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A Vanishing Theorem for Proper Direct Images

By

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Introduction

Let X be a complex analytic space and $\mathcal{F}$ a coherent analytic sheaf over X. Jf X is q-complete, then $H^r(X, \mathcal{F})$, the r-th cohomology of X with coefficients in $\mathcal{F}$, vanishes for $r \geq q$ (cf. Andreotti-Grauert [1]). If moreover X is a q-complete manifold of dimension n and $\mathcal{F}$ is locally free, then one has $H_0^{n-r}(X, \omega_X \otimes \mathcal{F}) = 0$ by Serre's duality, where ω_X denotes the canonical sheaf of X and H_0^{n-r} denotes the cohomology with compact supports.

In [2], A. Andreotti and E. Vesentini established an L^2-theory on non-compact complex manifolds and showed that the vanishing of $H_0^{n-r}(X, \omega_X \otimes \mathcal{F})$ is a direct consequence of certain a priori estimate as well as the vanishing of $H^r(X, \mathcal{F})$. Their approach is of interest since the cohomology vanishing is reduced to the solvability of a $\bar{\partial}$-equation with uniform estimates related to weighted L^2-norms of Carleman type.

The purpose of the present article is to give a relative version of their theory in the following situation.

Let $f: X \to S$ be a morphism between complex analytic spaces and $\mathcal{F}$ a coherent analytic sheaf over X. The q-th direct image sheaf $R^q f_* \mathcal{F}$ is a sheaf over S defined by $(R^q f_* \mathcal{F})_x := \lim_{\substack{\to \\ U}} H^q(f^{-1}(U), \mathcal{F})$, where U runs through the neighbourhoods of x. The q-th proper direct image sheaf $R^q f_! \mathcal{F}$ is defined by $(R^q f_! \mathcal{F})_x := \lim_{\substack{\to \\ U}} H^q_{\mathcal{P}}(f^{-1}(U), \mathcal{F})$, where $\mathcal{P}$ denotes the family of supports consisting of the subsets of X on which f is proper.

A recent result of K. Takegoshi shows that $R^q f_* \omega_X = 0$ for $q > 0$ in the above situation, if X is a complex manifold which is bimeromorphically equivalent to a Stein space.

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Similarly as above, Takegoshi's vanishing theorem implies that $R^q f_! \mathcal{O}_X = 0$ for $q < \dim X - \dim S$, by the duality theorem of Ramis and Ruget (cf. [9]).

Thus our task here shall be to give a direct proof to $R^q f_! \mathcal{O}_X = 0$ in the spirit of Andreotti-Vesentini. It will be reduced to finding two convex increasing functions on $\mathbb{R}$ which compose the weight functions controling the supports of the solutions of the $\bar{\partial}$-equation.

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§ 1. L^2-Cohomology Vanishing

Let (X, ds^2) be a Hermitian manifold of dimension n and (E, h) a holomorphic Hermitian vector bundle over X. We denote by $L^*_{\text{loc}}(E)$ the set of locally square integrable differential forms on X with values in E, and by $L^{p,q}_{\text{loc}}(E)$, or by $L^{p,q}_{\text{loc}}(X, E)$, the set of (p, q)-forms in $L^*_{\text{loc}}(E)$. For any C^∞ function $\psi: X \to \mathbb{R}$ we denote by $L^*(E)$ the set of $u \in L^*_{\text{loc}}(E)$ with

$$\int_X e^{-\psi} |u|^2 dV < \infty .$$

Here $|u|$ denotes the length of u and dV the volume form. We put

$$||u||_\psi := \{ \int_X e^{-\psi} |u|^2 dV \}^{1/2} .$$

Then $L^*(E)$ is a Hilbert space with norm $|| \ ||_\psi$. We put $L^{p,q}_\psi(E)$ $(= L^{p,q}_\psi(X, E)) := L^{p,q}_{\text{loc}}(E) \cap L^*_\psi(E)$. Let $\langle u, v \rangle$ be the pointwise inner product of u and v in $L^*_{\text{loc}}(E)$ with respect to the metrics ds^2 and h. Then the inner product of u and v is expressed as

$$(u, v)_\psi := \int_X e^{-\psi} \langle u, v \rangle dV .$$

The complex differentiation $\bar{\partial}: L^*_{\text{loc}}(E) \to L^*_{\text{loc}}(E)$ is defined in the sense of distribution. The domain of the restriction of $\bar{\partial}$ to $L^*(E)$ will be denoted by $D_{\bar{\partial}}$, shortly. Clearly $\bar{\partial}$ is a densely defined closed linear operator for any ψ. We denote by $\bar{\partial}^*_\psi$ the adjoint of $\bar{\partial}$ on $L^*_\psi(E)$. The domain of $\bar{\partial}^*_\psi$ is denoted by $D_{\bar{\partial}^*_\psi}$. The following is a direct consequence of Hahn-Banach's theorem. For the proof, the reader is referred to Hormander's book [6] (cf.

Chap. IV).

Proposition 1.1 *For any* $v\in L^2_\psi(E)$ *the following conditions* (i) *and* (ii) *are equivalent.*

(i) $|(u, v)_\psi| \le C\|\bar\partial^*_\psi u\|_\psi$, *for any* $u\in D_{\bar\partial^*_\psi}$ *with* $\bar\partial u=0$.

(ii) *There exists a* $w\in L^2_\psi(E)$ *satisfying* $\bar\partial w=v$ *and* $\|w\|_\psi\le C$.

Here C *is a positive number which depends on* v.

To study geometric conditions which imply the condition (i) above, the following lemma is helpful.

Lemma 1.2 (cf. [2]) *Let* (X, ds^2) *be a complete Hermitian manifold,* (E, h) *a Hermitian vector bundle over* X, A: $L^2_\psi(E)\to L^2_\psi(E)$ *a bounded linear operator, and* (p, q) *a pair of nonnegative integers. If the inequality* $\|Au\|^2_\psi\le\|\bar\partial u\|^2_\psi+\|\bar\partial^*_\psi v\|^2_\psi$ *holds for any compactly supported E-valued* C^∞ (p, q)*-form* u *on* X, *then the same inequality is valid for all* $u\in D_{\bar\partial}\cap D_{\bar\partial^*_\psi}\cap L^{p,q}_\psi(E)$. *Moreover, for any* $v\in\operatorname{Im} A$, (i) *holds with* $C=\|A^{-1}v\|_\psi$.

Proof. For the first part, see [2]. The second assertion follows from Cauchy-Schwartz inequality.

Before proceeding further, we explain the notion of Nakano seminegativity of the bundle (E, h).

By identifying the metric h with a C^∞ section of $\operatorname{Hom}(E, \bar E^*)$, we define the curvature of (E, h) by $\bar\partial\circ h^{-1}\circ\partial\circ h$, which is naturally identified with a C^∞ section, say Θ_h, of the bundle $T^*_X\otimes\bar T^*_X\otimes\operatorname{Hom}(E, E)$ or equivalently, of the bundle $\operatorname{Hom}(T_X, \bar T^*_X)\otimes\operatorname{Hom}(E, E)$, where T_X denotes the holomorphic tangent bundle of X. Thus $h\circ\bar\partial\circ h^{-1}\circ\partial\circ h$ is naturally identified with a section of $\operatorname{Hom}(E\otimes T_X, \bar E^*\otimes\bar T^*_X)$, which defines a quadratic form along the fibers of $E\otimes T_X$.

Definition. A Hermitian vector bundle (E, h) is said to be Nakano seminegative if $h\circ\bar\partial\circ h^{-1}\circ\partial\circ h$ is a negative semidefinite quadratic form on $F\otimes T_X$.

In what follows we shall regard Θ_h as a $(1, 1)$-form with values in $\operatorname{Hom}(E, E)$.

Lemma 1.3 *Let* (X, ds^2) *be a Kahler manifold of dimension* n, (E, h) *a Nakano seminegative vector bundle over* X, *and* ψ *a* C^∞ *real valued function on* X. *Let* $\gamma_1\ge\cdots\ge\gamma_n$ *be the eigenvalues of* $i\partial\bar\partial\psi$ *on* X. *Then,* $\|\bar\partial u\|^2_\psi+\|\bar\partial^*_\psi u\|^2_\psi\ge(-(\gamma_1+\cdots+\gamma_{n-q})u, u)$, *for any compactly supported E-valued* C^∞ $(0, q)$*-form* u *on* X.

Proof. By Bochner-Nakano's inequality,

$$||\bar{\partial}u||_{\psi}^{2}+||\bar{\partial}_{\psi}^{*}u||_{\psi}^{2}\geq(-i\Lambda((\Theta_{h}+\partial\bar{\partial}\psi)\wedge u),u)_{\psi},$$

for any compactly supported C^{∞} $(0, q)$-form u. Here Λ denotes the adjoint of the wedge multiplication by the fundamental form of ds^2. Since (E, h) is Nakano seminegative, $(-i\Lambda(\Theta_h\wedge u), u)\geq 0$ (cf. [7] p. 197 (14)). The rest is a straightforward computation.

Theorem 1.4 *Let (E, h) be a Nakano seminegative Hermitian vector bundle over a complex manifold X of dimension n, let Φ_1, Φ_2 be two C^{∞} plurisubharmonic functions on X and let r be an integer such that rank $\partial\bar{\partial}\Phi_2\leq r$ everywhere. Suppose that $i\partial\bar{\partial}(\Phi_1+\Phi_2)$ is a positive $(1, 1)$-form associated to a complete Kahler metric on X, say ds^2. Then, for any $q<n-r$ and $v\in L^{0,q}_{-(r+1)\Phi_1+\Phi_2}(E)$ with $\bar{\partial}v=0$, where the metric on X is ds^2, there exists a $w\in L^{0,q-1}_{-(r+1)\Phi_1+\Phi_2}(E)$ such that $\bar{\partial}w=v$ and $||w||_{-(r+1)\Phi_1+\Phi_2}\leq||v||_{-(r+1)\Phi_1+\Phi_2}$. Here we use a convention that $L^{0,q}_{\psi}(E)=0$ for $q<0$.*

Proof. Let $\Gamma_1\geq\cdots\geq\Gamma_n$ be the eigenvalues of $i\partial\bar{\partial}(-(r+1)\Phi_1+\Phi_2)$ with respect to the metric ds^2 $(=2\partial\bar{\partial}(\Phi_1+\Phi_2))$. By assumption, $-(\Gamma_1+\cdots+\Gamma_{n-q})\geq 1$ everywhere on X. Hence the assertion follows from Proposition 1.1, Lemma 1.2 and Lemma 1.3.

In the following paragraph we shall prove the following.

Theorem *Let X be a complex manifold of pure dimension n which is bimeromorphically equivalent to a Stein space, (E, h) a Nakono seminegative bundle over X, S a complex space of dimension s, and f a holomorphic map from X to S. Then $R^qf_!\mathcal{O}_X(E)=0$ for $q<n-s$.*

Since every holomorphic vector bundle admits a Nakano seminegative Hermitian structure on Stein manifolds, we have the following corollary.

Corollary *Let X be a Stein manifold of pure dimension n, S a complex analytic space of dimension s, and $f\colon X\to S$ a holomorphic map. Then, for any holomorphic vector bundle E over X, $R^qf_!\mathcal{O}_X(E)=0$ for $q<n-s$.*

Proof of Theorem

1. The Stein case: First we shall give the proof when X is a Stein manifold because it is completely independent of the other results which we need in the proof of the general case. Assume that we are given a Stein manifold

X of pure dimension n, a Nakano seminegative vector bundle (E, h) over X, a complex space S of dimension s and a holomorphic map $f: X \to S$. Let q be an integex with $q<n-s$ and $v \in L^{0,p}_{\mathrm{loc}}(E)$ a $\bar\partial$-closed form such that $f \mid \operatorname{supp} v$ is proper. Since the problem is local on S, we may assume that S is a Stein space and that there exists a C^∞ strictly plurisubharmonic function Φ_1 on X such that $\operatorname{supp} v \subset X_c := \{x \in X;\ \Phi_1(x) \leq c_0\}$ for some $c_0 \in \mathbb{R}$ and the restriction of f to X_c is proper for all $c \in \mathbb{R}$. Let $\varphi_\circ$ be a C^∞ plurisubharmonic exhaustion function on S and set $\varphi := \varphi_\circ \circ f$. Evidently, there exists a C^∞ convex increasing function $\lambda: \mathbb{R} \to \mathbb{R}$ such that the metric $ds^2 := 2\partial\bar\partial(\Phi_1 + \lambda(\varphi))$ is complete and that $v \in L^{0,q}_{-(s+1)\Phi_1+\lambda(\varphi)}(E)$ (with respect to ds^2). Let χ be a C^∞ convex increasing function on $\mathbb{R}$ such that $\chi(t)=0$ for $t \leq c_0$ and $\chi'(t)=1$ for $t \geq c_0+1$. Clearly $\|v\|_{-(s+1)\Phi_1+\lambda(\varphi)-\mu\chi(\Phi_1)} = \|v\|_{-(s+1)\Phi_1+\lambda(\varphi)}$ for any $\mu>0$. Since $\partial\bar\partial\chi(\Phi_1) \geq 0$ by Theorem 1.4 one can find a $w_\mu \in L^{0,q-1}_{-(s+1)\Phi_1+\lambda(\varphi)-\mu\chi(\Phi_1)}(E)$ satisfying $\bar\partial w_\mu = v$ and $\|w_\mu\|_{-(s+1)\Phi_1+\lambda(\varphi)-\mu\chi(\Phi_1)} \leq \|v\|_{-(s+1)\Phi_1+\lambda(\varphi)}$, for any $\mu>0$.
If $\mu \geqslant \nu$ one has

$$\|w_u\|_{-(s+1)\Phi_1+\lambda(\varphi)-\mu\chi(\Phi_1)} \geq \|w_\mu\|_{-(s+1)\Phi_1+\lambda(\varphi)-\nu\chi(\Phi_1)} \cdot$$

Therefore one can choose a subsequence of $\{w_\mu\}_{\mu=1}^\infty$ which converges weakly to some $w \in L^{0,q-1}_{\mathrm{loc}}(E)$. Clearly $\bar\partial w = v$ and $\operatorname{supp} w \subset X_{c_0}$, which implies that v represents the 0-class in $H^q_{\mathcal{P}}(X, \mathcal{O}_X(E))$ with $\mathcal{P} := \{A \subset X; f \mid A \text{ is proper}\}$.

2. General case. To prove our theorem in full generality we rely upon the desingularization theorem of Hironaka in the following form (cf [5]).

Theorem 2.1 *Let Y be a Stein space of dimension n and let X be a complex space with a proper holomorphic map $\pi: X \to Y$ such that $\pi \mid X \backslash \pi^{-1}(\mathrm{Sing}\ Y)$ is biholomorphic. Here Sing Y denotes the singular locus of Y. Then, for any point $y \in Y$, there exist a neighbourhood $U \ni y$ and a complex manifold $\tilde U$ with a proper holomorphic embedding $\iota: \tilde U \hookrightarrow U \times \mathbb{P}^N$ for some N such that the composite of ι with the projection to the first component factors through π, i.e., there exists a holomorphic map $p: \tilde U \to \pi^{-1}(U)$ such that the diagram*

$$\begin{array}{ccc} \tilde U & \xrightarrow{\ \iota\ } & U \times \mathbb{P}^N \\ {\scriptstyle p}\downarrow & & \downarrow \\ \pi^{-1}(U) & \xrightarrow{\ \pi\ } & U \end{array}$$

commutes.

Similarly as in Proposition 4.1 in [8], we may apply Hironaka's theorem to obtain the following:

Proposition 2.2 *Let X be a complex manifold of pure dimension n bimeromorphic to a Stein space and let $\pi: X \to \hat{X}$ be its Remmert reduction. Then, for any point $x \in \hat{X}$, there exist a neighbourhood U, an analytic subset $A \subset \pi^{-1}(U)$ and a plurisubharmonic function ψ on $\pi^{-1}(U)$ with $A = \{x; \psi(x) = -\infty\}$ which satisfies the following properties:*

(i) *ψ is C^∞ outside A and $2\partial\bar{\partial}\psi$ gives a complete Kahler metric on $\pi^{-1}(U) \backslash A$.*

(ii) *For any open set $V \Subset \pi^{-1}(U)$ a holomorphic function h (resp. a holomorphic n-form ω) on $V \backslash A$ is holomorphically extendable to V if and only if $h \in L^{0,0}_{-\psi}(V \backslash A)$ (resp. $\omega \in L^{n,0}_{\psi}(V \backslash A)$) with respect to the metric $2\partial\bar{\partial}\psi$.*

(iii) *For any $V \Subset \pi^{-1}(U)$ and any holomorphic n-form ω on V, the multiplication*

$$\omega \wedge : L^{0,q}_{-\psi}(V \backslash A) \to L^{n,q}_{\psi}(V \backslash A)$$

is continuous for all $q \geq 0$.

We shall omit the proof, because the construction of ψ is quite similar to that in [8] (cf. the function $K\varphi^2 - \log(1 + \rho \log \psi)$ in Proposition 4.1) and the properties above are immediate from the definition of ψ.

Our theorem is now contained in the following

Theorem 2.3 *Let X be a complex manifold of pure dimension n, (E, h) a Nakano seminegative vector bundle over X, $A \subset X$ an analytic subset, and ψ a plurisubharmonic function satisfying* (i), (ii) *and* (iii) *in Proposition* 2.2 *for $X = \pi^{-1}(U)$. Let Ψ and φ be C^∞ plurisubharmonic functions on X such that $\Psi + \varphi$ is exhausting. Assume that rank $\partial\bar{\partial}e^{\varphi} \leq s$ everywhere. Then, with respect to the family of supports $\mathcal{P} := \{U: \sup_U \Psi < \infty\}$,*

$$H^q_{\mathcal{P}}(X, \mathcal{O}_X(E)) = 0, \quad \text{for} \quad q < n - s.$$

Proof. First we assert that it suffices to show that for any $q < n - s$ and a C^∞ $(0, q)$-form v on X with $\bar{\partial}v = 0$ and supp $v \subset \{x; \Psi(x) \leq c\}$, there exists a $w \in L^{0,q-1}_{\text{loc}}(X \backslash A, E)$ with the following properties:

(1) $\quad \bar{\partial}w = v \quad$ on $\quad X \backslash A$.

(2) $\quad$ supp $w \subset \{x \in X \backslash A; \Psi(x) \leq c\}$.

(3) $\quad$ For any point $x \in X$ there exists a neighbourhood U such that $w \in L^{0,q-1}_{-\psi}(U \backslash A, E)$ with respect to the metric $2\partial\bar{\partial}\psi$.

To see the validity of this assertion, let $\mathcal{U}$ be a locally finite covering of X by relatively compact polydiscs such that $\sup\{|\Phi_1(x) - \Phi_1(y)|; x, y \in U\} < 1$

for any $U\in\mathcal{U}$. Let σ be a Čech q-cocycle associated to $\mathcal{U}$ such that $\operatorname{supp}\sigma\subset\{x;\Psi(x)\leq c\}$ for some c. Then, in a canonical way one can associate to σ a $\bar{\partial}$-closed C^∞ $(0,q)$-form v on X such that $\bar{\partial}v=0$ and $\operatorname{supp} v\subset\{x;\Psi(x)\leq c\}$. Suppose that there exists a w satisfying (1), (2) and (3). Then, from (iii) we can regard w locally as an element of $L_{\psi}^{n,q-1}(U\backslash A, E)$ for each $U\in\mathcal{U}$, by multiplying a holomorphic n-form ω_u with no zero on $\bar{U}$. Hence, applying repeatedly the L^2-vanishing theorem on $U\backslash A$ (cf. Theorem 2.8 in [8]) we arrive at a Čech $(q-1)$-cochain τ associated to $\mathcal{U}':=\{U\backslash A\}_{U\in\mathcal{U}}$ such that $\delta\tau=\sigma\,|\,X\backslash A$, $\tau_{1\cdots q}\omega_{U_1}\in L_{\psi}^{n,0}(U_1\cap\cdots\cap U_q\backslash A, E)$ for any $U_1,\cdots,U_q\in\mathcal{U}$ and that $\operatorname{supp}\tau\subset\{x\in X\backslash A;\Psi(x)<c+1\}$. Hence, by (ii) all $\tau_{1\cdots q}$ are holomorphically extendable on $U_1\cap\cdots\cap U_q$. Thus σ represents the 0-class in $H^q_{\Phi}(X, \mathcal{O}_X(E))$.

Now let v be as above and apply Theorem 1.4 as before by letting $\Phi_1=\psi+\mu\chi(\Psi)$, $\Phi_2=\lambda(\varphi)$ and $ds^2=2\partial\bar{\partial}(\psi+\lambda(\varphi))$, where λ is chosen as before. Letting $\mu\to\infty$ we obtain a $w\in L_{\mathrm{loc}}^{0,q-1}(X\backslash A, E)$ satisfying (1), (2) and (3) above.

q.e.d.

Remark Our proof of the cohomology vanishing is direct in the sense it is independent of the coherency theorem for the direct image sheaves (cf. Grauert [4] Siu [10] and Ermine [3]). Moreover it gives a slightly more general result. In fact, in Theorem 2.3, A need not be holomorphically convex. For instance one may take as A a pseudoconvex neighbourhood of the zero section of any topologically trivial line bundle over a compact Kahler manifold.

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