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On The Brownian Curve and its Circumscribing Sphere

By

Omer ADELMAN*

Abstract

The number of points the image of [0, t] under a planar Brownian motion has in common with its circumscribing circle does *not* have a zero-one law: it can be either two or three, both with a strictly positive probability. (This contradicts an assertion made by Paul Lévy.)

Let B be a standard d-dimensional Brownian motion, t a generic element of $]0, \infty[$, $S_t \equiv S(B[0, t])$ the sphere circumscribing[1] $B[0, t] \equiv B([0, t])$. Set

$$E_t = S_t \cap B[0, t],$$
$$K_t \equiv K(B[0, t]) = \text{Card}\, E_t,$$
$$N_t = \text{Card}\{s \in [0, t]/B_s \in S_t\}.$$

It is not hard to see that if $d=1$ then, almost surely, for all t, $K_t=2$ and $N_t \leqq 3$, and $\{t; N_t=3\}$ is countable and has no isolated point.

Set

$$I_k = \{t; K_t = k\},$$
$$p_k = P(K_1 = k).$$

Clearly, for all t, $p_k=P(K_t=k)$. Also,

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* Université Pierre et Marie Curie, Laboratoire de Probabilités, 75252 Paris, France.

[1] For a nonempty bounded $A \subset \mathbb{R}^d$, letting $\mathcal{D}(A)=\{D; D$ is a closed ball of $\mathbb{R}^d$, and $D \supset A\}$, one easily sees that $\mathcal{R}(A) \equiv \{r; r$ is the radius of some $D \in \mathcal{D}(A)\}$ is closed. Let

$$R(A) = \inf \mathcal{R}(A).$$

Since the intersection of two distinct closed balls of radius t is included in a ball of radius $<t$, it follows that there exists exactly one D in $\mathcal{D}(A)$ with radius $R(A)$. We denote it by $D(A)$. The circumscribing sphere of A (or: the sphere circumscribing A) is the boundary of $D(A)$. We denote it by $S(A)$. We set: $K(A)=\text{Card}(A \cap S(A))$.

It is easy to check that, for $D \in \mathcal{D}(A)$, the centre of D belongs to the convex hull of $\bar{A} \cap \partial D$ if, and only if, $D=D(A)$. ($\bar{A}, \partial A$ and $\hat{A}$ denote, respectively, the closure, the boundary and the convex hull of A.)

$$p_k>0 \Leftrightarrow P(\lambda(I_k)>0)=1$$
$$\Leftrightarrow P(\lambda(I_k)=\infty)=1\,.$$

(λ_d=the Lebesgue measure on $\mathbb{R}^d$, $\lambda\equiv\lambda_1$.) It is about as obvious that the law of $(tI_1, \cdots, tI_k, \cdots)$ is independent of t.[2]

Paul Lévy's studied[3] some of the stochastic properties of E_t. Among other things, he claimed that $P_{d+1}=1$. This is the conjunction of two statements:

$$(L) \qquad P(K_1>d+1)=0\,,$$

$$(L') \qquad P(K_1<d+1)=0\,.$$

Lévy's proof of (L) is correct, although realizing it requires quite cooperative a reader. As to (L'), Lévy's argumentation in its favour fails to be a proof[4], and, indeed, (L') is not valid. What *is* true is:

$$p_k>0 \text{ if, and only if, } \quad k\in\{2, \cdots, d+1\}\,.$$

This last assertion is a combination of (L) with the two following statements.

$$(*) \qquad \text{For all } k\in\{2, \cdots, d+1\},\, p_k>0\,.$$

$$(**) \qquad p_1=0\,.$$

Now, $(**)$ is immediate, since, for any t, $\{K_t=1\}=\{\mathrm{Card}(B[0, t])=1\}=\{B \text{ is constant on } [0, t]\}$, an event whose probability is obviously zero.

Thanks to (L), in order to show that $p_{d+1}>0$, it suffices to establish that $P(K_1\geqq d+1)>0$. Let us do it right away for $d=2$. Assuming B is a planar Brownian motion, we want to show that $P(K(B[0, 1])>2)>0$. Let T be an equilateral triangle centred at the origin. Assume $h>1$. Now, with a strictly positive probability, the convex hull of $B[0, 1]$ includes T and is included in hT. And if h is close enough to 1, then *any* compact subset A of the plane for which $T\subset\hat{A}\subset hT$ has at least three points in common with its circumscribing circle: Clearly, for a compact planar set X, if $K(X)\leqq 2$, then diam $S(X)$ =diam X. Noticing that, for planar X and $Y\supset X$, $S(\hat{X})=S(X)$ and diam $S(X)\leqq$diam $S(Y)$, we see that if A is as above and $K(A)\leqq 2$, then diam $S(T)$

[2] So each I_k is stochastically self-similar, and almost surely, if I_k is nonempty, then $\inf I_k=0$, $\sup I_k=\infty$.

[3] See [1], Section 5. (Our notation differs from Lévy's.)

[4] Lévy seems to assume implicitly that (L′) follows from the last paragraph on page 19 of [1]. Nothing is wrong with the mentioned paragraph, but (L') just does not follow.

In view of our present considerations, Lévy's remark in the last paragraph of 5, 7° (page 22 of [1]) is also to be modulated. The rest of Lévy's paper is not affected.

$\leqq$diam $S(\hat{A})$=diam $S(A)$=diam $A\leqq$diam $hT=h$ diam T. Combined with the fact that diam $S(T)=\frac{2}{\sqrt{3}}$ diam T, the above implies:

$$\frac{2}{\sqrt{3}} \text{ diam } T \leqq h \text{ diam } T\,.$$

But h may take a fancy to being strictly smaller than $2/\sqrt{3}\cdots$. (The adaption of all this for $d>2$ is straightforward.)

In I and II we show that if $d=2$, then $p_2>0$. One who fully understands I and II can hardly fail to realize that $p_2>0$ *independently* of d. Establishing (*) (for $d>2$) is somewhat more cumbersome, but necessitates no novel idea.

Once we get insight into matters, a number of other facts can be noticed. Some are summarized below. (Proofs are not even outlined.)

Observations

Almost surely,

(a) for all $k\in\{2, \cdots, d+1\}$, $\lambda(I_k)=\infty$,

(b) $]0, \infty[=I_2\cup\cdots\cup I_{d+2}$ (so, for all t, $K_t\leqq d+2$),

(c) $\lambda(I_{d+2})=0$,

(d) for all $k\in\{2, \cdots, d+2\}$, inf $I_k=0$ and sup $I_k=\infty$,

(e) no I_k has an isolated point,

(f) $\partial I_2\subsetneqq\cdots\subsetneqq\partial I_{d+1}=\partial I_{d+2}=\bar{I}_{d+2}=\{0\}\cup\{t;\ B_t\in S_t\}$ (so $\lambda(]0, \infty[\setminus(\mathring{I}_2\cup\cdots\cup\mathring{I}_{d+1}))=0$),

(g) for all k, any maximal proper interval[5] of I_k is left closed, right open, and its right endpoint belongs to I_{k+1},

(h) if $t<u<\infty$ and $K_u\neq K_t<k\leqq d+1$, then $]t, u[$ includes a maximal proper interval of I_k,

(i) E is constant on any maximal interval of any I_k,

(j) for all t which is the right endpoint of a maximal proper interval of I_{d+1}, there is exactly one point A_t in $E_t\setminus\{B_t\}$ such that

$$u\downarrow\downarrow t\Rightarrow\delta(E_u, E_t\setminus\{A_t\})\rightarrow 0\,^{6}\,,$$

(k) for all t *not* as above,

$$u\downarrow t\Rightarrow\delta(E_u, E_t)\rightarrow 0\,,$$

(l) if $d>1$, then, for all t, $N_t=K_t$.

The list of observation is continued in III.

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5 A proper interval is one with a nonempty interior.

6 δ is the Euclidean distance in $\mathbb{R}^d$.

I

Here, and in II below, we assume $d=2$; i.e., B is a planar Brownian motion, the balls D_t are discs, the spheres S_t are circles. Under this assumption, we set to prove the following

Theorem 1. $P(K_3=2)>0$.

Letting μ be a version of $P(\cdot/\sigma(B\,[0, 1]\cup[2, 3)])$, it is obvious that for (P-)almost all ω,

$$\mu(B]1, 2[\quad \text{does not encounter } S(B([0, 1]\cup[2, 3)]))\,(\omega)>0\,.$$

So, in order to prove Theorem 1, it is sufficient to show that with a strictly positive probability, $B([0, 1]\cup[2, 3])$ encounters its circumscribing circle at exactly two points.

Letting ν be a version of $P(\cdot/\sigma(B[0, 1], (B_\cdot-B_2)\,[2, 3]))$, it is obvious that, for almost all ω, $\nu(\omega)$ endows B_2 with a density which is continuous and strictly positive over the whole plane (and which is, therefore, equivalent to the planar Lebesgue measure λ_2). One concludes immediately that Theorem 1 is entailed by the fact that *with a strictly positive probability, the set*

$$V\equiv\{v\in\mathbb{R}^2;\ K(B[0, 1]\cup(v+B[2, 3]))=2\}$$

has a strictly positive planar Lebesgue measure[7].

This last claim, we shall see, has nothing to do with the stochastic properties specific to Brownian motion. It is an obvious by-product of

Theorem 2. *Let X and Y be nonempty compacs subsets of the plane. Then*

$$\lambda_2\{v\in\mathbb{R}^2;\ K(X\cup(v+Y))=2\}>0\,.$$

Here X plays the role of $B[0, 1]$, Y that of $B[2, 3]$ (or that of $B[2, 3]-B_2$). In case X and Y are singletons, Theorem 2 becomes a triviality. From now on, we assume that X and Y are arbitrary fixed compact subsets of the plane, that at least one of them is not a singleton, and that both include the origin.

Let W denote the set of elements w of the plane such that each of $X-w$,

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7 As long as measurability considerations (which, by the way, are not difficult) have not been carried out, the statement should perhaps read: The *inner* probability of $\{V$ has a strictly positive *inner* planar Lebesgue measure$\}$ is strictly positive.

In fact, questions about measurability could (and should?) have been raised earlier. In order not to annoy the reader (and the writer) with such matters, that arise every here and there, we go on, pretending there is no problem. Things will be settled in II.

$Y+w$ has exactly one point in common with the circumscribing circle of $(X-w)\cup(Y+w)$. Theorem 2 is (obviously) entailed by

Theorem 2′. $\lambda_2(W)>0$.

Let r be a generic element of $]0, \infty[$, θ a generic element of $[0, 2\pi[$. Set

$$\psi_r = \{\theta;\ re^{i\theta}\in W\}^{8}.$$

Since

$$\lambda_2(W) = \int_0^\infty \lambda(\psi_r) r\, dr ,$$

Theorem 2′ will be established once we show that, on a subset of $]0, \infty[$ of a strictly positive measure, $\lambda(\psi_\circ)$ is larger than some strictly positive constant. This will turn to be a corollary of the conjunction of Theorem 3 and Theorem 4, to follow in a moment.

Let C be a circle in the plane. Its centre is denoted by $\mathring{C}$. We denote by C_X the set of points x such that, for some v in the plane, $X+v$ is included in the closed disc having C for boundary, and $(X+v)\cap C=\{x\}$.

According to Theorem 3, if C is large compared to X, then, in term of length, C_X constitutes a large part of C. The length of the circle C is its perimeter, 2π times its radius. If C' is a subset of C, then $\mathrm{Arg}(C'-\mathring{C})\equiv\{\mathrm{Arg}(z-\mathring{C});\ z\in C'\}$ is a subset of $[0, 2\pi[$, and we define the length $L(C')$ of C' to be $\lambda(\mathrm{Arg}(C'-\mathring{C}))$ times the radius of C (iff $\mathrm{Arg}(C'-\mathring{C})$ is Lebesgue measurable).

Note that the perimeter[9] per $\hat{X}$ of the convex hull $\hat{X}$ of X is finite[10].

Theorem 3. $L(C\setminus C_X)\leqq\frac{\pi}{2}\cdot\mathrm{per}\ \hat{X}$.

(The proof is given in II.)

Let C^X denote the set of elements of C diametrically opposite to elements of C_X (so $C^X=2\mathring{C}-C_X$). We have:

$$L(C_X\cap C^Y) = L(C_X\setminus(C\setminus C^Y))\geqq L(C_X)\setminus L(C\setminus C^Y).$$

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8 We identify the Cartesian plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, in the standard way.

9 per $\hat{X}$ is the infimum of the set of perimeters of simply connected open sets including X. (So the perimeter of a straight segment is twice its length.)

10 The best bounds for per $\hat{X}$ are given by: $4R(X)\leqq\mathrm{per}\ \hat{X}\leqq\pi$ diam X. ($R(X)$ is the radius of the circumscribing circle of X.)

per $\hat{X}$ equals $4R(X)$ iff $\hat{X}$ is a straight segment. It equals π diam X iff the length of the orthogonal projection of $\hat{X}$ on a straight line in the plane does not depend on the line. (Discs are not the only figures satisfying this last condition.)

Clearly, $L(C^Y)=L(C_Y)$. By Theorem 3, $L(C_X)\geqq L(C)-\frac{\pi}{2}$ per $\hat{X}$ and $L(C\setminus C^Y)\leqq\frac{\pi}{2}$ per $\hat{Y}$. We deduce

Theorem 3′. $L(C_X\cap C^Y)\geqq L(C)-\frac{\pi}{2}$ (per $\hat{X}$+per $\hat{Y}$).

For an element v of the plane, set

$$\begin{aligned}
\Delta_v &= \sup \delta((X-v)\times(Y+v))^{11}\ (\equiv \sup_{x\in X-v,\, y\in Y+v}\delta(x, y))\,,\\
X_v &= \{x\in X-v;\ \text{for some } y\in Y+v,\ \delta(x, y) = \Delta_v\}\,,\\
Y_v &= \{y\in Y+v;\ \text{for some } x\in X_v,\ \delta(x, y) = \Delta_v\}\,,\\
\alpha(v) &= \inf \operatorname{Arg}(Y_v-X_v)\quad (\equiv \inf_{\substack{x\in X_v,\, y\in Y_v\\ \delta(x,y)=\Delta_v}}\operatorname{Arg}(y-x))\,,
\end{aligned}$$

and let $(x(v), y(v))$ be the unique $(x, y)\in X_v\times Y_v$ such that $\operatorname{Arg}(y-x)=\alpha(v)$[12].

Setting $\alpha_r(\theta)=\alpha(re^{i\theta})$, let α_r' denote the right lower derivative of α_r:

$$\alpha_r'(\theta) = \liminf_{h\downarrow\downarrow 0}\frac{1}{h}(\alpha_r(\theta+h)-\alpha_r(\theta))\,.$$

Observe that there exists a unique t in $[0, 2\pi[$ such that, for all $s\in[0, t[$ and $u\in[t, 2\pi[$, $\alpha_r(u)<\alpha_r(s)$, and such that α_r is strictly increasing on $[0, t[$ and on $[t, 2\pi[$. This assures that, (λ-)almost everywhere on $[0, 2\pi[$, α_r admits a derivative (which equals α_r' and is $\geqq 0$).

The measure $\lambda\circ\alpha_r$ admits a decomposition

$$\lambda\circ\alpha_r = m_a+m_s\,,$$

where m_a and m_s are positive measures on $[0, 2\pi[$, and where m_a (resp. m_s) is absolutely continuous (resp. singular) with respect to λ.

Now, m_a admits a Radon-Nikodym derivative with respect to λ, which almost everywhere on $[0, 2\pi[$, equals α_r'.

So, setting

$$a(r) = \inf_\theta \alpha_r'(\theta)\,,$$

we see that, for all Lebesgue measurable $\Theta\subset[0, 2\pi[$,

$$\lambda(\alpha_r(\Theta))\geqq m_a(\Theta) = \int_\Theta \alpha_r'(\theta)d\theta\geqq a(r)\lambda(\Theta)\,. \tag{1}$$

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11 Note that if the modulus of v is large enough (larger than $\sup_{v\in X\cup Y}\|v\|$, say) then $\Delta_v=\operatorname{diam}(X-v)\cup(Y+v)$.

12 Since X and Y are compact, so are X_v and Y_v, and $\alpha(v)=\min\operatorname{Arg}(Y_v-X_v)$. The fact that at least one of X, Y is not a singleton guarantees that $\Delta_v>0$, so $\alpha(v)$ is *not* defined as Arg 0, and is hence well determined.

It is not hard to see that if $v=re^{i\theta}$ and r is fixed, then for all but countably many values of θ *both* X_v and Y_v are singletons.

Assuming that r is the radius of the circle C, let ϕ_r be the subset of $[0, 2\pi[$ defined by

$$\phi_r = \mathrm{Arg}(C_X \cap C^Y - \mathring{C}) .$$

Given r, ϕ_r is clearly independent of $\mathring{C}$. Also, ϕ is increasing ($r<s<\infty \Rightarrow \phi_r \subset \phi_s$). In particular, denoting by r_θ the radius of the circle circumscribing $(X-re^{i\theta}) \cup (Y+re^{i\theta})$, we have:

$$r_\theta \equiv R((X-re^{i\theta}) \cup (Y+re^{i\theta})) \geqq R(\{-re^{i\theta}, re^{i\theta}\}) = r ,$$

so

$$\phi_r \subset \phi_{r_\theta} . \tag{2}$$

Observe that

$$\alpha_r(\theta) \in \phi_{r_\theta} \Rightarrow \theta \in \psi_r , \tag{3}$$

and that Theorem 3′ can be rewritten as:

$$\lambda(\phi_r) \geqq 2\pi - \frac{\pi}{2r} (\mathrm{per}\ \hat{X} + \mathrm{per}\ \hat{Y}) . \tag{4}$$

From the above four numbered relations we deduce that, if $a(r)>0$, then

$$\begin{aligned}
\lambda(\psi_r) &\geqq \lambda\{\theta;\ \alpha_r(\theta) \in \phi_{r_\theta}\} && \text{by (3)}\\
&\geqq \lambda\{\theta;\ \alpha_r(\theta) \in \phi_r\} && \text{by (2)}\\
&= 2\pi - \lambda\{\theta;\ \alpha_r(\theta) \in [0, 2\pi[\backslash \phi_r\} \\
&\geqq 2\pi - \frac{1}{a(r)} \lambda([0, 2\pi[\backslash \phi_r) && \text{by (1)}\\
&\geqq 2\pi - \frac{1}{a(r)} (2\pi - (2\pi - \frac{\pi}{2r} (\mathrm{per}\ \hat{X} + \mathrm{per}\ \hat{Y}))) , && \text{by (4)}
\end{aligned}$$

so

$$\lambda(\psi_r) \geqq 2\pi - \frac{\pi}{2r\, a(r)} (\mathrm{per}\ \hat{X} + \mathrm{per}\ \hat{Y}) . \tag{5}$$

Thiorem 4. $\liminf_{r\to\infty} a(r) > 0$.

(The proof is postponed to (II)).

This implies that, as $r\to\infty$, $1/(r\,a(r)) \to 0$, so, according to (5), $\lambda(\psi_r) \to 2\pi$. So, for some $q>0$, for all $r>q$, $\lambda(\psi_r)>5$. For such a q, we have:

$$\lambda_2(W) \geqq \int_q^\infty \lambda(\psi_r) r\, dr \geqq 5 \int_q^\infty r\, dr = \infty$$

(which is even better than Theorem 2′).

III

Before proving Theorem 3 and Theorem 4, let us settle the measurability problem raised in footnote 7. At the present stage, it is clear that all will be in order once we show that W is measurable[13]. Now, it is standard (and easy) to prove that

$$W_\varepsilon \equiv \{v \in \mathbb{R}^2;\ \operatorname{diam}(X-v) \cap S((X-v) \cup (Y+v)) < \varepsilon$$

and

$$\operatorname{diam}(Y+v) \cap S((X-v) \cup (Y+v)) < \varepsilon\}$$

is open (so Borel measurable). But W is nothing but the intersection of W_ε over the countably many strictly positive rationals ε.

Proof of Theorem 3.
(X a nonempty compact planar set, C a planar circle.)

Case 0. Assume X is a singleton. Then $C_X = C$ and $L(C \setminus C_X) = 0 = \operatorname{per} \hat{X}$.

Definition and observation. An arc of a circle is termed "small" if it is included in a half-circle. Observe that the length of a small arc does not exceed $\pi/2$ times its diameter.

Case 1. Assume $\hat{X}$ is a non-degenerate straight segment (i.e., X is not a singleton, but is in included in a straight line) and $\operatorname{diam} C > \operatorname{diam} S(X)$ $(= L(\hat{X}))$. Then $C \setminus C_X$ is the union of two small arcs of C, each having $L(\hat{X})$ for diameter. So

$$L(C \setminus C_X) \leqq 2 \cdot \frac{\pi}{2} L(\hat{X}) = \frac{\pi}{2} \operatorname{per} \hat{X}.$$

Case 2. Assume that X is not a singleton and that $\operatorname{diam} C \leqq \operatorname{diam} S(X)$.

The centre S of $S \equiv S(X)$ is included in the convex hull of $X \cap S$. So, in $X \cap S$, there are three (not necessarily distinct) points Z_0, Z_1, Z_2, such that S belongs to the (eventually degenerate) triangle whose vertices are exactly the Z_i's. Now, the Z_i's determine three disjoint small arcs $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$ of S,

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[13] On the other hand, had W been non-measurable, using the inequality

$$\lambda_2(W) \geqq \int_0^\infty \lambda(\psi_r) r\, dr$$

(λ_2 denoting the inner planar Lebesgue measure) would be suspicious (in case the right hand-side were strictly positive). There are classical constructions (relying on the Axiom of Choice) of planar sets whose inner planar Lebesgue measure is zero, but whose trace on any circle centred at the origin has only a countable complement (with respect to the circle).

whose union is S[14] and, correspondingly, disjoint arcs $\mathcal{S}'_i$ of $\partial\hat{X}$.

For all $i(\in\{0, 1, 2\})$, $\mathcal{S}_i$ being a small arc,

$$L(\mathcal{S}_i)\leqq\frac{\pi}{2}\operatorname{diam}\mathcal{S}_i=\frac{\pi}{2}\operatorname{diam}\mathcal{S}'_i\leqq\frac{\pi}{2}L(\mathcal{S}'_i)\,.$$

Summing of i yields:

$$L(S)\leqq\frac{\pi}{2}\operatorname{per}\hat{X}\,.$$

Since diam $C\leqq$diam S, we get:

$$L(C\setminus C_X)\leqq L(C)\leqq L(S)\leqq\frac{\pi}{2}\operatorname{per}\hat{X}\,.$$

Case 3. Here we assume that $\hat{X}$ is not a straight segment and that diam $C>$diam $S(X)$. Now, $L(C\setminus C_X)$ is invariant under translations of C (or of X). So, with no loss of generality, we assume: X is included in the open disc having C for boundary.

For each $\theta(\in[0, 2\pi[)$, let v_θ denote the unique $v=re^{i\theta}$ $(r\geqq0)$ such that $X+v$ encounters C and is included in $\bar{C}$ (=the closed disc having C for boundary).

Set

$$C_\theta=(X+v_\theta)\cap C\,.$$

Let $\mathcal{C}_\theta$ denote the shortest arc of C including C_θ. Observe that $\mathcal{C}_\theta$ is a small arc. Also, notice that $\mathcal{C}_\theta$ are mutually disjoint (closed) arcs (whose union is exactly C). The length of C being finite, this implies that the set

$$\Theta\equiv\{\theta;\ L(\mathcal{C}_\theta)>0\}\ (=\{\theta;\ \mathcal{C}_\theta \text{ is not a singleton}\})$$

is only countable.

Observe that $C\setminus C_X$ is exactly $\cup_{\theta\in\Theta}\mathcal{C}_\theta$[15].

Let $\mathcal{B}_\theta$ denote the orthogonal projection of $\mathcal{C}_\theta$ on $\hat{X}$. (This is an arc of $\partial\hat{X}$).

Let $\mathcal{A}_\theta$ denote the interior of $\mathcal{B}_\theta-v_\theta$ with respect to the trace of the usual plane topology on $\partial\hat{X}$. (If $\theta\in\Theta$, $\mathcal{A}_\theta$ is an open arc of $\partial\hat{X}$; if not, it is empty.) Note that the arcs $\mathcal{A}_\theta$ are mutually disjoint and that, for each θ, $\mathcal{A}_\theta$, $\mathcal{B}_\theta$ and

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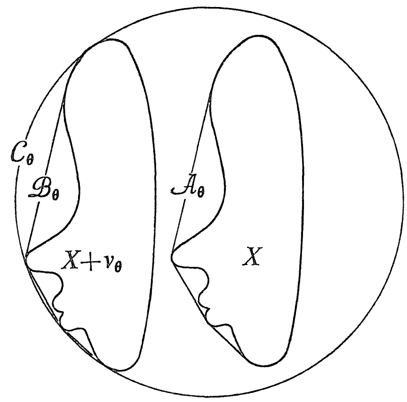
14 A precise definition does not seem indispensable. (One such would consist of letting $\mathcal{S}_i=f[i.\ i+1]\setminus f\{i\}$ for a continuous map f of $[0, 3]$ onto S such that $f(3)=f(0)=Z_0$, $f(1)=Z_1$, $f(2)=Z_2$ and such that, if $0\leqq s<t<3$ and $f(s)=f(t)$, then f is constant on $[s, t]$.) Note that at most two of the Z_i's coincide, so at most one of the $\mathcal{S}_i$'s is of zero length.

15 The fact that $C\setminus C_X$ is a union of countably many arcs of C (or, equivalently, that Arg $((C\setminus C_X)-\dot{C})$ is a union of countably many subintervals of $[0, 2\pi[)$ guarantees that $L(C\setminus C_X)$ is well defined.

C_θ have the same diameter[16].



Finally,

$$\begin{aligned}
L(C\setminus C_X) &= L(\cup_{\theta\in\Theta} C_\theta) \\
&\leqq \textstyle\sum_{\theta\in\Theta} L(C_\theta) \quad \text{(Here, in fact, equality holds.)} \\
&\leqq \textstyle\sum_{\theta\in\Theta} \frac{\pi}{2} \operatorname{diam} C_\theta \\
&= \frac{\pi}{2} \textstyle\sum_{\theta\in\Theta} \operatorname{diam} \mathcal{A}_\theta \\
&\leqq \frac{\pi}{2} \textstyle\sum_{\theta\in\Theta} L(\mathcal{A}_\theta) \\
&\leqq \frac{\pi}{2} L(\partial\hat{X}) \\
&= \frac{\pi}{2} \operatorname{per} \hat{X}\,.
\end{aligned}$$

This completes the proof.

Proof of Theorem 4.

(X, Y compact planar sets including the origin.)

Fix some $r>q\equiv 70 \operatorname{diam} X+80 \operatorname{diam} Y+90$ and some $\theta\in[0, 2\pi[$. We shall see that if $\tau>\theta$ is close enough to θ, then $\alpha_r(\tau)-\alpha_r(\theta)\geqq(\tau-\theta)/4$. (Obviously, this implies Theorem 4.)

Let $\tau>\theta$ be close enough to θ so that $\tau<2\pi$, $\tau<\theta+1/17$ (say), and α_r is (strictly) increasing on $[\theta, \tau]$.

Let

$$h=\tau-\theta \ (<1/17)\,,$$

[16] The empty set has diameter zero.

$$\begin{aligned}
0 &= \text{the origin},\\
E &= x(re^{i\theta})+re^{i\theta} && (\in X),\\
F &= y(re^{i\theta})+re^{i\theta} && (\in Y+2re^{i\theta}),\\
G &= x(re^{i\tau})+re^{i\tau} && (\in X),\\
H &= y(re^{i\tau})+re^{i\tau} && (\in Y+2re^{i\tau}),\\
I &= y(re^{i\theta})-re^{i\theta}+2re^{i\tau}\\
&= 2re^{i\tau}+F-2re^{i\theta} && (\in Y+2re^{i\tau}),\\
I' &= -F||I||/||F|| && (||v|| = \delta(0, v)),\\
J &= Fe^{ih},\\
K &= 2re^{i\tau}.
\end{aligned}$$

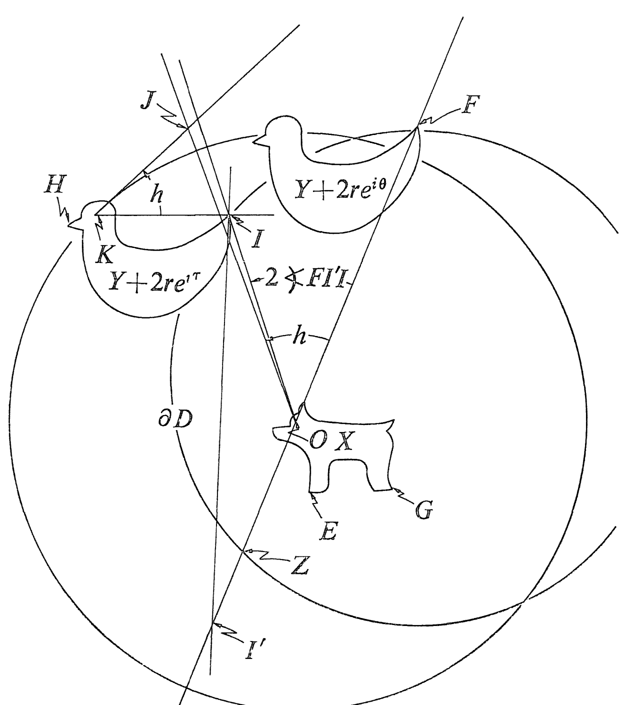
For three distinct points u, v, w in the plane, let

$$\measuredangle uvw = (\operatorname{Arg}(w-v)-\operatorname{Arg}(u-v)) \bmod 2\pi .$$

Observe that

$$\varepsilon \equiv ||J-K|| = ||I-K|| \leqq \operatorname{diam} Y < q/80$$

and that, if $J \neq I$, then $\measuredangle IKJ = h$, so, anyhow,



$$\delta(I, J) \leqq 2\varepsilon \sin \frac{h}{2},$$

and hence

$$\sin(\frac{1}{2} \measuredangle IOJ) \leqq \frac{\delta(I, J)/2}{\min\{||I||, ||J||\}} \leqq \frac{\varepsilon \sin(h/2)}{2q-\varepsilon} < \frac{\sin(h/2)}{159} < \frac{h}{318}.$$

Now, for all $t \in]0, h[$, $(\mathrm{Arcsin}\ t)/t$ is rather close to unity and is, certainly, strictly smaller than 318/100. So

$$\measuredangle IOJ \leqq 2 \ \mathrm{Arcsin}(h/318) \leqq 2 \cdot (318/100)\ (h/318) = h/50,$$

and, since $< FOJ = h$, we have:

$$\measuredangle FOI = \measuredangle FOJ - \measuredangle IOJ \geqq h - h/50 > h/2.$$

Note that

$$\begin{aligned} \alpha_r(\tau) - \alpha_r(\theta) &= \mathrm{Arg}(H-G) - \mathrm{Arg}(F-E) \\ &\geqq \mathrm{Arg}(I-E) - \mathrm{Arg}(F-E) \\ &= \measuredangle FEI \end{aligned}$$

and that our job will be done once we show that this last angle is $\geqq h/4$.

Since $||I'|| = ||I||$, we clearly have:

$$\measuredangle FI'I = \frac{1}{2} \measuredangle FOI > h/4.$$

Let D be a disc having F and I on its boundary, including X, but not I' (after having checked that such a disc exists).

Let Z be the unique point distinct from F which ∂D has in common with the straight segment joining F to I'.

We have

$$\measuredangle FI'I \leqq \vee FZI < \pi/2,$$

so, for all $\nu \in D \setminus \{F, I\}$,

$$\measuredangle F\nu I \geqq \measuredangle FZI.$$

Observing that $E \in D \setminus \{F, I\}$, we get

$$\measuredangle FEI \geqq \measuredangle FZI \geqq \measuredangle FI'I > h/4,$$

and Theorem 4 is established.

IIII

Before concluding, we introduce some additional notation and list some more observations (without proofs).

For all t, let

$$\begin{aligned}\underline{\underline{t}} &= \inf\{s\in[0, t];\ S_s = S_t\}\ ,\\ \underline{t} &= \sup\{s\leqq t;\ B_s\in S_s = S_t\}\ ,\\ \bar{t} &= \inf\{u\geqq t;\ B_u\in S_u = S_t\}\ ,\\ \bar{\bar{t}} &= \sup\{u\geqq t;\ S_u = S_t\}\ .\end{aligned}$$

Clearly, $\underline{\underline{t}}\leqq\underline{t}\leqq t\leqq\bar{t}\leqq\bar{\bar{t}}$.

Observations (continued).

Almost surely,

(m) for (Lebesgue-)almost all t, $\underline{\underline{t}}=\underline{t}<t<\bar{t}=\bar{\bar{t}}$,

(n) for all t, if $\underline{\underline{t}}<\underline{t}$, then $t=\bar{t}$, and if $\bar{\bar{t}}>\bar{t}$, then $t=\underline{t}$,

(o) the sets $\{t;\ \underline{t}<t=\bar{t}\}$ and $\{t;\ \underline{t}=t<\bar{t}\}$ are countable and have no isolated points,

(p) for all t, $\underline{t}=\bar{t}$ if, and only if, $t\in S_t$.

Exhausting all observation of this kind does not seem of the utmost interest, but one who enjoys it can find the perspective promising.

A number of variations on our subject can be conceived.

Example. Let A be a convex compact subset of $\mathbb{R}^d$ with a nonempty interior. Its boundary $\partial\equiv\partial A$ is said to be a *circumscribing form* if, for all nonempty bounded $X\subset\mathbb{R}^d$, there is exactly one transformation of $\mathbb{R}^d$ of the form $x\rightarrow T(x)=ax+b$ ($a\geqq 0$, $b\in\mathbb{R}^d$) such that $T(A)$ includes X and, subject to this, a is minimal[17]; in this case, $T(\partial)$ is the *circumscribing ∂-form* of X. It is quite clear that, if ∂ is a smooth circumscribing form, then all we did above can be adjusted into a coherent text if one wishes to replace circumscribing spheres by circumscribing ∂-forms.

Example. For an unbounded $X\subset\mathbb{R}^d$, define $S(X)$ to be the empty set. Let $\mathcal{P}$ be the set of nonempty subsets of $[0, \infty[$. Observe that, since B is continuous, for any $A\in\mathcal{P}$, $S_A\equiv S(B(A))=S(\overline{B(A)})=S(B(\bar{A}))$ is a well defined random variable.

Letting $K_A=\mathrm{Card}(S_A\cap\overline{B(A)})$, study the random families

$$(K_A)_{A\in\mathcal{P}}\ \text{and}\ (\{A\in\mathcal{P};\ K_A = k\})_k\ \text{a cardinal number}\cdots$$

[17] A sufficient condition in order that ∂ be a circumscribing form is: A is strictly convex (i.e., ∂ is the set of extremal points of A). A necessary and sufficient condition is: no two distinct maximal convex subsets of ∂ which are not reduced to points are parallel.

Example. Replace B by some diffusion, or fractionary Brownian motion, on $\mathbf{R}^d$ or on some other appropriate manifold$\cdots$

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Reference

[1] Lévy, P., La mesure de Hausdorff de la courbe de mouvement brownien, *Giornale dell'Instituto Italiano degli Attuari*, **16**, (1953), 1–37.