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On the Dirichlet Problem for Quasilinear Elliptic Equations with Degenerate Coefficients

By

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§ 1. Introduction and Results

In this paper we consider the weak solution u of the Dirichlet boundary value problem for a certain quasilinear elliptic equation, whose coefficients degenerate on the boundary. Our aim is to study the regularity behavior of u near the boundary.

Let Ω be a bounded domain in $\boldsymbol{R}^n$ with boundary $\partial\Omega$. We suppose that for a function $\phi\in C^3(\boldsymbol{R}^n)$

$$\Omega=\{x\in\boldsymbol{R}^n\,;\,\phi(x)>0\}\,,\quad \partial\Omega=\{x\in\boldsymbol{R}^n\,;\,\phi(x)=0\}$$

and

$$d\phi(x)\neq 0 \quad \text{for } x\in\partial\Omega\,,$$

where $d\phi$ is the differential of ϕ.

It is assumed that the usual function spaces $C^k(\bar{\Omega})$, $C_0^k(\Omega)$, $L^q(\Omega)$ are known. For real numbers μ and q with $1\leqq q<\infty$ we define

$$L_\mu^q(\Omega)=\{u\,;\,\phi^{\mu/q}u\in L^q(\Omega)\}$$

and we write

$$\|u\|_q=\left(\int_\Omega |u|^q dx\right)^{1/q},\quad \|u\|_{L_\mu^q}=\|\phi^{\mu/q}u\|_q\,.$$

If $q>1$, the space $L_\mu^q(\Omega)$ is a separable and reflexive Banach space. It is seen that the dual space of $L_\mu^q(\Omega)$ is $L_{-\mu/(q-1)}^{q^*}(\Omega)$, where $q^*=q/(q-1)$. Denoting by ∇u the gradient of u, we define

$$W_\mu^{1,q}(\Omega)=\{u\in L_\mu^q(\Omega)\,;\,\|\phi^{\mu/q}\nabla u\|_q<\infty\}$$

and we write

$$\|u\|_{W_\mu^{1,q}}=\|u\|_{L_\mu^q}+\|\nabla u\|_{L_\mu^q}\,.$$

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Then $W^{1,q}_{\mu}(\Omega)$ is also a separable and reflexive Banach space, which was studied by P. Grisvard [6]. We denote by $\mathring{W}^{1,q}_{\mu}(\Omega)$ and $\widetilde{W}^{1,q}_{\mu}(\Omega)$ the completion of $C^{\infty}_{0}(\Omega)$ and $C^{\infty}(\bar{\Omega})$, respectively with respect to the norm $\| \ \|_{W^{1,q}_{\mu}}$.

Throughout this paper let us suppose that $p>2$ and $0\leqq\alpha<p-2$. And we consider the following boundary value problem

$$(1.1)\qquad \begin{cases} -\nabla\cdot(\phi|\nabla u|^{p-2}\nabla u)+|u|^{\alpha}u=f & \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega. \end{cases}$$

If f is in the dual space of $\mathring{W}^{1,p}_{1}(\Omega)$, we can find a unique weak solution u of (1.1) belonging to $\mathring{W}^{1,p}_{1}(\Omega)$ (see Lemma 2.3 and part (b) of Lemma 2.2).

Let θ be a vector field of class C^{1} tangent to $\partial\Omega$, namely,

$$(1.2)\qquad \theta\in[C^{1}(\bar{\Omega})]^{n} \quad \text{and} \quad \theta\cdot\boldsymbol{n}=0 \quad \text{on } \partial\Omega,$$

where $\boldsymbol{n}$ is the outer normal of $\partial\Omega$ with respect to Ω. We write $\theta=(\theta_{1},\cdots,\theta_{n})$ and $\theta\cdot\nabla=\sum_{i=1}^{n}\theta_{i}\partial_{x_{i}}$, which is a tangential differential operator of first order.

Our aim is to prove the following theorems.

Theorem 1. *Let $f\in\widetilde{W}^{1,p*}_{-1/(p-1)}(\Omega)$. If $u\in\mathring{W}^{1,p}_{1}(\Omega)$ is a weak solution of (1.1). Then it holds that*

$$\phi^{1/2}(\theta\cdot\nabla)|\nabla u|^{p/2}\in L^{2}(\Omega)$$

and

$$\|\phi^{1/2}(\theta\cdot\nabla)|\nabla u|^{p/2}\|_{2}^{2}\leqq C[\|f\|_{W^{1,p*}_{-1/(p-1)}}+(\|f\|_{p*})^{(1+\alpha)/(p-1)}]^{p*},$$

where C is independent of f.

Theorem 2. *Let $0<\beta<1$. Under the assumptions in Theorem 1 it holds that*

$$\phi^{-\beta/p}(\theta\cdot\nabla)u\in L^{p}(\Omega)$$

and

$$(\|\phi^{-\beta/p}(\theta\cdot\nabla)u\|_{p})^{p}\leqq C(\beta)[\|f\|_{W^{1,p*}_{-1/(p-1)}}+(\|f\|_{p*})^{(1+\alpha)/(p-1)}]^{p*},$$

where $C(\beta)$ is a constant depending on β and not on f.

Theorem 3. *Let $\gamma>1/(p-1)$. Under the assumptions in Theorem 1 it holds that*

$$\phi^{\gamma/p}\nabla u\in L^{p}(\Omega)$$

and

$$(\|\phi^{\gamma/p}\nabla u\|_{p})^{p}\leqq C(\gamma)[\|f\|_{W^{1,p*}_{-1/(p-1)}}+(\|f\|_{p*})^{(1+\alpha)/(p-1)}]^{p*},$$

where $C(\gamma)$ is a constant depending on γ and not on f.

The interior regularity for the equation

$$(1.3)\qquad -\nabla\cdot(|\nabla u|^{p-2}\nabla u)=f$$

was studied by several authors. For example, L.C. Evans [3] proved that weak solutions of (1.3) are of class $C^{1+\delta}$ $(0<\delta<1)$ if f is smooth. For more general equations $C^{1+\delta}$-regularity was shown by P. Tolksdorf [11], where detailed references are given.

Secondly we consider the Dirichlet boundary value problem for (1.3) under Dirichlet data 0. More explicitely,

$$\begin{cases} -\nabla\cdot(|\nabla u|^{p-2}\nabla u)=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Let Ω and ϕ be the domain and the function, respectively, in the beginning of this Section. It is well-known that the existence of a weak solution u for (1.4) is shown by the "monotone" method (cf., e.g., [8]). The global regularity of u gives rise to a question. By the result of I.M. Vishik [12] it is known that

$$\phi\partial_{x_i}(|\nabla u|^{(p-2)/2}\partial_{x_j}u)\in L^2(\Omega), \quad i, j=1, \cdots, n,$$

if f and $\nabla f\in L^2(\Omega)$. His method is the use of Galerkin procedure. G.N. Jakolev extended the above result to more general equations in a series of his papers (cf., [7]), where the method of difference quotients is used. J. Simon [10] also proved the global regularity of u by estimating the fractional derivatives of u in Besov spaces. If we proceed along the line of [10], it is unnecessary to prepare a coordinates transformation for the estimation of normal derivatives of u. However we require an adequate coordinates transformation in this paper (see Section 3).

Now we shift our attention to the degenerate linear elliptic equation

$$-\nabla\cdot(\phi\nabla u)=f, \tag{1.5}$$

which was studied by M.S. Baouendi and C. Goulaouic [1]. They showed the global regularity for the weak solution $u\in W_1^{1,2}(\Omega)$ of (1.5). In particular $\nabla u\in L^2(\Omega)$ results from [1]. Recently C. Goulaouic and N. Shimakura [5] have proved that (1.5) gives an isomorphism from $C^{2+\delta}(\bar{\Omega})$ onto $C^{\delta}(\bar{\Omega})$, for any δ with $0<\delta<1$. In connection with (1.5), J.P. Dias [2] treated the variational inequality for the equation

$$-\nabla\cdot(\phi^{\mu}|\nabla u|^{p-2}\nabla u)=f, \tag{1.6}$$

where it is assumed that $0\leqq\mu<\min(p-1, p/n)$. He proved the global boundedness for weak solutions under some assumptions.

From the viewpoint of mathematical physics, the simplest unsteady two-dimensional equation related to (1.6) appears in J.R. Philip's work [9, p. 2], where transfer processes were treated. Thus it seems to us that the Dirichlet problem (1.1) is meaningful to study. Finally we give an example showing that the conclusion of Theorem 3 is sharp for $p>1+\sqrt{2}$.

Example. Let $n=1$ and Ω be the open interval $(0, 1)$. Thus $\phi\in C^3(\boldsymbol{R}^1)$ is a function such that $\phi(t)>0$ for $0<t<1$, $\phi(0)=\phi(1)=0$ and $\phi'(0)$, $\phi'(1)\neq 0$.

There is a positive constant c such that $c^{-1}t\leqq\phi\leqq ct$, if $0\leqq t\leqq 1/2$. Since $t^{-1}\phi(t)=\int_0^1\phi'(ts)ds$, we have

$$(t^{-1}\phi)'\in C^1(\boldsymbol{R}^1). \tag{1.7}$$

We take a function $\zeta\in C^\infty(\boldsymbol{R}^1)$ in such a way that

$$\zeta(t)=\begin{cases}1 & (-\infty<t<1/2),\\ 0 & (t>1).\end{cases}$$

Let us set $u(t)=\zeta(t)t^{(p-2)/(p-1)}$. It is easily seen that $u\in W_1^{1,p}((0, 1))$. From the condition with $u(0)=u(1)=0$ we conclude that $u\in\mathring{W}_1^{1,p}((0, 1))$, in virtue of [6, p. 262] (see Lemma 2.2 in this paper).

From (1.7), $-(\phi|u'|^{p-2}u')'\in C^1$ in a neighborhood of $t=0$. Let t_0 $(0<t_0<1)$ be a zero point of $u'=0$ with its order N. Then near $t=t_0$

$$|(\phi|u'|^{p-2}u')''|\leqq C|t-t_0|^{N(p-1)-2}\leqq C|t-t_0|^{p-3}.$$

Since $p^*(p-3)>-1$ for $p>1+\sqrt{2}$, it follows that $-(\phi|u'|^{p-2}u')'\in W^{1,p^*}$ in a neighborhood of $t=t_0$. Hence $-(\phi|u'|^{p-2}u')'\in W^{1,p^*}_{-1/(p-1)}((0, 1))$, moreover we can easily verify that $-(\phi|u'|^{p-2}u')'\in\widetilde{W}^{1,p^*}_{-1/(p-1)}((0, 1))$. If $\gamma>1/(p-1)$, $\phi^{\gamma/p}u'\in L^p((0, 1))$. And we see that $\phi^{\gamma/p}u'\notin L^p((0, 1))$ if $\gamma<1/(p-1)$.

§2. Preliminaries

We use the notations in Section 1. Throughout this paper the notation "$\rightharpoonup$" means the weak convergence.

Lemma 2.1 (J. L. Lions [8, p. 12]). *Let $u\in L^q(\Omega)$ $(1<q<\infty)$ and suppose that $\{\|u_j\|_q\}$ is uniformly bounded and $u_j\to u$ pointwise a.e. in Ω. Then $u_j\rightharpoonup u$ in $L^q(\Omega)$.*

If u is a function in Ω and the trace of u on $\partial\Omega$ exists, it is written by γu. We denote by $\langle\ \rangle_q$ the norm in $L^q(\partial\Omega)$.

Lemma 2.2 (P. Grisvard [6]). *Let $0\leqq\mu<q-1$. Then the following assertions hold:*

(a) *If $u\in W_\mu^{1,q}(\Omega)$, then $\gamma u\in L^q(\partial\Omega)$ and*

$$\langle\gamma u\rangle_q\leqq C\|u\|_{W_\mu^{1,q}}.$$

(b) *The space $\mathring{W}_\mu^{1,q}(\Omega)$ consists of all $u\in W_\mu^{1,q}(\Omega)$ with $\gamma u=0$.*

(c) *If $u\in\mathring{W}_\mu^{1,q}(\Omega)$, then $u\in L^q_{\mu-q}(\Omega)$ and*

$$\|u\|_{L^q_{\mu-q}} \leqq C\|\nabla u\|_{L^q_\mu}.$$

The above constant C are all independent of u.

From now on we assume that $0 \leqq \alpha < p-2$ and $0 \leqq \mu < p-1$. The norm and inner product in $L^2(\Omega)$ are simply denoted by $\| \ \|$ and $(,)$, respectively. We set $V = \mathring{W}^{1,p}_\mu(\Omega)$. Thus $\| \ \|_V$ is the norm in $\mathring{W}^{1,p}_\mu(\Omega)$.

For $u \in V$ we define $A(u)$ as follows:

$$\langle A(u), v\rangle = (\phi^\mu |\nabla u|^{p-2}\nabla u, \nabla v) + (|u|^\alpha u, v), \quad v \in V.$$

Then by Hölder's inequality

$$|\langle A(u), v\rangle| \leqq (\|u\|_V)^{p-1}\|v\|_V + \||u|^{1+\alpha}\|_{p^*}\|v\|_p.$$

Since $1+\alpha < p-1$, we have from part (c) of Lemma 2.2

$$\||u|^{1+\alpha}\|_{p^*}\|v\|_p \leqq C(\|u\|_V)^{1+\alpha}\|v\|_V.$$

Hence A is a mapping from V into its dual space V'. And denoting by $\| \ \|_{V'}$ the norm in V', we have

$$\|A(u)\|_{V'} \leqq C[(\|u\|_V)^{p-1} + (\|u\|_V)^{1+\alpha}], \quad u \in V, \tag{2.1}$$

where C is a constant independent of u.

For any given $f \in V'$ we consider the equation

$$A(u) = f, \quad u \in V. \tag{2.2}$$

By using part (b) of Lemma 2, we see that (2.2) is equivalent to

$$\begin{cases} -\nabla\cdot(\phi^\mu |\nabla u|^{p-2}\nabla u) + |u|^\alpha u = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2$'$}$$

Now we have

Lemma 2.3. *The equation* (2.2) *has a unique solution* $u \in V$ *and it holds that*

$$\|u\|_V \leqq C(\|f\|_{V'})^{1/(p-1)}, \tag{2.3}$$

where C is independent of u.

Proof. In order to prove the existence of solutions of (2.2), it is enough to show the following properties for A (cf., e.g., [8]):

(i) A is bounded; (ii) A is hemicontinuous; (iii) A is monotone; (iv) A is coercive.

First property (i) is (2.1) itself. We prove property (ii). For $u, v, w \in V$ and $\lambda \in \boldsymbol{R}$

$$\langle A(u+\lambda v),\, w\rangle=(\phi^{\mu}|\nabla(u+\lambda v)|^{p-2}\nabla(u+\lambda v),\, \nabla w)$$
$$+(|u+\lambda v|^{\alpha}(u+\lambda v),\, w).$$

And for λ with $|\lambda|\leqq\lambda_0$ there is a constant C independent of λ such that

$$\phi^{\mu}|\nabla(u+\lambda v)|^{p-1}|\nabla w|\leqq C\phi^{\mu}(|\nabla u|^{p-1}+|\nabla v|^{p-1})|\nabla w|\,,$$

$$|u+\lambda v|^{1+\alpha}|w|\leqq C(|u|^{1+\alpha}+|v|^{1+\alpha})|w|\,.$$

In the same way as deriving (2.1), we see that each term on the right-hand side of these inequalities is integrable in Ω. Thus $(A(u+\lambda v), w)$ is continuous with the variable λ by Lebesgue's theorem, which implies property (ii).

Next we easily see that there is a positive constant c_1 such that for $u, v\in V$

$$\langle A(u)-A(v),\, u-v\rangle\geqq c_1(\|\phi^{\mu/p}\nabla(u-v)\|_p)^p.$$

Thus from part (c) of Lemma 2.2 it holds that

$$\langle A(u)-A(v),\, u-v\rangle\geqq c_2(\|u-v\|_V)^p \tag{2.4}$$

for another positive constant c_2. Hence property (iii) is correct. Setting $v=0$ particularly in (2.4), we have

$$c_2(\|u\|_V)^{p-1}\leqq\langle A(u),\, u\rangle/\|u\|_V$$

from which $\langle A(u), u\rangle/\|u\|_V\to\infty$ as $\|u\|_V\to\infty$ and (iv) is established.

The uniqueness of solutions of (2.2) also follows from (2.4). The inequality (2.3) is clear from (2.4). Q. E. D.

For $\varepsilon>0$ we consider the Dirichlet boundary value problem

$$\begin{cases} -\nabla\cdot((\varepsilon+\phi)(\varepsilon+|\nabla u|^2)^{(p-2)/2}\nabla u)+|u|^{\alpha}u=g & \text{in } \Omega\,, \\ u=0 \quad \text{on } \partial\Omega\,. \end{cases} \tag{2.5}$$

This is an elliptic regularization of (1.1). From our assumptions $\partial\Omega$ is of class $C^{2+\delta}$ for any δ with $0<\delta<1$. The following lemma is due to D. Gilberg and N. S. Trudinger [4, Chap. 14].

Lemma 2.4. ([4]). *If $g\in C^{\delta}(\bar{\Omega})$ for δ with $0<\delta<1$, then there is a solution $u\in C^{2+\delta}(\bar{\Omega})$ of* (2.5).

Let $f\in\widetilde{W}^{-1,p^*}_{-1/(p-1)}(\Omega)$ and let us take an approximating sequence $\{f_j\}\subset C^{\infty}(\bar{\Omega})$ such that $f_j\to f$ in $W^{-1,p^*}_{-1/(p-1)}(\Omega)$ as $j\to\infty$. Further let $\{\varepsilon_j\}$ be a sequence of positive numbers tending to zero. By Lemma 2.4 there is a solution $u_j\in C^{2+\delta}(\bar{\Omega})$ for each j satisfying

$$\begin{cases} -\nabla\cdot((\varepsilon_j+\phi)(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}\nabla u_j)+|u_j|^{\alpha}u_j=f_j & \text{in } \Omega\,, \\ u_j=0 \quad \text{on } \partial\Omega\,, \end{cases} \tag{2.6}$$

where δ is any number with $0<\delta<1$. Integrating by parts, we have from (2.6)

$$\int_{\Omega}(\varepsilon_j+\phi)(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}|\nabla u_j|^2 dx \leqq (f_j, u_j).$$

By Hölder's inequality and part (c) of Lemma 2.2 it follows that

$$|(f_j, u_j)| \leqq C\|f_j\|_{p*}\|u_j\|_{W_1^{1,p}}.$$

We denote by the same C all constants independent of j. Combining the above inequalities we have

$$\|u_j\|_{W_1^{1,p}} \leqq C(\|f_j\|_{p*})^{1/(p-1)}. \tag{2.7}$$

Therefore it follows that

$$\int_{\Omega}(\varepsilon_j+\phi)(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}|\nabla u_j|^2 dx \leqq C(\|f_j\|_{p*})^{p*}$$

and

$$\int_{\Omega}(\varepsilon_j+\phi)(\varepsilon_j+|\nabla u_j|^2)^{p/2} dx \leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{p*})^{p*}]. \tag{2.8}$$

§ 3. Coordinates Transformation

As stated in the first section we are obliged to take an adequate coordinates transformation, in order to estimate the normal derivative of weak solutions of (1.1). Thus we prepare such a coordinates transformation.

Lemma 3.1. *Let $\mathscr{D}$ be a domain in $\boldsymbol{R}^n$. Let $\boldsymbol{v}$ be a real-valued vector function belonging to $[C^m(\mathscr{D})]^n$. Then there is a set of functions $\{u_j\}_{j=1}^{n-1}$ such that $u_j \in C^m(\mathscr{D})$, $|\nabla u_j| \neq 0$, $\nabla u_j \cdot \boldsymbol{v}=0$ and $\nabla u_i \cdot \nabla u_j=0$ in $\mathscr{D}$ if $i \neq j$.*

Proof. If $\boldsymbol{v}=(0, \cdots, 0, v_n)$ particularly, it is enough to take $u_j=x_j$.

For the general case it is easily seen that there is an orthogonal matrix (a_{ij}) of order n such that $a_{ij} \in C^m(\mathscr{D})$ and

$$\sum_j a_{jk} v_j = 0 \quad \text{in } \mathscr{D}, \quad k=1, \cdots, n-1, \tag{3.1}$$

where $\boldsymbol{v}=(v_1, \cdots, v_n)$. We define $\boldsymbol{e}_i=(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0)$ and

$$\boldsymbol{e}_i' = \sum_j a_{ji} \boldsymbol{e}_j, \quad i=1, \cdots, n. \tag{3.2}$$

Then

$$\boldsymbol{e}_i = \sum_j a_{ij} \boldsymbol{e}_j'.$$

Denoting by ∂_i' the differentiation in the direction $\boldsymbol{e}_i'$, we have

$$(\partial'_i f)(x)=\lim_{h\to 0}\frac{1}{h}[f(x+he'_i)-f(x)]$$
$$=\sum_j a_{ji}(\partial_{x_j} f)(x).$$

Hence it holds that

$$\sum_i \partial_{x_i} f\cdot e_i=\sum_i \partial'_i f\cdot e'_i,$$

from which $\nabla f\cdot\nabla g$ is invariant for any two functions f, g under the coordinates transformation (3.2). On the other hand we can write $v=\sum_i v_i a_{in} e'_n$. Therefore the assertion is reduced to the first simple case. This completes the proof.

Q. E. D.

Let ϕ and Ω be the function and the domain in the first Section, respectively. Let P be any fixed point in $\partial\Omega$ and let U be a sufficiently small neighborhood of P, which will be determined later. From our assumption $\nabla\phi\neq 0$ in U. We take the set $\{u_j\}_{j=1}^{n-1}$ in Lemma 3.1, by setting $\mathscr{D}=U$, $v=\nabla\phi$ and $m=2$.

We define the following mapping from U into R^n

$$\Phi:\begin{cases} y_1=u_1(x),\\ \cdots\cdots\cdots\cdots\cdots\\ y_{n-1}=u_{n-1}(x),\\ y_n=\phi(x).\end{cases} \tag{3.3}$$

Then Φ is a one-to-one mapping. Further Φ and Φ^{-1} are of class C^2. The coordinates system $(y_1(x), \cdots, y_n(x))$ defines that of orthogonal curvilinear coordinates. We set $g_{ij}=\sum_k \partial_{y_i} x_k\cdot\partial_{y_j} x_k$ for the original coordinate system $(x_1(y), \cdots, x_n(y))$. Then it is easily seen that $g_{ij}=0$ $(i\neq j)$ and $g_{ii}>c$ for some positive constant c. And we have

$$\nabla_x f\cdot\nabla_x h=\sum_j (g_{jj})^{-1}\partial_{y_j} f\cdot\partial_{y_j} h$$

for any functions f and h. In addition, the Jacobian of Φ^{-1} is written as

$$\left|\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)}\right|=\sqrt{g_{11}\cdots g_{nn}}.$$

We set

$$\theta^{(i)}=(\partial_{y_i} x_1, \cdots, \partial_{y_i} x_n), \quad i=1, \cdots, n-1. \tag{3.4}$$

Then $\theta^{(i)}\in[C^1(U)]^n$. $\theta^{(i)}\neq O$ and $\theta^{(i)}\cdot\nabla_x\phi=0$ in U. Hence $\{\theta^{(i)}\}_{i=1}^{n-1}$ is a vector field tangent to $\partial\Omega$ and it is an orthogonal system.

Let v be in $C_0^1(U)$ and $v=0$ on $\partial\Omega$. Integrating by parts, we have from (2.6)

$$((\varepsilon_j+\phi)(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}\nabla u_j, \nabla v)+(|u_j|^\alpha u_j, v)=(f_j, v). \tag{3.5}$$

Rewriting this with $(y_1, \cdots, y_n)$-variables, we see that

$$\int_{y_n\geq 0} d(\varepsilon_j+y_n)(\varepsilon_j+\sum_{k=1}^n a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\Big(\sum_{k=1}^n a_k\partial_{y_k}u_j\cdot\partial_{y_k}v\Big)dy$$
$$+\int_{y_n\geq 0} d|u_j|^\alpha u_j v\,dy=\int_{y_n\geq 0} df_j v\,dy,$$

where $d=\sqrt{g_{11}\cdots g_{nn}}$ and $a_k=(g_{kk})^{-1}$. We note that $d,\ a_k\in C^1(\Phi(U))$ and d, $a_k>0$ in $\Phi(U)$. From now on we denote by $(\ ,\ )_y$ the inner product of $L^2(\{y_n\geqq 0\})$ with respect to $(y_1,\cdots,y_n)$-variables. The above equality is again rewritten as follows:

$$((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2},\ \sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}v)_y$$
$$+(d|u_j|^\alpha u_j,\ v)_y=(df_j,\ v)_y. \tag{3.6}$$

Here $b_k=da_k$ and v is an arbitrary function in $C_0^1(\Phi(U))$ such that $v=0$ on $y_n=0$.

§ 4. Propositions

Let U be the neighborhood in the previous Section and $\{\theta^{(i)}\}_{i=1}^{n-1}$ be the orthogonal system in (3.4). Then we have

Proposition 4.1. *Let $\eta\in C_0^1(U)$, and let u_j be the solution of* (2.6). *Then it holds that*

$$\|\eta(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}(\theta^{(i)}\cdot\nabla)\nabla u_j\|^2$$
$$\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}],$$

where $i\neq n$ and C is a constant independent of j.

Proof. As we have remarked in the previous Section, (2.6) is reduced to (3.6) in $U\cap\Omega$. We take a neighborhood V of $\Phi(P)$ such that $\Phi(U\cap\Omega)=V\cap\{y_n>0\}$. And we define the following function space with $(y_1,\cdots,y_n)$-variables

$$\dot{C}^m(\overline{V}\cap\{y_n\geqq 0\})=\{u\,;\,u\in C^m(\overline{V}\cap\{y_n\geqq 0\})\ \text{ and }\ u=0\ \text{ on }\ y_n=0\}.$$

From now on we denote by $\|\ \|_y$ the norm in $L^2(\{y_n\geqq 0\})$. Let $\zeta\in C_0^\infty(V)$ with $\zeta\geqq 0$ and $v\in\dot{C}^1(\overline{V}\cap\{y_n\geqq 0\})$. The test function v in (3.6) can be replaced with $-\zeta\partial_{y_i}w$, where $i\neq n$ and $w\in\dot{C}^2(\overline{V}\cap\{y_n\geqq 0\})$. We write $\partial_{y_i}w$ simply by $\partial' w$. Then (3.6) becomes

$$-((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2},\ \sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}(\zeta\partial' w))_y$$
$$-(d|u_j|^\alpha u_j,\ \zeta\partial' w)_y=-(df_j,\ \zeta\partial' w)_y \tag{4.1}$$

Now we calculate each term on the both sides of (4.1). First we see

$$(4.2)\qquad -((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}, \sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}(\zeta\partial' w))_y$$
$$=-((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta, \sum_k b_k\partial_{y_k}u_j\cdot\partial'\partial_{y_k}w)_y$$
$$-((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\partial' w, \sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}\zeta)_y.$$

By integration by parts

$$(4.3)\qquad -((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta, \sum_k b_k\partial_{y_k}u_j\cdot\partial'\partial_{y_k}w)_y$$
$$=((\varepsilon_j+y_n)\partial'(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}, \zeta\sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}w)_y$$
$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\partial'\zeta, \sum_k b_k\partial_{y_k}u_j\cdot\partial_{y_k}w)_y$$
$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta, \sum_k \partial' b_k\cdot\partial_{y_k}u_j\cdot\partial_{y_k}w)_y$$
$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta, \sum_k b_k\partial'\partial_{y_k}w)_y.$$

In addition

$$(4.4)\qquad -(d|u_j|^\alpha u_j, \zeta\partial' w)_y=(1+\alpha)(d|u_j|^\alpha\partial' u_j, \zeta w)_y$$
$$+(d|u_j|^\alpha u_j, \partial'\zeta\cdot w)_y+(\partial' d\cdot|u_j|^\alpha u_j, \zeta w)_y$$

and

$$(4.5)\qquad -(df_j, \zeta\partial' w)_y=(d\partial' f_j, \zeta w)_y+(df_j, \partial'\zeta\cdot w)_y+(\partial' d\cdot f_j, \zeta w)_y.$$

Here we remember that $w\in\dot{C}^2(\overline{V}\cap\{y_n\geqq 0\})$. However, at most first derivatives only appear for w in each term on the right-hand sides of (4.2)-(4.5). Hence it is enough to assume that $w\in\dot{C}^1(\overline{V}\cap\{y_n\geqq 0\})$, if we take an approximating sequence of w. This implies that we can put $w=\partial' u_j$. By an easy computation

$$\partial'(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}$$
$$=(p-2)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-4)/2}(\sum_k a_k\partial_{y_k}u_j\cdot\partial'\partial_{y_k}u_j)$$
$$+\frac{1}{2}(p-2)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-4)/2}(\sum_k \partial' a_k\cdot(\partial_{y_k}u_j)^2).$$

Noting that $b_k=da_k$, we obtain the following equality from the above mentioned

$$(4.6)\qquad -((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\partial'^2 u_j, \sum_k b_k\partial_{y_k}\zeta\cdot\partial_{y_k}u_j)_y$$
$$+\frac{1}{2}(p-2)((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-4)/2}$$
$$\cdot(\sum_k \partial' a_k\cdot(\partial_{y_k}u_j)^2)\zeta, \sum_k b_k\partial_{y_k}u_j\cdot\partial'\partial_{y_k}u_j)_y$$
$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\partial'\zeta, \sum_k b_k\partial_{y_k}u_j\cdot\partial'\partial_{y_k}u_j)_y$$
$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta, \sum_k \partial' b_k\cdot\partial_{y_k}u_j\cdot\partial'\partial_{y_k}u_j)_y$$

$$+((\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}\zeta,\ \sum_k b_k(\partial'\partial_{y_k}u_j)^2)_y$$
$$+(d|u_j|^\alpha u_j,\ \partial'\zeta\cdot\partial'u_j)_y+(\partial'd\cdot|u_j|^\alpha u_j,\ \zeta\partial'u_j)_y$$
$$\leqq(d\partial'f_j,\ \zeta\partial'u_j)_y+(df_j,\ \partial'\zeta\cdot\partial'u_j)_y+(\partial'd\cdot f_j,\ \zeta\partial'u_j)_y.$$

We set the left-hand side of (4.6)$=\sum_{i=1}^7 I_i$. And we put $\zeta=\eta^2$, where $\eta(y)\in C_0^1(V)$, namely, $\eta(x)\in C_0^1(U)$. Let us estimate each I_i.

It is easily seen that

$$\begin{aligned}&|I_1|,\ |I_2|,\ |I_3|,\ |I_4|\\&\leqq C\|\eta(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\partial'\nabla u_j\|_y\\&\qquad\cdot\|(\eta+|\nabla\eta|)(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|_y,\end{aligned}$$

where $\nabla=(\partial y_1,\cdots,\partial y_n)$. Denoting by $\|\ \|_{q,y}$ the norm in $L^q(\{y_n\geqq 0\})$, we have

$$|I_6|,\ |I_7|\leqq C\|\eta y_n^{-1/p}|u_j|^{1+\alpha}\|_{p*,y}\|(\eta+|\nabla\eta|)y_n^{1/p}\nabla u_j\|_{p,y}.$$

Further there is a positive constant c_0 such that

$$c_0\|\eta(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\partial'\nabla u_j\|_y^2\leqq I_5.$$

Next estimating each term on the right-hand side of (4.6), we have

$$\begin{aligned}&|(d\partial'f_j,\ \zeta\partial'u_j)_y|+|(df_j,\ \partial'\zeta\cdot\partial'u_j)_y|+|(\partial'd\cdot f_j,\ \zeta\partial'u_j)_y|\\&\leqq C(\|\eta y_n^{-1/p}f_j\|_{p*,y}+\|\eta y_n^{-1/p}\nabla f_j\|_{p*,y})\|(\eta+|\nabla\eta|)y_n^{1/p}\nabla u_j\|_{p,y}.\end{aligned}$$

Combining the above inequalities with (4.6), we obtain

$$\begin{aligned}(4.7)\qquad&\|\eta(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\partial'\nabla u_j\|_y^2\\&\leqq C[\|(\eta+|\nabla\eta|)(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|_y^2\\&\quad+\|(\eta+|\nabla\eta|)y_n^{1/p}\nabla u_j\|_{p,y}(\|\eta y_n^{-1/p}|u_j|^{1+\alpha}\|_{p*,y}\\&\quad+\|\eta y_n^{-1/p}f_j\|_{p*,y}+\|\eta y_n^{-1/p}\nabla f_j\|_{p*,y})].\end{aligned}$$

Coming back to the original $(x_1,\cdots,x_n)$-space, we use (2.7) and (2.8). Then it follows that

$$\|(\eta+|\nabla\eta|)y_n^{1/p}\nabla u_j\|_{p,y}\leqq C(\|f_j\|_{p*})^{1/(p-1)},$$
$$\|(\eta+|\nabla\eta|)(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|_y^2\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{p*})^{p*}].$$

And obviously

$$\|\eta y_n^{-1/p}f_j\|_{p*,y}+\|\eta y_n^{-1/p}\nabla f_j\|_{p*,y}\leqq C\|f_j\|_{W^{1,p*}_{-1/(p-1)}}.$$

Further we have

$$\|\eta y_n^{-1/p}|u_j|^{1+\alpha}\|_{p*,y}\leqq C(\|\phi^{-1/(p(1+\alpha))}u_j\|_{(1+\alpha)p*})^{1+\alpha}.$$

From our assumptions on p and α we see that $(1+\alpha)p^*<p$ and $(1/p)-1<-(1/p(1+\alpha))$. Thus by (2.7) and part (c) of Lemma 2.2 we have

$$\|\phi^{-1/(p(1+\alpha))}u_j\|_{(1+\alpha)p^*}\leqq C\|\phi^{(1-p)/p}u_j\|_p\leqq C(\|f_j\|_{p^*})^{1/(p-1)}.$$

Combining the above inequalities with (4.7), we conclude that

$$\begin{aligned}(4.8)\qquad &\|\eta(\varepsilon_j+y_n)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\partial'\nabla u_j\|_y^2\\ &\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].\end{aligned}$$

Since $\partial'=\partial_{y_i}=\theta^{(i)}\cdot\nabla$ $(i\neq n)$, the proof is completed with the aid of (2.8) and (4.8). Q.E.D.

We repeat the proof of Proposition 4.1 without reducing (2.6) to (3.6). However we replace $\eta\in C_0^1(U)$ with $\eta\in C_0^1(\Omega)$. And as a test function we take $-\eta^2\partial_{x_i}w$, where $w=\partial_{x_i}u_j$ with $1\leqq i\leqq n$. Then the following proposition is easily obtained:

Proposition 4.2. *Let $\eta\in C_0^1(\Omega)$, and let u_j be the solution of (2.6). Then it holds that*

$$\begin{aligned}&\|\eta(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\partial_{x_k}\partial_{x_l}u_j\|^2\\ &\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}],\quad 1\leqq k,\ l\leqq n,\end{aligned}$$

where C is a constant depending on η and not on j.

§5. Proof of Theorem 1

Let $\{\theta^{(i)}\}_{i=1}^{n-1}$ be the vector fields in (3.4). We supplement $\theta^{(n)}=(\partial_{y_n}x_1,\cdots,\partial_{y_n}x_n)$ to them. Then $\{\theta^{(i)}\}_{i=1}^{n}$ are linearly independent in U. Hence there are functions $\xi_i(x)\in C^1(\bar\Omega\cap U)$ $(1\leqq i\leqq n)$ such that

$$\theta=\sum_{i=1}^{n}\xi_i(x)\theta^{(i)}\quad\text{in } \bar\Omega\cap U.$$

From the assumption on θ we see that $\theta\cdot\nabla\phi=0$ on $\partial\Omega$. On the other hand $\theta^{(i)}\cdot\nabla\phi=0$ for $i\neq n$ and $\theta^{(n)}\cdot\nabla\phi=1$ in U. Thus $\xi_n(x)=0$ on $\partial\Omega\cap U$.

Let $\tilde\theta=\sum_{i=1}^{n-1}\xi_i(x)\theta^{(i)}$ and $\eta\in C_0^1(U)$. Then we have the following inequality by Proposition 4.1:

$$\begin{aligned}(5.1)\qquad &\|\eta(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}(\tilde\theta\cdot\nabla)\nabla u_j\|^2\\ &\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].\end{aligned}$$

In the proof Proposition 4.1 we replace by $-\eta^2\xi_n^2\partial_{y_n}w$ the test function v in (3.6), where $w\in C^2(\bar V\cap\{y_n\geqq 0\})$. By taking an approximating sequence, we can take w from $C^1(\bar V\cap\{y_n\geqq 0\})$. We put next $w=\partial_{y_n}u_j$ particularly. Since $\partial_{y_n}=\theta^{(n)}\cdot\nabla$, it is easy to see that

(5.2)
$$\|\eta(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}\xi_n(\theta^{(n)}\cdot\nabla)\nabla u_j\|^2$$
$$\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].$$

From (5.1) and (5.2) it follows that

$$\|\eta(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}(\theta\cdot\nabla)\nabla u_j\|^2$$
$$\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].$$

Hence by Proposition 4.2 and by a partition of unity in Ω we obtain

(5.3)
$$\|(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/4}(\theta\cdot\nabla)\nabla u_j\|^2$$
$$\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].$$

From now on we denote by the same $\{u_{j'}\}$ any subsequence of $\{u_j\}$. And we write simply by ∂ any differential ∂_{x_i} $(1\leqq i\leqq n)$. We omit sometimes the notation of sums with respect to i. Obviously

$$|\nabla((\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}\partial u_j)|\leqq C(\varepsilon_j+|\nabla u_i|^2)^{(p-2)/2}|\nabla\partial u_j|.$$

Let Ω' be a subdomain of Ω with $\bar{\Omega}'\subset\Omega$ such that $\partial\Omega'$ is appropriately smooth. Since $p^*(p-2)/(2-p^*)=p$, we get by Hölder's inequality

$$\int_{\Omega'}|\nabla((\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}\partial u_j)|^{p^*}dx$$
$$\leqq C\Big(\int_{\Omega'}(\varepsilon_j+|\nabla u_j|^2)^{p/2}dx\Big)^{(2-p^*)/2}$$
$$\cdot\Big(\int_{\Omega'}(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}|\nabla\partial u_j|^2dx\Big)^{p^*/2}.$$

We write the space $W_0^{1,p^*}(\Omega')$ simply by $W^{1,p^*}(\Omega')$, where W_0^{1,p^*} is in the sense of W_μ^{1,p^*} with $\mu=0$. Thus the norm $\|\ \|_{W_0^{1,p^*}(\Omega')}$ equals $\|\ \|_{W^{1,p^*}(\Omega')}$. Combining the above inequality, (2.8) and Proposition 4.2, we obtain

(5.4)
$$\|(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}\nabla u_j\|_{W^{1,p^*}(\Omega')}\leqq C(\Omega'),$$

where $C(\Omega')$ is a constant depending on Ω' and not on j.

By (5.4) and Sobolev's compact imbedding theorem there are $\{g_i\}_{i=1}^n\subset L^{p^*}_{\mathrm{loc}}(\Omega)$ such that for any subdomain Ω' of Ω with $\bar{\Omega}'\subset\Omega$

(5.5)
$$(\varepsilon_{j'}+|\nabla u_{j'}|^2)^{(p-2)/2}\partial_{x_i}u_{j'}\longrightarrow g_i\quad\text{in } L^{p^*}(\Omega'),$$

which implies

$$(\varepsilon_{j'}+|\nabla u_{j'}|^2)^{(p-2)/2}\partial_{x_i}u_{j'}\longrightarrow g_i\quad\text{a.e. in } \Omega.$$

Accordingly there are $\{h_i\}_{i=1}^n$ satisfying

(5.6)
$$\partial_{x_i}u_{j'}\longrightarrow h_i\quad\text{a.e. in } \Omega$$

and

$$g_i=(\sum_k h_k^2)^{(p-2)/2}h_i.$$

Since $\sum_i g_i^2=(\sum_i h_i^2)^{p-1}$, we have $h_i\in L_{\mathrm{loc}}^p(\Omega)$.

On the other hand each u_j is in $\mathring{W}_1^{1,p}(\Omega)$ by part (b) of Lemma 2.2. And $\{\|u_j\|_{W_1^{1,p}}\}$ are uniformly bounded by (2.7). Hence there is a function $u\in\mathring{W}_1^{1,p}(\Omega)$ satisfying

$$u_{j'}\longrightarrow u \quad \text{in } \mathring{W}_1^{1,p}(\Omega).$$

And by Sobolev's compact imbedding theorem

$$u_{j'}\longrightarrow u \quad \text{in } L^p(\Omega') \tag{5.7}$$

for any Ω' with $\bar{\Omega}'\subset\Omega$. By virtue of Lemma 2.1 we see that $h_i=\partial_{x_i}u$. Thus $g_i=|\nabla u|^{p-2}\partial_{x_i}u$.

From (5.5) we have

$$(\phi(\varepsilon_{j'}+|\nabla u_{j'}|^2)^{(p-2)/2}\nabla u_{j'},\ \nabla v)\longrightarrow(\phi|\nabla u|^{p-2}\nabla u,\ \nabla v),\quad v\in C_0^1(\Omega).$$

Since $(\alpha+1)p^*<p$, it follows from part (c) of Lemma 2.2 that

$$\begin{aligned}\||u_j|^\alpha u_j\|_{p^*}&\leqq C(\|u_j\|_p)^{1+\alpha}\\ &\leqq C(\|u_j\|_{W_1^{1,p}})^{1+\alpha},\end{aligned} \tag{5.8}$$

so that $\{\||u_j|^\alpha u_j\|_{p^*}\}$ are uniformly bounded. Since $u\in L^p(\Omega)$, $|u|^\alpha u\in L^{p^*}(\Omega)$. Thus it holds from Lemma 2.1 that

$$(|u_{j'}|^\alpha u_j,\ v)\longrightarrow(|u|^\alpha u,\ v),\quad v\in C_0^1(\Omega).$$

And naturally

$$(f_j,\ v)\longrightarrow(f,\ v),\quad v\in C_0^1(\Omega).$$

From the above and (2.6) it follows that for any $v\in C_0^1(\Omega)$

$$(\phi|\nabla u|^{p-2}\nabla u,\ \nabla v)+(|u|^\alpha u,\ v)=(f,\ v). \tag{5.9}$$

We show that (5.9) is valid for any $v\in\mathring{W}_1^{1,p}(\Omega)$. We take an approximating sequence $\{v_j\}\subset C_0^\infty(\Omega)$ such that $v_j\to v$ in $W_1^{1,p}(\Omega)$. From (5.9)

$$(\phi|\nabla u|^{p-2}\nabla u,\ \nabla v_j)+(|u|^\alpha u,\ v_j)=(f,\ v_j). \tag{5.10}$$

Since

$$|(\phi|\nabla u|^{p-2}\nabla u,\ \nabla(v_j-v))|\leqq C(\|u\|_{W_1^{1,p}})^{p-1}\|v_j-v\|_{W_1^{1,p}},$$

the first term on the left-hand side of (5.10) tends to $(\phi|\nabla u|^{p-2}\nabla u,\ \nabla v)$. Similarly as in (5.8) we have

$$|(|u|^\alpha u,\ v_j-v)|\leqq C(\|u\|_{W_1^{1,p}})^{1+\alpha}\|v_j-v\|_{W_1^{1,p}}.$$

Hence the second term on the left-hand side of (5.10) tends to $(|u|^\alpha u,\ v)$. And the inequality

$$|(f, v_j-v)|\leqq C\|f\|_{p*}\|v_j-v\|_{W_1^{1,p}}$$

yields that $(f, v_j)\to(f, v)$. From the above mentioned we conclude that $u\in\mathring{W}_1^{1,p}(\Omega)$ is a weak solution of (2.2) with $\mu=1$.

Now by using the coordinates transformation (3.3), we have the following inequality from the assumption on ϕ

$$|(\theta\cdot\nabla)\phi|\leqq C\phi \quad \text{in } \Omega. \tag{5.11}$$

We consider again the solution u_j of (2.6). From (5.11)

$$\begin{aligned}&\|(\theta\cdot\nabla)[(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}]\|\\&\leqq C[\|(\varepsilon_j+\phi)^{1/2}(\theta\cdot\nabla)(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|\\&\qquad+\|(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|].\end{aligned}$$

Thus the family $\{\|(\theta\cdot\nabla)[(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}]\|\}$ is uniformly bounded by (5.3) and (2.8). Accordingly there is a function $w\in L^2(\Omega)$ such that

$$((\theta\cdot\nabla)[(\varepsilon_{j'}+\phi)^{1/2}(\varepsilon_{j'}+|\nabla u_{j'}|^2)^{p/4}], v)\longrightarrow(w, v), \quad v\in C_0^1(\Omega). \tag{5.12}$$

On the other hand

$$\begin{aligned}&((\theta\cdot\nabla)[(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}], v)\\&=-((\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}, (\theta\cdot\nabla)v)\\&\quad-((\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}, (\nabla\cdot\theta)v)\end{aligned} \tag{5.13}$$

and the family $\{\|(\varepsilon_j+\phi)^{1/2}(\varepsilon_j+|\nabla u_j|^2)^{p/4}\|\}$ is uniformly bounded from (2.8). Further $u\in\mathring{W}_1^{1,p}(\Omega)$ and $\partial_{x_i}u_j\to\partial_{x_i}u$ a.e. in Ω from (5.6). Therefore by Lemma 2.1 we see that the first term (the second term) on the right-hand side of (5.13)$\to-(\phi^{1/2}|\nabla u|^{p/2}, (\theta\cdot\nabla)v)$ $(-(\phi^{1/2}|\nabla u|^{p/2}, (\nabla\cdot\theta)v))$, which implies that

$$w=(\theta\cdot\nabla)(\phi^{1/2}|\nabla u|^{p/2})$$

from (5.12) and (5.13). Thus we obtain

$$\|(\theta\cdot\nabla)(\phi^{1/2}|\nabla u|^{p/2})\|\leqq\varliminf_{j'\to\infty}\|(\theta\cdot\nabla)[(\varepsilon_{j'}+\phi)^{1/2}(\varepsilon_{j'}+|\nabla u_{j'}|^2)^{p/4}]\|.$$

Combining (5.3), (5.11), (2.8) and this inequality, we have completed the proof of Theorem 1. Q.E.D.

§6. Proof of Theorems 2 and 3

First we prepare the following lemma:

Lemma 6.1. *Let* $0<\beta<1$. *Then for* $v\in C^1(\bar{\Omega})$

$$\int_\Omega \phi^{-\beta} v^2 dx \leqq C(\beta) \int_\Omega \phi^{2-\beta}(v^2 + |\nabla v|^2) dx,$$

where $C(\beta)$ is a constant depending on β and not on v.

Proof. For $P \in \partial\Omega$ we take the neighborhood U of P such that (3.3) is defined. It is enough to show that for $\eta \in C_0^1(U)$

$$\int_\Omega \phi^{-\beta} (\eta v)^2 dx \leqq C \int_\Omega \phi^{2-\beta} |\nabla(\eta v)|^2 dx.$$

For this sake it is sufficient to prove that

$$\int_0^\infty t^{-\beta} w(t)^2 dt \leqq C \int_0^\infty t^{2-\beta} w'(t)^2 dt, \tag{6.1}$$

where $w \in C^1([0, \infty))$ and $w(t)=0$ for large t. By an integration by parts

$$\int_0^\infty t^{-\beta} w(t)^2 dt = \frac{2}{\beta-1} \int_0^\infty t^{1-\beta} w w' dt.$$

Using Schwarz inequality, we have

$$\int_0^\infty t^{-\beta} w(t)^2 dt \leqq C\left(\int_0^\infty t^{-\beta} w(t)^2 dt\right)^{1/2} \left(\int_0^\infty t^{2-\beta} w'(t)^2 dt\right)^{1/2},$$

from which (6.1) follows. Q. E. D.

Proof of Theorem 2. First we see that

$$\begin{aligned} &|\nabla(\varepsilon_j + |(\theta\cdot\nabla)u_j|^2)^{p/4}| \\ &\leqq C[(\varepsilon_j + |\nabla u_j|^2)^{(p-2)/4} |(\theta\cdot\nabla)\nabla u_j| + (\varepsilon_j + |\nabla u_j|^2)^{p/4}]. \end{aligned}$$

Hence

$$\begin{aligned} &\|(\varepsilon_j+\phi)^{1/2} \nabla(\varepsilon_j + |(\theta\cdot\nabla)u_j|^2)^{p/4}\|^2 \\ &\leqq C[\|(\varepsilon_j+\phi)^{1/2}(\varepsilon_j + |\nabla u_j|^2)^{(p-2)/4}(\theta\cdot\nabla)\nabla u_j\|^2 \\ &\qquad + \|(\varepsilon_j+\phi)^{1/2}(\varepsilon_j + |\nabla u_j|^2)^{p/4}\|^2]. \end{aligned}$$

Combining this inequality, (2.8) and (5.3), we obtain

$$\begin{aligned} &\|(\varepsilon_j+\phi)^{1/2} \nabla(\varepsilon_j + |(\theta\cdot\nabla)u_j|^2)^{p/4}\|^2 \\ &\leqq C[\varepsilon_j^{p/2} + (\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*} + (\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}]. \end{aligned} \tag{6.2}$$

Therefore it follows from (2.8) and Lemma 6.1 that

$$\begin{aligned} &(\|\phi^{-\beta/p}(\varepsilon_j + |(\theta\cdot\nabla)u_j|^2)^{1/2}\|_p)^p \\ &\leqq C[\varepsilon_j^{p/2} + (\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*} + (\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}] \end{aligned} \tag{6.3}$$

for β with $0<\beta<1$.

By (6.2) and Sobolev's compact imbedding theorem there is a function $w\in L^2_{\mathrm{loc}}(\Omega)$ such that

$$(\varepsilon_{j'}+|(\theta\cdot\nabla)u_{j'}|^2)^{p/4}\longrightarrow w \quad \text{in } L^2(\Omega')$$

for any subdomain Ω' with $\bar{\Omega}'\subset\Omega$. On the other hand from (5.6)

$$(\varepsilon_{j'}+|(\theta\cdot\nabla)u_{j'}|^2)^{p/4}\longrightarrow |(\theta\cdot\nabla)u|^{p/2} \quad \text{a.e. in } \Omega.$$

Hence we have

$$(\varepsilon_{j'}+|(\theta\cdot\nabla)u_{j'}|^2)^{p/4}\longrightarrow |(\theta\cdot\nabla)u|^{p/2} \quad \text{in } L^2(\Omega').$$

Combining this with (6.3), we obtain

$$\begin{aligned}\int_{\Omega'}\phi^{-\beta}|(\theta\cdot\nabla)u|^p dx&=\lim_{j'\to\infty}\int_{\Omega'}\phi^{-\beta}(\varepsilon_{j'}+|(\theta\cdot\nabla)u_{j'}|^2)^{p/2}dx\\ &\leqq C[(\|f\|_{W^{1,p^*}_{-1/(p-1)}})^{\chi^*}+(\|f\|_{p^*})^{\chi^*(1+\alpha)/(p-1)}],\end{aligned}$$

where C is independent of Ω' and f. Since Ω' is an arbitrary subdomain of Ω with $\bar{\Omega}'\subset\Omega$, we complete the proof of Theorem 2. Q.E.D.

Before proving Theorem 3 we prepare the following proposition:

Proposition 6.1. *Let u_j be the solution of* (2.6). *If $\gamma>1/(p-1)$, it holds that*

$$\varepsilon_j^{1+\gamma}\int_{\partial\Omega}(\varepsilon_j+|\nabla u_j|^2)^{p/2}dS\longrightarrow 0 \quad \text{as } j\to\infty,$$

where dS is the surface element of $\partial\Omega$.

Proof. Taking the new coordinates $(y_1,\cdots,y_n)$ defined in (3.3), we consider in $(y_1,\cdots,y_n)$-space. Let U be the neighborhood of $P\in\partial\Omega$ such that (3.3) is defined. We take $\eta\in C^1_0(U)$ and denote $y'=(y_1,\cdots y_{n-1})$. Then it is sufficient to prove that

$$\varepsilon_j^{1+\gamma}\int_{y_n=0}\eta^2(\varepsilon_j+|\nabla u_j|^2)^{p/2}dy'\longrightarrow 0 \quad \text{as } j\to\infty. \tag{6.4}$$

From (3.6) we can write

$$\begin{aligned}&\partial_{y_n}(\eta(\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}b_n\partial_{y_n}u_j)\\ &=-\eta(\varepsilon_j+y_n)\sum_{k\neq n}\partial_{y_k}((\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}b_k\partial_{y_k}u_j)\\ &\qquad+\eta d|u_j|^\alpha u_j-\eta df_j\\ &\quad+\partial_{y_n}\eta\cdot(\varepsilon_j+y_n)(\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2)^{(p-2)/2}b_n\partial_{y_n}u_j.\end{aligned} \tag{6.5}$$

Setting

(6.6) $$F_j=\varepsilon_j+\sum_k a_k(\partial_{y_k}u_j)^2,$$

we have

$$|\partial_{y_k}(F_j^{(p-2)/2}b_k\partial_{y_k}u_j)|\leqq C(F_j^{(p-1)/2}+F_j^{(p-2)/2}|\partial_{y_k}\nabla u_j|).$$

By integrating both sides of (6.5) with y_n, we obtain therefore

(6.7) $$\begin{aligned}&\varepsilon_j\eta(y',0)F_j(y',0)^{(p-2)/2}|(\partial_{y_n}u_j)(y',0)|\\&\leqq C\Big[\int_0^\infty(\eta+|\nabla\eta|)(\varepsilon_j+y_n)F_j^{(p-1)/2}dy_n\\&\qquad+\int_0^\infty\eta(\varepsilon_j+y_n)F_j^{(p-2)/2}(\sum_{k\neq n}|\partial_{y_k}\nabla u_j|)dy_n\\&\qquad+\int_0^\infty\eta|u_j|^{1+\alpha}dy_n+\int_0^\infty\eta|f_j|dy_n\Big].\end{aligned}$$

In general it holds that for ε, $A\geqq 0$

(6.8) $$\varepsilon^{p^*}(\varepsilon+A^2)^{p/2}\leqq C[\varepsilon^{p^*(p+1)/2}+\varepsilon^{p^*}(A(\varepsilon+A^2)^{(p-2)/2})^{p^*}],$$

where C is a constant independent of ε and A. In fact

$$\varepsilon(\varepsilon+A^2)^{(p-1)/2}\leqq C[\varepsilon^{3/2}(\varepsilon+A^2)^{(p-2)/2}+\varepsilon A(\varepsilon+A^2)^{(p-2)/2}].$$

And by Young's inequality we get

$$\begin{aligned}\varepsilon^{3/2}(\varepsilon+A^2)^{(p-2)/2}&=\varepsilon^{(p+1)/(2(p-1))}\cdot\varepsilon^{(p-2)/(p-1)}(\varepsilon+A^2)^{(p-2)/2}\\&\leqq\delta\varepsilon(\varepsilon+A^2)^{(p-1)/2}+C(\delta)\varepsilon^{(p+1)/2},\quad \delta>0.\end{aligned}$$

Thus (6.8) is correct. From (6.7) and (6.8) it follows that

(6.9) $$\begin{aligned}&\varepsilon_j^{p^*}\int_{y_n=0}\eta^{p^*}(\varepsilon_j+|\partial_{y_n}u_j|^2)^{p/2}dy'\\&\leqq C\Big[\varepsilon_j^{p^*(p+1)/2}+\int_{y_n=0}(\varepsilon_j\eta F_j^{(p-2)/2}|(\partial_{y_n}u_j)|)^{p^*}dy'\Big]\\&\leqq C\Big[\varepsilon_j^{p^*(p+1)/2}+\int_{y_n\geqq 0}(\eta+|\nabla\eta|)(\varepsilon_j+y_n)^{p^*}F_j^{p/2}dy\\&\qquad+\int_{y_n\geqq 0}\eta^{p^*}(\varepsilon_j+y_n)^{p^*}F_j^{p^*(p-2)/2}(\sum_{k\neq n}|\partial_{y_k}\nabla u_j|^{p^*})dy\\&\qquad+\int_{y_n\geqq 0}\eta|u_j|^{p^*(1+\alpha)}dy+\int_{y_n\geqq 0}\eta|f_j|^{p^*}dy\Big].\end{aligned}$$

Using Hölder's inequality, we have for $k\neq n$

$$\begin{aligned}&\int_{y_n\geqq 0}\eta^{p^*}(\varepsilon_j+y_n)^{p^*}F_j^{p^*(p-2)/2}|\partial_{y_k}\nabla u_j|^{p^*}dy\\&\leqq C\int_{\{y_n\geqq 0\}\cap\operatorname{supp}\eta}(\varepsilon_j+y_n)^{p^*/2}F_j^{p^*(p-2)/4}\\&\qquad\cdot\eta^{p^*}(\varepsilon_j+y_n)^{p^*/2}F_j^{p^*(p-2)/4}|\partial_{y_k}\nabla u_j|^{p^*}dy\end{aligned}$$

$$\leqq C\Big(\int_{\{y_n\geqq 0\}\cap \operatorname{supp}\eta}(\varepsilon_j+y_n)^{p/(p-2)}(\varepsilon_j+|\nabla u_j|^2)^{p/2}dy\Big)^{(2-p^*)/2}$$
$$\cdot\Big(\int_{y_n\geqq 0}\eta^2(\varepsilon_j+y_n)(\varepsilon_j+|\nabla u_j|^2)^{(p-2)/2}|\partial_{y_k}\nabla u_j|^2dy\Big)^{p^*/2},$$

where we have used the equality $p^*(p-2)/(2-p^*)=p$. And similarly as in (5.8)

$$\int_{y_n\geqq 0}\eta|u_j|^{p^*(1+a)}dy\leqq C(\|u_j\|_{W_1^{1,p}})^{p^*(1+a)}.$$

Combining the above, (6.9), (2.8) and Proposition 4.1, we obtain

$$\varepsilon_j^{p^*}\int_{y_n=0}\eta^{p^*}(\varepsilon_j+|\partial_{y_n}u_j|^2)^{p/2}dy'\leqq C.$$

By using Theorem 2 we can prove more easily that for $k\neq n$

$$\varepsilon_j^{p^*}\int_{y_n=0}\eta^2(\varepsilon_j+|\partial_{y_k}u_j|^2)^{p/2}dy'\leqq C.$$

Therefore we conclude that

$$\varepsilon_j^{p^*}\int_{y_n=0}\eta^2(\varepsilon_j+|\nabla u_j|^2)^{p/2}dy'\leqq C,$$

which implies (6.4), because $1+\gamma>p^*$. Thus we have finished the proof.

Q. E. D.

Proof of Theorem 3. We consider in $(y_1, \cdots, y_n)$-space defined in (3.3). Let U be the neighborhood of $P\in\partial\Omega$ where (3.3) is defined. Let $\eta\in C_0^1(U)$ and F be the function in (6.6).

By an integration by parts

$$\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^\gamma F_j^{p/2}dy=-\frac{\varepsilon_j^{1+\gamma}}{1+\gamma}\int_{y_n=0}\eta F_j^{p/2}dy'$$
$$-\frac{p}{2(1+\gamma)}\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_n}F_jdy$$
$$-\frac{1}{1+\gamma}\int_{y_n\geqq 0}\partial_{y_n}\eta\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{p/2}dy.$$

Since

$$\partial_{y_n}F_j=2\sum_k a_k\partial_{y_k}u_j\cdot\partial_{y_k}\partial_{y_n}u_j+\sum_k\partial_{y_n}a_k\cdot(\partial_{y_k}u_j)^2,$$

we see that

$$\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_n}F_jdy$$
$$=2\int_{y_n\geqq 0}\eta a_n(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_n}u_j\cdot\partial_{y_n}^2u_jdy$$

$$+2\sum_{k\neq n}\int_{y_n\geq 0}\eta a_k(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_k}u_j\cdot\partial_{y_k}\partial_{y_n}u_j dy$$
$$+\sum_k\int_{y_n\geq 0}\eta\partial_{y_n}a_k\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}(\partial_{y_k}u_j)^2dy$$

and

$$\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_n}u_j\cdot\partial_{y_n}^2u_j dy$$
$$=-\varepsilon_j^{1+\gamma}\int_{y_n=0}\eta a_nF_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy'$$
$$-\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{\gamma}\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)\partial_{y_n}u_j dy$$
$$-\gamma\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{\gamma}F_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy$$
$$-\int_{y_n\geq 0}\partial_{y_n}(\eta a_n)\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy\,.$$

Combining the above inequalities we obtain

$$\int_{y_n\geq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{p/2}dy \tag{6.10}$$
$$=-\frac{\varepsilon_j^{1+\gamma}}{1+\gamma}\int_{y_n=0}\eta F_j^{p/2}dy'$$
$$+\frac{p}{1+\gamma}\varepsilon_j^{1+\gamma}\int_{y_n=0}\eta a_nF_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy'$$
$$+\frac{p}{1+\gamma}\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{\gamma}\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)\cdot\partial_{y_n}u_j dy$$
$$+\frac{p\gamma}{1+\gamma}\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{\gamma}F_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy$$
$$+\frac{p}{1+\gamma}\int_{y_n\geq 0}\partial_{y_n}(\eta a_n)\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy$$
$$-\frac{p}{1+\gamma}\sum_{k\neq n}\int_{y_n\geq 0}\eta a_k(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}\partial_{y_k}u_j\cdot\partial_{y_k}\partial_{y_n}u_j dy$$
$$-\frac{p}{2(1+\gamma)}\sum_k\int_{y_n\geq 0}\eta\partial_{y_n}a_k\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-2)/2}(\partial_{y_k}u_j)^2dy$$
$$-\frac{1}{1+\gamma}\int_{y_n\geq 0}\partial_{y_n}\eta\cdot(\varepsilon_j+y_n)^{1+\gamma}F_j^{p/2}dy\,.$$

On the right-hand side of (6.10) the integral of the fourth term is rewritten as follows:

$$\int_{y_n\geq 0}\eta a_n(\varepsilon_j+y_n)^{\gamma}F_j^{(p-2)/2}(\partial_{y_n}u_j)^2dy$$
$$=\int_{y_n\geq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{p/2}dy-\varepsilon_j\int_{y_n\geq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{(p-2)/2}dy$$

$$-\sum_{k\neq n}\int_{y_n\geqq 0}\eta a_k(\varepsilon_j+y_n)^\gamma F_j^{(p-2)/2}(\partial_{y_k}u_j)^2dy\,.$$

We insert this in the fourth term on the right-hand side of (6.10). Then using the inequality $p\gamma/(1+\gamma)>1$, we find

$$\begin{aligned}(6.11)\qquad &\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^\gamma F_j^{p/2}dy\\ &\leqq C\Big[\varepsilon_j^{1+\gamma}\int_{y_n=0}\eta F_j^{p/2}dy'\\ &\quad+\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^\gamma|\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)|\,|\nabla u_j|\,dy\\ &\quad+\varepsilon_j\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^\gamma F_j^{(p-2)/2}dy\\ &\quad+\sum_{k\neq n}\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^\gamma F_j^{(p-2)/2}(\partial_{y_k}u_j)^2dy\\ &\quad+\int_{y_n\geqq 0}(\eta+|\nabla\eta|)(\varepsilon_j+y_n)^{1+\gamma}F_j^{p/2}dy\\ &\quad+\sum_{k\neq n}\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-1)/2}|\partial_{y_k}\nabla u_j|\,dy\Big]\\ &\equiv C\sum_{l=1}^{6}J_l,\ \text{say}.\end{aligned}$$

First we have by Proposition 6.1

$$J_1\longrightarrow 0\quad\text{as } j\to\infty\,.$$

Next

$$J_2\leqq\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma p}|\nabla u_j|^p dy\Big)^{1/p}\cdot\Big(\int_{y_n\geqq 0}\eta|\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)|^{p*}dy\Big)^{1/p*}.$$

Since $\gamma p>1$, it follows from (2.8) that

$$\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma p}|\nabla u_j|^p dy\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{p*})^{p*}]\,.$$

From (3.6)

$$\begin{aligned}b_n\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)=&-\partial_{y_n}b\cdot(\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j\\ &-(\varepsilon_j+y_n)\sum_{k\neq n}\partial_{y_k}(F_j^{(p-2)/2}b_k\partial_{y_k}u_j)\\ &+d|u_j|^\alpha u_j-df_j\,.\end{aligned}$$

Hence

$$\begin{aligned}&|\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)|^{p*}\\ &\leqq C[(\varepsilon_j+y_n)F_j^{p/2}+(\varepsilon_j+y_n)\sum_{k\neq n}F_j^{p*(p-2)/2}|\partial_{y_k}\nabla u_j|^{p*}\\ &\qquad+|u_j|^{(1+\alpha)p*}+|f_j|^{p*}]\,.\end{aligned}$$

Therefore using the equality $p^*(p-2)/(2-p^*)=p$, we obtain by Hölder's inequality

$$\begin{aligned}&\int_{y_n\geqq 0}\eta|\partial_{y_n}((\varepsilon_j+y_n)F_j^{(p-2)/2}\partial_{y_n}u_j)|^{p^*}dy\\&\leqq C\Big[\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)F_j^{p/2}dy\\&\quad+\sum_{k\neq n}\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)F_j^{p/2}dy\Big)^{(2-p^*)/2}\\&\qquad\cdot\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)F_j^{(p-2)/2}|\partial_{y_k}\nabla u_j|^2dy\Big)^{p^*/2}\\&\quad+\int_{y_n\geqq 0}\eta|u_j|^{(1+\alpha)p^*}dy+\int_{y_n\geqq 0}\eta|f_j|^{p^*}dy\Big].\end{aligned}$$

Further we use (2.7), (2.8), Proposition 4.1 and the similar inequality as in (5.8). Then we conclude that

$$J_2\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].$$

By Young's inequality it is obvious that

$$J_3\leqq\delta\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{p/2}dy+C(\delta)\varepsilon_j^{p/2},\quad \delta>0.$$

If $k\neq n$,

$$\begin{aligned}&\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{(p-2)/2}(\partial_{y_k}u_j)^2dy\\&\leqq\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{p\gamma/(p-2)}F_j^{p/2}dy\Big)^{(p-2)/p}\Big(\int_{y_n\geqq 0}\eta|\partial_{y_k}u_j|^pdy\Big)^{2/p}\\&\leqq\delta\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{p/2}dy+C(\delta)\int_{y_n\geqq 0}\eta|\partial_{y_k}u_j|^pdy.\end{aligned}$$

Hence it follows from (6.3) that

$$\begin{aligned}J_4\leqq&\delta\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{\gamma}F_j^{p/2}dy\\&+C(\delta)[\varepsilon_j^{p/2}+(\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*}+(\|f_j\|_{p^*})^{p^*(1+\alpha)/(p-1)}].\end{aligned}$$

We have immediately from (2.8)

$$J_5\leqq C[\varepsilon_j^{p/2}+(\|f_j\|_{p^*})^{p^*}].$$

Lastly we estimate J_6. If $k\neq n$,

$$\begin{aligned}&\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)^{1+\gamma}F_j^{(p-1)/2}|\partial_{y_k}\nabla u_j|dy\\&\leqq\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)F_j^{p/2}dy\Big)^{1/2}\\&\qquad\cdot\Big(\int_{y_n\geqq 0}\eta(\varepsilon_j+y_n)F_j^{(p-2)/2}|\partial_{y_k}\nabla u_j|^2dy\Big)^{1/2}.\end{aligned}$$

Thus we get from (2.8) and Proposition 4.1

$$J_6 \leqq C[\varepsilon_j^{p/2} + (\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*} + (\|f_j\|_{p^*})^{p^*(1+a)/(p-1)}].$$

Combining the above inequalities with (6.11), we conclude that

$$\int_{y_n \geqq 0} \eta(\varepsilon_j + y_n)^\gamma F_j^{p/2} dy$$
$$\leqq C[\mu_j + (\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*} + (\|f_j\|_{p^*})^{p^*(1+a)/(p-1)}],$$

where $\mu_j \to 0$ as $j \to \infty$. Therefore it follows by partition of unity for Ω that

$$\int_\Omega \phi^\gamma |\nabla u_j|^p dx \leqq C[\mu_j + (\|f_j\|_{W^{1,p^*}_{-1/(p-1)}})^{p^*} + (\|f_j\|_{p^*})^{p^*(1+a)/(p-1)}]. \tag{6.12}$$

Without loss of generality we may assume that $\gamma < p-1$. From part (b) of Lemma 2.2 we see that $u_j \in \mathring{W}^{1,p}_\gamma(\Omega)$. Moreover, the family $\{\|u_j\|_{W^{1,p}_\gamma}\}$ is uniformly bounded by virtue of (6.12) and part (c) of Lemma 2.2. Hence there is a function $v \in \mathring{W}^{1,p}_\gamma(\Omega)$ such that $u_{j'} \to v$ in $\mathring{W}^{1,p}_\gamma(\Omega)$. From this and (5.7) we have $v = u$, where u is the solution of (1.1). Therefore we obtain

$$\|u\|_{W^{1,p}_\gamma} \leqq \varliminf_{j' \to \infty} \|u_{j'}\|_{W^{1,p}_\gamma}.$$

Combining this inequality with (6.12), we complete the proof of Theorem 3.

Q. E. D.

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