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Type I Orbits in the Pure States of a C^*-Dynamical System. II

By

Akitaka KISHIMOTO*

Abstract

For a C^*-algebra A with an action of a locally compact abelian group G, one considers the pure states of A with the associated action. Type I orbits are defined and studied in the previous paper [4]. We continue this study; in particular, we shall show that if A is separable and simple and if there is a type I orbit through a pure state f with trivial stabilizer $\{t\in G: \pi_f\circ\alpha_t\sim\pi_f\}=\{0\}$, then there is a type I orbit with stabilizer equal to any given closed subgroup of G.

Let A be a separable C^*-algebra and let α be a continuous action of a separable locally compact abelian group G on A. Let $P(A)$ denote the set of pure states of A and let $f\in P(A)$. We call the orbit $o_f=\{f\circ\alpha_t: t\in G\}$ through f in $P(A)$ type I if the representation

$$\rho_f\equiv\int_G^{\oplus}\pi_f\circ\alpha_t dt$$

of A on $L^2(G, \mathcal{H}_f)$ is of type I.

We denote by $\bar{\alpha}$ the extension of α to an action on $\rho_f(A)''$; in other words, $\bar{\alpha}_t\circ\rho_f=\rho_f\circ\alpha_t$, $t\in G$. Since $\bar{\alpha}$ is ergodic on the center Z of $\rho_f(A)''$, $\mathrm{Sp}(\bar{\alpha}|Z)$ is a closed subgroup of $\hat{G}$, which we denote by $\Delta(\pi_f)$ or $\Delta(\pi_f, \alpha)$. Let G_f be the set of $s\in G$ such that $\pi_f\circ\alpha_s$ is equivalent to π_f. If o_f is type I, then $G_f=\Delta(\pi_f)^{\perp}$ (see 0.1 in [4]).

In this case there is a weakly continuous action β of G_f on $\pi_f(A)''=B(\mathcal{H}_f)$ such that $\beta_t\circ\pi_f=\pi_f\circ\alpha_t$, $t\in G_f$, but in general π_f may not be $\alpha|G_f$-covariant, i.e., β may not be implemented by a unitary representation of G_f. If in addition π_f is $\alpha|G_f$-covariant, we call the orbit o_f *regular type I*.

For the system (A, G, α) we defined $\Gamma_1(\alpha)$, a subset of $\hat{G}$, in [4] as follows: $p\in\Gamma_1(\alpha)$ if for any non-zero $x\in A$, any compact neighbourhood U of p, and

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* Department of Mathematics, College of General Education, Tohoku University, Sendai, Japan.

any $\varepsilon>0$, there is an $a\in A^{\alpha}(U)$ such that $\|a\|=1$ and $\|xax^*\|\geqq(1-\varepsilon)\|x\|^2$. Let us now define another technical spectrum $\Gamma_2(\alpha)$ as follows: $p\in\Gamma_2(\alpha)$ if for any non-zero $x\in A$, any compact neighbourhood U of p, and any $\varepsilon>0$, there is an $a\in A^{\alpha}(U)$ such that $\|a\|=1$ and $\|x(a+a^*)x^*\|\geqq 2(1-\varepsilon)\|x\|^2$.

Now our results are as follows when the C^*-algebra A is simple and unital. If there is a regular type I orbit o_f such that the Connes spectrum $\Gamma(\alpha|G_f)$ equals $\hat{G}_f$, then for any closed subgroup H of G there is a regular type I orbit o_f with $G_f=H$ (Theorem 2). In particular if (A, G, α) is asymptotically abelian, there is always a covariant irreducible representation (see 2.3 and 3.1 in [4]). If there is a covariant irreducible representation, then $\Gamma_2(\alpha)=\Gamma(\alpha)$ (Theorem 7). When G is a connected Lie group and α is not uniformly continuous, there is always a non-type I orbit (Theorem 9).

We first prove the following properties of $\Gamma_2(\alpha)$.

1. Proposition. *Let A be a separable C^*-algebra and let α be a continuous action of a separable locally compact abelian group G on A. Then the following properties hold:*

(i) *For any faithful family F of irreducible representations of A, $\Gamma_2(\alpha)$ includes $\bigcap_{\pi\in F}\Delta(\pi)$.*

(ii) *There exists a faithful family F of irreducible representations of A such that $\Gamma_2(\alpha)=\bigcap_{\pi\in F}\Delta(\pi)$. In particular, $\Gamma_2(\alpha)$ is a closed subgroup of $\hat{G}$.*

(iii) *$\Gamma_2(\alpha)\subset\Gamma_1(\alpha)$, and if $\Gamma_1(\alpha)=\hat{G}$ then $\Gamma_2(\alpha)=\hat{G}$.*

Proof. If $p\in\Delta(\pi)$, there is a sequence $\{x_n\}$ in A of spectrum p such that $\|x_n\|\leqq 1$ and $\lim\pi(x_n)=1$ ([4]). Here $\{x_n\}$ is of spectrum p if for any neighbourhood U of p there is an N such that $x_n\in A^{\alpha}(U)$ for any $n\geqq N$. This immediately implies (i).

To prove (ii) we adapt the proof of (3)⇒(5) in 3.1 in [4]. We take for $\{U_n\}$ the subsequence of $\{U_n\}$ consisting of those which intersect $\Gamma_2(\alpha)$. Since we are not assuming the primeness of A here, we simply take for $\{I_n\}$ a constant sequence consisting of a non-zero ideal of A. By the same procedure as in [4] in this setting we obtain a pure state f of A such that $\|f|I\|=1$, and for any $p\in\Gamma_2(\alpha)$ and any unit vector $\xi\in\mathcal{H}_f$ there is a $Q\in\mathcal{M}(p)$ such that $\|Q\|\leqq 1$ and $\mathrm{Re}\langle Q\xi, \xi\rangle\geqq 1$, or in fact $\|Q\|=1=\langle Q\xi, \xi\rangle$. (See Section 1 of [4] for the definition of $\mathcal{M}(p)$.) Then by an argument given in the proof of 3.1 in [4] one can conclude that $\mathcal{M}(p)\ni 1$ for $p\in\Gamma_2(\alpha)$, or equivalent $\Delta(\pi_f)\supset\Gamma_2(\alpha)$. Taking for F the set of π_f for all non-zero ideals of A, one obtains that $\Gamma_2(\alpha)\subset\bigcap_{\pi\in F}(\pi)$. Hence the equality follows by (i). Since each $\Delta(\pi)$ is a closed subgroup of $\hat{G}$, so is $\Gamma_2(\alpha)$.

As for (iii), it is obvious that $\Gamma_2(\alpha)\subset\Gamma_1(\alpha)$. Suppose $\Gamma_1(\alpha)=\hat{G}$. Then since the same procedure as above applies, one can conclude that $\Gamma_2(\alpha)=\hat{G}$ (see 3.1

in [4]).

Now we state our first main result:

2. Theorem. *Let A be a separable prime C^*-algebra and let α be a continuous action of a separable locally compact abelian group G on A. Let H be an arbitrary closed subgroup of G. Then the following conditions are equivalent:*

(i) *$\Gamma(\alpha)=\hat{G}$ and there exists an α-covariant irreducible representation of A such that the corresponding representation of the crossed product $A\times_\alpha G$ is faithful.*

(i′) *$\Gamma(\alpha)=\hat{G}$ and there exists a family F of irreducible representations of A such that $\bigcap_{\pi\in F}\ker\pi=(0)$ and π is α-covariant for $\pi\in F$*

(ii) *$\Gamma(\alpha)=\hat{G}$ and $\Gamma_2(\hat{\alpha})=G$.*

(iii) *$\Gamma_2(\alpha)=\hat{G}$.*

(iv) *$\Gamma(\alpha|H)=\hat{H}$ and there exists an $\alpha|H$-covariant irreducible representation π of A such that the corresponding representation of $A\times_{\alpha|H}H$ is faithful and $\Delta(\pi)=H^\perp$.*

(iv′) *$\Gamma(\alpha|H)=\hat{H}$ and there exists a family F of irreducible representations of A such that $\bigcap_{\pi\in F}\ker\pi=(0)$, and π is $\alpha|H$-covariant and $\Delta(\pi)=H^\perp$ for $\pi\in F$.*

Moreover, if G is discrete the above conditions are equivalent to

(v) *α_t is properly outer for each $t\in G\setminus\{0\}$.*

Proof. It is trivial that (i) implies (i′) and (iv) does (iv′). Let F be as in (i′) and for each $\pi\in F$ let u be the unitary representation of G which implements α, so that $\tilde{\pi}=\pi\times u$ is the corresponding representation of $A\times_\alpha G$. Since $\pi(A)''=\tilde{\pi}(A\times_\alpha G)''$ and the spectrum of the action on the quotient $A\times_\alpha G/(\ker\pi)\times_\alpha G$ induced by $\hat{\alpha}$ is G, 1.3 in [4] implies that $\Delta(\tilde{\pi},\hat{\alpha})=G$ (note that the faithfulness assumption in 1.3 in [4] was needed only for ρ instead of π; in this case this amounts to the property that $\bigcap_{p\in\hat{G}}\ker\tilde{\pi}\circ\hat{\alpha}_p=(\ker\pi)\times_\alpha G$). Hence (i′) implies (ii) by 1(i).

To prove (ii)⇒(iii) we first give the following result:

3. Proposition. *Let A be a separable prime C^*-algebra and let α be a continuous action of a separable locally compact abelian group G on A. Suppose that $\Gamma_2(\alpha)=\hat{G}$. Then there exists an α-covariant irreducible representation of A such that the corresponding representation of $A\times_\alpha G$ is faithful.*

Proof. To prove this it suffices to show that $\Gamma(\alpha)=\hat{G}$ and $\Gamma_2(\hat{\alpha})=G$. Because, if this is the case, $A\times_\alpha G$ is separable and prime ([5]), and hence due to 3.1 in [4] and the lemma below there exists a faithful irreducible representation π of $A\times_\alpha G$ such that $\pi(A\times_\alpha G)''=\bar{\pi}(A)''$, where $\bar{\pi}$ is the extension of π to the multiplier algebra $M(A\times_\alpha G)$. Thus $\bar{\pi}|A$ has the desired properties.

4. Lemma. *If π is a representation of $A\times_\alpha G$ such that $\Delta(\pi, \hat{\alpha})=G$, then $\bar{\pi}(A)''=\pi(A\times_\alpha G)''$.*

Proof. Define a representation ρ of $A\times_\alpha G$ by

$$\rho=\int_{\hat{G}}^{\oplus}\pi\circ\hat{\alpha}_p dp$$

on $L^2(\hat{G}, \mathcal{H}_\pi)=L^2(\hat{G})\otimes\mathcal{H}_\pi$ and let β be the extension of $\hat{\alpha}$ to an action on $\mathfrak{N}=\rho(A\times_\alpha G)''$. It suffices to prove that $\mathfrak{N}^\beta=\bar{\rho}(A)''$ (see 1.1 in [4]). It is obvious that $\mathfrak{N}^\beta\supset\bar{\rho}(A)''$. For $f\in L^1(G)\cap L^2(G)$ it is known ([5]) that $\bar{\rho}(\lambda(f))Q\bar{\rho}(\lambda(f)^*)$ is β-integrable for $Q\in\mathfrak{N}$ and

$$\int\beta_p(\bar{\rho}(\lambda(f))Q\bar{\rho}(\lambda(f)^*))dp\in\bar{\rho}(A)''.$$

By a limiting procedure in f one can conclude that $Q\in\bar{\rho}(A)''$ for $Q\in\mathfrak{N}^\beta$.

Going back to the proof of the proposition, it is obvious that $\Gamma(\hat{\alpha})=\hat{G}$. To prove $\Gamma_2(\hat{\alpha})=G$ we first note

5. Lemma. *Under the assumption of the above proposition let H be a closed subgroup of G and suppose that there exists a faithful irreducible representation π of A such that π is $\alpha|H$-covariant and $\Delta(\pi)=H^\perp$. Then there exists an irreducible representation Φ of $A\times_\alpha G$ such that Φ is $\hat{\alpha}|H^\perp$-covariant, $\Delta(\Phi, \hat{\alpha})=H$, and $\bigcap_{p\in\hat{G}}\ker\Phi\circ\hat{\alpha}_p=(0)$. In particular, $\Gamma_2(\hat{\alpha})\supset H$.*

Proof. Let φ be a measurable function of G/H into G such that $\varphi(t)+H=t$, $t\in G/H$, and let v be a continuous unitary representation of H such that $\pi\circ\alpha_t=\operatorname{Ad}v(t)\circ\pi$. Define a representation Φ of $A\times_\alpha G$ on $L^2(G/H, \mathcal{H}_\pi)$ by

$$\bar{\Phi}(a)=\int_{G/H}^{\oplus}\pi\circ\alpha_{\varphi(t)}(a)dt, \quad a\in A,$$

$$\bar{\Phi}(\lambda(g))=\Big(\int_{G/H}^{\oplus}v(g+\varphi(s)-\varphi(s+\dot{g}))ds\Big)u_{\dot{g}}, \quad g\in G,$$

where $\bar{\Phi}$ is the extension of Φ to $M(A\times_\alpha G)$, λ is the canonical unitary representation of G in $M(A\times_\alpha G)$, $\dot{g}=g+H\in G/H$, and u is the unitary representation of G/H defined by

$$(u_s\xi)(t)=\xi(t+s), \quad \xi\in L^2(G/H, \mathcal{H}_\pi).$$

As is easily shown, Φ is in fact well defined, and since $\Phi(A)''=L^\infty(G/H)\otimes B(\mathcal{H}_\pi)$ and $u_s\in\Phi(A\times_\alpha G)''$, Φ is irreducible. Define a unitary representation w of $(G/H)^\wedge=H^\perp$ on $L^2(G/H, \mathcal{H}_\pi)$ by

$$w_p=\int_{G/H}^{\oplus}\langle s,\ p\rangle ds .$$

Then since $\Phi\circ\hat{\alpha}_p=\mathrm{Ad}\, w_p\circ\Phi$, $p\in H^{\perp}$, Φ is $\hat{\alpha}|H^{\perp}$-covariant. Since $\bar{\Phi}(\lambda(g))\in\bar{\Phi}(A)''$ for $g\in H$, one obtains that $\bar{\Phi}(A\lambda(g))^{-w}\ni 1$, for $g\in H$, i.e., $\Delta(\bar{\Phi},\ \hat{\alpha})=H$. Since $\bigcap_{p\in\hat{G}}\ker\Phi\circ\hat{\alpha}_p=(0)$ as π is faithful, this implies that $\Gamma_2(\hat{\alpha})\supset H$.

6. Lemma. *Under the assumption of the above proposition let H be a compact subgroup of G. Then there exists an $\alpha|H$-covariant irreducible representation π of A such that the corresponding representation of $A\times_{\alpha|H}H$ is faithful and $\Delta(\pi)=H^{\perp}$.*

Proof. Let $\beta=\alpha|H$. Then $\Gamma_2(\beta)=\hat{H}=\Gamma(\beta)$, and so $A\times_{\beta}H$ is prime. By the proof of 3.3 in [4], it suffices to show that the following two conditions are satisfied: For any neighbourhood U of any $p\in H^{\perp}$, and any $x\in A\times_{\beta}H$,

$$\sup\{\|x(a+a^*)x^*\|\ ;\ a\in A^{\alpha}(U)_1\}=2\|x\|^2,$$

where $A^{\alpha}(U)_1$ denotes the unit ball of $A^{\alpha}(U)$; for any neighbourhood U of any $s\in H$, and any $x\in A\times_{\beta}H$,

$$\sup\{\|x(a+a^*)x^*\|\ ;\ a\in(A\times_{\beta}H)^{\hat{\beta}}(U)_1\}=2\|x\|^2$$

or equivalently $\Gamma_2(\hat{\beta})=H$. The former can be proved as in 3.3, [4], and the latter can be proved by using the fact that there is a β-covariant faithful irreducible representation of A (see [1], [4]).

To complete the proof of Proposition 3, we have to show that $\Gamma_2(\hat{\alpha})=G$. By the previous two lemmas and 3.3 in [4], it follows that $\Gamma_2(\hat{\alpha})\supset H$ for any compact or discrete closed subgroup H of G. Since any compactly generated subgroup of G is of the form $K\times Z^l\times R^m$ (where K is a compact group and l, m are non-negative integers) and $\Gamma_2(\hat{\alpha})$ is a closed subgroup of G, it easily follows that $\Gamma_2(\hat{\alpha})=G$.

Proof of Theorem 2. To prove (ii)$\Rightarrow$(iii) we apply Proposition 3 to $(A\times_{\alpha}G,\ \hat{G},\ \hat{\alpha})$ to yield an $\hat{\alpha}$-covariant faithful irreducible representation of $A\times_{\alpha}G$, which in turn gives a faithful irreducible representation π of A with $\Delta(\pi)=\hat{G}$. This implies that $\Gamma_2(\alpha)=\hat{G}$.

The proof of (iii)$\Rightarrow$(iv) goes in exactly the same way as the proof of Lemma 6 or 3.3 in [4] as we know by Proposition 3 that there is an α-covariant faithful irreducible representation of A.

Suppose that (iv′) holds. Then applying (i′)$\Rightarrow$(iii) for $\beta=\alpha|H$ one obtains that $\Gamma_2(\beta)=\hat{H}=\hat{G}/H^{\perp}$. One also knows that $\Gamma_2(\alpha)\supset H^{\perp}$. Using these two properties, as in the proof of 3.3 in [4], one obtains a faithful irreducible representation π of A such that $\Delta(\pi,\ \beta)=\hat{G}/H^{\perp}$ and $\Delta(\pi,\ \alpha)\supset H^{\perp}$. From this follows that $\Delta(\pi, \alpha)=\hat{G}$ or (iii) $\Gamma_2(\alpha)=\hat{G}$. (By considering the representation ρ of A defined by

$$\rho=\int_G^{\oplus}\pi\circ\alpha_t dt\cong\int_{G/H}^{\oplus}\int_H^{\oplus}\pi\circ\alpha_{t+f(s)}dtds$$

with f a measurable section of G/H into H, one can conclude that both the integrals are central and so the center of $\rho(A)''$ equals $L^\infty(G)\otimes 1$.) Then by taking G for H in the implication (iii)⇒(iv), one obtains (i).

In general (i) implies (v) via (iv) with $H=(0)$. If G is discrete, (v)⇒(i) follows from 3.4 in [4] by using [1]. This completes the proof of Theorem 2.

7. **Theorem.** *Let A be a separable prime C^*-algebra and let α be a continuous action of a separable locally compact abelian group G on A. If there is a faithful family of α-covariant irreducible representations of A, then for any closed subgroup H of G with $H\supset\Gamma(\alpha)^\perp$, there exists a faithful irreducible representation π of A such that π is $\alpha|H$-covariant and $\Delta(\pi, \alpha)=H^\perp$. In particular, $\Gamma_2(\alpha)=\Gamma(\alpha)$.*

Proof. Let F be the family of irreducible representations in the theorem. For $\pi\in F$ let u be the implementing unitary representation of G and let $\tilde{\pi}=\pi\times u$ be the corresponding representation of $A\times_\alpha G$. Then $\Delta(\tilde{\pi}\circ\hat{\alpha}_p, \hat{\alpha})=G$ for each $\pi\in F$ and $p\in\hat{G}$, and the set F_1 of $\tilde{\pi}\circ\hat{\alpha}_p$, $\pi\in F$, $p\in\hat{G}$, is a faithful family of irreducible representations of $A\times_\alpha G$. Thus $\Gamma_2(\hat{\alpha})=G$, by 1(i).

Let $\beta=\hat{\alpha}|H^\perp$. Since $H^\perp\subset\Gamma(\alpha)$, one has that $I\cap\beta_t(I)\neq(0)$ for any $t\in H^\perp$ and for any non-zero ideal I of $A\times_\alpha G$. We assert:

8. **Lemma.** *For each $\rho\in F_1$, there is a primitive ideal P of $A\times_\alpha G$ such that P is β-invariant, $P\subset\ker\rho$ and $\Gamma_2(\beta/P)=(H^\perp)^\wedge=G/H$, where β/P denotes the action on the quotient $A\times_\alpha G/P$ induced by β.*

Proof. Let $\mathcal{P}$ be the set of primitive ideals P of $A\times_\alpha G$ such that $P\subset\ker\rho$ and there is an irreducible representation π of $A\times_\alpha G$ satisfying $\ker\pi=P$ and $\Delta(\pi, \hat{\alpha})=G$. We define an order on $\mathcal{P}$ by inclusion.

For a totally ordered set $\{P_\nu\}$ in $\mathcal{P}$ we claim that there is a P_1 in $\mathcal{P}$ such that $P_1\subset\bigcap_{p\in H^\perp}\beta_p(P_\nu)$ for all ν. Once this is proved, we simply take a minimal one in $\mathcal{P}$ for P in the lemma.

Let $\{P_\nu\}$ be as above and let $P_0=\bigcap_\nu P_\nu$. Since $A\times_\alpha G$ is separable we may assume that the index set $\{\nu\}$ is the positive integers. (For example, let $\{x_n\}$ be a dense sequence in $A\times_\alpha G$ and let ν_n be such that $\|x_n+P_{\nu_n}\|\geqq\|x_n+P_0\|/2$ and $\nu_n\geqq\nu_{n-1}$, and set $P_n=P_{\nu_n}$.) For each n let $\{I_{nk}\}$ be a decreasing sequence of ideals such that I_{nk} is not contained in P_n and for any non-zero ideal J not contained in P_n there is an I_{nk} with $J\supset I_{nk}$. (For example, for the primitive ideal $P=P_n$ of $A\times_\alpha G$, let $\{x_k\}$ be a dense sequence in $A\times_\alpha G\setminus P$ and let J_k be the smallest ideal of $A\times_\alpha G$ such that $\|x_k+J_k\|\leqq\|x_k+P\|/2$, and set $I_{nk}=J_1\cap\cdots\cap J_k$.)

Let $\{p_l\}$ be a dense sequence in $H^\perp$. We consider the set $\mathcal{S}=\{\beta_{p_l}(I_{nk}): n, k, l=1, 2, \cdots\}$.

We want to prove that if $J_i\in\mathcal{S}$, $i=1, \cdots, m$, then $\bigcap_{i=1}^m J_i$ is essential in J_1. First, if $J_1=I_{nk}$ and $J_2=I_{n'k'}$, then we may assume that $J_1\subset J_2$ or $J_1\supset J_2$ and if $J_1\supset J_2$, J_2 is essential in J_1 because $P_{n'}$ is primitive and $J_1\supset J_2\not\subset P_{n'}$. Second, if $J_1=I_{nk}$ and $J_2=\beta_{p_l}(I_{nk})$, and if $J\subset J_1$ is an ideal orthogonal to $J_1\cap J_2$, one must have that $J\cap\beta_{p_l}(J)=(0)$ which contradicts that $\Gamma(\hat{\alpha})\supset H^\perp$ unless $J=(0)$. Thus $J_1\cap J_2$ is essential in J_1. Since if $J_1\subset J_2$ and J_1 is essential in J_2, then $J_1\cap J$ is essential in $J_2\cap J$ for any ideal J, combining these two cases we get the assertion.

Let $\{J_n\}$ be an enumeration of $\mathcal{S}$; we may assume that $\{J_n\}$ is decreasing, replacing J_n by $J_1\cap\cdots\cap J_n$. From what we have proved above it follows that J_m is essential in J_n for $m\geqq n$.

Now we use the procedure in the proof of 3.3 in [4] for $(A\times_\alpha G, \hat{G}, \hat{\alpha})$ with $\{J_n\}$ instead of $\{I_n\}$. Then we obtain an irreducible representation π of $A\times_\alpha G$ such that $\Delta(\pi, \hat{\alpha})=G$ and $\pi|J_n\neq(0)$ for any n. If $\ker\pi\not\subset\beta_{p_l}(P_0)$, then $\ker\pi\not\subset\beta_{p_l}(P_n)$ for large n and then $\ker\pi\supset\beta_{p_l}(I_{nk})$ for large k, a contradiction. Thus $\ker\pi\subset\beta_{p_l}(P_0)$ for any l and so $\ker\pi\subset\cap\{\beta_p(P_0): p\in H^\perp\}$.

Now we resume the proof of Theorem 7. Let $\mathcal{P}$ be the set of primitive ideals P of $A\times_\alpha G$ such that P is β-invariant and $\Gamma_2(\beta/P)=G/H$. For each $P\in\mathcal{P}$, Theorem 2 is applicable to the system $(A\times_\alpha G/P, H^\perp, \beta)$. Thus $\tilde{P}=P\times_\beta H^\perp$ is a primitive ideal of $A\times_\alpha G\times_\beta H^\perp$, and for the quotient system $(A\times_\alpha G\times_\beta H^\perp/\tilde{P}, G/H, \hat{\beta}/\tilde{P})$, $\Gamma_2(\hat{\beta}/\tilde{P})=H^\perp$ follows. Since $\cap\{\tilde{P}, P\in\mathcal{P}\}=(0)$, this implies that $\Gamma_2(\hat{\beta})=H^\perp$.

On the other hand, by using $\Delta(\pi, \hat{\alpha})=G$ for $\pi\in F_1$ as in the begining of the proof, we obtain that for any $t\in H$, any neighbourhood U of t, and any $x\in A\times_\alpha G\times_\beta H^\perp$,

$$\sup\{\|x(a+a^*)x^*\| : a\in(A\times_\alpha G)^{\hat{\alpha}}(U)_1\}=2\|x\|^2.$$

By using this with $\Gamma_2(\hat{\beta})=H^\perp$ we obtain an irreducible representation ρ of $A\times_\alpha G\times_\beta H^\perp$ such that $\bar{\rho}(A\times_\alpha G)''=\rho(A\times_\alpha G\times_\beta H^\perp)''$ and $\Delta(\pi, \hat{\alpha})=H$ for $\pi=\bar{\rho}|A\times_\alpha G$. Moreover we can easily assume that $\bar{\pi}|A$ is faithful. By Lemma 5 or its proof we obtain an irreducible representation Φ of $A\times_\alpha G\times_{\hat{\alpha}} G\cong A\otimes K$, where K is the compact operators on $L^2(G)$, such that Φ is $\hat{\hat{\alpha}}|H$-covariant, $\Delta(\Phi, \hat{\hat{\alpha}})=H^\perp$, and $\bar{\Phi}|A$ is faithful. This completes the proof by the duality for crossed products.

For a C^*-algebra A we denote by $P_f(A)$ the set of pure states φ of A with $\ker\pi_\varphi=(0)$.

9. Theorem. *Let A be a separable prime C^*-algebra and let α be a continuous action of an abelian Lie group G on A. Then the following conditions are equivalent:*

(i) *α^* on $P_f(A)$ is strongly continuous.*

(ii) *For each $\varphi \in P_f(A)$, $\alpha|G_0$ extends to a σ-weakly continuous action on $\pi_\varphi(A)''=B(\mathcal{H}_\varphi)$, where G_0 is the connected component of the identity in G.*

(iii) *For each $\varphi\in P_f(A)$, $\rho_\varphi(A)''$ is of type I with atomic center, where ρ_φ is defined by $\rho_\varphi=\int^\oplus \pi_\varphi\circ\alpha_t dt$.*

(iv) *Every orbit in $p_f(A)$ is of type I.*

(v) *α_t is not properly outer for any $t\in G_0$, where G_0 is defined in* (ii).

10. *Remark.* In the above theorem if in addition A is simple and unital, α is uniformly continuous (cf. [3]).

Proof. Suppose that (i) holds. Then for $\varphi\in P_f(A)$ there is an open neighbourhood U of $0\in G$ such that

$$\|\varphi\circ\alpha_t-\varphi\|<2, \quad t\in U,$$

which implies that α_t is weakly inner in π_φ for $t\in U$. Since G_0 is generated by $G_0\cap U$, it follows that α_t is weakly inner in π_φ for any $t\in G_0$. Hence there is an action β on $\pi_\varphi(A)''$ such that $\beta_t\circ\pi_\varphi=\pi_\varphi\circ\alpha_t$, $t\in G_0$. Since β is automatically σ-weakly continuous (e. g. [4]), β is the desired action on $\pi_\varphi(A)''$ in (ii).

Suppose that (ii) holds. For $\varphi\in P_f(A)$

$$\rho_1=\int^\oplus_{G_0}\pi_\varphi\circ\alpha_t dt$$

is quasi-equivalent to π_φ and so $\rho_1(A)''$ is a type I factor. Since G/G_0 is discrete, (iii) follows immediately.

(iii)$\Rightarrow$(iv) is trivial.

Suppose that (iv) holds and set

$$H=\{s\in G:\alpha_s \text{ is not properly outer}\}.$$

If H is not closed, let $s\in\bar{H}\setminus H$. Then there is a $\varphi\in P_f(A)$ such that $\pi_\varphi\circ\alpha_s \overset{|}{\circ} \pi_\varphi$ (cf. [2]). Then the orbit o_φ through φ cannot be of type I. Because if o_φ were of type I, then $G_\varphi=\{t\in G:\pi_\varphi\circ\alpha_t\sim\pi_\varphi\}$ would be closed, but $G_\varphi\supset H$ and $G_\varphi\not\supset\bar{H}$. Hence H must be closed.

If H does not include G_0, then G/H has a closed subgroup which is isomorphic to $\boldsymbol{T}$ or $\boldsymbol{R}$. By using this fact one can easily find subgroups D_1, D_2 of G/H such that $D_1\cong\boldsymbol{Z}\oplus\boldsymbol{Z}\cong D_2$, $D_1\cap D_2=(0)$, and $\bar{D}_1=\bar{D}_2$, and then subgroups D_1', D_2' of G such that $D_1'\cap H=(0)=D_2'\cap H$, $q(D_i')=D_i$, $i=1, 2$, where q is the quotient map of G onto G/H. By Theorem 2 applied to the discrete subgroup $D_1'+D_2'$, there is a $\varphi\in P_f(A)$ such that $\pi_\varphi\circ\alpha_t\sim\pi_\varphi$ for $t\in D_1'$, and $\pi_\varphi\circ\alpha_t \overset{\perp}{\circ} \pi_\varphi$ for $t\in D_2'$. Thus $G_\varphi\supset q^{-1}(D_1)$ and $G_\varphi^c\supset q^{-1}(D_2)$. Since $\bar{D}_1=\bar{D}_2$, it follows that $\overline{q^{-1}(D_1)}=\overline{q^{-1}(D_2)}$. This implies that G_φ is not closed and hence the orbit o_φ is not type I.

Suppose that (v) holds. Then for $\varphi\in P_f(A)$, α_t is weakly inner in π_φ for $t\in G_0$. Then as in the proof of (i)⇒(ii), one can show that the action β of G_0 defined by $\beta_t\circ\pi_\varphi=\pi_\varphi\circ\alpha_t$ is continuous. Hence $t\mapsto\alpha_t^*\varphi$ is norm continuous.

Let P be a primitive ideal of A. Then a pure state of the quotient C^*-algebra A/P is naturally regarded as a pure state of A. Thus $P(A)$ is regarded as the disjoint union of $P_f(A/P)$ with P running over the primitive ideals of A.

11. Corollary. *Let A be a separable C^*-algebra and let α be a continuous action of an abelian Lie group G on A. Then the following conditions are equivalent:*

(i) *Every orbit in $P(A)$ is of type I.*

(ii) *For any primitive ideal P of A the induced action $(\alpha|G_P)^*$ on $P_f(A/P)$ is strongly continuous, where $G_P=\{t\in G:\alpha_t(P)=P\}$.*

Proof. Suppose that (ii) holds. Let $\varphi\in P(A)$. With $P=\ker\pi_\varphi$, φ belongs to $P_f(A/P)$. By the previous theorem

$$\rho_1=\int_{G_P}^{\oplus}\pi_\varphi\circ\alpha_t dt$$

is of type I. Let f be a measurable function of G/G_P into G such that $f(s)+G_P=s$, $s\in G/G_P$. Then ρ_φ is equivalent to

$$\int_{G/G_P}^{\oplus}\rho_1\circ\alpha_{f(s)}ds$$

and we assert that

$$\rho_\varphi(A)''=L^\infty(G/G_P)\otimes\rho_1(A)'',$$

which implies (i). To prove this it suffices to show that $N\equiv\mathrm{Sp}(\alpha|Z)\supset G_P^\perp$ where $\bar{\alpha}$ is the extension of α to an action on $\rho_\varphi(A)''$ and Z is the center of $\rho_\varphi(A)''$. If $s\in N^\perp$, then it follows that for any neighbourhood U of $0\in G$

$$\int_U^{\oplus}\pi_\varphi\circ\alpha_t dt \quad\text{and}\quad \int_U^{\oplus}\pi_\varphi\circ\alpha_{s+t}dt$$

are mutually quasi-equivalent. In particular the kernels of these representations are equal:

$$\bigcap_{t\in U}\alpha_{-t}(P)=\bigcap_{t\in U}\alpha_{-s-t}(P).$$

Since this is true for any neighbourhood U of $0\in G$, one obtains that $P=\alpha_{-s}(P)$, i.e., $s\in G_P$. Hence $N^{\perp}\subset G_P$.

Suppose that (ii) does not hold. Then there is a primitive ideal P of A such that $(\alpha|G_P)^*$ on $P_f(A/P)$ is not strongly continuous. Then by Theorem 9 there is a $\varphi\in P_f(A/P)$ such that the orbit through φ under $(\alpha|G_P)^*$ is non-type I. Then as in the proof of (i)$\Rightarrow$(ii), $\rho_\varphi(A)''=L^\infty(G/G_P)\otimes\rho_1(A)''$ and hence the orbit through φ under α^* is non-type I.

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