

# On Confluence of Singular Fibers in Elliptic Fibrations

By

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## Introduction

In a deformation of elliptic surfaces one will often observe several singular fibers  $F_1, F_2, \dots, F_r$  of nearby surfaces flowing together into one singular fiber  $F$  at some special member of the deformation. Such a phenomenon is called a confluence of singular fibers in this article. We associate the monodromy matrices  $M_1, M_2, \dots, M_r, M$  with  $F_1, F_2, \dots, F_r, F$  in the usual way. If the paths to define  $M_i, M$  are suitably chosen, then one has the identity  $M = M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(r)}$  for some permutation  $\sigma$  of indices. In particular we have

$$\mathrm{tr}(M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(r)}) = \mathrm{tr}(M)$$

which we call the trace equation of the confluence. Since, for elliptic surfaces, the trace is a strong invariant in determining conjugacy classes of local monodromies, this equation naturally controls the confluence to a great extent. In case where  $r=3$  and  $F_1, F_2, F_3$  are of Types  $I_a, I_b, I_c$ , it leads to a Diophantine equation of the form

$$bcx^2 + acy^2 + abz^2 - abcxyz = d \quad (d: = 2 - \mathrm{tr}(M))$$

which we call the generalized Markov-Mordell equation. Here  $x, y, z$  are the  $SL(2, \mathbb{Z})$ -invariants associated with  $M_1, M_2, M_3$ ; they are unique up to the even sign changes of  $x, y, z$ . (See Section 3 for the definition.) These invariants of course depend on the choice of paths taken for the introduction of  $M_1, M_2, M_3$ . Since the ambiguity consisting in this kind of choice is described by the braid group of three strings, this group operates on the solution space of the Diophantine equation. We let  $\hat{B}$  denote the transformation

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group generated by this and the even sign changes. If one succeeds in giving a system of representatives to the  $\widehat{B}$ -orbits in the space of (geometric) solutions of the trace equation, then we can say that one has given the canonical forms to the monodromy matrices  $M_1, M_2, M_3$  in this type of confluences. We can in fact recover the matrices from the invariants  $x, y, z$  almost uniquely up to the  $SL(2, \mathbb{Z})$ -conjugation, provided that  $(x, y, z)$  satisfies some additional conditions to be geometric. In this article we attempt to classify  $(M_1, M_2, M_3)$  up to the ambiguity arising from the choice of paths and the  $SL(2, \mathbb{Z})$ -conjugation, through the arithmetic study of the Diophantine equation above. The result is immediately generalized to the case where one of  $M_i$  is of Type  $I_\alpha^*$  and the others remain to be of Types  $I_\beta, I_\gamma$  for some  $\alpha, \beta, \gamma$ . We remark that, as a byproduct, we proved the finiteness theorem for the  $\widehat{B}$ -orbits of *regular* solutions of the trace equation. We also note that the theory exceptionally well behaves in the case where  $r=3$  and that the generalization to the case where  $r>3$  does not seem easy at all (see Section 8).

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### §1. Preliminaries

By Kodaira [2] the local monodromies around singular fibers of elliptic surfaces are classified as some non-trivial conjugacy classes in  $SL(2, \mathbb{Z})$ . For them the absolute value of the trace is not greater than 2, but the converse is not true. The reason is that we always choose paths on the base curve *counter-clockwise* to determine the monodromies. To state in an invariant way what classes they are, we need the following two lemmas.

**Lemma 1.1.** *Let*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

*be an element of  $SL(2, \mathbb{Z})$  such that  $|\alpha + \delta| \leq 2$ . Then we have  $\beta\gamma \leq 0$ ;  $\beta\gamma = 0$  implies  $|\alpha + \delta| = 2$ .*

*Proof.* Note that we have

$$(1.1) \quad \begin{aligned} (\alpha - \delta)^2 + 4\beta\gamma &= (\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma) \\ &= (\alpha + \delta)^2 - 4 \leq 0. \end{aligned}$$

In particular we have  $\beta\gamma \leq -(\alpha - \delta)^2/4 \leq 0$ . The last statement is obvious.

**Lemma 1.2.** *Suppose that we are given an identity of the form between elements of  $SL(2, \mathbb{Z})$ :*

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If  $|\alpha + \delta| (= |\bar{\alpha} + \bar{\delta}|) \leq 2$ , then we have

$$(1.2) \quad \beta\bar{\beta} \geq 0, \quad \gamma\bar{\gamma} \geq 0, \quad \beta\bar{\gamma} \leq 0, \quad \gamma\bar{\beta} \leq 0.$$

In the case where  $|\alpha + \beta| \leq 1$ , we have the strict inequalities in (1.2). We have the four equalities in (1.2) if and only if  $\beta = \gamma = 0, \alpha = \delta = \pm 1$ .

*Proof.* By a direct computation we obtain

$$(1.3) \quad 4\beta\bar{\beta} = \{2\beta a + (\delta - \alpha)b\}^2 - \{(\delta - \alpha)^2 + 4\beta\gamma\} b^2$$

But, by (1.1), the right hand side is non-negative. The other inequalities in

Table 1

Type	$A$ is similar to	Sign condition
$I_b$	$\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$	$\beta > 0$ or $\gamma < 0$
$I_b^*$	$\begin{pmatrix} -1 & -b \\ & -1 \end{pmatrix}$	$\beta \leq 0, \gamma \geq 0$
$II$	$\begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix}$	$\beta > 0$ ( $\gamma < 0$ )
$II^*$	$\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}$	$\beta < 0$ ( $\gamma > 0$ )
$III$	$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\beta > 0$ ( $\gamma < 0$ )
$III^*$	$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$	$\beta < 0$ ( $\gamma > 0$ )
$IV$	$\begin{pmatrix} & 1 \\ -1 & -1 \end{pmatrix}$	$\beta > 0$ ( $\gamma < 0$ )
$IV^*$	$\begin{pmatrix} -1 & -1 \\ & 1 \end{pmatrix}$	$\beta < 0$ ( $\gamma > 0$ )

(1.2) follow similarly. The last assertion of Lemma 1.1 proves the second statement. The last assertion is also obvious.

From these lemmas we obtain several useful sign conditions for non hyperbolic conjugacy classes of  $SL(2, \mathbb{Z})$ . For example, that either  $\beta > 0$  or  $\gamma < 0$  holds for  $A$  such that  $|\alpha + \delta| \leq 2$  is an invariant property under the conjugation. We now pick up here only the arithmetic part of the table in [2] with some necessary conditions for local monodromies which we need later. We let the matrix  $A$  above stand for the monodromy in Table 1.

If we have  $\alpha + \delta = 2$  and  $\beta > 0$  or  $\gamma < 0$ , then  $A$  is of Type  $I_b$  for some  $b = 1, 2, \dots$ . If we have  $\alpha + \delta = -2$  and  $\beta \leq 0$  and  $\gamma \geq 0$ , there  $A$  is of Type  $I_b^*$  for some  $b = 0, 1, 2, \dots$ . For the other types, the trace  $\alpha + \delta$  and the sign of  $\beta$  (or  $\gamma$ ) completely determine the conjugacy class to which  $A$  belongs.

### §2. Confluence of Singular Fibers in Deformations of Elliptic Surfaces

Let  $S$  be a non-singular elliptic surface with the base curve  $\mathcal{A}$  and the projection  $\pi: S \rightarrow \mathcal{A}$ . Let  $\mathcal{E}$  be the set of points over which the singular fibers lie and  $\mathcal{A}^*$  denote the complement  $\mathcal{A} - \mathcal{E}$ . We choose a point  $p_0$  on  $\mathcal{A}^*$  as the reference point for the fundamental group  $\pi_1(\mathcal{A}^*) = \pi_1(\mathcal{A}^*, p_0)$ . We then obtain the monodromy representation  $\pi_1(\mathcal{A}^*) \rightarrow \text{Aut}(H^1(\pi^{-1}(p_0), \mathbb{Z}), \langle, \rangle)$  where  $\langle, \rangle$  is the intersection form on the cohomology group. To get a homomorphism of  $\pi_1(\mathcal{A}^*)$  into  $SL(2, \mathbb{Z})$  we still have to choose a symplectic isomorphism of  $H^1(\pi^{-1}(p_0), \mathbb{Z})$  to the lattice on which  $SL(2, \mathbb{Z})$  acts. We set

$$L := M_{(2,1)}(\mathbb{Z}) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix}; p, q \in \mathbb{Z} \right\}$$

$$L^* := M_{(1,2)}(\mathbb{Z}) = \{(p, q); p, q \in \mathbb{Z}\} .$$

The matrix multiplication  $L^* \times L \rightarrow \mathbb{Z}$  makes  $L, L^*$  the dual spaces to each other. We set further

$$u^* := (-q, p) \quad \text{for } u = \begin{pmatrix} p \\ q \end{pmatrix} \in L$$

to introduce the symplectic form

$$\langle u, v \rangle = u^* \cdot v \quad u, v \in L .$$

We have thus  $SL(2, \mathbb{Z}) = \text{Aut}(L, \langle, \rangle)$ . Now let us choose a symplectic isomorphism

$$\alpha_0: (L, \langle, \rangle) \rightarrow H^1(\pi^{-1}(p_0), \mathbb{Z})$$

to induce the isomorphism  $\alpha_0^*: \text{Aut}(H^1(\pi^{-1}(p_0), \mathbb{Z}), \langle, \rangle) \rightarrow SL(2, \mathbb{Z})$ . By composing the above with this we obtain the monodromy representation in the usual sense:

$$\rho: \pi_1(\mathcal{A}^*) \rightarrow SL(2, \mathbb{Z}).$$

Now let  $p \in \Sigma$ . If one chooses a closed path  $\gamma$  on  $\mathcal{A}^*$  issuing from  $p_0$ , without self-intersection and encircling  $p$  once counterclockwise, then  $\rho([\gamma])$  is called the local monodromy (with respect to  $\gamma$ ) associated with the singular fiber  $\pi^{-1}(p)$  ( $[\gamma]$ : the class of  $\gamma$  in  $\pi_1(\mathcal{A}^*)$ ). We use the following notation

$$M_\gamma := \rho([\gamma]).$$

The ambiguity in the choice of isomorphism  $\alpha_0$  exactly corresponds to the conjugation in  $SL(2, \mathbb{Z})$  in the expression of  $M_\gamma$ . Different choices of the path  $\gamma$  are also absorbed in a similar way in the conjugation. Thus the conjugacy class of  $M_\gamma$  is intrinsically associated with the singular fiber  $\pi^{-1}(p)$ , which is the precise definition of the local monodromy assumed in the previous section.

Now suppose that we are given a commutative diagram of the form

$$\begin{array}{ccc} & & E \\ & \swarrow \pi & \downarrow \eta \\ D & & T \\ & \searrow \delta & \end{array}$$

where  $E, D, T$  are complex analytic manifolds of dimension  $n+2, n+1, n$  and  $\pi, \delta, \eta$  are proper surjections. We set

$$S_t := \eta^{-1}(t) \quad \mathcal{A}_t = \delta^{-1}(t) \quad t \in T.$$

**Definition 2.1.** The data  $(E, D, T; \pi, \delta, \eta)$  above is called an  $n$ -parameter deformation of elliptic surfaces if the following conditions are satisfied:

- (i)  $\delta$  and  $\eta$  are locally trivial  $C^\infty$ -fibrations.
- (ii) The restriction  $\pi: S_t \rightarrow \mathcal{A}_t$  is an elliptic surface.

We assume (i) and (ii) and denote by  $\Sigma_t$  the set of points on  $\mathcal{A}_t$  over which the singular fibers of  $S_t$  lie. We set

$$\begin{aligned} \Sigma &:= \bigcup_{t \in T} \Sigma_t & \mathcal{A}_t^* &:= \mathcal{A}_t - \Sigma_t \\ D^* &:= D - \Sigma. \end{aligned}$$

$\Sigma$  is obviously a closed analytic subset of  $D$  and the restriction  $\pi: \Sigma \rightarrow T$  is a finite proper map. Since we are interested in local deformations, we assume that  $T$  is contractible. This assures us that there is a  $C^\infty$ -section of  $\delta$  into  $D^*$

$$p_0: T \rightarrow D^* \quad \delta \circ p_0 = id_T$$

and that we can identify, once for all, the stalks of  $R^1\pi_*\mathbb{Z}$  along  $p_0(T)$  with the standard lattice  $(L, \langle, \rangle)$  in a flat manner. Thus we obtain the parametrized monodromy representation:

$$(2.1) \quad \rho_t: \pi_1(D_t^*, p_0(t)) \rightarrow SL(2, \mathbb{Z}) \quad t \in T.$$

Now take two points  $t_0, t$  on the parameter space  $T$  and choose points  $q_0 \in \Sigma_{t_0}; q_1, q_2, \dots, q_r \in \Sigma_t$ .

**Definition 2.2.** We say that the singular fibers  $\pi^{-1}(q_i) \ i=1, 2, \dots, r$  of  $S_t$  flow together to the singular fiber  $\pi^{-1}(q_0)$  if there are closed paths  $\gamma_0$  on  $D_{t_0}$  and  $\gamma_1, \gamma_2, \dots, \gamma_r$  on  $D_t$  for which the following three conditions are satisfied:

- (i)  $\gamma_i$  does not have self-intersections and it encircles only  $q_i$  once counter-clockwise for  $i=0, 1, 2, \dots, r$ .
- (ii) There is a permutation  $\sigma$  of  $\{1, 2, \dots, r\}$  such that the composition  $\gamma_{\sigma(r)} \cdots \gamma_{\sigma(2)} \gamma_{\sigma(1)}$  can be deformed to a closed path in  $D_t$  without self-intersection and surrounding all  $q_1, q_2, \dots, q_r$  in its inside.
- (iii) This composed path and  $\gamma_0$  belong to the same class in the group  $\pi_1(D^* - \Sigma, p_0(T))$ .

We say then that an  $r$ -branched confluence occurs at  $q_0$  or at  $\pi^{-1}(q_0)$ . Its type is defined to be  $(C_1, C_2, \dots, C_r; C_0)$  where  $C_0, C_i \ (i=1, 2, \dots, r)$  denote the conjugacy classes of the monodromy matrices  $M_0 = \rho_{t_0}([\gamma_0])$ ,  $M_i = \rho_t([\gamma_i]) \ (i=1, 2, \dots, r)$ .  $M_i \ (i=0, 1, 2, \dots, r)$  are called the monodromy matrices of the confluence.

Since the direct image sheaf  $R^1\pi_*\mathbb{Z}$  is locally constant over  $D^* = D - \Sigma$ , we have the monodromy representation  $\pi_1(D^*, p_0(T)) \rightarrow SL(2, \mathbb{Z})$ . The representation  $\rho_t$  in (2.1) is induced from this by the inclusion  $(D_t^*, p_0(t)) \hookrightarrow (D^*, p_0(T))$ . Therefore the condition (iii) in Definition 2.2 implies the following important identity:

$$(2.2) \quad M_0 = M_{\sigma(1)} M_{\sigma(2)} \cdots M_{\sigma(r)}.$$

### §3. Generalized Markov-Mordell Equation as Trace Equation

In this section we restrict ourselves to confluences for types of the form

$(I_a, I_b, I_c; C)$ . We will obtain a Diophantine equation which holds between some  $SL(2, \mathbb{Z})$ -invariants associated with the confluence and which is very useful to control the monodromy matrices. We will further describe the action of the braid group arising from the ambiguity of the choice of the paths  $\gamma_1, \gamma_2, \gamma_3$ . We begin with the following observation. Recall that  $L$  is the lattice of integral column 2-vectors with the symplectic form  $\langle u, v \rangle = u^* \cdot v$ . Through the natural identification  $\text{End}_{\mathbb{Z}}(L) = L \otimes L^*$ , the matrix product  $u \cdot v^* = u \otimes v^*$  means the endomorphism  $L \ni w \mapsto \langle v, w \rangle u \in L$ . (One should not confuse  $u \cdot v^*$  and  $v^* \cdot u = \langle v, u \rangle$ .) The composition rule for such endomorphisms is very simple:

$$(3.1) \quad (u \otimes v^*) \circ (u' \otimes (v')^*) = \langle v, u' \rangle u \otimes (v')^*$$

in particular we have  $(u \otimes u^*)^2 = 0$ . Thus the matrix

$$M(u, b) := I + bu \otimes u^* = \begin{pmatrix} 1 - bpq & bp^2 \\ -bq^2 & 1 + bpq \end{pmatrix}$$

is in  $SL(2, \mathbb{Z})$  and it is of Type  $I_b$  provided that  $u$  is *primitive* i.e. not divisible by any  $2 \leq m \in \mathbb{Z}$ . As is easily proved, any matrix in the conjugacy class  $I_b$  is written in this form. (Geometrically  $M(u, b)$  is nothing other than the Picard-Lefschetz monodromy associated with the vanishing cycle  $\pm bu^*$ .) We note that

$$(3.2) \quad M(u, b) = M(-u, b).$$

By  $\text{tr}(u \otimes v) = v^* \cdot u = \langle v, u \rangle$  and (3.1), the following formula is immediately derived:

$$(3.3) \quad \begin{aligned} &\text{tr}(M(u, a) M(v, b) M(w, c)) \\ &= 2 - bc \langle v, w \rangle^2 - ac \langle w, u \rangle^2 - ab \langle u, v \rangle^2 \\ &\quad + abc \langle v, w \rangle \langle w, u \rangle \langle u, v \rangle. \end{aligned}$$

Now suppose that we are given a confluence of singular fibers of type  $(I_a, I_b, I_c; C)$  with the monodromy matrices  $(M_1, M_2, M_3; M)$ . Let  $\sigma$  be the permutation of  $\{1, 2, 3\}$  such that  $M_{\sigma(1)} M_{\sigma(2)} M_{\sigma(3)} = M$ . We write  $M_i$  in the following form:

$$M_1 = M(u, a) \quad M_2 = M(v, b) \quad M_3 = M(w, c)$$

and we then introduce  $SL(2, \mathbb{Z})$ -invariants:

$$x := \text{sgn}(\sigma) \langle v, w \rangle$$

$$y := \text{sgn}(\sigma)\langle w, u \rangle$$

$$z := \text{sgn}(\sigma)\langle u, v \rangle.$$

By (3.3) we now obtain the fundamental equation:

$$(3.4) \quad bcx^2 + acy^2 + abz^2 - abcxyz = d$$

where we have put

$$d := 2 - \text{tr}(M).$$

Note that the sign changes of  $u, v, w$  do not affect the matrices  $M_1, M_2, M_3$  by (3.2) but the  $SL(2, \mathbb{Z})$ -invariants  $x, y, z$ ; they induce the even sign changes of  $x, y, z$ , which further induce transformations among the solutions of the Diophantine equation (3.4). This ambiguity is easy to manage. But recall that, in Definition 2.2, the matrices  $M_i$  ( $i=1, 2, \dots, r$ ) depend strongly on the choice of the paths  $\gamma_i$  ( $i=1, 2, \dots, r$ ). Thus  $x, y, z$  should also depend on the choice of paths  $\gamma_1, \gamma_2, \gamma_3$  satisfying the condition (ii) in Definition 2.2, assuming that the confluence we are discussing arises from the situation of the definition with  $r=3$ . Since the braid group of three strings, denoted here by  $B=B_3$ , controls this kind of path-choices,  $B$  naturally operates on the set of solutions of (3.4). We will now show that this operation is nothing other than the classical elementary operations of the Markov chain [3], [4]. Since  $M_{\sigma(1)}M_{\sigma(2)}M_{\sigma(3)}$  and  $M_1M_2M_3$  are conjugate in  $SL(2, \mathbb{Z})$  if  $\sigma$  is even, it suffices to discuss only the cases where  $\sigma$  is trivial or the transposition of (1, 2). Assume first that  $\sigma$  is trivial and consider the following braid action:

$$(\gamma_1, \gamma_2, \gamma_3) \mapsto (\gamma_1^{-1} \gamma_2 \gamma_1, \gamma_1, \gamma_3).$$

This transforms  $(M_1, M_2, M_3; M)$  to  $(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3; \tilde{M}) := (M_1, M_1M_2M_1^{-1}, M_3; M)$ . (See Figure 1.)

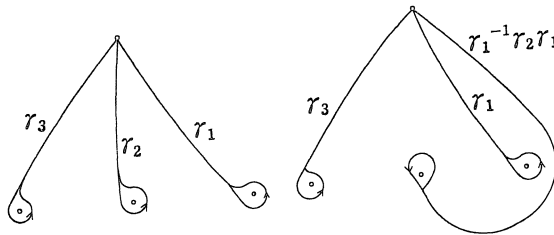


Figure 1

Thus, as the transform of  $(u, v, w)$ , we can take



$$\tilde{u} = u, \quad \tilde{v} = v + a\langle u, v \rangle u, \quad \tilde{w} = w.$$

Since  $\tilde{M}_2\tilde{M}_1\tilde{M}_3 = \tilde{M}$ , the transform  $(\tilde{x}, \tilde{y}, \tilde{z})$  of  $(x, y, z)$  is given by

$$\begin{aligned} \tilde{x} &= -\langle v, w \rangle - a\langle u, v \rangle \langle u, w \rangle = ayz - x \\ \tilde{y} &= -y, \quad \tilde{z} = -z \quad (\because \langle u, u \rangle = 0). \end{aligned}$$

By composing this with the even sign change  $y \rightarrow -y, z \rightarrow -z$ , we obtain the classical elementary transformation:

$$(x, y, z) \mapsto (ayz - x, y, z).$$

The other elementary transformations are

$$\begin{aligned} (x, y, z) &\mapsto (x, bxz - y, z) \\ (x, y, z) &\mapsto (x, y, cxy - z) \end{aligned}$$

and they are also obtained by the action of the other generators of  $B$  composed with suitable sign changes. We have thus proved:

**Proposition 3.1.** *The group generated by the even sign changes and the elementary transformations above exactly corresponds to the ambiguity coming from the sign changes of  $u, v, w$  and the action of the braid group  $B = B_3$  on the path-triplets  $(r_1, r_2, r_3)$ .*

We denote the group in the proposition by  $\hat{B}$ ; we regard it as a transformation group on the set of (integral) solutions of (3.4). The main problem is now to ask whether there are only finitely many  $\hat{B}$ -orbits in the solution space. But, in our geometric situation, we do not have to observe all solutions of (3.4). Suppose that  $(x, y, z)$  is induced from primitive vectors:

$$u = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad v = \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \quad w = \begin{pmatrix} p_3 \\ q_3 \end{pmatrix}.$$

This means that  $(x, y, z)$  are, up to sign, the (2,2)-minors of the matrix

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}.$$

Since  $p_i$  and  $q_i$  are coprime for  $i=1, 2, 3$ , we see that the following condition is fulfilled:

$$(3.5) \quad x \in \mathbb{Z}y + \mathbb{Z}z, \quad y \in \mathbb{Z}x + \mathbb{Z}z, \quad z \in \mathbb{Z}x + \mathbb{Z}y.$$

We call a solution of (3.4) for which (3.5) is fulfilled *regular*. The regularity is an invariant notion under  $\widehat{B}$ . As we will see later, there are only a finite number of  $\widehat{B}$ -orbits of regular solutions. The condition (3.5) is not sufficient to guarantee the existence of primitive vectors  $(u, v, w)$  above  $(x, y, z)$ . In this connection we mention the following:

**Proposition 3.2.** *The  $SL(2, \mathbb{Z})$ -equivalence classes of triplets  $(u, v, w)$  of primitive vectors above  $(x, y, z)$  are in one to one correspondence with the set*

$$(U_s)^3 \cap \{(\alpha, \beta, \gamma) \in (\mathbb{Z}/s\mathbb{Z})^3; \tilde{x}\alpha + \tilde{y}\beta + \tilde{z}\gamma = 0\} / U_s$$

where  $s = \text{gcd}(x, y, z)$ ,  $\tilde{x} = x/s$ ,  $\tilde{y} = y/s$ ,  $\tilde{z} = z/s$  and  $U_s$  is the unit group of the finite ring  $\mathbb{Z}/s\mathbb{Z}$ .

In particular, if  $s=2$ ,  $\tilde{x} \equiv \tilde{y} \equiv \tilde{z} \equiv 1 \pmod{2}$ , then the set in the proposition is empty. We call  $(x, y, z)$  *liftable* if the set above is non-empty.

*Proof.* We let  $V$  denote the  $\mathbb{Z}$ -module of 3 dimensional integral row vectors:  $V = \sum_{i=1}^3 \mathbb{Z}e_i$ ,  $e_1 = (1, 0, 0)$ ,  $e_2 = (0, 1, 0)$ ,  $e_3 = (0, 0, 1)$ . Then we have also  $V^* = \sum_{i=1}^3 \mathbb{Z}e_i^*$  where  $(e_1^*, e_2^*, e_3^*)$  is the dual base to  $(e_1, e_2, e_3)$ . Naturally we obtain the identification  $\overset{2}{\wedge} V \cong V^*$  by  $\zeta \rightarrow \zeta \lrcorner e_1 \wedge e_2 \wedge e_3$  ( $\zeta \in V^*$ ); we have  $e_i^* = e_j \wedge e_k$  for any even permutation  $(i, j, k)$  of  $(1, 2, 3)$ . Note that under this convention

$$(x, y, z) = \mathbf{p} \wedge \mathbf{q}$$

where we have assumed that  $(x, y, z)$  is induced from  $(u, v, w)$ , and put  $\mathbf{p} = (p_1, p_2, p_3)$ ,  $\mathbf{q} = (q_1, q_2, q_3)$ . This implies that  $\hat{\xi} := (x, y, z)$  is a generator of the  $\mathbb{Z}$ -module  $\overset{2}{\wedge} W$  of rank 1 if we put

$$W = \mathbb{Z}\mathbf{p} + \mathbb{Z}\mathbf{q} \subseteq V.$$

Now we obtain two subspaces  $\overset{2}{\wedge} W, W^\perp$  of  $V^* = \overset{2}{\wedge} V$  ( $W^\perp$ : the annihilator of  $W$ ) and we have the inclusion  $\overset{2}{\wedge} W \subseteq W^\perp$ . We have further  $[W^\perp : \overset{2}{\wedge} W] = s$  and see that  $\hat{\xi} := (\tilde{x}, \tilde{y}, \tilde{z})$  generates  $W^\perp$  over  $\mathbb{Z}$ , with  $s, \tilde{x}, \tilde{y}, \tilde{z}$  being defined above. Now, if we put  $\widehat{W} := (W^\perp)^\perp$ , then we obtain

$$s\widehat{W} \subseteq W \subseteq \widehat{W} \quad [\widehat{W} : W] = s \quad (= [W : s\widehat{W}]).$$

One can easily check that  $W/s\widehat{W}$  is a free submodule of rank 1 of the  $(\mathbb{Z}/s\mathbb{Z})$ -module  $\widehat{W}/s\widehat{W}$ . Note also that  $\widehat{W}/s\widehat{W} = \{(\alpha, \beta, \gamma) \in (\mathbb{Z}/s\mathbb{Z})^3; \tilde{x}\alpha + \tilde{y}\beta + \tilde{z}\gamma = 0\}$ . Recall that the regularity condition (3.5) implies that the restriction  $e_i^* : W \rightarrow \mathbb{Z}$  is surjective for each  $i=1, 2, 3$ . This further implies that  $e_1^*, e_2^*, e_3^*$  induce

surjective mappings of  $W/s\widehat{W}$  onto  $\mathbb{Z}/s\mathbb{Z}$ . By this we see in particular that any generator of  $W/s\widehat{W}$  lies in  $(U_s)^3$ . Since the generators for this  $(\mathbb{Z}/s\mathbb{Z})$ -module are unique up to the multiplication of  $U_s$ , we have thus associated an element of the set in the proposition with the primitive vectors  $u, v, w$  originally given. Conversely suppose that a regular triplet  $(x, y, z)$  is given and take an element  $\eta$  of the coset space in the proposition. Then there obviously exists a unique free  $(\mathbb{Z}/s\mathbb{Z})$ -submodule  $H$  of rank 1 of  $\widehat{W}/s\widehat{W}$  which contains  $\eta$ , where  $\widehat{W}$  is defined to be the annihilator  $(\tilde{x}, \tilde{y}, \tilde{z})^\perp = \{(A, B, C) \in \mathbb{Z}^3; \tilde{x}A + \tilde{y}B + \tilde{z}C = 0\}$ . We should then define  $W$  to be the inverse image of  $H$  under  $\widehat{W} \rightarrow \widehat{W}/s\widehat{W}$ . Since  $[\widehat{W}: W] = s$ , we have  $(x, y, z) = s(\tilde{x}, \tilde{y}, \tilde{z}) = \pm \mathfrak{p} \wedge \mathfrak{q}$  for any  $\mathbb{Z}$ -base  $(\mathfrak{p}, \mathfrak{q})$  of  $W$ . The  $\mathbb{Z}$ -bases  $(\mathfrak{p}, \mathfrak{q})$  such that  $(x, y, z) = \mathfrak{p} \wedge \mathfrak{q}$  are unique up to the action of  $SL(2, \mathbb{Z})$ . We also easily see that (3.5) implies that  $p_i, q_i$  are coprime for  $i=1, 2, 3$  with  $\mathfrak{p} = (p_1, p_2, p_3), \mathfrak{q} = (q_1, q_2, q_3)$ . It suffices that we simply set  $u = {}^t(p_1, q_1), v = {}^t(p_2, q_2), w = {}^t(p_3, q_3)$ . Q.E.D.

We moreover have to mention an important geometric condition which rules all confluence phenomena in the situation of Definition 2.2: For a confluence of type  $(C_1, C_2, \dots, C_r; C_0)$ , the sum of Euler numbers of the fibers of Types  $C_1, C_2, \dots, C_r$  should equal the Euler number of the fiber of Type  $C_0$ . We will call this the *Conservation Law* for Euler numbers. The proof is left to the reader.

*Remark.* We want to call a Diophantine equation of the form (3.4) the generalized Markov-Mordell equation. Markov [3] treats the case where  $a=b=c=3, d=0$ ; Mordell [4] discusses the case where  $a=b=c$  and  $d$  is divisible by  $a^2$  generally. For informations about similar kinds of equations we refer to the related references given in [4].

*Remark.* The invariant  $(x, y, z)$  is an equivalent datum to the intersection matrix between the vanishing cycles  $(au, bv, cw)$ . Brieskorn [1] beautifully describes how this kind of intersection matrix behaves under the action of the braid group  $B_\mu$  for the versal deformations of some good classes of isolated singularities ( $\mu$ : the Milnor number). The above action of  $B=B_3$  on the solution space naturally resembles to his description in the case of  $A_2$ .

#### §4. Arithmetic Study of the Trace Equation

In this section we will give a reasonable fundamental set to the action of the extended braid group  $\widehat{B}$  over the set of integral regular solutions of (3.4)

i.e. a set in which one can find representatives for all orbits. As is remarked in the previous section, the regularity assumption is natural from the geometric point of view, and it makes the theory simpler in many places. We note also that, except for only a few cases, all solutions are regular.

Now recall that the equation (3.4) depends on the integral parameters  $0 < a, b, c; 0 \leq d \leq 4$ . We introduce the following notations:

$$\text{Sol}(a, b, c; d) := \{(x, y, z) \in \mathbb{Z}^3; \quad (3.4) \\ bcx^2 + acy^2 + abz^2 - abcxyz = d\}$$

$$\text{Sol}^*(a, b, c; d) := \{(x, y, z) \in \text{Sol}(a, b, c; d); \\ (x, y, z): \text{ regular}\} .$$

If  $d=0$ , then (3.4) has the obvious solution  $x=y=z=0$  which we call the *commutative* solution. In fact this means geometrically that the monodromy matrices  $M_1, M_2, M_3$  (and  $M$ ) commute. This solution itself forms an orbit of  $\hat{B}$ .

**Definition 4.1.** A solution  $(x, y, z)$  of (3.4) for which we have  $xyz=0$  is called *reducible*.

Geometrically the reducibility implies that some two of  $M_1, M_2, M_3$  commute. We set

$$R := \{\text{reducible solutions of (3.4)}\} \\ R^* := R \cap \text{Sol}^*(a, b, c; d) .$$

**Definition 4.2.** A solution  $(x, y, z)$  of (3.4) is called *positive* if  $x, y, z$  are *positive*; we write then  $0 < (x, y, z)$  or  $(x, y, z) > 0$ . A positive solution  $(x, y, z)$  is called *exceptional* if one of the following inequalities is satisfied:

$$(i) \quad x \geq ayz \quad (ii) \quad y \geq bxz \quad (iii) \quad z \geq cxy .$$

We note that the exceptional ones are exactly those solutions which can be brought to a non-positive solutions by one elementary transformation. We set

$$E := \{\text{exceptional solutions of (3.4)}\} . \\ E^* := E \cap \text{Sol}^*(a, b, c; d) .$$

Now note that, if  $xyz > 0$  for a solution  $(x, y, z)$ , then we can transform it to a positive solution by applying a suitable even sign change of  $x, y, z$ . On

the other hand we call a solution  $(x, y, z)$  *essentially negative* if  $xyz < 0$ . The following proposition is immediate:

**Proposition 4.1.** *If the equation (3.4) has an essentially negative solution  $(x, y, z)$ , then we have*

$$a = b = c = 1, d = 4, xyz = -1, |x| = |y| = |z| = 1.$$

The  $\widehat{B}$ -orbit of this solution consists of sixteen elements, among which there are exceptional solutions  $(2, 1, 1), (1, 2, 1), (1, 1, 2)$ .

Now we list in the following table all reducible and exceptional solutions  $(x, y, z)$  for which  $x \geq 0, y \geq 0, z \geq 0$ :

Table 2

Parameter system $(a, b, c; d)$	Reducible solutions $(x, y, z) \geq 0$	Exceptional solutions $(x, y, z) \geq 0$
$(1, 1, 1; 2)$	$(1, 0, 0) + \alpha, (0, 1, 1)^* + \alpha$	$(1, 1, 1)^*$
$(1, 1, 1; 4)$	$(2, 0, 0) + \alpha$	$(2, 1, 1)^* + \alpha$
$(1, 1, 2; 3)$	$(0, 1, 1)^* + \alpha$	$(1, 1, 1)^*$
$(1, 1, 2; 4)$	$(0, 0, 2), (1, 1, 0)^*$	$(1, 1, 2)^*$
$(1, 1, 3; 4)$	$(0, 1, 1)^* + \alpha, (0, 0, 2)$	$(1, 1, 1)^*$
$(1, 2, 2; 4)$	$(0, 1, 1)^*, (0, 0, 1) + \alpha$	$(1, 1, 1)^*$
$(1, 1, l; 1)$	$(0, 0, 1)$	empty
$(1, 1, l; 4)$	$(0, 0, 2)$	empty
$(2, 2, l; 4)$	$(0, 0, 1)$	empty
$(1, l, m; l) \ 1 \leq l \leq 4, 1 \leq m$	$(0, 0, 1) + \alpha$	empty

Here we have excluded the commutative one. The solutions with suffix \* are regular; the adjunction of  $+\alpha$  to a solution means that one should add the solutions obtained from it by the symmetry of  $(a, b, c)$  in case it exists. We also note that, in Table 2, we have classified the cases and the solutions only up to the simultaneous permutations of  $(x, y, z)$  and  $(a, b, c)$ .

To give the fundamental set promised at the beginning of this section we now introduce the following two sets of positive solutions:

$$F := \{(x, y, z) \in \text{Sol}(a, b, c; d);$$

$$0 < x \leq ayz/2, \ 0 < y \leq bxz/2, \ 0 < z \leq cxy/2\}$$

$$F^* := F \cap \text{Sol}^*(a, b, c; d).$$

We further set

$$R_+ := \{(x, y, z) \in R; x \geq 0, y \geq 0, z \geq 0\}$$

$$R_+^* := R_+ \cap R^*$$

$$E_+ := \{(x, y, z) \in E; x \geq 0, y \geq 0, z \geq 0\}$$

$$E_+^* := E_+ \cap E^*.$$

**Proposition 4.2.** *With the notations above, the natural mappings*

$$R_+ \cup E_+ \cup F \rightarrow \text{Sol}(a, b, c; d)/\widehat{B}$$

$$R_+^* \cup E_+^* \cup F^* \rightarrow \text{Sol}^*(a, b, c; d)/\widehat{B}$$

are onto. In particular, we see that  $\text{Sol}(a, b, c; d)/\widehat{B}$  resp.  $\text{Sol}^*(a, b, c; d)/\widehat{B}$  is a finite set if  $F$  resp.  $F^*$  is a finite set.

*Proof.* By the  $\widehat{B}$ -invariance of the regularity it suffices to prove the surjectivity of the first mapping. Following [4], we first define for a positive solution  $(x, y, z)$  its height to be the sum  $x+y+z$ . Now let  $\mathcal{O}$  be a  $\widehat{B}$ -orbit in the space  $\text{Sol}(a, b, c; d)$ . We want to show that  $\mathcal{O} \cap (R_+ \cup E_+ \cup F)$  is non-empty. If  $\mathcal{O}$  does not contain any positive solution, then it must only consist of reducible solutions by Proposition 4.1 and we are done in this case. We may thus assume that  $\mathcal{O}$  contains at least one positive solution. Let now  $(x, y, z)$  be a positive solution in  $\mathcal{O}$  with the minimal height. If it is an exceptional solution, then we are done. If it is not, then any elementary transformation brings it to a positive solution in  $\mathcal{O}$  with a greater or equal height, which exactly means that  $(x, y, z)$  belongs to  $F$ . Q.E.D.

Our next aim is to show that  $F^*$  is always a finite set, by estimating  $x, y, z$  for  $(x, y, z) \in F$ . The following argument generalizes the method of Mordell [4] to the case of the equation (3.4) which has more parameters:

**Proposition 4.3.** *For  $(x, y, z) \in F$  we have*

$$(4.1) \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \min(x, y, z)$$

where the right hand side denotes the minimum of  $x, y, z$ .

*Proof.* By the symmetry we can assume that  $x \leq y \leq z$ . By replacing  $x, y$  by larger  $z$  in the first two terms in (3.4) we have the estimate  $z \{(bc+ac$

$+ab)z - abcxy\} \geq d \geq 0$ . By dividing by  $z > 0$  we have  $z \geq A$  where we put  $A := abcxy/(bc + ac + ab)$ . By  $(x, y, z) \in F$  we have  $B \geq z$  where  $B := cxy/2$ , and we thus obtain  $B \geq z \geq A$  which implies that

$$|z - B| \leq B - A.$$

Since  $x \leq y$  and since (3.4) can be written in the form  $c(bx^2 + ay^2) + ab\{(z - B)^2 - B^2\} = d \geq 0$ , we deduce further  $0 \leq c(a + b)y^2 - abA(2B - A)$  from which it follows

$$y^2 \left\{ 1 - \left[ \frac{abc}{bc + ac + ab} \right]^2 x^2 \right\} \geq 0.$$

Since  $y^2 > 0$ , we have obtained the desired inequality.

The estimate (4.1) already implies something remarkable:

**Corollary 4.1.** *If we have the inequality*

$$(4.2) \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

*then, in any confluence of type  $(I_a, I_b, I_c; C)$ , the monodromy matrices commute and  $C = I_{a+b+c}$ .*

*Proof.* The inequalities (4.1), (4.2) imply that  $F$  is empty. We further note that none of the cases in Table 2 occurs if (4.2) is satisfied. This implies that there is not any exceptional solution or reducible solution except the commutative one, which proves the corollary.

To prove the finiteness for  $F^*$  we will also need the following:

**Proposition 4.4.** *For  $(x, y, z) \in F$  we have the inequalities*

$$(4.3) \quad bcx^2 \geq 4, \quad acy^2 \geq 4, \quad abz^2 \geq 4.$$

*If  $bcx^2 > 4$ , then we have the further estimation:*

$$(4.4) \quad \begin{cases} \frac{cx^2(bc x^2 - d)}{a(bc x^2 - 4)} \geq z^2 \\ \frac{bx^2(bc x^2 - d)}{a(bc x^2 - 4)} \geq y^2. \end{cases}$$

*Proof.* From the definition of the set  $F$  we have  $0 < x \leq ayz/2, 0 < y \leq bxz/2, 0 < z \leq cxy/2$ ; in particular

$$\frac{2}{cx} z \leq y \leq \frac{bx}{2} z$$

from which we obtain  $bcx^2 \geq 4$ . The other inequalities in (4.3) are similarly proved. From the estimate above, we can also deduce

$$|y - bxz/2| \leq \frac{bcx^2 - 4}{2cx} z.$$

Since  $bcx^2 - d + ac(y - bxz/2)^2 + ab(1 - bcx^2/4)z^2 = 0$ , we obtain the estimate  $0 \leq bcx^2 - d + acz^2(bc x^2 - 4)^2 / (4c^2 x^2) + ab(1 - bcx^2/4)z^2 = bcx^2 - d - a(bc x^2 - 4)z^2 / cx^2$ , from which the first inequality in (4.4) follows. The other one in (4.4) is proved similarly. Q.E.D.

To prove the finiteness theorem for  $\hat{B}$ -orbits of solutions, we still have to list one more exceptional class of parameter systems  $(a, b, c; d)$  which will appear in the study of confluences to Types  $I_b^*$ :

Table 3

Parameter system $(a, b, c; d)$	Special solutions $(x, y, z)$ in $F$
$(m, 1, 1; 4)$	$(2, y, y) \quad y \geq 1$
$(m, 2, 2; 4)$	$(1, y, y) \quad y \geq 1$
$(m, 1, 4; 4)$	$(1, y, 2y) \quad y \geq 1$

The meaning of “special” in Table 3 is explained in the following proposition:

**Proposition 4.5.** *Assume that  $(x, y, z) \in F$ , that  $x = \min(x, y, z)$ , and that  $bcx^2 = 4$ . Then  $(a, b, c; d)$  is one of the parameter systems in Table 3 and  $(x, y, z)$  is one of the special solutions there.*

*Proof.* From the proof of Proposition 4.3 it is clear that we have  $d = 4$  and  $y = bxz/2 = 2z/cx$  under the assumption above. From this one easily obtains the conclusion of the proposition.

We will of course extend the speciality of solutions in Table 3 to an invariant notion under the simultaneous permutations of  $(a, b, c)$  and  $(x, y, z)$ . Now the main result in this section is the following:

**Theorem 4.1.** *The orbit space*



$$\text{Sol}^*(a, b, c; d)/\hat{B}$$

is a finite set.

*Proof.* By Propositions 4.3–4.5 we have the estimates  $0 < x \leq 1/a + 1/b + 1/c$  and (4.4) for non-special solutions  $(x, y, z)$  in  $F$ , by permuting  $x, y, z$  and  $a, b, c$  simultaneously. This implies that there are only finitely many non-special solutions in  $F$ . But we also see that there are only one or two regular special solutions for each case in Table 3. Thus the set  $F^*$  is always a finite set. Now Proposition 4.2 proves the theorem.

§5. Confluence to Singular Fibers of Type  $I_\beta^*$

We begin this section by giving a supplement to Proposition 4.4. We assume here that

$$(5.1) \quad d = 4$$

which exactly means that we are dealing with the case where  $C = I_\beta^*$  for some  $\beta$  in the confluence of type  $(I_a, I_b, I_c; C)$ . We can then rewrite the equation (3.4) in the form:

$$(5.2) \quad ab(z - cxy/2)^2 - \{c(bx^2 + ay^2)/4 - 2\}^2 + \{c(bx^2 - ay^2)/4\}^2 = 0.$$

**Lemma 5.1.** *Suppose that we have  $bx^2 = ay^2$  for  $(x, y, z) \in \text{Sol}(a, b, c; 4)$ . Then  $\sqrt{ab}$  is an integer and, we have either (i)  $\sqrt{ab}z = 2$  or (ii)  $\sqrt{ab}z = c \cdot \sqrt{ab}xy - 2$ . If  $(x, y, z)$  belongs to the fundamental set  $F$ , then (i) and (ii) are equivalent.*

*Proof.* The first assertion of the lemma follows from (5.2) immediately if one notes that  $bx^2 = ay^2 = \sqrt{ab}xy$  under the assumption. From the definition of  $F$ , we have  $0 < z \leq cxy/2$  for  $(x, y, z) \in F$ . Thus, if (ii) is satisfied, then we have

$$4 \geq c\sqrt{ab}xy = bcx^2 = acy^2.$$

On the other hand, we have by Proposition 4.4  $bcx^2 \geq 4, acy^2 \geq 4$ . We thus obtain  $c\sqrt{ab}xy = 4$  which shows the equivalence of (i) and (ii). Q.E.D.

**Proposition 5.1.** *Under the assumption (5.1) the set  $F$  consists only of special solutions i.e. we have one of the cases in Table 3 for  $(a, b, c)$  and  $(x, y, z) \in F$  up to the simultaneous permutations.*

*Proof.* Suppose that, in  $bcx^2 \geq 4, acy^2 \geq 4, abz^2 \geq 4$ , we have two equalities, say  $bcx^2 = 4 = acy^2$ . Then the assumption of Lemma 5.1 is satisfied and we obtain  $\sqrt{ab}z = 2, \sqrt{ab}$  being an integer. Thus, we have either  $ab = 1, z = 2$  or  $ab = 4, z = 1$ . Since we also have the identity  $bx^2 = ay^2$ , we are led to one of the cases of Table 3. Now suppose that, in  $bcx^2 \geq 4, acy^2 \geq 4, abz^2 \geq 4$ , we have two strict inequalities, say  $bcx^2 > 4, acy^2 > 4$ . Then, by Proposition 4.4 we obtain four inequalities  $cx^2 \geq az^2, bx^2 \geq ay^2; ay^2 \geq bx^2, cy^2 \geq bz^2$ . Thus we are led again to the assumption  $bx^2 = ay^2$  of Lemma 5.1.

*Remark.* In the first case of Table 3 we have exactly two regular solutions  $(2, 1, 1), (2, 2, 2)$ . But  $(2, 2, 2)$  is not liftable in the sense of Section 3 (see Proposition 3.2). Thus we should exclude this solution in the study of actual confluences. We have exactly one geometric special solution for each case of Table 3.

§6. Confluences to Types  $I_\beta, II, III, IV$

Suppose that we are given a confluence of Type  $(I_a, I_b, I_c; C)$  with the monodromy matrices  $(M_1, M_2, M_3)$  such that the class  $C$  is one of the types in the title. We also assume that  $M = M_1 M_2 M_3$ , so that, according to the convention of Section 3, we have

$$x = \langle v, w \rangle \quad y = \langle w, u \rangle \quad z = \langle u, v \rangle$$

where we put

$$(6.0) \quad M_1 = M(u, a) \quad M_2 = M(v, b) \quad M_3 = M(w, c).$$

To include the case of confluences with less than three branches, we allow some of the vectors  $u, v, w$  to be zero; we have of course assumed that every vector is primitive unless it is zero. Now we begin with the following reformulation of the sign condition in Table 1.

**Proposition 6.1.** *Let the notation and the assumption be as above. If  $u$  is not zero, then*

$$(6.1) \quad cy^2 + bz^2 - bcxyz \geq 0$$

where we have the strict inequality except for the case where  $C = I_\beta$  for some  $\beta$ .

*Proof.* Since  $u$  is primitive, we can assume that  $u, v, w$  are given as follows:

$$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} p \\ z \end{pmatrix} \quad w = \begin{pmatrix} \bar{p} \\ -y \end{pmatrix}.$$

We have then  $x = -(py + \bar{p}z)$  and, by calculating directly, we see that the (2,1)-entry of  $M = M_1 M_2 M_3$  is equal to  $-(cy^2 + bz^2 - bcyz)$ . Thus the sign condition in Table 1 implies the desired inequality of the proposition.

**Proposition 6.2.** *If  $C = I_\beta$  for some  $\beta > 0$ , then the monodromy matrices  $M_i$  commute. If  $C = II$ , then the last case in Table 2 occurs with  $l = 1$ . If  $C = III$ , then either we have  $abc = xyz = 1$  or the last case in Table 2 occurs with  $l = 2$ . If  $C = IV$ , then either we have  $abc = 2, xyz = 1$  or the first case of Table 2 occurs or the last case occurs with  $l = 3$ .*

*Proof.* We write the trace equation in the form

$$(6.2) \quad bcx^2 + a(cy^2 + bz^2 - bcxyz) = d$$

where  $d = 0, 1, 2, 3$  according to  $C = I_\beta, II, III, IV$ . Thus, in case where  $C = I_\beta$ , we have by Proposition 6.1  $x = 0$  provided that  $u \neq 0$ . Note that  $u = 0$  implies  $y = z = 0$  and also  $x = 0$  since  $d = 0$  in this case. By the symmetry we have proved  $x = y = z = 0$  for  $C = I_\beta$ . Now suppose that  $d = 1$  and that none of  $u, v, w$  is equal to 0. Then, by the strict form of (6.1), we obtain  $x = y = z = 0$ , which is obviously a contradiction. But, if  $d = 1$  and one of  $u, v, w$ , say  $w$  is equal to 0, then we have  $x = y = 0, abz^2 = 1$ , which leads to the last case of Table 2 with  $l = 1$ . The remaining two statements of the proposition are proved similarly.

**§7. Confluences to Singular Fibers of Types II\*, III\*, IV\***

As in the previous section we assume that we are given a confluence of Type  $(I_a, I_b, I_c; C)$  where  $C$  is one of the types above. We let also the associated monodromy matrices  $M_1, M_2, M_3$  be given by (6.0) and let  $x = \langle v, w \rangle, y = \langle w, u \rangle, z = \langle u, v \rangle$ . We of course assume that  $M_1 M_2 M_3$  is in conjugacy class  $C$ . But we note that, for the present exceptional cases, the sign condition does not give such a strong restriction as (6.1); so we will instead rely on the conservation law for Euler numbers. We shall namely impose the following condition which gives only a finite number of possibilities for  $(a, b, c)$ :

$$(7.1) \quad a + b + c = 10, 9, 8 \quad \text{according to } C = II^*, III^*, IV^*.$$

Recall also that, in the equation (3.4), we have  $d = 1, 2, 3$  according to  $C = II^*,$

III\*, IV\*. Our strategy of classification is now simple: We can assume that  $(x, y, z)$  is in the set  $F$  in Section 4. But then Proposition 4.3 gives us the inequality to be satisfied for  $a, b, c$ ;  $x, y, z$ :  $1/a+1/b+1/c \geq \min(x, y, z) = 1, 2$ . Without loss of generality we can assume that  $x = \min(x, y, z) = 1$  or 2. This, together with (7.1) and the congruence  $bcx^2 \equiv d \pmod{a}$ , reduces greatly the number of possibilities of  $(a, b, c)$  to consider. We can further use the estimates (4.4) of Proposition 4.4 to look for the non-special solutions in  $F$ , while in the present cases, any special solution does not appear. Now we simply list the result of this procedure in the following table:

Table 4

$C$	$(a, b, c)$	$(x, y, z)$	$C$	$(a, b, c)$	$(x, y, z)$
II*	(1, 1, 8)	(1, 1, 3)	IV*	(1, 1, 6)	(1, 1, 3)
	(1, 2, 7)	(1, 1, 2)		(1, 2, 5)	(1, 1, 2)
	(1, 4, 5)	(1, 1, 2)		(2, 3, 3)	(1, 1, 1)
	(2, 3, 5)	(1, 1, 1)			
III*	(1, 1, 7)	(1, 1, 3)			
	(1, 2, 6)	(1, 1, 2)			
	(1, 3, 5)	(1, 1, 2)			
	(2, 3, 4)	(1, 1, 1)			

### §8. Concluding Remarks

So far we have discussed only confluences of Types  $(I_a, I_b, I_c; C)$  where  $C$  is an arbitrary class in Table 1. But the same method can be applied without any essential change to the study of confluences of Types  $(I_a^*, I_b, I_c; C)$  since the conjugacy classes in Table 1 are transposed among themselves by the involution  $M \leftrightarrow -M$  in  $SL(2, \mathbf{Z})$  and  $I_a^* = -I_a$ . For example we obtain the monodromy matrices  $M_1, M_2, M_3$  for confluences of Types  $(I_1^*, I_1, I_1; III^*)$ ,  $(I_2^*, I_1, I_1; IV^*)$ ,  $(I_2, I_1^*, I_1; IV^*)$  etc. from those for confluences of Types  $(I_1, I_1, I_1; III)$ ,  $(I_2, I_1, I_1; IV)$  etc. But we should remark here that the inverse procedure is not allowed by the conservation law for Euler numbers. To sum up results, we found that there is a principle which controls the confluence-phenomena observed so far. We formulate this principle in the following way: We assign  $I_{k+1}, I_{k-4}^*, IV^*, III^*, II^*$  to the Dynkin diagrams  $A_k (k \geq 1)$ ,  $D_k (k \geq 4)$ ,  $E_6, E_7, E_8$  by comparing the singular fibers of these types and the exceptional sets in minimal resolutions of simple singularities. We call these types of singular fibers or the corresponding conjugacy classes in  $SL(2, \mathbf{Z})$

the *Lie types*. We will also call  $I_1$  the extra-Lie type. We associate the empty set with this as the Dynkin-diagram. Then our principle asserts that, in any confluence of Type  $(C_1, C_2, C_3; C)$  where  $C$  is a Lie-type and  $C_1, C_2, C_3$  are Lie or extra-Lie types, the disjoint union of the Dynkin-diagrams of  $C_1, C_2, C_3$  is obtained from the Dynkin-diagram of  $C$  by removing some vertices and the bonds issuing from them. This behavior of confluences is exactly like the unfolding procedure of simple singularities except at the extra-point that one should supplement fibers of Type  $I_1$  to fill out the defect of Euler numbers. This seems to suggest the existence of some universal objects of deformation for (local) elliptic surfaces.

It was too restrictive that we assumed the properness and local triviality for the mappings  $\delta, \eta$  in Definition 2.1. In fact [2] classifies the local monodromies of local one-parameter family of elliptic curves. We think that the appropriate local version of confluence of singular fibers might be clear for the reader. Our results on monodromies receive anyway no change at all.

If we want to observe confluences with more than three branches, then we are faced to the difficulty that there are non-trivial algebraic relations among the intersection numbers of vanishing cycles. For example, suppose that we are given four matrices of the form  $M_i = M(u_i, a_i)$  ( $1 \leq i \leq 4$ ). Then the trace of  $M_1 M_2 M_3 M_4$  can in fact be expressed again by  $a_i$  ( $1 \leq i \leq 4$ ) and the six quantity  $x_{ij} := \langle u_i, u_j \rangle$  ( $x_{ij} + x_{ji} = 0$ ). But we also receive the Plücker relation  $x_{12}x_{34} + x_{13}x_{42} + x_{14}x_{23} = 0$  besides the trace equation. We will moreover have another difficulty i.e. the braid group  $B_4$  does not preserve the form of the trace equation in general. This might also make the theory more cumbersome. There still remain many things to overcome for the arithmetic study of confluences with more than three branches.

To close this article, we explain how to construct the monodromy matrices  $M_1 = M(u, a)$ ,  $M_2 = M(v, b)$ ,  $M_3 = M(w, c)$  from the data  $(a, b, c; x, y, z)$  which we have classified above. As is done in Section 6, we can assume

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v = \begin{pmatrix} p \\ z \end{pmatrix} \quad w = \begin{pmatrix} \bar{p} \\ -y \end{pmatrix}$$

$$x = -(py + \bar{p}z) \quad (x, y, z > 0).$$

By applying the conjugacy of the stabilizer subgroup of  $u$  if necessary, we can further assume that

$$0 \leq p \leq z - 1.$$

Table 5

Type	$M_1$	$M_2$	$M_3$
$(I_1, I_1, I_8; II^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$	$\begin{pmatrix} -7 & 8 \\ -8 & 9 \end{pmatrix}$
$(I_1, I_2, I_7; II^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$	$\begin{pmatrix} -6 & 7 \\ -7 & 8 \end{pmatrix}$
$(I_1, I_4, I_5; II^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -7 & 4 \\ -16 & 9 \end{pmatrix}$	$\begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix}$
$(I_2, I_3, I_5; II^*)$	$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix}$
$(I_1, I_1, I_7; III^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$	$\begin{pmatrix} -6 & 7 \\ -7 & 8 \end{pmatrix}$
$(I_1, I_2, I_6; III^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$	$\begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix}$
$(I_1, I_3, I_5; III^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -5 & 3 \\ -12 & 7 \end{pmatrix}$	$\begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix}$
$(I_2, I_3, I_4; III^*)$	$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$
$(I_1, I_1, I_6; IV^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -5 & 4 \\ -9 & 7 \end{pmatrix}$	$\begin{pmatrix} -5 & 6 \\ -6 & 7 \end{pmatrix}$
$(I_1, I_2, I_5; IV^*)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -3 & 2 \\ -8 & 5 \end{pmatrix}$	$\begin{pmatrix} -4 & 5 \\ -5 & 6 \end{pmatrix}$
$(I_2, I_3, I_3; IV^*)$	$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -3 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$
$(I_a, I_1, I_1; I_a^* - 4)$ $a \geq 4$	$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}$
$(I_a, I_2, I_2; I_a^* - 2)$ $a \geq 2$	$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$
$(I_a, I_1, I_4; I_a^*)$ $a \geq 1$	$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$	$\begin{pmatrix} -3 & 4 \\ -4 & 5 \end{pmatrix}$
$(I_1, I_1; II)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	
$(I_2, I_1, III)$	$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	
$(I_1, I_1, I_1; III)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -1 & 2 \end{pmatrix}$
$(I_3, I_1; IV)$	$\begin{pmatrix} 1 & 3 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	
$(I_2, I_1, I_1; IV)$	$\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -1 & 2 \end{pmatrix}$

Since  $v, w$  should be primitive, we have the conditions  $\gcd(p, z) = \gcd(\bar{p}, y) = 1$ . Now we can check that, in all our cases of classification, one finds a unique  $p$  in the above region with  $\gcd(p, z) = 1$  such that  $x + py$  is divisible by  $z$ , and that the condition  $\gcd(\bar{p}, y) = 1$  is satisfied with  $\bar{p} = -(x + py)/z$ . Thus the vectors  $u, v, w$  are determined and we obtain the desired matrices  $M_1, M_2, M_3$ . We sum up the result in Table 5.

### References

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