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Entire Functions of Several Complex Variables Bounded Outside a Set of Finite Volume

By

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Introduction

In this paper, we generalize two theorems of A. Edrei and P. Erdös [2] for the 1 dimensional case to the n ($\geqq 2$) dimensional case.

The first one is the following:

Theorem 1. *Let $f(z)$ be a nonconstant holomorphic function on $\boldsymbol{C}^n$ (i.e., entire function of n complex variables) such that*

$$\text{(A)}\quad \liminf_{r\to+\infty}\frac{\log\log\log M(r)}{\log r}<2n$$

$$\left(M(r)=\max_{\|z\|=r}|f(z)|,\ \|z\|=(|z_1|^2+\cdots+|z_n|^2)^{1/2}\right).$$

Then, for every positive constant B, it satisfies the condition

$$\text{(B)}\quad m_{2n}(\{z\in\boldsymbol{C}^n \mid |f(z)|>B\})=+\infty,$$

where m_{2n} denotes the $2n$ dimensional Lebesgue measure.

(*Remark.* The case $n=1$ is the Edrei-Erdös theorem.)

The second result of [2] is the construction of an example which shows that, when $n=1$, the constant $2n=2$ in the right side of the inequality (A) is the largest possible in order to ensure the condition (B).

Let us recall this example. Let Ω be the domain in $\boldsymbol{C}$ defined by

$$(\Omega.1)\quad \Omega=\left\{w=x+iy\,\middle|\, e^2<x,\ -\frac{\pi}{2x(\log x)^2}<y<\frac{\pi}{2x(\log x)^2}\right\}.$$

Note that we have

$$(\Omega.2)\quad m_2(\Omega)=\frac{\pi}{2}<+\infty,$$

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where m_2 denotes the 2 dimensional Lebesgue measure. Then, the holomorphic function $\Phi(w)$ on $\boldsymbol{C}$, which we call the Edrei-Erdös function in this paper, has the following two properties:

(E. 1) $\Phi(w)$ remains bounded for $w \in \boldsymbol{C}-\Omega$,

(E. 2) $\Phi(w)-\exp(\exp(w \log w)^2)$ remains bounded for $w \in \Omega$, where the branch of $\log w$ is determined so as to take real values for real $w \in \Omega$.

Especially, combining (Ω. 1), (Ω. 2), (E. 1) and (E. 2), we have

(E. 3) $\displaystyle \lim_{r\to+\infty} \frac{\log \log \log M(r)}{\log r}=2 \quad \left(M(r)=\max_{|w|=r} |\Phi(w)|\right)$,

(E. 4) $\{w \in \boldsymbol{C} \mid |\Phi(w)|>B_1\} \subset \Omega$ and hence $m_2(\{w \in \boldsymbol{C} \mid |\Phi(w)|>B_1\})<+\infty$ for some suitable $B_1>0$.

In order to state our second result, we need to introduce some special polynomials. For each integer $k \geqq 1$, we define the polynomials $Q_{n,k}(z)$ of n variables $z=(z_1, \cdots, z_n)$ inductively on n $(\geqq 2)$ by

(Q. 1) $Q_{2,k}(z_1, z_2)=z_1{}^2+z_2{}^k$

$Q_{n,k}(z_1, z_2, \cdots, z_n)=Q_{2,k}(z_1, Q_{n-1,k}(z_2, \cdots, z_n))$ $(n \geqq 3)$.

Observe the following properties:

(Q. 2) degree of $Q_{n,k}(z)=k^{n-1}$,

(Q. 3) $Q_{n,k}(0, \cdots, 0, r)=r^{k^{n-1}}$.

The following theorem will be proved in § 2.

Theorem 2. *If* $k \geqq 4$, *we have*

(ΩQ. 1) $m_{2n}(\{z \in \boldsymbol{C}^n \mid Q_{n,k}(z) \in \Omega\})<+\infty$.

Consider the composition $\Phi_{n,k}=\Phi \circ Q_{n,k}$ of the Edrei-Erdös function Φ and the polynomial $Q_{n,k}$. Then, combining (E. 3), (E. 4), (Q. 2), (Q.3) and Theorem 2, we have

(Φ. 1) $\displaystyle \lim_{r\to+\infty} \frac{\log \log \log M(r)}{\log r}=2k^{n-1} \quad \left(M(r)=\max_{\|z\|=r} |\Phi_{n,k}(z)|\right)$,

(Φ. 2) $\{z \in \boldsymbol{C}^n \mid |\Phi_{n,k}(z)|>B_1\} \subset \{z \in \boldsymbol{C}^n \mid Q_{n,k}(z) \in \Omega\}$, hence $m_{2n}(\{z \in \boldsymbol{C}^n \mid |\Phi_{n,k}(z)|>B_1\})<+\infty$ when $k \geqq 4$, where B_1 is the constant in (E. 4).

Especially, when $k=4$, we have

Corollary. *The holomorphic function* $\Phi_{n,4}(z)$ *on* $\boldsymbol{C}^n$ $(n \geqq 2)$ *has the following two properties*:

(C) $\displaystyle \lim_{r\to+\infty} \frac{\log \log \log M(r)}{\log r}=2^{2n-1} \quad \left(M(r)=\max_{\|z\|=r} |\Phi_{n,4}(z)|\right)$,

(D) $m_{2n}(\{z \in \boldsymbol{C}^n \mid |\Phi_{n,4}(z)|>B_1\})<+\infty$, *where* B_1 *is the constant in* (E. 4).

Remark. Compare this Corollary with Theorem 1. Then, because of the difference between $2n$ in (A) and 2^{2n-1} in (C), our results leave something to be improved.

Such problems as are dealt with in this paper were also treated by A. A. Gol'dberg [4] and L. J. Hansen [5] for the case $n=1$. Moreover, G. A. Camera [1] considered the case of subharmonic functions in R^m (see Remark to Theorem 1 in § 1).

In § 1, we prove Theorem 1. For that purpose, for each point x in the unit sphere $S=\{z \mid \|z\|=1\}$ in C^n, we consider the so-called slice function $f_x(t)=f(tx)$ which is a holomorphic function in $t\in C$. We adapt the argument of [2] for these slice functions $\{f_x \mid x\in S\}$.

In § 2, after making some observations about the polynomials $\{Q_{n,k}(z)\}$, we prove Theorem 2.

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§ 1. Proof of Theorem 1

In [2], Edrei and Erdös used a lemma of Borel type to prove Theorem 1 when $n=1$. In our proof of Theorem 1 when $n\geqq 2$, we use a lemma of the same type. So we prepare it first of all.

Lemma 1. *Let $U(r)$ be a positive continuous nondecreasing function on $[1, \infty)$ such that*

$$U(1)>e. \tag{1.1}$$

Let $\delta>0$ be an arbitrarily chosen constant. Then, there exists a closed subset $\mathcal{E}\subset[1, \infty)$ such that

(a) *for every $r\notin\mathcal{E}$ with $r\geqq 1$,*

$$U(r+r(\log U(r))^{-(1+\delta)})<eU(r); \tag{1.2}$$

(b) *for every s, t with $1\leqq s<t<\infty$ and with*

$$\delta(\log U(s)-1)^{\delta}>4, \tag{1.3}$$

we have

$$m_1(\mathcal{E}(s, t))<\frac{t}{4} \tag{1.4}$$

where $\mathcal{E}(s, t)=\{r \mid s\leqq r\leqq t,\ r\in\mathcal{E}\}$ and m_1 denotes the 1 dimensional Lebesgue measure.

Remark. A sharper form of Lemma 1 is found in the paper of Edrei and Fuchs [3]. But in their statement, the extent of s (the above condition (1.3)) to ensure the estimate (1.4) is implicit. As we need (1.3) later on, we give the proof of Lemma 1.

Proof. Put $\phi(x)=\log U(e^x)$ $(0\leqq x<\infty)$. Since $U(r)$ is nondecreasing, (1.1) implies $\phi(x)>1$ $(0\leqq x<\infty)$. Consider the function $h(x)=x^{-(1+\delta)}$ $(0<x<\infty)$, and put $H(x)=\phi(x+h(\phi(x)))-\phi(x)-1$ $(0\leqq x<\infty)$. Then $H(x)$ is well-defined and continuous. Put $E=\{x \mid 0\leqq x,\ H(x)\geqq 0\}$ and $\mathcal{E}=\{r \mid r=e^x,\ x\in E\}$. Then E and $\mathcal{E}$ are closed subsets of $[0, \infty)$ and $[1, \infty)$ respectively. Using the inequality $e^\alpha\geqq\alpha+1$, we can see that (a) holds.

In order to prove (b), put $y=\log s$, $z=\log t$ and $E(y, z)=E\cap[y, z]$. We claim

$$m_1(E(y, z))\leqq\int_{\phi(y)-1}^{\phi(z)} h(x)dx .$$

In fact, we define $y_1, y_1', y_2, y_2', \cdots$ inductively by

$$y_1=\min E(y, z) \qquad y_1'=y_1+h(\phi(y_1))$$
$$y_n=\min E(y_{n-1}', z) \qquad y_n'=y_n+h(\phi(y_n)) \quad (n\geqq 2)$$

where $E(y_{n-1}', z)=E\cap[y_{n-1}', z]$. Since $\phi(y_n')-\phi(y_n)\geqq 1$, there exists some integer $N\geqq 1$ such that $y_N'>z$ or $y_N'\leqq z$ with $E(y_N', z)=\emptyset$. Hence $E(y, z)\subset\bigcup_{n=1}^N[y_n, y_n']$. Noting that $\phi(y_n)-\phi(y_{n-1})\geqq 1$ $(n\geqq 2)$, we have

$$\begin{aligned}
m_1(E(y, z))&\leqq\sum_{n=1}^N (y_n'-y_n)\\
&=\sum_{n=1}^N h(\phi(y_n))\\
&\leqq h(\phi(y_1))+\sum_{n=2}^N h(\phi(y_n))(\phi(y_n)-\phi(y_{n-1}))\\
&\leqq h(\phi(y_1))+\int_{\phi(y_1)}^{\phi(y_N)} h(x)dx\\
&\leqq\int_{\phi(y_1)-1}^{\phi(z)} h(x)dx\\
&\leqq\int_{\phi(y)-1}^{\phi(z)} h(x)dx .
\end{aligned}$$

Then, in view of (1.3), a calculation yields

$$\begin{aligned}
m_1(\mathcal{E}(s, t))&\leqq t\delta^{-1}(\log U(s)-1)^{-\delta}\\
&\leqq\frac{t}{4}.
\end{aligned}$$

This proves Lemma 1.

Let C^n be the n dimensional complex Euclidean space, and let S be the unit sphere $\{z \mid \|z\|=1\}$ in C^n. Especially, the unit circle (when $n=1$) is denoted by $T: T=\{e^{i\theta} \mid 0\leqq\theta\leqq 2\pi\}$. We denote by dm_{2n} the $2n$ dimensional Lebesgue measure on C^n, by dS the rotation-invariant area element of S and by $d\theta$ the line element of T. Considering the identification $C^n-\{0\}=(0, \infty)\times S$, we obtain $dm_{2n}=r^{2n-1}drdS$.

Definition 1. Let $f(z)$ be a holomorphic function on C^n $(n\geqq 2)$. For $r>0$ and $x\in S$, define

$$M_f(r, x)=\max_{e^{i\theta}\in T}|f(re^{i\theta}x)|$$

$$T_f(r, x)=\frac{1}{2\pi}\int_T \log^+|f(re^{i\theta}x)|\,d\theta$$

where $\log^+ t=\max(\log t, 0)$ $(t>0)$. Then these two functions are continuous in $(0, \infty)\times S$.

Concerning the functions $M_f(r, x)$ and $T_f(r, x)$, the following facts are well known.

Lemma 2. *Let $f(z)$ be a holomorphic function on C^n $(n\geqq 2)$. Then $M_f(r, x)$ and $T_f(r, x)$ are related by the following inequalities:*

$$T_f(r, x)\leqq \log^+ M_f(r, x)\leqq \frac{s+r}{s-r}T_f(s, x) \quad (0<r<s). \tag{1.5}$$

For $x\in S$ such that the slice function $f_x(t)=f(tx)$ $(t\in C)$ is not a constant function, $M_f(r, x)$ and $T_f(r, x)$ are unbounded continuous increasing function in r.

Definition 2. Let $f(z)$ be a holomorphic function on C^n $(n\geqq 2)$. For $r>0$ and $x\in S$, we put

$$\Lambda_f(r, x)=\left\{e^{i\theta}\in T \,\middle|\, \log|f(re^{i\theta}x)|>\frac{1}{2}T_f(r, x)\right\}$$

$$\Lambda_f(r)=\left\{y\in S \,\middle|\, \log|f(ry)|>\frac{1}{2}T_f(r, y)\right\}$$

$$\Lambda_f=\left\{z\in C^n-\{0\} \,\middle|\, \log|f(z)|>\frac{1}{2}T_f(\|z\|, \|z\|^{-1}z)\right\}.$$

Then $\Lambda_f(r, x)$, $\Lambda_f(r)$ and Λ_f are open subsets of T, S and $C^n-\{0\}$ respectively.

A subset $M\subset S$ is said to be circular if $e^{i\theta}x\in M$ whenever $x\in M$ and $e^{i\theta}\in T$.

Lemma 3. *Let $f(z)$ be a holomorphic function on C^n $(n\geqq 2)$. We put $l_f(r, x)=\int_{\Lambda_f(r, x)}d\theta$ on $(0, \infty)\times S$. Let M be a measurable circular subset of S.*

(a) *$l_f(r, x)$ is a measurable function in* $(0, \infty)\times S$.

(b) *For* $r>0$, $\int_{M\cap\Lambda_f(r)} dS(x)=\frac{1}{2\pi}\int_M l_f(r, x)dS(x)$.

Proof. Consider the characteristic function $g(z)$ of the open subset Λ_f of $\boldsymbol{C}^n-\{0\}$ (i.e., $g(z)=0$ for $z\in\boldsymbol{C}^n-(\{0\}\cup\Lambda_f)$; $g(z)=1$ for $z\in\Lambda_f$). Consider the mapping $\tau: (0, \infty)\times S\times T\to\boldsymbol{C}^n-\{0\}$ defined by $\tau(r, x, e^{i\theta})=re^{i\theta}x$. Then we have

$$l_f(r, x)=\int_T g\circ\tau(r, x, e^{i\theta})d\theta .$$

So, from Fubini's theorem applied to the function $g\circ\tau(r, x, e^{i\theta})$ on $(0, \infty)\times S\times T$, the assertion (a) follows.

Next, consider the characteristic function $h(x)$ of the subset $\Lambda_f(r)$ of S, and the mapping $\rho: S\times T\to S$ defined by $\rho(x, e^{i\theta})=e^{i\theta}x$. Then according to Fubini's theorem, we have

$$\int_{M\times T} h\circ\rho(x, e^{i\theta})dS(x)d\theta$$

$$=\int_T d\theta\int_M h\circ\rho(x, e^{i\theta})dS(x).$$

Since dS is rotation-invariant and M is circular, the inner integral in the last integral is independent of $e^{i\theta}$ and is equal to $\int_{M\cap\Lambda_f(r)} dS(x)$. On the other hand,

$$\int_{M\times T} h\circ\rho(x, e^{i\theta})dS(x)d\theta$$

$$=\int_M dS(x)\int_T h\circ\rho(x, e^{i\theta})d\theta$$

$$=\int_M l_f(r, x)dS(x).$$

Consequently, the assertion (b) is proved.

Proof of Theorem 1 *when* $n\geqq 2$. Let $f(z)$ be a nonconstant holomorphic function on $\boldsymbol{C}^n$ satisfying the condition (A). We fix a positive constant B. For simplicity, we assume

$$|f(0)|>\exp(e). \tag{1.6}$$

Note that this assumption is always fulfilled if we replace $f(z)$ by $f(z-z_0)$ for a suitable $z_0\in\boldsymbol{C}^n$, and that this replacement does not change the assumption (A) nor the conclusion (B).

According to the assumption (A), there exist a constant $\eta>0$ and an infinite set I of positive numbers with $\sup\{r \mid r\in I\}=+\infty$ such that

(1.7) $$\log\log M(r)<r^{2n(1-\eta)} \quad \text{for every } r\in I.$$

We take $a>0$ which is big enough to yield

(1.8) $$\eta(a-1)^{\eta}>4.$$

Next, define the subset $Q\subset S$ by

$$Q=\{x\in S \mid T_f(r, x) \text{ is not bounded as } r\to\infty\}.$$

Since $S-Q=\{x\in S \mid f_x(t) \text{ is identically equal to } f(0)\}$, $S-Q$ is a closed subset of measure 0. So Q is an open subset, and putting $c_n=\int_S dS$, we have

(1.9) $$\int_Q dS=c_n.$$

For $r>0$, let $Q(r)$ be the open subset of Q defined by

(1.10) $$Q(r)=\{x\in Q \mid T_f(r, x)>\max(e^a, 2\log B)\}.$$

Then from $Q(r)\subset Q(r')$ $(r<r')$ and $Q=\bigcup_{r>0}Q(r)$, (1.9) yields

$$\lim_{r\to+\infty}\int_{Q(r)}dS=c_n.$$

Hence we can choose a constant $r_1>0$ such that

(1.11) $$\int_{Q(r_1)}dS\geqq\frac{1}{2}c_n.$$

For $R\geqq r_1$, we define the open subset $G(R)$ of $\mathbf{C}^n$ by

$$G(R)=\left\{z\in\mathbf{C}^n \,\middle|\, R<\|z\|<2R,\ \log|f(z)|>\frac{1}{2}T_f(\|z\|, \|z\|^{-1}z),\ \|z\|^{-1}z\in Q(r_1)\right\}.$$

Then for every $z\in G(R)$, in view of (1.10), we have

$$\log|f(z)|>\frac{1}{2}T_f(\|z\|, \|z\|^{-1}z)>\frac{1}{2}T_f(r_1, \|z\|^{-1}z)>\log B,$$

which implies

(1.12) $$\{z\in\mathbf{C}^n \mid |f(z)|>B\}\supset G(R).$$

In order to prove Theorem 1, we shall estimate from below the measure $m_{2n}(G(R))$. We start with the following obvious inequality:

$$T_f(r, x)\leqq\frac{1}{2\pi}\int_{A_f(r,x)}\log M_f(r, x)d\theta+\frac{1}{2\pi}\int_T\frac{1}{2}T_f(r, x)d\theta.$$

From this inequality, we have

$$l_f(r,\ x)\geqq\frac{\pi T_f(r,\ x)}{\log M_f(r,\ x)}. \tag{1.13}$$

In order to estimate the right side of (1.13), we apply Lemma 1. In view of (1.6), $U(r)=T_f(r,\ x)$ $(x\in S)$ satisfies (1.1). Take the number η in (1.7) as δ in Lemma 1. Note that, for $x\in Q(r_1)$, (1.8) and (1.10) implies $\eta(\log T_f(r_1,\ x)-1)^{\eta}>4$, which corresponds to the condition (1.3). Hence for each $x\in Q(r_1)$, there exists a closed subset $\mathcal{E}(x)\subset[1,\ \infty)$ such that

(a) for every $r\notin\mathcal{E}(x)$ with $r\geqq 1$,

$$T_f(r+r(\log T_f(r,\ x))^{-(1+\eta)})<eT_f(r,\ x), \tag{1.14}$$

(b) for every $R\geqq r_1$

$$m_1(\mathcal{E}(R,\ 2R\ ;\ x))<\frac{R}{2}, \tag{1.15}$$

where $\mathcal{E}(R,\ 2R\ ;\ x)=\{r\mid R\leqq r\leqq 2R,\ r\in\mathcal{E}(x)\}$.

Especially, for $r\geqq 1$ with $r\notin\mathcal{E}(x)$, (1.5) and (1.14) yield

$$\log M_f(r,\ x)\leqq 3e(\log T_f(r,\ x))^{1+\eta}T_f(r,\ x)$$

and hence, in view of (1.13)

$$l_f(r,\ x)\geqq e^{-1}(\log T_f(r,\ x))^{-(1+\eta)}. \tag{1.16}$$

Note that $Q(r_1)$ is a circular subset of S. Hence we can apply Lemma 3. According to Fubini's theorem, (1.15) and (1.16), we have

$$\begin{aligned} m_{2n}(G(R)) &= \int_R^{2R} r^{2n-1}dr\int_{Q(r_1)\cap A_f(r)} dS(x) \\ &= \int_R^{2R} r^{2n-1}dr\frac{1}{2\pi}\int_{Q(r_1)} l_f(r,\ x)dS(x) \\ &\geqq R^{2n-1}\int_{Q(r_1)} dS(x)\frac{1}{2\pi}\int_{[R,\,2R]-\varepsilon(R,\,2R\,;\,x)} l_f(r,\ x)dr \\ &\geqq \frac{1}{4\pi e}R^{2n}\int_{Q(r_1)}(\log T_f(2R,\ x))^{-(1+\eta)}dS(x). \end{aligned}$$

If we choose R such that $2R\in I$, then (1.5), (1.7) and (1.11) yield

$$m_{2n}(G(R))\geqq\frac{c_n}{8\pi e}R^{2n}(2R)^{-2n(1-\eta^2)}=\tilde{c}_n R^{2n\eta^2}.$$

Hence letting $R\to+\infty$ with $2R\in I$, we find that $\limsup_{R\to+\infty} m_{2n}(G(R))=+\infty$. In view of (1.12) we conclude (B).

Remark. Let $u(x)$ be a subharmonic function in $\boldsymbol{R}^m$ $(m\geqq 2)$. We put $B(r)=\max_{\|x\|=r} u^+(x)$ and $T(r)=\frac{1}{c_m}\int_{x\in S} u^+(rx)dS(x)$, where dS is the area element of $S=\{\|x\|=1\}$, $c_m=\int_{x\in S} dS(x)$ and $u^+=\max(u, 0)$. Then they are related as follows:

$$T(r)\leqq B(r)\leqq \frac{s^{m-2}(s+r)}{(s-r)^{m-1}}T(s) \quad (0<r<s).$$

By a direct adaptation of the argument of [2], using the above inequalities, we can prove the following:

If $u(x)$ is not bounded above and satisfies

$$\liminf_{r\to+\infty}\frac{\log\log B(r)}{\log r}<\frac{m}{m-1}, \tag{1.17}$$

then, for every real constant B,

$$m_m(\{x\in \boldsymbol{R}^m \mid u(x)>B\})=+\infty.$$

As was shown in [1], the constant $m(m-1)^{-1}$ on the right side of (1.17) is the largest possible. On the other hand, the constant on the right side of (A) in Theorem 1 is not $2n(2n-1)^{-1}$ but $2n$. This improvement seems to come from the fact that slice function $\varphi_x(t)=\varphi(tx)$ $(x\in S,\ t\in \boldsymbol{C})$ of a plurisubharmonic function φ in $\boldsymbol{C}^n$ is subharmonic (or $\equiv-\infty$), while the restriction to a proper linear subspace of a subharmonic function in $\boldsymbol{R}^m$ is not necessarily subharmonic.

§2. Proof of Theorem 2

Recall the polynomials $Q_{n,k}(z)$ which were introduced in Introduction. We shall make some preparations in order to prove the property (ΩQ. 1) (Theorem 2).

For integers $n\geqq 2$, $k\geqq 1$ and $N\geqq 1$, we put

$$J_{n,k}(N)=m_{2n}(\{z\in \boldsymbol{C}^n \mid N-1\leqq |Q_{n,k}(z)|\leqq N\}).$$

We shall estimate $J_{n,k}(N)$.

First of all, we confine ourselves to the case $n=2$. For integers $k\geqq 1$ and $N\geqq 1$, and for $z_2\in \boldsymbol{C}$, we pose

$$\begin{aligned} s_k(z_2, N)&=m_2(\{z_1\in \boldsymbol{C} \mid N-1\leqq |Q_{2,k}(z_1, z_2)|\leqq N\})\\ &=m_2(\{z_1\in \boldsymbol{C} \mid N-1\leqq |z_1^2+z_2^k|\leqq N\}). \end{aligned}$$

Put $r=|z_2|$. Then $s_k(z_2, N)$ depends only on r:

$$s_k(z_2, N)=s_k(r, N)=m_2(\{z_1\in \boldsymbol{C} \mid N-1\leqq |z_1^2+r^k|\leqq N\}).$$

Lemma 4. *For $k\geqq 1$ and $N\geqq 1$, we have*

$$(2.1) \qquad s_k(r,\ N) \leqq \alpha_1 N^{1/2} \qquad if\ r^k \leqq N$$
$$s_k(r,\ N) \leqq \alpha_2 N^{3/2} r^{-k} \qquad if\ r^k \geqq N$$

where α_1 and α_2 are absolute constants.

Proof. Changing the notations for simplicity, we put

$$D=\{z \in \boldsymbol{C} \mid \operatorname{Re} z>0,\ a<|z^2-c|<b\} \quad (0 \leqq a<b,\ c \geqq 0).$$

We estimate the area of the domain D. Consider the holomorphic function $w=z^2$ and the annulus

$$D'=\{w \in \boldsymbol{C} \mid a<|w-c|<b\}.$$

Then by this function, D is mapped conformally onto the domain $D'-\{$the negative part of the real axis$\}$. Consequently, taking the polar coordinates $w=\rho e^{i\phi}$,

$$\begin{aligned} m_2(D) &= \int_D dm_2(z) \\ &= \int_{D'} \left| \frac{dz}{dw} \right|^2 dm_2(w) \\ &= \int_{D'} \frac{1}{4|w|} dm_2(w) \\ &= \frac{1}{4} \int_{D'} d\rho d\phi . \end{aligned}$$

Denote by $l(\phi)$ the length of the intersection of the half line $\{\arg w=\phi\}$ with D' $(0 \leqq \phi \leqq 2\pi)$. Then, we have

$$m_2(D)=\frac{1}{4}\int_0^{2\pi} l(\phi)d\phi .$$

We estimate $l(\phi)$ by examining the two cases below separately.

(i) When $b \geqq c$, we have $l(\phi)+l(\phi+\pi) \leqq 2(b^2-a^2)^{1/2}$ $(0 \leqq \phi \leqq \pi)$. Hence, $m_2(D) \leqq \frac{\pi}{2}(b^2-a^2)^{1/2}$.

(ii) When $b \leqq c$, we have $l(\phi)>0$ only when $|\phi|<\operatorname{Arc} \sin \frac{b}{c}$. We also have $l(\phi) \leqq 2(b^2-a^2)^{1/2}$. It follows that

$$m_2(D) \leqq (b^2-a^2)^{1/2} \operatorname{Arc} \sin \frac{b}{c} \leqq \frac{\pi}{2}(b^2-a^2)^{1/2} \frac{b}{c}.$$

Thus, putting $a=N-1$ and $b=N$, we obtain (2.1).

Lemma 5. *If $k \geqq 3$, then for every $N \geqq 1$, we have*

$$(\mathrm{Q}.4)' \qquad J_{2,k}(N) \leqq c N^{(k+4)/2k}$$

where $c>0$ is some absolute constant.

Proof. For a positive integer M, we put

$$S_k(M, N)=\max_{M-1\leqq r\leqq M} s_k(r, N).$$

Then, we have

$$J_{2,k}(N)\leqq 2\pi\sum_{M=1}^{\infty} MS_k(M, N). \tag{2.2}$$

It can be easily deduced from Lemma 4 that

$$\begin{aligned} S_k(M, N)&\leqq\beta_1 N^{1/2} && \text{if } M\leqq N^{1/k}+2 \\ S_k(M, N)&\leqq\beta_2 N^{3/2}(M-1)^{-k} && \text{if } M\geqq N^{1/k}+2 \end{aligned} \tag{2.3}$$

where $\beta_1=\max(\alpha_1, \alpha_2)$ and $\beta_2=\alpha_2$.

When $k\geqq 3$, according to (2.3) and the following estimates

$$\sum M\leqq(N^{1/k}+2)^2\leqq 9N^{2/k}$$

$$\sum M(M-1)^{-k}\leqq\int_{N^{1/k}+1}^{\infty} x(x-1)^{-k}dx\leqq 2N^{(-k+2)/k}$$

where the first and the second summations extend over integers with $1\leqq M\leqq N^{1/k}+2$ and $N^{1/k}+2\leqq M<+\infty$ respectively, we obtain

$$2\pi\sum_{M=1}^{\infty} MS_k(M, N)\leqq cN^{(k+4)/2k} \tag{2.4}$$

in which we put $c=2(9\beta_1+2\beta_2)$. In view of (2.2) and (2.4), Lemma 5 is proved.

Now, we return to the general case $n\geqq 2$.

Lemma 6. *Let $n\geqq 2$ and $N\geqq 1$ be integers. If $k\geqq 4$,*

$$J_{n,k}(N)\leqq c^{n-1}N^{(k+4)/2k} \tag{Q.4}$$

where $c>0$ is the absolute constant in Lemma 5.

Proof. We shall proceed by induction on n. The case $n=2$ was proved in Lemma 5. Assume that $n\geqq 3$. Then observing

$$\begin{aligned} &m_{2n}(\{z\in \boldsymbol{C}^n \mid N-1\leqq |z_1{}^2+Q_{n-1,k}(z_2, \cdots, z_n)^k|\leqq N, \\ &\qquad M-1\leqq |Q_{n-1,k}(z_2, \cdots, z_n)|\leqq M\}) \\ &\leqq m_{2n-2}(\{(z_2, \cdots, z_n)\in \boldsymbol{C}^{n-1} \mid M-1\leqq |Q_{n-1,k}(z_2, \cdots, z_n)|\leqq M\})S(M, N) \\ &=J_{n-1,k}(M)S_k(M, N), \end{aligned}$$

we obtain

$$J_{n,k}(N) \leqq \sum_{M=1}^{\infty} J_{n-1,k}(M) S_k(M, N). \tag{2.5}$$

According to the induction hypothesis, (2.4) and (2.5), when $k \geqq 4$,

$$\begin{aligned} J_{n,k}(N) &\leqq c^{n-2} \sum_{M=1}^{\infty} M^{(k+4)/2k} S_k(M, N) \\ &\leqq c^{n-2} 2\pi \sum_{M=1}^{\infty} M S_k(M, N) \\ &\leqq c^{n-1} N^{(k+4)/2k}. \end{aligned}$$

Thus, Lemma 6 is proved.

For $e^{i\theta} \in T$, let $R_\theta : \boldsymbol{C} \to \boldsymbol{C}$ be the rotation defined by $R_\theta(w) = e^{i\theta} w$. For a subset $E \subset \boldsymbol{C}$, the image of E under R_θ will be denoted by $R_\theta(E)$.

Lemma 7. *For $e^{i\theta} \in T$, we have*

$$m_{2n}(\{z \mid Q_{n,k}(z) \in E\}) = m_{2n}(\{z \mid Q_{n,k}(z) \in R_\theta(E)\}) \tag{Q.5}$$

where both sides may be infinite simultaneously.

Proof. For simplicity, we shall prove Lemma 7 only when $n=2$, writing $Q(z_1, z_2)$ in place of $Q_{2,k}(z_1, z_2)$. The proof for the general case is similar. Define a unitary transformation U of $\boldsymbol{C}^2$ by

$$U(z_1, z_2) = \left(\exp\left(-\frac{i\theta}{2}\right) z_1, \exp\left(-\frac{i\theta}{k}\right) z_2\right).$$

Then we have $Q \circ U(z_1, z_2) = e^{-i\theta} Q(z_1, z_2)$. The invariance of the Lebesgue measure under the unitary transformations yields

$$\begin{aligned} &m_4(\{(z_1, z_2) \mid Q(z_1, z_2) \in R_\theta(E)\}) \\ &= m_4(\{(z_1, z_2) \mid Q \circ U(z_1, z_2) \in E\}) \\ &= m_4(\{(z_1, z_2) \mid Q(z_1, z_2) \in E\}). \end{aligned}$$

In addition, we prepare the following lemma concerning the domain Ω.

Lemma 8. *There exists an absolute constant β such that, for every integer $N > e^2$,*

$$\{w \in \boldsymbol{C} \mid N-1 \leqq |w| \leqq N,\ w \in \Omega\} \subset \{w \in \boldsymbol{C} \mid N-1 \leqq |w| \leqq N,\ |\arg(w)| \leqq \beta (N \log N)^{-2}\}. \tag{Ω.3}$$

Proof is quite obvious.

Now, we can prove Theorem 2.

Proof. According to Lemma 7 and Lemma 8,

$$m_{2n}(\{z\in\mathbf{C}^n \mid N-1\leqq|Q_{n,k}(z)|\leqq N,\ Q_{n,k}(z)\in\Omega\})$$
$$\leqq\pi^{-1}\beta(N\log N)^{-2}J_{n,k}(N)\quad(N>e^2). \tag{2.6}$$

Consequently, when $k\geqq 4$, Lemma 6 yields

$$m_{2n}(\{z\in\mathbf{C}^n \mid Q_{n,k}(z)\in\Omega\})$$
$$\leqq\pi^{-1}\beta c^{n-1}\sum_{N=2}^{\infty}N^{(-3k+4)/2k}(\log N)^{-2}$$
$$\leqq\pi^{-1}\beta c^{n-1}\sum_{N=2}^{\infty}N^{-1}(\log N)^{-2}<+\infty$$

where we used the fact that the measure on the left in (2.6) is equal to 0 when $N<e^2$. This proves Theorem 2.

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