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Type I Orbits in the Pure States of a C^*-dynamical System

By

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0. Introduction

Let A be a C^*-algebra and let α be an action of a locally compact abelian group G on A such that α is continuous, i.e., $t\mapsto\alpha_t(x)$ is continuous on G for each $x\in A$. Let $P(A)$ denote the set of pure states of A and α^* the action of G on $P(A)$ defined by $\alpha_t^* f=f\circ\alpha_t$, $t\in G$, which is continuous in the sense that $(t, f)\mapsto\alpha_t^* f$ is jointly continuous if $P(A)$ is equipped with the weak* topology. Thus, with the C^*-dynamical system (A, G, α) one associates the dynamical system $(P(A), G, (\alpha^*)$. (If A is abelian, $P(A)$ is a locally compact space and A is identified with the continuous functions on $P(A)$ vanishing at infinity.)

Let $f\in P(A)$. We call the orbit $\{\alpha_t^* f, t\in G\}$ through f of *type I* if the representation

$$\rho_f = \int_G^{\oplus} \pi_f\circ\alpha_t\, dt$$

of A on $L^2(G, \mathcal{H}_f)=L^2(G)\otimes\mathcal{H}_f$ is of type I, where π_f is the GNS representation of A associated with f, and dt is a Haar measure on G. If A is abelian, or more generally of type I, or if α^* is strongly continuous, then every orbit in $P(A)$ is of type I. In general it is not even true that a system has a type I orbit.

In this paper we shall show that there exist type I orbits for C^*-dynamical systems satisfying a certain spectrum condition. This is defined in terms of a spectrum similar to the Connes spectrum.

Before we state our main results more concretely, we look at the representations ρ_f closely. Given $f\in P(A)$, we denote by u the unitary representa-

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tion of G on $L^2(G, \mathcal{H}_f)$ defined by

$$(u(s)\,\xi)\,(t) = \xi\,(t+s)\,, \quad \text{a,a, } t$$

for $\xi \in L^2(G, \mathcal{H}_f)$. Since $\rho_f \circ \alpha_s = \mathrm{Ad}\, u(s) \circ \rho_f$, α extends to an action $\bar{\alpha}$ on $\rho_f(A)''$ by $\bar{\alpha}_s = \mathrm{Ad}\, u(s)$. Let $Z = \rho_f(A)'' \cap \rho_f(A)'$ and let $N = \mathrm{Sp}(\bar{\alpha}\,|\,Z)$. Then since Z is a von Neumann subalgebra of $L^\infty(G) \otimes 1$, $\bar{\alpha}$ is ergodic on Z. Hence the von Neumann algebra $\rho_f(A)''$ is homogeneous, e.g., purely of type I, II, or III, and N is a closed subgroup of $\hat{G}$. It easily follows that

$$Z = \{p : p \in N\}'' \otimes 1\,.$$

Let $H = \{s \in G : \pi_f \circ \alpha_s \sim \pi_f\}$ where $\sim$ denotes (quasi-) equivalence. Then H is a subgroup of G and it follows that $H \subset N^\perp$ in general.

0.1. Proposition. Let A be a separable C^*-algebra and let α be a continuous action of a separable (i.e., second-countable) locally compact abelian group G on A, and adopt the notation above. Then the following conditions are equivalent:

1. $\rho_f(A)''$ is of type I.
2. $H = N^\perp$.

Now one of our results runs as follows: In the same situation as above suppose that A is simple and unital, α is faithful, and that there is an automorphism σ of A such that $\sigma \circ \alpha_t = \alpha_t \circ \sigma$, $t \in G$ and $[\sigma^n(x), y] \to 0$ as $n \to \infty$ for $x, y \in A$ (which is a situation where the spectrum condition referred to before is satisfied). Then for each closed discrete subgroup H of G there exists a pure state f as described in 0.1 (see 2.3 and 3.3). When the group G is compact we can remove, by using [1], the condition 'discreteness' on H in the above result (see 3.5), and furthermore we could say much more about the pure states in connection with the action α^* due to the Glimm's type theorem [3].

We conclude this section by giving the proof of 0.1.

Suppose (1) and let $s \in N^\perp$. Then $\bar{\alpha}_s$ leaves each element of the center Z invariant, and hence it is inner, i.e., there is a unitary V in $\rho_f(A)''$ such that $\alpha_s = \mathrm{Ad}\, V$. Let $\{V_t : t \in G\}$ be a measurable family of unitaries such that

$$V = \int_G^\oplus V_t\, dt\,.$$

Then there is a $t \in G$ such that

$$\pi_f \circ \alpha_{s+t}(x) = V_t\, \pi_f \circ \alpha_t(x)\, V_t^*\,, \quad x \in A$$

and hence $\pi_f \circ \alpha_s \sim \pi_f$ or $s \in H$. Conversely if $s \in H$, $\bar{\alpha}_s$ is implemented by by a unitary of the form $1\otimes u$ and hence it is trivial on Z, or $s\in N^{\perp}$. Thus (2) follows.

Suppose (2). For each s in the closed subgroup H of G, α_s extends to an automorphism β_s of $\pi_f(A)''=B(\mathcal{H}_f)$: $\beta_s\circ\pi_f=\pi_f\circ\alpha_s$. Since A is separable, it follows that $s\mapsto\beta_s(Q)$ is weakly* measurable for $Q\in B(\mathcal{H}_f)$, i.e., $s\mapsto\varphi\circ\beta_s$ is weakly measurable for $\varphi\in B(\mathcal{H}_f)_*$. Hence, as $B(\mathcal{H}_f)_*$ is separable, $s\mapsto(\beta_s)_*$ is strongly measurable, and so strongly continuous. For the representation

$$\pi_H=\int_H^{\oplus}\pi_f\circ\alpha_s\,ds$$

of A on $L^2(H, \mathcal{H}_f)$, one has that π_H is quasi-equivalent to π_f: For any $Q\in\pi_H(A)''$ there is a unique $Q_1\in\pi_f(A)''$ such that

$$Q=\int_H^{\oplus}\beta_s(Q_1)\,ds$$

and, for any $Q_1\in B(\mathcal{H}_f)$, the above direct integral defines an element of $\pi_H(A)''$. Since ρ_f is unitarily equivalent to

$$\int_{G/H}^{\oplus}\pi_H\circ\alpha_{f(s)}\,ds\,,$$

where f: $G/H\to G$ is a measurable function such that $f(s)+H=s$, $s\in G/H$, it follows by (2) that the above integral is a central decomposition of ρ_f. Hence $\rho_f(A)''=L^{\infty}(G/H)\otimes\pi_f(A)''$, which is of type I.

§1. Spectral Subspaces

Let A be a C^*-algebra and let α be a continuous action of a locally compact abelian group G on A. For a subset U of $\hat{G}$, $A^{\alpha}(U)$ denotes the set of $x\in A$ with $\mathrm{Sp}_{\alpha}(x)\subset U$. Let $\{x_{\mu}\}$ be a bounded net in A and let $p\in G$. If for any neighbourhood U of $0\in\hat{G}$ there exists a μ_0 such that $x_{\mu}\in A^{\alpha}(p+U)$ for all $\mu\geq\mu_0$, then $\{x_{\mu}\}$ is said to be a bounded net of spectrum p.

Let π be a representation of A and let $\mathcal{M}=\pi(A)''$. For each $p\in\hat{G}$ let $\mathcal{M}(p)$ denote the set of elements of $\mathcal{M}$ which are obtained as weak limit points of $\{\pi(x_{\mu})\}$ for all bounded nets $\{x_{\mu}\}$ in A of spectrum p. (Note that $\mathcal{M}(p)$ also depends on π and A.) Then it is obvious that $\mathcal{M}(p)$ is a subspace of $\mathcal{M}$ and that $\mathcal{M}(p)^*=\mathcal{M}(-p)$ and $\mathcal{M}(p)\,\mathcal{M}(q)\subset\mathcal{M}(p+q)$ for p, $q\in\hat{G}$. If there is a continuous action $\bar{\alpha}$ of G on $\mathcal{M}$ such that $\bar{\alpha}_t\circ\pi=\pi\circ\alpha_t$, $t\in G$, it follows that $\mathcal{M}(p)=\mathcal{M}^{\bar{\alpha}}(p)$ where

$$\mathcal{M}^{\bar{\alpha}}(p)=\{Q\in\mathcal{M}:\ \bar{\alpha}_t(Q)=\langle t,p\rangle\,Q,\ t\in G\}\ .$$

1.1. Proposition. Let A be a C^*-algebra and let α be a continuous action of a locally compact abelian group G on A. Let π be a representation of A and define a representation ρ of A by

$$\rho = \int_G^{\oplus} \pi\circ\alpha_t \, dt$$

on $L^2(G, \mathcal{H}_\pi)=L^2(G)\otimes\mathcal{H}_\pi$ where dt is a Haar measure. Let $\mathcal{M}=\pi(A)''$ and $\mathcal{N}=\rho(A)''$, and let $\bar{\alpha}$ be the action of G on $\mathcal{N}$ induced by α. Then for any $p\in\hat{G}$ it follows that $\mathcal{N}^{\bar{\alpha}}(p)=p\otimes\mathcal{M}(p)$ and that for each $Q\in\mathcal{M}(p)$ there is a bounded net $\{x_\mu\}$ in A of spectrum p such that $\|x_\mu\|\leq\|Q\|$ and $\lim \pi(x_\mu)=Q$. In particular $\mathcal{M}(p)$ is a closed subspace of $\mathcal{M}$, and $\mathcal{M}(0)$ is a von Neumann subalgebra of $\mathcal{M}$.

Proof. Let $Q\in\mathcal{M}(p)$, and let $\{x_\mu\}$ be a bounded net of spectrum p such that $\pi(x_\mu)$ converges weakly to Q. Since $\|\alpha_t(x_\mu)-\langle t, p\rangle x_\mu\|\to 0$ uniformly on each compact subset of $t\in G$ ([8], 8.1.7) one has that

$$\langle\pi\circ\alpha_t(x_\mu)\,\xi, \eta\rangle \to \langle t, p\rangle\langle Q\xi, \eta\rangle$$

uniformly on each compact subset of G for any ξ, $\eta\in\mathcal{H}_\pi$. Thus it follows that $\lim \rho(x_\mu)=p\otimes Q$ which belongs to $\mathcal{N}^{\bar{\alpha}}(p)$.

Let $Q'\in\mathcal{N}^{\bar{\alpha}}(p)$. Since $Q'\in L^\infty(G)\otimes B(\mathcal{H}_\pi)$, and Q' satisfies that $\bar{\alpha}_t(Q')=\langle t, p\rangle Q'$, there is a bounded operator Q on $\mathcal{H}_\pi$ such that $Q'=p\otimes Q$. We have to show that $Q\in\mathcal{M}(p)$.

By Kaplansky's density theorem there is a net $\{x_\mu\}$ in A such that $\|x_\mu\|\leq\|Q'\|=\|Q\|$ and $\lim \rho(x_\mu)=Q'$. Let g be a positive continuous integrable function on G such that supp $\hat{g}$ is compact and $\hat{g}(0)=1$. For such a g and μ let

$$x_{\mu,g} = \int \alpha_t(x_\mu)\,\overline{\langle t, p\rangle}\, g(t)\, dt\,.$$

Then $\|x_{\mu,g}\|\leq\|x_\mu\|\,\|g\|_1\leq\|Q'\|$ and $\rho(x_{\mu,g})$ converges to Q' as $\mu\to\infty$ for each g. Define an order on the pairs (μ, g) by:

$$(\mu_1, g_1)\geq(\mu_2, g_2) \quad \text{if} \quad \mu_1\geq\mu_2,\ \operatorname{supp}\hat{g}_1\subset\operatorname{supp}\hat{g}_2\,.$$

Then $\{x_{\mu,g}\}$ is a net and Q' is a weak limit point of $\{\rho(x_{\mu,g})\}$. This shows that there is a bounded net $\{y_\mu\}$ in A of spectrum p such that $\|y_\mu\|\leq\|Q'\|$ and $\lim \rho(y_\mu)=Q'$. Since $\{y_\mu\}$ is bounded, there is a subnet $\{y'_\mu\}$ of $\{y_\mu\}$ such that $\pi(y'_\mu)$ converges weakly, say to Q_1. Since $\|\alpha_t(y'_\mu)-\langle t, p\rangle y'_\mu\|\to 0$ for $t\in G$, it follows that $\lim \rho(y'_\mu)=p\otimes Q_1$. Hence $Q=Q_1$ and thus $Q\in\mathcal{M}(p)$.

Consider the same situation as in 1.1. Let N be a closed subgroup of $\hat{G}$ and let $p\in\hat{G}$. A bounded net $\{x_\mu\}$ in A is said to be of spectrum $p+N$ if for any neighbourhood U of $0\in\hat{G}$ there exists a μ_0 such that $x_\mu\in A^\alpha(p+N+U)$ for all $\mu\geq\mu_0$, or equivalently if it is a bounded net of spectrum $p+N$ $(\in\hat{G}/N)$ with respect to the action defined by restricting α to $H=N^\perp$. Let $\mathcal{M}(p+N)$ be the set of elements of $\mathcal{M}$ which are obtained as weak limit points of $\{\pi(x_\mu)\}$ for all bounded nets $\{x_\mu\}$ in A of spectrum $p+N$. Then one immediately obtains

1.2. Corollary. In the same situation as in 1.1, let N be a closed subgroup of $\hat{G}$. Then $\mathcal{M}(p+N)^*=\mathcal{M}(-p+N)$, $\mathcal{M}(p+N)\,\mathcal{M}(q+N)\subset\mathcal{M}(p+q+N)$, and $\mathcal{M}(p+N)$ is a closed subspace of $\mathcal{M}$. Moreover for any $Q\in\mathcal{M}(p+N)$ there exists a bounded net $\{x_\mu\}$ in A of spectrum $p+N$ such that $\|x_\mu\|\leq\|Q\|$ and $\lim\pi(x_\mu)=Q$.

1.3. Proposition. Let A be a prime C^*-algebra and let α be a continuous action of a locally compact abelian group G on A. Let π be a faithful representation of A and define the representation ρ of A as in 1.1. Let $\mathcal{M}=\pi(A)''$ and let $\mathcal{N}=\rho(A)''$. If $\mathcal{M}$ is a factor, the following conditions are equivalent:

1. $\mathcal{M}(0)=\mathcal{M}$, and α is faithful.
2. $\mathcal{M}(p)=\mathcal{M}$ for all $p\in\hat{G}$.
3. $\mathcal{M}(p)\ni 1$ for all $p\in\hat{G}$.
4. $\mathcal{N}\supset L^\infty(G)\otimes 1$.

Proof. Suppose (1). Then $\mathcal{M}(p)$ is a closed ideal of $\mathcal{M}$, i.e., $\mathcal{M}(p)=(0)$ or $\mathcal{M}$, since $\mathcal{M}$ is a factor. The set of $p\in\hat{G}$ with $\mathcal{M}(p)=\mathcal{M}$ is a closed subgroup of $\hat{G}$; it is a group since $\mathcal{M}(p)^*=\mathcal{M}(-p)$ and $\mathcal{M}(p)\,\mathcal{M}(q)\subset\mathcal{M}(p+q)$, and it is closed since $\{p\in\hat{G}\colon \mathcal{M}(p)\ni 1\}$ is closed.

Let $\bar{\alpha}$ be the action of G on $\mathcal{N}$ induced by α. Since $\bar{\alpha}$ is faithful, we only have to show that $\mathcal{M}(p)\neq(0)$, or $\mathcal{N}^{\bar{\alpha}}(p)\neq(0)$ by 1.1, for each $p\in\mathrm{Sp}(\bar{\alpha})$. For any compact neighbourhood U of $p\in\mathrm{Sp}(\bar{\alpha})$, $\mathcal{N}^{\bar{\alpha}}(U)$ is not zero. Let $Q\in\mathcal{N}^{\bar{\alpha}}(U)$ with $\|Q\|=1$ and let $t\mapsto Q(t)$ be a norm continuous map of G into $\mathcal{M}$ which represents Q (see [9], IV.7.17 and note that the continuity requirement is satisfied since $\mathrm{Sp}_{\bar{\alpha}}(Q)$ is compact). By replacing Q by $\bar{\alpha}_s(Q)$ with $s\in G$ if necessary, we suppose that $\|Q(0)\|>1/2$. Since $1\otimes Q(0)\in\mathcal{N}^{\bar{\alpha}}(0)$, we further suppose that $Q(0)\geq 0$ by replacing Q by $Q\cdot 1\otimes Q(0)^*/\|Q(0)\|$. In a similar way we may eventually suppose that $Q(0)=1/2$ by using $1\otimes\mathcal{M}\subset\mathcal{N}^{\bar{\alpha}}(0)$. For each compact neighbourhood U of p we choose $Q_U\in\mathcal{N}^{\bar{\alpha}}(U)$ such that $\|Q_U\|\leq 1$ and $Q_U(0)=1/2$. Then since the family $t\mapsto Q_U(t)$ is equi-continuous as U shrinks to $\{p\}$, any weak limit point of $\{Q_U\}$ is not zero and belongs to $\mathfrak{N}^{\bar{\alpha}}(p)$.

By 1.1 the implications (2)⇒(3)⇒(4) are immediate.

Suppose (4) and let $x\in A$ and $\varepsilon>0$. Let U be an open neighbourhood of 0 such that $\|\alpha_t(x)-x\|<\varepsilon$, $t\in U$. Let K be a compact subset of G and take a finite covering $\{U+t_i: i=1, \cdots, n\}$ of K. Let

$$Q=\sum_{i=1}^{n}(\chi_{U_i}\otimes 1)\,\rho(\alpha_{-t_i}(x))$$

where $U_i=(U+t_i)\cap K\setminus\bigcup_{i=1}^{i-1}U_j$, $U_1=(U+t_1)\cap K$. Then $Q\in\mathfrak{N}$ and $\|Q-\chi_K\otimes\pi(x)\|<\varepsilon$ which implies that $\chi_K\otimes\pi(x)\in\mathfrak{N}$ and so it follows that $1\otimes\pi(x)\in\mathfrak{N}$. Thus $\mathfrak{N}\supset L^\infty(G)\otimes\mathcal{M}$ i.e., $\mathcal{M}(0)=\mathcal{M}$. Since $\mathrm{Sp}(\alpha)=\hat{G}$ follows trivially this concludes the proof.

§2. Spectrum

Let α be a continuous action of a locally compact abelian group G on a C^*-algebra A. The Connes spectrum $\Gamma(\alpha)$ of α is defined as follows (cf. [8]): $p\in\Gamma(\alpha)$ if for any non-zero $x\in A$ and any compact neighbourhood U of p there are $t\in G$ and $a\in A^\alpha(U)$ such that $xa\alpha_t(x^*)\neq 0$. We define a subset $\Gamma_1(\alpha)$ of $\hat{G}$ as follows: $p\in\Gamma_1(\alpha)$ if for any non-zero $x\in A$, any compact neighbourhood U of p, and any $\varepsilon>0$, there is an $a\in A^\alpha(U)$ such that $\|a\|=1$ and $\|xax^*\|\geq(1-\varepsilon)\|x\|^2$. Clearly $\Gamma_1(\alpha)$ is a closed subset of $\Gamma(\alpha)$ and satisfies that $\Gamma_1(\alpha)\ni 0$ and $-\Gamma_1(\alpha)=\Gamma_1(\alpha)$.

2.1. Proposition. $\Gamma_1(\alpha)$ is a closed subgroup of $\hat{G}$.

Proof. It suffices to show that $\Gamma_1(\alpha)$ is closed under multiplication. Let $p, q\in\Gamma_1(\alpha)$, $x\in A$, and $\varepsilon>0$. For any compact neighbourhood U of $0\in\hat{G}$ there is an $a\in A^\alpha(p+U)$ such that $\|a\|=1$ and $\|xax^*\|\geq(1-\varepsilon)\|x\|^2$. Let $y=ax^*$. And then $\|y\|\leq\|x\|$ and $\|xy\|\geq(1-\varepsilon)\|x\|\,\|y\|$. By the following lemma we have a $b\in A^\alpha(q+U)$ such that $\|b\|=1$ and $\|xby\|\geq\varphi_1(1-\varepsilon)\|xy\|$, where $\varphi_1(t)\uparrow 1$ as $t\uparrow 1$. Since $ba\in A^\alpha(p+q+U+U)$ and $\|xbax^*\|\geq(1-\varepsilon)\varphi_1(1-\varepsilon)\|x\|^2$, one gets the conclusion.

2.2. Lemma. There exists an increasing function φ on [0, 1] such that $\lim_{t\uparrow 1}\varphi(t)=\varphi(1)=1$ and for any $p\in\Gamma_1(\alpha)$, any neighbourhood U of p and any non-zero $x, y\in A$, $t\in[0, 1]$ with $\|xy\|=t\|x\|\,\|y\|$, it follows that

$$\sup\{\|xay\|: a\in A^\alpha(U)_1\}\geq\varphi(t)\|x\|\,\|y\|$$

where $A^\alpha(U)_1$ denotes the unit ball of $A^\alpha(U)$.

Proof. Let $x, y \in A$, $t \in [0, 1]$ be as in the above lemma, and let $x_1 = \|x\|^{-2}x^*x$, and $y_1 = \|y\|^{-2}yy^*$. Then $\|x_1\| = \|y_1\| = 1$, $x_1 \geq 0$, $y_1 \geq 0$ and $\|x_1 y_1\| \geq t^3$. Since $\|xay\| \geq \|x_1 a y_1\|\, \|x\|\, \|y\|$ and $t^3 \uparrow 1$ as $t \uparrow 1$ we now assume that $x \geq 0$, $y \geq 0$ and that $\|x\| = 1 = \|y\|$. For any $\varepsilon > 0$, there is an $a \in A^\alpha(U)$ such that $\|a\| = 1$ and $\|xyayx\| > (1-\varepsilon)\, t^2$. There exists a pure state f of A such that

$$f(xyayx^2ya^*yx) > (1-\varepsilon)^2\, t^4 .$$

Since $xyayx^2\, ya^*yx \leq t^2\, xy^2x$, one has that $f(xy^2x) > (1-\varepsilon)^2\, t^2$, and so

$$\begin{aligned} f((xy-x)\,(xy-x)^*) &= f(xy^2x) - 2f(xyx) + f(x^2) \\ &\leq f(x^2) - f(xy^2x) < 1 - (1-\varepsilon)^2\, t^2 . \end{aligned}$$

One calculates:

$$\begin{aligned} f(xay^2\, a^*x) &\geq f(xayx^2\, ya^*x) \\ &= f(xyayx^2\, ya^*yx) + f((x-xy)\, ayx^2\, ya^*yx) + f(xayx^2\, ya^*(x-yx)) \\ &> (1-\varepsilon)^2\, t^4 - 2t^2(1-(1-\varepsilon)^2\, t^2)^{1/2} . \end{aligned}$$

Hence

$$\sup\, \{\|xay\| : a \in A^\alpha(U)_1\} \geq t(t^2 - 2(1-t^2)^{1/2})^{1/2}$$

and thus the lemma is proved.

2.3. Proposition. Let A be a simple unital C^*-algebra and let α be a continuous action of a locally compact abelian group G on A. Suppose that there is an automorphism σ of A such that $\sigma \circ \alpha_t = \alpha_t \circ \sigma$, $t \in G$ and $[x, \sigma^n(y)]$ goes to zero as $n \to \infty$ for any $x, y \in A$. Then $\Gamma_1(\alpha) = \Gamma(\alpha) = \mathrm{Sp}(\alpha)$.

Proof. This follows easily since $\|x\, \sigma^n(y)\| \to \|x\|\, \|y\|$ as $n \to \infty$ [7].

2.4. Proposition. Let A be a simple unital C^*-algebra and let α be a continuous action of a compact abelian group G on A such that $A^\alpha(0) = \mathbf{C}1$. Then $\Gamma_1(\alpha) = \{0\}$ and $\Gamma(\alpha) = \mathrm{Sp}(\alpha)$.

Proof. It is trivial that $\Gamma(\alpha) = \mathrm{Sp}(\alpha)$, and we may assume that $\mathrm{Sp}(\alpha) = \hat{G}$. For each $p \in \hat{G}$, $A^\alpha(p)$ is one-dimensional and contains a unitary, say u_p. There is an injective homomorphism φ of $\hat{G}$ into G such that $\alpha_{\varphi(p)} = \mathrm{Ad}\, u_p$, $p \in \hat{G}$ (and the range is dense in G). If $p \neq 0$, there is a $q \in \hat{G}$ such that $\langle \varphi(p), q \rangle \neq 1$, and so there is a non-zero positive x in the C^*-algebra generated by u_q such that $x\, \alpha_{\varphi(p)}(x) = 0$, i.e., $xu_px = 0$, which shows that $p \notin \Gamma_1(\alpha)$.

For the C^*-dynamical system in 2.4, every orbit in the pure states is of type II_1, or, for any pure state f, the representation

$$\int_G^{\oplus} \pi_f \circ \alpha_t \, dt$$

is quasi-equivalent to the GNS representation associated with the unique invariant (tracial) state.

§3. Type I Orbits

The following result generalizes part of [2], Theorem 2.1, where Condition 5 below is derived for some asymptotically abelian systems.

3.1. Theorem. Let A be a separable prime C^*-algebra and α a faithful continuous action of a separable locally compact abelian group G on A. Then the following conditions are equivalent:

1. There exists a $\delta \in (0, 1]$ such that for any $x, y \in A$ and any compact neighbourhood U of $0 \in \hat{G}$ it follows that

$$\sup \{||xay||: a \in A^{\alpha}(U)_1\} \geq \delta ||x|| \, ||y|| \, .$$

2. Condition 1 holds with $\delta = 1$.

3. There exists a $\delta \in (0, 1]$ such that for any $x \in A$ and any non-empty open subset U of $\hat{G}$ it follows that

$$\sup \{||xax^*||: a \in A^{\alpha}(U)_1\} \geq \delta ||x||^2 \, .$$

4. Condition 3 holds with $\delta = 1$, or $\Gamma_1(\alpha) = \hat{G}$.

5. There exists a pure state f of A such that π_f is faithful and for the representation ρ of A defined by

$$\rho = \int_G^{\oplus} \pi_f \circ \alpha_t \, dt$$

on $L^2(G, \mathcal{H}_f) = L^2(G) \otimes \mathcal{H}_f$, $\rho(A)''$ is of type I with center $L^{\infty}(G) \otimes 1$.

3.2. *Remarks.* A condition similar to (1) above was considered in [1] in the case G is compact. From the result there we may conjecture that (1) is equivalent to

1'. For any non-zero $x, y \in A$ and any neighbourhood U of $0 \in \hat{G}$ it follows that $xA^{\alpha}(U)\, y \neq (0)$.

A similar remark applies to Condition 3 or 4. Condition 5 was first considered in [6] and from the result there it follows that (5) is equivalent to

5'. There exists a pure state f of $A \times_{\alpha} G$ such that π_f is faithful and is covariant under the dual action $\hat{\alpha}$ of $\hat{G}$.

Proof of 3.1. If (5) is satisfied, then for $\mathcal{M}=\pi_f(A)''$ one has that $\mathcal{M}(p)=\mathcal{M}=B(\mathcal{H}_f)$ by 1.3, from which one immediately obtains the other conditions by using 1.1; e.g., to obtain (4) note that for any $p\in\hat{G}$ there is a bounded net $\{x_\mu\}$ in A of spectrum p such that $\|x_\mu\|\leq 1$ and $\pi(x_\mu)\to 1$, and hence one obtains that for any $x\in A$, $\|xx_\mu x^*\|$ converges to $\|x\|^2$.

It is trivial that (2) implies (1) and (4) implies (3) We shall show that (1) implies (3). Let p, $q\in \mathrm{Sp}(\alpha)$. Then for any compact neighbourhood U of $0\in\hat{G}$, Condition 1 implies that

$$A^\alpha(p+U)\,A^\alpha(U)\,A^\alpha(q+U)\neq(0)\,,$$

which then implies that $p+q\in\mathrm{Sp}(\alpha)$. Since α is faithful, it follows that $\mathrm{Sp}(\alpha)=\hat{G}$. Then it is straightforward to prove that (1) with δ implies (3) with δ^2 in place of δ.

Now we have to show that (3) implies (5). Let $\{u_n\}$ be a dense sequence in the unitaries of A (or $A+\boldsymbol{C}1$ if $A\not\ni 1$), and let $\{U_n\}$ be a countable basis for the open subsets of $\hat{G}$, where we suppose that each isolated point set appears infinitely often in $\{U_n\}$. We enumerate $\{(u_k, U_m): k, m=1, 2, \cdots\}$ and let $\{(u_n, U_n)\}$ be the resulting sequence. Let $\{I_n\}$ be a sequence of non-zero ideals of A such that for any non-zero ideal J of A there is an n with $J\supset I_n$. (This is possible because A is separable and prime.) Define T be the set of $x\in A$ such that $x\geq 0$, $\|x\|=1$, and $B(x)\equiv\{a\in A: xa=ax=a\}$ is non-zero. Note that $B(x)$ is a hereditary C^*-subalgebra of A and the open projection $p(x)$ corresponding to $B(x)$ is majorized by the (closed) spectral projection of x corresponding to the eigenvalue 1 (in A^{**}).

Fix $e_1\in T\cap I_1$ and let $p_1=p(e_1)$. Let

$$\lambda_1=\sup\{\|p_1u_1^*(b+b^*)u_1p_1\|,\ b\in A^\alpha(U_1)_1\}\ .$$

Then it follows from (3) that $\lambda_1\geq\delta$ since for $a\in T\cap B(e_1)$

$$\sup\{\|au_1^*bu_1a\|,\ b\in A^\alpha(U_1)_1\}\geq\delta$$

and $a\,p_1=a$. And then we find $a_1\in T$ and $b_1\in A^\alpha(U_1)_1$ such that $a_1e_1=a_1$ and

$$\sup\mathrm{Spec}(y_1)>\lambda_1-\delta/2$$

where $\mathrm{Spec}(y_1)$ is the spectrum of y_1 and $y_1=a_1u_1^*(b_1+b_1^*)u_1a_1$. Define a continuous function f_1 on $\boldsymbol{R}$ by

$$f_1(t)=\begin{cases}0 & t\leq 0\\ 1 & t\geq\lambda_1-\delta/2\end{cases}$$

and by linearity elsewhere, and note that $f_1(y_1)\in T$. Then as A is prime, $B(f_1(y_1))\cap I_2$ is a non-zero hereditary C^*-subalgebra of A. We choose $e_2\in T\cap B(f_1(y_1))\cap I_2$.

We repeat this procedure. Namely, if e_n is defined, let $p_n=p(e_n)$ and let

$$\lambda_n = \sup \{\|p_n u_n^*(b+b^*) u_n p_n\| : b\in A^\alpha(U_n)_1\} .$$

We find $a_n\in T\cap B(e_n)$ and $b_n\in A^\alpha(U_n)_1$ such that

$$\sup \mathrm{Spec}(y_n)>\lambda_n-\delta/2n$$

where $y_n=a_n u_n^*(b_n+b_n^*) u_n a_n$. Define a continuous function f_n on $\boldsymbol{R}$ by

$$f_n(t) = \begin{cases} 0 & t\leq 0 \\ 1 & t\geq \lambda_n-\delta/2n \end{cases}$$

and by linearity elsewhere, and choose $e_{n+1}\in T\cap B(f_n(y_n))\cap I_n$.

Since $f_n(y_n)\, e_n=f_n(y_n)$ and $e_{n+1}f_n(y_n)=e_{n+1}$, $\{e_n\}$ forms a decreasing sequence in T. Let f be a pure state of A such that $f(e_n)=1$ for all n; we assert that f satisfies the desired properties.

The representation π_f is faithful because $\|f\,|\,I_n\|=1$ for all n.

We want to show that $f(a_n^2)\to 1$. Let $z_n=u_n^*(b_n+b_n^*)\, u_n$ and compute:

$$\begin{aligned}\lambda_n-\delta/2n\leq f(a_n z_n a_n) &= f(a_n p_n z_n p_n a_n) \\ &\leq \|p_n z_n p_n\|\, f(a_n^2)\leq \lambda_n f(a_n^2) .\end{aligned}$$

Since $\lambda_n\geq\delta>0$ and $f(a_n^2)\leq 1$, this implies that $f(a_n^2)\to 1$.

Let $p\in\hat{G}$ and let u be a unitary of A (or $A+\boldsymbol{C}1$) and choose a subsequence $\{n_k\}$ such that $\|u_{n_k}-u\|\to 0$ and $\{U_{n_k}\}$ forms a basis for the neighbourhoods of p. Since

$$\mathrm{Re}\, f(a_n u_n^* b_n u_n a_n)>(\lambda_n-\delta/2n)/2 ,$$

$\lambda_n\geq\delta$, and $\|\pi_f(a_n)\,\Omega_f-\Omega_f\|\to 0$, any weak limit point Q of $\{\pi_f(b_{n_k})\}$ on $\mathcal{H}_f$ satisfies that

$$\mathrm{Re}\,\langle Q\pi_f(u)\,\Omega_f,\ \pi_f(u)\,\Omega_f\rangle\geq\delta/2 .$$

Note that $\|Q\|\leq 1$, and Q is the weak limit of a bounded net in $\pi_f(A)$ of spectrum p, i.e., $Q\in\mathcal{M}(p)$ where $\mathcal{M}=\pi_f(A)''=B(\mathcal{H}_f)$. Thus, we have shown that for any $p\in\hat{G}$ and any unit vector ξ of $\mathcal{H}_f$ there is a $Q\in\mathcal{M}(p)$ such that $\|Q\|\leq 1$ and

$$\mathrm{Re}\,\langle Q\xi,\ \xi\rangle\geq\delta/2 .$$

From section 1 it follows that $\mathcal{M}(p)$ is a weakly closed subspace of $\mathcal{M}$ such that $\mathcal{M}(0)\,\mathcal{M}(p)\,\mathcal{M}(0)\subset\mathcal{M}(p)$, $\mathcal{M}(p)\mathcal{M}(p)^*\subset\mathcal{M}(0)$, andth at $\mathcal{M}(0)$ is a von Neumann subalgebra of $\mathcal{M}$. Since the initial and final supports of $\mathcal{M}(p)$ are 1, the weak linear span of $\mathcal{M}(p)\,\mathcal{M}(p)^*$ and $\mathcal{M}(p)^*\,\mathcal{M}(p)$ respectively is $\mathcal{M}(0)$. Thus it follows that $\mathcal{M}(p)$ has an element $u(p)$ such that at least one of $u(p)\,u(p)^*$ and $u(p)^*\,u(p)$ is 1. If $u(p)\,u(p)^*=1$, then $\mathcal{M}(p)=u(p)\,\mathcal{M}(0)$ and if $u(p)^*\,u(p)=1$, then $\mathcal{M}(p)=\mathcal{M}(0)\,u(p)$. Since the central support of $e=u(p)\,u(p)^*$ is 1 in $\mathcal{M}(0)$, the reduction $\mathcal{M}(0)'\to\mathcal{M}(0)'e$ is an isomorphism. We define a map β_p of $\mathcal{M}(0)'$ by

$$\beta_p(Q)\,e=u(p)\,Qu(p)^*,\ Q\in\mathcal{M}(0)'\,.$$

To show that this is well-defined we have to prove that $u(p)\,Qu(p)^*\in\mathcal{M}(0)'e$. But this follows since Ad $u(p)$ is an isomorphism of $u(p)^*\,u(p)\,\mathcal{M}(0)\,u(p)^*\,u(p)$ onto $e\,\mathcal{M}(0)e$. It easily follows that β_p is an automorphism of $\mathcal{M}(0)'$ and that it does not depend on the choice of $u(p)$: if $v\in e\,\mathcal{M}(p)$, then $\beta_p(Q)\,v=vQ$. Since $\mathcal{M}(p)\,\mathcal{M}(q)\subset\mathcal{M}(p+q)$ and moreover the weak linear span of $\mathcal{M}(p)\mathcal{M}(q)$ is equal to $\mathcal{M}(p+q)$, it follows that β is an action of G on $\mathcal{M}(0)'$ (without any continuity asserted).

If β is trivial, then $\mathcal{M}(p)\subset\mathcal{M}(0)''=\mathcal{M}(0)$, i.e., $\mathcal{M}(p)$ is a weakly closed ideal of $\mathcal{M}(0)$. Since the support of $\mathcal{M}(p)$ is 1, it follows that $\mathcal{M}(p)=\mathcal{M}(0)$. Hence $\mathcal{M}(p)\ni 1$ for all $p\in\hat{G}$, which implies Condition 5 by 1.3.

Suppose that there exist a $p\in\hat{G}$ and a non-zero projection $E\in\mathcal{M}(0)'$ such that $\|E\,\beta_p(E)\|<\delta/2$. Then $\|E(1-\beta_p(E))\,E\|>1-\delta^2/4$, which implies $\|E(uu^*-uEu^*)\,E\|>1-\delta^2/4$ for $u=u(p)$. Hence

$$\|Eu(1-E)\|>(1-\delta^2/4)^{1/2}\,.$$

From this it follows that there exist unit vectors $\xi,\ \eta\in\mathcal{H}_f$ such that $(1-E)\xi=\xi$, $E\,\eta=\eta$, and

$$\langle u\,\xi,\eta\rangle = |\langle u\,\xi,\eta\rangle|\equiv\lambda>(1-\delta^2/4)^{1/2}\,.$$

Furthermore, there exists a $Q\in\mathcal{M}(-p)$ such that $\|Q\|\leq 1$ and

$$\langle Q\,\eta,\eta\rangle=|\langle Q\,\eta,\eta\rangle|\equiv\mu\geq\delta/2\,,$$

When one writes $u\,\xi=\lambda\,\eta+\zeta$, one obtains that $\langle\eta,\zeta\rangle=0$ and $\|\zeta\|<\delta/2$. Then $Qu\in\mathcal{M}(0)$ and

$$\begin{aligned}\mathrm{Re}\,\langle Qu\,\xi,\eta\rangle &= \mathrm{Re}\,\lambda\langle Q\,\eta,\eta\rangle+\mathrm{Re}\,\langle\zeta,Q^*\eta\rangle\\ &\geq\lambda\,\mu-\|\zeta\|\,\|Q^*\eta-\mu\,\eta\|\\ &>\frac{\delta}{2}\left(1-\frac{\delta^2}{4}\right)^{1/2}-\frac{\delta}{2}\,(1-\mu^2)^{1/2}\geq 0\end{aligned}$$

which is a contradiction since $EQu(1-E)=0$. Hence for any $p\in\hat{G}$ and any non-zero projection E of $\mathcal{M}(0)'$ one has that $\|E\,\beta_p(E)\|\geq\delta/2$. Hence it follows (cf. [4]) that β_p is inner and

$$\mathrm{Sp}(\beta_p)\subset\{e^{i\theta};\ |\theta|\leq 2\,\theta_0\} \tag{*}$$

where $\theta_0=\cos^{-1}\delta/2\in(0,\pi/2)$.

If G is compact, then $\mathcal{M}(p)$ is the weak closure of $\pi_f(A^{\alpha}(p))$ and so the family $\mathcal{M}(p)$ with $p\in\hat{G}$ generates $\mathcal{M}=B(\mathcal{H}_f)$. Thus β is an ergodic action of the discrete group $\hat{G}$ on $\mathcal{M}(0)'$ and $\mathcal{M}(0)'$ is a factor since β_p is inner for each $p\in\hat{G}$. If $v(p)$ is a unitary of $\mathcal{M}(0)'$ which implements β_p, one has that $\|Ev(p)E\|\geq\delta/2$ for any non-zero projection E of $\mathcal{M}(0)'$, and hence one may assume that

$$\mathrm{Spec}(v(p))\subset\{e^{i\theta}:\ |\theta|\leq\theta_0\} \tag{**}$$

by multiplying a complex number of modulus 1 if necessary (cf. [5]). Since $\beta_q(v(p))$ also implements β_p for any $q\in\hat{G}$, one has that $\beta_q(v(p))\,v(p)^*\in\boldsymbol{C}1$, and concludes that there exists a $t\in G$ such that $\beta_q(v(p))=\langle t, q\rangle\, v(p)$, $q\in\hat{G}$. For this to be compatible with (**), one must have $t=0$ or $v(p)\in\boldsymbol{C}1$ which implies that β_p is the identity map. Thus β is trivial and so one gets the conclusion.

Suppose that G equals $K\times Z^l\times R^m$ where K is a compact group and l, m are non-negative integers. First we apply the previous argument to the system $(A, K, \alpha|K)$ to obtain that the π_f restricted to $A^{\alpha|K}(0)$ is irreducible. Then we consider the system $(A^{\alpha|K}(0), Z^l\times R^m, \alpha|Z^l\times R^m)$ knowing that the properties of π_f described for (A, G, α) are still satisfied for this new system. Hence we now suppose that $G=Z^l\times R^m$. Let $N=\ker\beta$, which is a subgroup of $\hat{G}=T^l\times R^m$. Since $N=\{p\in\hat{G}: \mathcal{M}(p)\ni 1\}$, N is closed. If $N\neq\hat{G}$, then the quotient group $\hat{G}/N$ has an element of infinite order. Let $p\in\hat{G}$ be such that $np\notin N$ for any $n\neq 0$. It follows that any $\lambda\in\mathrm{Sp}(\beta_p)$ is of finite order in $\boldsymbol{T}$ and $\mathrm{Sp}(\beta_p)$ is discrete since otherwise $\mathrm{Sp}(\beta_p^n)$ must meet $\{e^{i\theta}:\ \pi\geq|\theta|>2\theta_0\}$ for some n, which contradicts (*). Thus $\mathrm{Sp}(\beta_p)$ is a finite set whose elements are all of finite order, which implies that $\mathrm{Sp}(\beta_p^n)$ is $\{1\}$ for some n, a contradiction. Hence $N=\hat{G}$ or β is trivial.

In general let $\mathcal{G}$ be the set of compactly generated open subgroups of G, i.e., $H\in\mathcal{G}$ if there is an open neighbourhood U of $0\in G$ such that $\bar{U}$ is compact, $U=-U$, and H is generated by U as a group. $\mathcal{G}$ is a directed set under inclusion. For $H\in\mathcal{G}$, $H^{\perp}$ is a compact subgroup of $\hat{G}$ since $H^{\perp}$ is the dual of the discrete group G/H. For any open neighbourhood V of $0 \in \hat{G}$ there exists an $H \in \mathcal{G}$ such that $H^\perp \subset V$ (by the definition of the topology of $\hat{G}$).

Let $p \in \hat{G}$. Since H is of the form $K \times Z^l \times R^m$ for $H \in \mathcal{G}$, it follows that $\mathcal{M}(p+H^\perp) \ni 1$ for any $H \in \mathcal{G}$. Since $\{p+H^\perp+V: H \in \mathcal{G}, V$ is an open neighbourhood of $0 \in \hat{G}\}$ forms a basis for the neighbourhoods of p, it easily follows that $\mathcal{M}(p) \ni 1$, and thus β is trivial. This concludes the proof.

3.3. Theorem. Let A be a separable prime C^*-algebra and α a continuous action of a separable locally compact abelian group G on A such that $\Gamma_1(\alpha) = \hat{G}$. Let H be a closed discrete subgroup of G and let $N = H^\perp$. Then there exists a pure state φ of A such that π_φ is faithful and for the representation ρ_φ of A defined by

$$\rho_\varphi = \int_G^\oplus \pi_\varphi \circ \alpha_t \, dt$$

on $L^2(G, \mathcal{H}_\varphi) = L^2(G) \otimes \mathcal{H}_\varphi$, $\rho_\varphi(A)''$ is of type I with center $\{p: p \in N\}'' \otimes 1$.

Proof. By 3.1 there is a pure state f of A such that π_f is faithful and $\rho_f(A)'' \cap \rho_f(A)' = L^\infty(G) \otimes 1$. We define a representation Φ on $l^2(H, \mathcal{H}_f)$ of the crossed product $A \times_\beta H$ with $\beta = \alpha \mid H$ by

$$(\Phi(a)\,\xi)(t) = \pi_f(\alpha_t(a))\,\xi(t), \quad a \in A\,,$$
$$(\bar{\Phi}(\lambda(s))\,\xi)(t) = \xi(t+s), \quad s \in H\,,$$

for $\xi \in l^2(H, \mathcal{H}_f)$ where λ is the canonical unitary representation of H in the multiplier algebra $M(A \times_\beta H)$ and $\bar{\Phi}$ is the unique extension of Φ to $M(A \times_\beta H)$. Then Φ is a faithful irreducible representation of $A \times_\beta H$ since π_f is faithful and $\pi_f \circ \alpha_s$ is disjoint from π_f for $s \in H \setminus \{0\}$. (In particular, $A \times_\beta H$ is prime.) For $p \in N$ there is a bounded net $\{x_\mu\}$ in A of spectrum p such that $\|x_\mu\| \leq 1$ and $\lim \pi_f(x_\mu) = 1$. Since $\|\alpha_s(x_\mu) - x_\mu\| \to 0$ for $s \in H$, one obtains that $\lim \Phi(x_\mu) = 1$. Thus it follows that for any neighbourhood U of $p \in N$ and any $x \in A \times_\beta H$,

$$\sup\{\|x(a+a^*)\,x^*\|: \ a \in A^\alpha(U)_1\} = 2\|x\|^2\,.$$

Consider the dual action $\hat{\beta}$ of the compact abelian group $\hat{H}$ on the prime C^*-algebra $A \times_\beta H$. By 3.4 below one can apply [1] to this system to conclude that there is a faithful irreducible representation π of $A \times_\beta H$ such that the restriction of π to A is also irreducible. Let $\mathcal{M} = \pi(A \times_\beta H)''$. In the notation in Section 1 one has that $\mathcal{M}(s) = \mathcal{M}$ for $s \in H$. Hence it follows that for any $x \in A \times_\beta H$ and $s \in H$,

$$\sup \{\|x(a+a^*)x^*\|: \ a\in A\lambda(s)\} = 2\|x\|^2. \qquad (**)$$

(In particular $\Gamma_1(\hat{\beta})=H$ for the dual action $\hat{\beta}$ of $\hat{H}$ on $A\times_\beta H$.)

We now apply a procedure similar to the one in the proof of 3.1 by using (*) and (**) simultaneously (instead of the condition:

$$\sup \{\|x(a+a^*)x^*\|: \ a\in A^\alpha(U)_1\} \geq \delta\|x\|^2$$

for any non-empty open set U of $\hat{G}$ and any $x\in A$). Since the procedure is quite similar, we omit the details. The result is that one obtains a pure state f of $A\times_\beta H$ such that π_f is faithful and for any unit vector $\xi\in\mathcal{H}_f$, any $p\in N$, and any $s\in H$, there are a bounded net $\{a_\mu\}$ in A of spectrum p and a bounded net $\{b_\mu\}$ in A such that $\|a_\mu\|\leq 1$, $\|b_\mu\|\leq 1$, and

$$\lim \langle\pi_f(a_\mu)\xi, \xi\rangle = 1, \quad \lim \langle\pi_f(b_\mu\lambda(s))\xi, \xi\rangle = 1\,.$$

From the second estimates one concludes that $\pi_f(A)''=\pi_f(A\times_\beta H)''$ in exactly the same way as in 3.1. Thus the restriction π of π_f to A is irreducible. On setting $\mathcal{M}=\pi(A)''$ for the system (A, G, α), the first estimates imply that $\mathcal{M}(p)\ni 1$ for $p\in N$. (Since we now know that the value corresponding t6 λ_n in the proof of 3.1 is 2, we can conclude that $\|E\,\beta_p(E)\|=1$ for any non-zero projection E of $\mathcal{M}(0)'$, which implies that β_p is the identity.) Since α_s is weakly inner in π for $s\in H$, if $\mathcal{M}(p)\ni 1$ then $\langle s, p\rangle=1$ for $s\in H$, i.e., $p\in H^\perp$. Thus it follows that $\mathcal{M}(p)\ni 1$ if, and only if $p\in N$. For the representation ρ of A defined by

$$\rho = \int_G^\oplus \pi\circ\alpha_t \, dt\,,$$

it follows by 1.1 that $\rho(A)''\cap\rho(A)' = \{p: p\in N\}''\otimes 1$. Thus $\rho(A)''$ is of type I and this completes the proof with $\varphi=f\,|\,A$.

3.4. Lemma. Let A be a prime C^*-algebra and β an action of a discrete group H on A such that β_t is properly outer for each $t\in H\setminus\{0\}$. Let $A\times_\beta H$ be the reduced crossed product of A by β. Then for any non-zero x, $y\in A\times_\beta H$ it follows that $xAy\neq(0)$.

Proof. There is a faithful conditional expectation Φ of $A\times_\beta H$ onto A such that $\Phi(a)=a$ for $a\in A$, and $\Phi(a\,\lambda(s))=0$ for $a\in A$, $s\in H\setminus\{0\}$. Let $x=\sum x(s)\,\lambda(s)$, $y=\sum y(s)\,\lambda(s)$ be positive elements of $A\times_\beta H$ such that the summations are finite. We shall show that

$$\sup \{\|xay\|: a\in A_1\} \geq \|\Phi(x)\|\,\|\Phi(y)\|\,. \qquad (*)$$

Since those elements x, y are dense in the positive part of $A\times_\beta H$, this is enough to conclude that (*) holds for any positive $x, y\in A\times_\beta H$. From this we get the conclusion.

To prove (*) we proceed as in [5]. First for any $\varepsilon>0$ one finds positive $e, f\in A$ such that $||e||=1=||f||$, and

$$||exe-ex(0)e||<\varepsilon, \quad ||ex(0)e||>(1-\varepsilon)||x(0)||,$$
$$||fyf-fy(0)f||<\varepsilon, \quad ||fy(0)f||>(1-\varepsilon)||y(0)||.$$

Then one finds a $b\in A$ such that $||b||=1$, and

$$||ex(0)\,ebfy(0)f||>(1-\varepsilon)||ex(0)e||\;||fy(0)f||.$$

Thus one obtains that for $a=ebf\in A$,

$$||xay||\geq||exebfyf||>(1-\varepsilon)^3||x(0)||\;||y(0)||-2\varepsilon.$$

Since $\Phi(x)=x(0)$ etc., this concludes the proof.

3.5. Theorem. Let A be a separable prime C^*-algebra and α a faithful continuous action of a (separable) compact abelian group G on A. Let H be an arbitrary closed subgroup of G. Then the following conditions are equivalent:

1. A^G is prime and there exists a G-invariant pure state f of A such that π_f is faithful.

2. A^H is prime and there exists an H-invariant pure state φ of A such that π_φ is faithful and $\rho_\varphi(A)''\cap\rho_\varphi(A)'=\{p\colon p\in H^\perp\}''\otimes 1$, where $A^H=A^{\alpha|H}(0)$ etc. and ρ_φ is defined as in 3.3.

Proof. Suppose (1). By using the state f in (1) one can define a representation of $A\times_\alpha G$ by extending π_f on the same space $\mathcal{H}_f$. Hence it follows from 3.1 that $\Gamma_1(\hat{\alpha})=G$ for the dual action $\hat{\alpha}$ on $A\times_\alpha G$. In the same way for the dual action $\hat{\beta}$ on $A\times_\beta H$ with $\beta=\alpha|H$ it follows that $\Gamma_1(\hat{\beta})=H$, or rather more: For any $x\in A\times_\beta H$ and any non-empty open subset U of H,

$$\sup\{||x(a+a^*)x^*||\colon a\in(A\times_\beta H)^{\hat{\beta}}(U)_1\}=2||x||^2.$$

On the other hand one can conclude as in the proof of 3.3 that for any $x\in A\times_\beta H$ and any neighbourhood U of $p\in H^\perp$,

$$\sup\{||x(a+a^*)x^*||\colon a\in A^\alpha(U)_1\}=2||x||^2.$$

Using these two conditions we proceed in exactly the same way as in 3.3 to obtain a pure state f of $A\times_\beta H$ such that π_f is faithful, $\pi_f(A)''=\pi_f(A\times_\beta H)''$, and with $\mathcal{M}=\pi_f(A)''$ for the system (A, G, α), $\mathcal{M}(p)\ni 1$ if, and only if $p\in H^{\perp}$. Since $\pi_f|A$ is β-covariant there is a unit vector in $\mathcal{H}_f$ which defines a β-invariant state φ of A. Since $\pi_f|A=\pi_\varphi$, φ has the desired properties.

Suppose (2). It follows from [1] that α_s is properly outer for each $s\in H\setminus\{0\}$. If $s\notin H$, then α_s induces an automorphism of $\rho_\varphi(A)''$ (with φ in (2)) which is non-trivial on the center, and so it is properly outer. Thus α_s is properly outer for any $s\in G\setminus\{0\}$. Now we shall show that A^α is prime, concluding the proof by [1].

We restrict π_φ to $B=A^H$, which we denote by π and consider the action β of G/H on B induced by α. Let $\mathcal{M}=\pi(B)''$ for $(B, G/H, \beta)$. Then (2) implies that $\mathcal{M}(p)\ni 1$ for $p\in(G/H)^\wedge=H^{\perp}$.

Let u be the unitary representation of H on $\mathcal{H}_\varphi$ defined by $u_s\pi_\varphi(x)\Omega_\varphi=\pi_\varphi\circ\alpha_s(x)\Omega_\varphi$, $x\in A$, and let E_p be the spectral projection of u corresponding to the character $p\in\hat{H}$. Then since $\pi_\varphi(B)''=\{E_p\colon p\in H\}'$, $\pi_p=\pi|E_p\mathcal{H}_\varphi$ is an irreducible representation of B for any $p\in\hat{H}$. Since the condition that $\mathcal{M}(p)\ni 1$ for $p\in H^{\perp}$ is inherited by π_p, $p\in\hat{H}$, it follows from 1.3 that $\pi_p(B)''=\pi_p(B^\beta)''$. Thus the family $\{\pi_p\colon p\in\hat{H}\}$ of irreducible representations of B satisfies that $\pi_p(B^\beta)''=\pi_p(B)''$ and $\bigoplus_{p\in\hat{H}}\pi_p$ is faithful. From this it follows that $xB^\beta y\neq(0)$ for any non-zero x, $y\in B$ since B is prime. Thus in particular $B^\beta=A^G$ is prime.

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