_d-(|\alpha_d|-k_d)\rho_\theta(\alpha_d, j_d)\}$ s in $\{a_d\}_{1\leqq d\leqq n}$. If we recall (7.4) and (7.5), then

$$\begin{aligned}I &\equiv (a_1+\cdots+a_n)+(b_1^1+\cdots+b_n^{|\alpha_n|})-(|\alpha_1|+\cdots+|\alpha_n|)\kappa \\ &= \{m-j_1-(|\alpha_1|-k_1)\rho_\theta(\alpha_1, j_1)\}+\cdots+\{m-j_i-(|\alpha_i|-k_i)\rho_\theta(\alpha_i, j_i)\} \\ &\quad+\{m-j_{i+1}-h(\alpha_{i+1}, j_{i+1})-1+k_{i+1}\}+\cdots+\{m-j_n-h(\alpha_n, j_n)-1+k_n\} \\ &\quad+(\kappa-1)\rho_\theta(\alpha_1, j_1)k_1+\kappa\rho_\theta(\alpha_1, j_1)(|\alpha_1|-k_1) \\ &\quad+\cdots+(\kappa-1)\rho_\theta(\alpha_i, j_i)k_i+\kappa\rho_\theta(\alpha_i, j_i)(|\alpha_i|-k_i) \\ &\quad+(\kappa-1)k_{i+1}+\kappa\{h(\alpha_{i+1}, j_{i+1})-k_{i+1}+1\}+\cdots+(\kappa-1)k_n+\kappa\{h(\alpha_n, j_n)-k_n+1\} \\ &\quad-(|\alpha_1|+\cdots+|\alpha_n|)\kappa \\ &= \{m-j_1-|\alpha_1|\rho_\theta(\alpha_1, j_1)+|\alpha_1|\kappa\rho_\theta(\alpha_1, j_1)-|\alpha_1|\kappa\} \\ &\quad+\cdots+\{m-j_i-|\alpha_i|\rho_\theta(\alpha_i, j_i)+|\alpha_i|\kappa\rho_\theta(\alpha_i, j_i)-|\alpha_i|\kappa\} \\ &\quad+\{m-j_{i+1}-h(\alpha_{i+1}, j_{i+1})-1+\kappa h(\alpha_{i+1}, j_{i+1})+\kappa-|\alpha_{i+1}|\kappa\} \\ &\quad+\cdots+\{m-j_n-h(\alpha_n, j_n)-1+\kappa h(\alpha_n, j_n)+\kappa-|\alpha_n|\kappa\}.\end{aligned}$$

Now recalling (6.2) and (6.4), then

$$\begin{aligned}&\{m-j-h(\alpha, j)-1+\kappa h(\alpha, j)+\kappa-|\alpha|\kappa\}-\{m-j-|\alpha|\rho_\theta(\alpha, j)+|\alpha|\kappa\rho_\theta(\alpha, j)-|\alpha|\kappa\} \\ &\quad= (\kappa-1)\{h(\alpha, j)+1-|\alpha|\rho_\theta(\alpha, j)\} \\ &\quad= (\kappa-1)\times\begin{cases} p(\alpha, j)+1-\nu(\alpha, j)/\mu & \text{if} \quad l\nu(\alpha, j)<\mu s(\alpha, j) \\ q(\alpha, j)+1-|\alpha|s(\alpha, j)/(|\alpha|l+\theta) & \text{if} \quad l\nu(\alpha, j)\geqq\mu s(\alpha, j)\end{cases} \\ &\quad\geqq 0.\end{aligned}$$

Let us set

$$\rho_\theta = \max_{|\alpha|+j\leqq m-1} \{(m-j-|\alpha|\rho_\theta(\alpha, j))/(m-j-|\alpha|)\}.$$

If $1\leqq\kappa<\rho_\theta/(\rho_\theta-1)$, then we find that

$$\begin{aligned} I &\geqq \{m-j_1-|\alpha_1|\rho_\theta(\alpha_1, j_1)+|\alpha_1|\kappa\rho_\theta(\alpha_1, j_1)-|\alpha_1|\kappa\} \\ &\quad +\cdots+\{m-l_n-|\alpha_n|\rho_\theta(\alpha_n, j_n)+|\alpha_n|\kappa\rho_\theta(\alpha_n, j_n)-|\alpha_n|\kappa\} \\ &= (m-j_1-|\alpha_1|)[(m-j_1-|\alpha_1|\rho_\theta(\alpha_1, j_1))/(m-j_1-|\alpha_1|) \\ &\quad -\{(m-j_1-|\alpha_1|\rho_\theta(\alpha_1, j_1))/(m-j_1-|\alpha_1|)-1\}\kappa] \\ &\quad +\cdots+(m-j_n-|\alpha_n|)[(m-j_n-|\alpha_n|\rho_\theta(\alpha_n, j_n))/(m-j_n-|\alpha_n|) \\ &\quad -\{(m-j_n-|\alpha_n|\rho_\theta(\alpha_n, j_n))/(m-j_n-|\alpha_n|)-1\}\kappa] \\ &\geqq n\{\rho_\theta-(\rho_\theta-1)\kappa\}>qn, \quad \text{where} \quad q>0. \end{aligned}$$

If we note that for fixed κ such that $1\leqq\kappa<\rho/(\rho-1)$, we can choose $0<\theta\leqq 1$ such that $1\leqq\kappa<\rho_\theta/(\rho_\theta-1)\leqq\rho/(\rho-1)$, then this completes the proof. Q.E.D.

Corollary 7.1. *If* $1\leqq\kappa<\rho/(\rho-1)$, *the formal solution*

$$u(x, t) = \sum_{j=0}^{\infty} u_j(x, t)$$

converges in $\mathscr{B}([0, T], \Gamma^{(\kappa)})$.

Proof. If we devide $u(x, t)$ as

$$u(x, t) = \sum_{j=0}^{N-1} u_j(x, t)+\sum_{j=N}^{\infty} u_j(x, t),$$

then this corollary immidiately follows from Lemma 7.2 and Lemma 7.3. Q.E.D.

Therefore we get the existence of solutions.

Next we shall show the uniqueness of solutions.

Lemma 7.4. *If* $u(x, t)\in\mathscr{B}([0, T], \Gamma^{(\kappa)})$ *is a solution of the Cauchy problem*:

$$\begin{cases} Lu(x, t) = 0 \\ D_t^i u(x, t)|_{t=0} = 0, \quad 0\leqq i\leqq m-k-1, \end{cases}$$

where $1\leqq\kappa<\rho/(\rho-1)$, *then* $u(x, t)\equiv 0$.

Proof. We may assume that for sufficiently large s, there exist constants $c, R>0$ such that

$$||\Lambda^r u||\leqq cw_r(s, t, R) \qquad \text{for any} \quad r\geqq 0.$$

therefore similar to the proof of Lemma 7.3, we can obtain that

$$||A^r u|| \leqq c\tilde{A}B^n n^{-qn} w_r(s, t, \tilde{R}) \quad \text{for some constant } \tilde{R}.$$

Let $n \to \infty$, then we find that $u(x, t) \equiv 0$. Q.E.D.

Finally we shall prove assertion (2°).

Lemma 7.5. *If* $\operatorname{supp} u^i(x) \subset K$, $0 \leqq i \leqq m-k-1$ *and* $\operatorname{supp} f(x, t) \subset C_l(K)$ *for compact set* $K \subset \mathbb{R}^n$, *then* $\operatorname{supp} u(x, t) \subset C_l(K)$, *where* $u(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ *is a solution of* (1.6).

Proof. From (2°) in Proposition 2.1 and $(2.5)_0$, $\operatorname{supp} u_0(x, t) \subset C_l(K)$. *any* Next if we note how to make Q_1 and that Q_1 is a differential operator, then

$$-Q_1 u_0(x, t) = t^{m-k} f_1(x, t),$$

where $f_1(x, t) \in \mathcal{B}([0, T], \Gamma^{(\kappa)})$ and $\operatorname{supp} f_1(x, t) \subset C_l(K)$. Hence using (2°) in Proposition 2.1 again, $\operatorname{supp} u_1(x, t) \subset C_l(K)$. Repeating these steps, we obtain that $\operatorname{supp} u_j(x, t) \subset C_l(K)$ for any $j \geqq 0$. Thus from the convergence of the formal solution, we find that $\operatorname{supp} u(x,t) \subset C_l(K)$. Q.E.D.

This completes the proof of Theorem 2.1.

Appendix

Following Igari [3] and Uryu [17], we introduce a certain class of pseudo-differential operators.

Definition A.1. (1) For any $m \in \mathbb{R}$ and $\kappa > 1$, we denote by $S^m(\kappa)$ the set of functions $h(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the property that for any α, β, there exist constants c_α and R such that

$$|\partial_\xi^\alpha D_x^\beta h(x, \xi)| \leqq c_\alpha R^{|\beta|} |\beta|!^\kappa \langle\xi\rangle^{m-|\alpha|} \quad \text{for} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(2) For any $h(x, \xi) \in S^m(\kappa)$, we shall define a semi-norm of $h(x, \xi)$ such that for any integer $l \geqq 0$,

$$|h(x, \xi)|_l = \max_{|\alpha+\beta| \leqq l} \sup_{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n} |\partial_\xi^\alpha D_x^\beta h(x, \xi)| \langle\xi\rangle^{-m+|\alpha|}.$$

Now we can define a pseudo-differential operator with a symbol $h(x, \xi) \in S^m(\kappa)$ as follows:

$$H(x, D_x)u(x) = (2\pi)^{-n} \int \exp(ix \cdot \xi) h(x, \xi) \hat{u}(\xi) d\xi.$$

Lemma A.1. (*see Igari* [3]). *Let* $h(x, \xi) \in S^m(\kappa)$ *and* $r \geqq 0$. *Then*

$$\sigma(\Lambda^r H)=\sum_{j=1}^{N-1}\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^r D_x^\alpha h(x,\xi)+r_N(x,\xi),$$

where $N=r^*+m$. *And for any integer* $l\geqq 0$, *there exist constants* c_l, $R>0$ *such that*

$$|D_x^\alpha h(x,\xi)\langle\xi\rangle^{-m}|_l\leqq c_l R^{|\alpha|-m}(|\alpha|-m)!^\kappa$$

and

$$|r_N(x,\xi)|_l\leqq c_l R^r r!^\kappa.$$

The following lemma is well-known.

Lemma A.2. *For any* $h(x,\xi)\in S^0$, *there exist a constant c and non-negative integer* l *dependent only on dimension n such that*

$$\|H(x,D_x)u\|\leqq c|h(x,\xi)|_l\|u\|.$$

Lemma A.3. (*see Uryu* [17] *and Igari* [3]). *Under the assumptions of Lemma A.1, if we denote* $h_j(x,\xi)$ *by*

$$h_j(x,\xi)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha\langle\xi\rangle^r D_x^\alpha h(x,\xi),$$

then there exist $\hat{c}$, $\hat{R}>0$ *such that*

$$\|H_j(x,D_x)u\|\leqq \hat{c}\hat{R}^{j-m}(j-m)!^\kappa\binom{r^*}{j}\|\Lambda^{m+r-j}u\| \quad for \quad 1\leqq j\leqq r^*,$$

$$\|H_j(x,D_x)u\|\leqq \hat{c}\hat{R}^{j-m}(j-m)!^\kappa\|\Lambda^{m+r-j}u\| \quad for \quad r^*+1\leqq j\leqq N-1,$$

and

$$\|R_N(x,D_x)u\|\leqq \hat{c}\hat{R}^r r!^\kappa\|u\|.$$

Lemma A.4. *Let* $\{i_1,\cdots,i_n\}$ *be a subset of non-negative numbers* $a_1,\cdots,a_m$, *then there exist constants* A_1, $R_1>0$ *such that*

$$n^{i_1+\cdots+i_n}\leqq A_1R_1^n 1^{i_1}2^{i_2}\cdots n^{i_n}.$$

Proof. Set $S=n^{i_1+\cdots+i_n}/1^{i_1}\cdots n^{i_n}$. Then

$$\begin{aligned}S&=(n/1)^{i_1}\cdots(n/n)^{i_n}\\&\leqq(n/1)^a\cdots(n/n)^a\\&=(n^a/n!)^a,\quad\text{where}\quad a=\max\{a_1,\cdots,a_n\}.\end{aligned}$$

Using Stirling's formula, we can get the desired inequality. Q.E.D.

Lemma A.5. *Let* $\{i_1,\cdots,i_n\}\subset\{1,\cdots,m-1\}$, *then there exist constants* A_2, $R_2>0$ *such that*

$$(i_1+\cdots+i_n)!\leqq A_2R_2^n n^{i_1+\cdots+i_n}.$$

Proof. By Stirling's formula, there exists $R_3>0$ such that

$$\begin{aligned}(i_1+\cdots+i_n)! &\leqq A_2R_3^{i_1+\cdots+i_n}(i_1+\cdots+i_n)^{i_1+\cdots+i_n}\\ &\leqq A_2\{R_3(m-1)n\}^{i_1+\cdots+i_n}\\ &\leqq A_2\{R_3(m-1)\}^{(m-1)n}n^{i_1+\cdots+i_n}.\end{aligned}$$ Q.E.D.

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