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Splitting Theorems for PO-Complexes

By

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Introduction

In this paper we formulate two splitting criteria for PO-complexes (2.4, 4.2), being inspired by the work [6] of V. P. Palamodov. These criteria play an important part in the solution of the local moduli problem for proper holomorphic maps and strictly pseudoconvex spaces, see [1, 2] and [3]. Detailed proofs are given in [3], Kap. II. Also, one can use these results to show an improved version of the direct image theorem for proper holomorphic maps and coherence theorems for families of elliptic complexes.

§ 1. Splittings of Complexes

1.1. Let $\mathcal{C}$ be an additive category. By $\mathbb{G}(\mathcal{C})$ we denote the graded additive category defined as follows. An object of $\mathbb{G}(\mathcal{C})$ is a family $K=(K^{\nu})_{\nu\in Z}$ of objects of $\mathcal{C}$. For two objects K and L of $\mathbb{G}(\mathcal{C})$ and an integer $\nu\in Z$ let $\mathrm{Hom}(K, L)^{\nu}:=\prod_{\mu\in Z}\mathrm{Hom}_{\mathcal{C}}(K^{\mu}, L^{\mu+\nu})$ be the group of morphisms of degree ν from K to L and $\mathrm{Hom}(K, L):=\coprod_{\nu\in Z}\mathrm{Hom}(K, L)^{\nu}$.

A *complex* in $\mathbb{G}(\mathcal{C})$ is a pair $K=(K, d)$ consisting of an object K of $\mathbb{G}(\mathcal{C})$ and an endomorphism d from $\mathrm{End}_{\mathbb{G}(\mathcal{C})}(K)^1$ such that $d^2=0$. The complexes in $\mathbb{G}(\mathcal{C})$, endowed with the morphisms of $\mathbb{G}(\mathcal{C})$, form again a graded additive category $\mathbb{K}(\mathcal{C})$. For a complex K and integers $m\leqq r$ we denote by $K^{(m,r)}$ the complex defined by $(K^{(m,r)})^{\nu}:=K^{\nu}$ for $m\leqq\nu\leqq r$ and $(K^{(m,r)})^{\nu}:=0$ otherwise, the differential being induced by that of K.

1.2. Let $K=(K, d)$ be a complex from $\mathbb{K}(\mathcal{C})$. A *splitting* of K is an element h of $\mathrm{End}_{\mathbb{K}(\mathcal{C})}(K)^{-1}$ such that $dhd=d$ and $dhhd=0$. We say that K *splits*, if a splitting exists. If h is a splitting of K, then $h':=hdh$ is a splitting of K satisfying the additional relations $(h')^2=0$ and $h'dh'=h'$. With these notions, it's easy to check the following proposition.

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Lemma 1.3. *Suppose that kernels and cokernels in $\mathcal{C}$ always exist. Then:*

(1) *If h is a splitting of K, then $1_K-[d, h]$ induces maps $f:\mathrm{H}(K)\to K$ and $f':K\to\mathrm{H}(K)$ which are compatible with the differentials and satisfy $f'f=1_{\mathrm{H}(K)}$ and $ff'=1_K-[d, h]$.*[1]

(2) *Conversely, if $f:\mathrm{H}(K)\to K$ and $f':K\to\mathrm{H}(K)$ are maps of degree 0 in $\mathbf{K}(\mathcal{C})$ being compatible with the differentials and h is an element of $\mathrm{End}(K)^{-1}$ such that $f'f=1_{\mathrm{H}(K)}$ and $ff'=1_K-[d, h]$, then h is a splitting of K.*

1.4. Let $u:M\to N$ be a morphism in $\mathcal{C}$. We say that u *splits*, if there is a morphism $h:N\to M$ in $\mathcal{C}$ such that $uhu=u$. If both $\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$ exist in $\mathcal{C}$, then 1_M-hu resp. uh is a projection onto $\mathrm{Ker}(u)$ resp. $\mathrm{Im}(u)$. Obviously u splits if and only if the complex defined by u splits in $\mathbf{K}(\mathcal{C})$.

§ 2. The Absolute Splitting Criterion

2.1. Following V. P. Palamodov [6], by a PO-*space* we understand a vector space E over $\boldsymbol{C}$ endowed with a family $\|\cdot\|_\lambda$, $\lambda\in]0, 1[$, of seminorms satisfying the relation $\|\cdot\|_\lambda\leqq\|\cdot\|_{\lambda'}$ for $\lambda\leqq\lambda'$. Let E, F be two PO-spaces and ε an element of $]0, 1]$. A $\boldsymbol{C}$-linear mapping $u:E\to F$ is called a PO_ε-*map*, if there exists a constant $C>0$ such that $\|u(x)\|_\lambda\leqq C\|x\|_\lambda$ holds for all elements x of E and λ of $[1-\varepsilon, 1[\setminus\{0\}$. By $\mathrm{PO}_\varepsilon(\boldsymbol{C})$ we denote the additive category formed by the PO-spaces as objects and by the PO_ε-maps as morphisms. Further we put $\mathrm{PO}(\boldsymbol{C}):=\varinjlim_\varepsilon \mathrm{PO}_\varepsilon(\boldsymbol{C})$.

2.2. Let E be a PO-space. For an element λ of $]0, 1[$ we denote by $\bar{E}_\lambda$ the completion of E with respect to the seminorm $\|\cdot\|_\lambda$. Then $E_t:=\varprojlim_{\lambda<t}\bar{E}_\lambda$ is a Fréchet space for t from $]0, 1]$, which has a natural PO-structure given by $\|\cdot\|_\lambda^t:=\|\cdot\|_{\lambda t}$ for λ from $]0, 1[$. For $0<t\leqq t'\leqq 1$ we have canonical homomorphisms $i_{tt'}:E_{t'}\to E_t$ and $E\to E_1$ in $\mathrm{PO}_1(\boldsymbol{C})$, and $(E_t, i_{tt'})_{0<t\leqq 1}$ is an inverse system of PO-spaces. Obviously $E_1=\hat{E}$ is the completion of E, considered as a topological vector space over $\boldsymbol{C}$.

We say that E is *nuclear*, if the canonical maps $\bar{E}_{\lambda'}\to\bar{E}_\lambda$ are nuclear for all $\lambda<\lambda'<1$. In this case E_t is an FN-space for every t, and $i_{tt'}:E_{t'}\to E_t$ is a nuclear map for $t<t'\leqq 1$.

Let ε be an element of $]0, 1]$. For abbreviation we call E ε-*good*, if E is complete and the maps $\bar{E}_{\lambda'}\to\bar{E}_\lambda$ are injective for all elements λ, λ' of $]1-\varepsilon, 1[$ satisfying $\lambda\leqq\lambda'$.

Let $u:E\to F$ be a morphism in $\mathrm{PO}_\varepsilon(\boldsymbol{C})$. Then u induces in a natural way

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[1] Here $\mathrm{H}(K)$ is considered as a complex via the zero differential.

compatible continuous linear maps $\bar{u}_\lambda : \bar{E}_\lambda \to \bar{F}_\lambda$, $\lambda \in [1-\varepsilon, 1[$, and hence PO-maps $u_t : E_t \to F_t$, $t \in]1-\varepsilon, 1]$, forming an inverse system.

In an obvious way one further introduces the notion of a PO-*algebra* and the notion of a PO-*module* over a PO-algebra.

Example 2.3. Let $x \in \boldsymbol{C}^n$ be a point. By a *general open polycylinder* in $\boldsymbol{C}^n$ with center x we understand a set $P = P(x; (a, b))$ of the form

$$P = \{z \in \boldsymbol{C}^n : a_i < |z_i - x_i| < b_i \quad \text{for } 1 \leqq i \leqq n\}$$

with $a_i \in \boldsymbol{R}^*$ and $b_i \in \boldsymbol{R}^*_+$. If moreover $a_i < 0$ holds for all i, we say that P is *special*. For any t from $]0, 1]$ the set $P^{(t)} := P(x; (a/t, tb))$ is again a general polycylinder with center x being called the shrinking of P with respect to t. Let now f be a function holomorphic on P, $f = \sum_{\nu \in \boldsymbol{Z}^n} f_\nu \cdot (z-x)^\nu$ its Laurent series and λ be an element of $]0, 1[$. We put

$$|f|_\lambda := \sup\{|f(z)| : z \in P^{(\lambda)}\}$$

if $P^{(\lambda)} \neq \emptyset$ and $|f|_\lambda := 0$ otherwise, and

$$\|f\|_\lambda := \sum_{\nu \in \boldsymbol{Z}^n} |f_\nu| \, |(z-x)^\nu|_\lambda .$$

Then $\Gamma(P, \mathcal{O}_{\boldsymbol{C}^n})$ is a nuclear PO-algebra with respect to the family $\|\cdot\|_\lambda$, $\lambda \in]0, 1[$, of seminorms, the underlying topology being the canonical FN-topology. For an element t of $]0, 1]$ we have an equality $\Gamma(P, \mathcal{O}_{\boldsymbol{C}^n})_t = \Gamma(P^{(t)}, \mathcal{O}_{\boldsymbol{C}^n})$ as PO-algebras. Finally, if ε is an element from $]0, 1]$ such that $P^{(\lambda)} \neq \emptyset$ for all $\lambda > 1-\varepsilon$, then $\Gamma(P, \mathcal{O}_{\boldsymbol{C}^n})$ is obviously ε-good.

Theorem 2.4. *Let $E = (E, d)$ be a nuclear* $\mathrm{PO}_\varepsilon(\boldsymbol{C})$*-complex, t_0 be an element of $]1-\varepsilon, 1[$ and p an integer with the following properties:*

(1) *$i_{t_0 1} : E_1 \to E_{t_0}$ is a quasiisomorphism in dimensions p and $p+1$.*

(2) *For $j = p, p+1$ there exist* PO*-morphisms*

$$\zeta^j : \mathrm{Z}^j(E_{t_0}) \longrightarrow \mathrm{Z}^j(E_1), \quad \rho^j : \mathrm{Z}^j(E_{t_0}) \longrightarrow E_{t_0}^{j-1}$$

such that $1 = i_{t_0 1}\zeta^j + d_{t_0}\rho^j$ holds on $\mathrm{Z}^j(E_{t_0})$.

Then the cohomology vector spaces $\mathrm{H}^j(E_{t_0})$, $j = p, p+1$, are finite dimensional, and the morphism $d_{t_0} : E_{t_0}^{p-1} \to E_{t_0}^p$ splits in the category $\mathrm{PO}(\boldsymbol{C})$.

The *proof* of this result uses standard methods of functional analysis (the theorems of Banach and Schwartz ([4], [8]) and the theory of nuclear maps (see [7] for instance)).

§ 3. PO-ringed Spaces

3.1. Let A be a PO-algebra and M be a module over A, endowed with a family $\|\cdot\|_{\mu,\lambda}$, $\mu, \lambda \in]0, 1[$, of seminorms. We call M a *special* PO-*module* over A, if the conditions $\|am\|_{\mu,\lambda} \leqq \|a\|_{\mu}\|m\|_{\mu,\lambda}$ and $\|m\|_{\mu,\lambda} \leqq \|m\|_{\mu',\lambda'}$ hold for all elements a of A, m of M and $\mu, \mu', \lambda, \lambda'$ of $]0, 1[$ such that $\mu \leqq \mu'$ and $\lambda \leqq \lambda'$. Then the completion $\overline{M}_{\mu,\lambda}$ of M with respect to the seminorm $\|\cdot\|_{\mu,\lambda}$ is a Banach module over the Banach algebra $\overline{A}_{\mu}$. If r, t are two elements of $]0, 1]$, then $M_{r,t} := \varprojlim_{\mu<r, \lambda<t} \overline{M}_{\mu,\lambda}$ is again a special PO-module over the PO-algebra A_r in a natural way.

Let $\varphi: A \to B$ be a *continuous* homomorphism of PO-algebras, M resp. N a special PO-module over A resp. B and $\varepsilon \in]0, 1]$. A φ-linear mapping $u: M \to N$ is a $\mathrm{PO}^{\varepsilon}$-*map*, if there exists for any μ from $]0, 1[$ a $\tau(\mu)$ from $]0, 1[$ and a constant $C_{\mu} > 0$ satisfying $\|u(m)\|_{\mu,\lambda} \leqq C_{\mu}\|m\|_{\tau(\mu),\lambda}$ for all elements m of M and λ of $[1-\varepsilon, 1[$. Suppose now that φ is even a morphism of PO-algebras. Further let $\varepsilon, \varepsilon'$ be two elements of $]0, 1]$. We call u a $\mathrm{PO}_{\varepsilon',\varepsilon}$-*map*, if there exists a constant $C > 0$ such that $\|u(m)\|_{\mu,\lambda} \leqq C\|m\|_{\mu,\lambda}$ holds for all elements $m \in M$, $\mu \in [1-\varepsilon', 1[$ and $\lambda \in [1-\varepsilon, 1[$.

Example 3.2. Let A be a PO-algebra and E be a PO-space. Then $A \hat{\otimes}_{C} E$ is a special PO-module over A with respect to the family $\|\cdot\|_{\mu,\lambda} := \|\cdot\|_{\mu} \hat{\otimes}_{\pi} \|\cdot\|_{\lambda}$, $\mu, \lambda \in]0, 1[$, of seminorms. For any two elements r, t of $]0, 1]$ we have $(A \hat{\otimes}_{C} E)_{r,t} = A_r \hat{\otimes}_{C} E_t$.

3.3. Let S be a locally compact $\boldsymbol{C}$-ringed space with countable topology such that for every point $s \in S$ there are given an ordered set $\Gamma(s)$ and a family $\mathcal{B}(s) = (U_{\gamma})_{\gamma \in \Gamma(s)}$ of open subsets of S satisfying $U_{\gamma} \subseteqq U_{\gamma'}$ if $\gamma \leqq \gamma'$ and containing a neighborhood basis of s. We call $S = (S, (\mathcal{B}(s))_{s \in S})$ a PO-*ringed space*, if the following conditions are satisfied:

(1) $\mathcal{O}_S(U_{\gamma})$ is a complete nuclear PO-algebra for every point s of S and every element γ of $\Gamma(s)$.

(2) If $\gamma \in \Gamma(s)$ and $\gamma' \in \Gamma(s')$ are two elements such that $U_{\gamma} \subseteqq U_{\gamma'}$, then the restriction $\mathcal{O}_S(U_{\gamma'}) \to \mathcal{O}_S(U_{\gamma})$ is continuous.

(3) For any two elements γ, γ' of $\Gamma(s)$ such that $\gamma \leqq \gamma'$, the restriction $\mathcal{O}_S(U_{\gamma'}) \to \mathcal{O}_S(U_{\gamma})$ is a morphism of PO-algebras.

Then the structure sheaf $\mathcal{O}_S$ of S is a sheaf of FN-algebras in a canonical way. Let now $\mathcal{M}$ be a module over $\mathcal{O}_S$ such that $\mathcal{M}(U_{\gamma})$ is a PO-module over $\mathcal{O}_S(U_{\gamma})$ for every element γ of $\Gamma(s)$ and every point $s \in S$. We call $\mathcal{M}$ a PO-*module* over $\mathcal{O}_S$, if the restrictions $\mathcal{M}(U_{\gamma'}) \to \mathcal{M}(U_{\gamma})$ are continuous resp. PO_1-maps in the situation of (2) resp. (3).

Let $\mathcal{M}$ and $\mathcal{N}$ be two PO-modules over $\mathcal{O}_S$, $u:\mathcal{M}\to\mathcal{N}$ an $\mathcal{O}_S$-linear map and let ε be an element of $]0, 1]$. If $u(U_\gamma)$ is a PO$_\varepsilon$-map for every $\gamma\in\Gamma(s)$, $s\in S$, we say that u is a PO$_\varepsilon$-*map*. Endowed with the PO$_\varepsilon$-maps as morphisms, the PO-modules over $\mathcal{O}_S$ form an additive category being denoted by PO$_\varepsilon(\mathcal{O}_S)$. Further we put $\mathrm{PO}(\mathcal{O}_S):=\varinjlim_\varepsilon \mathrm{PO}_\varepsilon(\mathcal{O}_S)$.

Let $\mathcal{J}$ be an ideal of $\mathcal{O}_S$ satisfying $\mathrm{H}^1(U_\gamma, \mathcal{J})=0$ for all γ, such that $\mathcal{J}(U_\gamma)$ is closed in $\mathcal{O}_S(U_\gamma)$ for every γ. Then the closed subspace T of S defined by $\mathcal{J}$ is again a PO-ringed space with respect to the family $\mathcal{B}(s)\cap T:=(U_\gamma\cap T)_{\gamma\in\Gamma(s)}$, $s\in T$, in a natural way.

Examples 3.4. (1) Let $U\subseteqq \boldsymbol{C}^n$ be an open subspace and $\mathcal{B}(x)=\mathcal{B}(U, x)$ for a point $x\in U$ be the family of all general open polycylinders P with center x and $\bar{P}\subseteqq U$, ordered by inclusion. Then $\mathcal{O}_U(P)$ is a complete nuclear PO-algebra for $P\in\mathcal{B}(x)$, see 2.3. If $P\in\mathcal{B}(x)$ and $P'\in\mathcal{B}(x')$ are two polycylinders such that $P\subseteqq P'$, then the restriction map $\mathcal{O}_U(P')\to\mathcal{O}_U(P)$ is continuous and in case $x=x'$ even a morphism of PO-algebras. Endowed with these data, U is a PO-ringed space. If $S\subseteqq U$ is a closed (complex) subspace, then S is a PO-ringed space again with respect to the families $\mathcal{B}(S, s):=(P\cap S)_{P\in\mathcal{B}(U,s)}$, $s\in S$.

(2) Let S be as in (1) and let $\mathcal{F}$ be a coherent module over $\mathcal{O}_S$ together with an $\mathcal{O}_S$-linear surjection $\mathcal{O}_S^k\to\mathcal{F}$. Then the PO-module structure of $\mathcal{O}_S^k$ induces a structure of the same type on $\mathcal{F}$. The PO-module structure on $\mathcal{F}$ obtained in this way is independent of the choice of the surjection up to PO$_1$-equivalence. Any such structure is called *canonical*.

3.5. Let S be a PO-ringed space and $\mathcal{M}$ be a module over $\mathcal{O}_S$ such that $\mathcal{M}(U_\gamma)$ is a special PO-module over $\mathcal{O}_S(U_\gamma)$ for every γ. We call $\mathcal{M}$ a *special* PO-*module* over $\mathcal{O}_S$, if the following conditions hold:

(1) If $\gamma\in\Gamma(s)$ and $\gamma'\in\Gamma(s')$ are two elements such that $U_\gamma\subseteqq U_{\gamma'}$, then the restriction $\mathcal{M}(U_{\gamma'})\to\mathcal{M}(U_\gamma)$ is a PO^1-map in the sense of 3.1.

(2) For any two elements γ, γ' of $\Gamma(s)$ such that $\gamma\leqq\gamma'$, the restriction $\mathcal{M}(U_{\gamma'})\to\mathcal{M}(U_\gamma)$ is a PO$_{1,1}$-map in the sense of 3.1.

Let $\mathcal{M}$ and $\mathcal{N}$ be two special PO-modules over $\mathcal{O}_S$ and ε, ε' be two elements of $]0, 1]$. An $\mathcal{O}_S$-linear homomorphism $u:\mathcal{M}\to\mathcal{N}$ is called a PO$_{\varepsilon',\varepsilon}$-*map*, if $u(U_\gamma)$ is a PO$_{\varepsilon',\varepsilon}$-map in the sense of 3.1 for every γ. Endowed with the PO$_{\varepsilon',\varepsilon}$-maps as morphisms, the special PO-modules over $\mathcal{O}_S$ form an additive category being denoted by PO$_{\varepsilon',\varepsilon}(\mathcal{O}_S)$. Further we put $\mathrm{PO}_{\varepsilon',\cdot}(\mathcal{O}_S):=\varinjlim_\varepsilon \mathrm{PO}_{\varepsilon',\varepsilon}(\mathcal{O}_S)$.

Let $\mathcal{M}$ be a special PO-module over $\mathcal{O}_S$ and t be an element of $]0, 1]$. By $\mathcal{M}_{1,t}$ we denote the module over $\mathcal{O}_S$ associated with the premodule $U_\gamma\mapsto\mathcal{M}(U_\gamma)_{1,t}$. For $t\leqq t'$ we have canonical $\mathcal{O}_S$-linear maps $\mathcal{M}_{1,t'}\to\mathcal{M}_{1,t}$. If the equality

$\mathcal{M}_{1,t}(U_\gamma)=\mathcal{M}(U_\gamma)_{1,t}$ holds for every γ, then $\mathcal{M}_{1,t}$ is a special PO-module over $\mathcal{O}_S$ in a natural way.

Examples 3.6. (1) Let S be a PO-ringed space and E be a PO-space. Then $\mathcal{O}_S\hat{\otimes}_C E$ is a special PO-module over $\mathcal{O}_S$ with respect to the seminorms defined in 3.2. For any element t of $]0, 1]$ we have $(\mathcal{O}_S\hat{\otimes}_C E)_{1,t}=\mathcal{O}_S\hat{\otimes}_C E_t$. A special PO-module $\mathcal{M}$ over $\mathcal{O}_S$ is called *nuclear free*, if $\mathcal{M}$ is isomorphic in $\mathrm{PO}_{1,1}(\mathcal{O}_S)$ to a special PO-module of the form $\mathcal{O}_S\hat{\otimes}_C E$, the PO-space E being nuclear.

(2) A PO-module over $\mathcal{O}_S$ can be considered as a special PO-module over $\mathcal{O}_S$ in a natural way.

§4. The Relative Splitting Criterion

4.1. Let $U\subseteq \boldsymbol{C}^n$ be an open subspace and $S\subseteq U$ a closed subspace of U, being considered as a PO-ringed space according to 3.4 (1), and let $s\in S$ be a fixed point.

For a module $\mathcal{M}$ over $\mathcal{O}_S$ we denote as usual by $\mathcal{M}(s)$ the vector space $k(s)\otimes_{\mathcal{O}_{S,s}}\mathcal{M}_s$ over $\boldsymbol{C}$. If $\mathcal{M}$ is a nuclear free PO-module over $\mathcal{O}_S$, we can consider $\mathcal{M}(s)$ as a PO-space in a natural way. Let now $\mathcal{N}$ be another nuclear free PO-module over $\mathcal{O}_S$ and let $w:\mathcal{M}(s)\to\mathcal{N}(s)$ be a homomorphism in $\mathrm{PO}_\varepsilon(\boldsymbol{C})$. Then w defines in a natural way a morphism $\mathcal{M}\to\mathcal{N}$ in $\mathrm{PO}_{1,\varepsilon}(\mathcal{O}_S)$, which we denote by w again. Finally we put

$$\mathrm{PO}_{\varepsilon',\cdot}(\mathcal{O}_{S,s}):=\varinjlim_{V}\mathrm{PO}_{\varepsilon',\cdot}(\mathcal{O}_{S|V})$$

for $\varepsilon'\in]0, 1]$; here the limit is taken over all open neighborhoods V of s *in* U, and $S|V$ is considered as a closed subspace of V.

Theorem 4.2. *Let the assumptions and notations be as in* 4.1. *Further let* $\mathcal{E}=(\mathcal{E}, d)$ *be a complex of nuclear free modules over* $\mathcal{O}_S$ *in* $\mathbf{K}(\mathrm{PO}_{1,\varepsilon}(\mathcal{O}_S))$ *and* t_0 *be an element of* $]1-\varepsilon, 1[$ *with the following properties:*

(1) *The natural map* $\mathcal{E}(s)\to\mathcal{E}_{1,t_0}(s)$ *is a quasiisomorphism.*

(2) $\mathcal{E}_{1,t_0}(s)$ *splits in* $\mathbf{K}(\mathrm{PO}(\boldsymbol{C}))$.

Then there exists a complex $\mathcal{L}$ *in* $\mathbf{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_{S,s}))$, *the* $\mathcal{L}^p$ *being finitely generated free* PO-*modules over* $\mathcal{O}_S$ *with* $\mathrm{rank}(\mathcal{L}^p)=\dim_C(\mathrm{H}^p(\mathcal{E}(s)))$, *and a homotopy equivalence* $\mathcal{L}\to\mathcal{E}_{1,t_0}$ *in* $\mathbf{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_{S,s}))$.

Sketch of proof. After replacing $\mathcal{E}$ by $\mathcal{E}_{1,t_0}$, we may assume that $\mathcal{E}(s)$ splits in the category $\mathbf{K}(\mathrm{PO}(\boldsymbol{C}))$ and that the PO-structure induced by $\mathcal{E}(s)$ on the (finite dimensional) cohomology vector spaces $\mathrm{H}^p(\mathcal{E}(s))$, $p\in\boldsymbol{Z}$, is equivalent to the canonical PO-structure. Let $\mathcal{L}$ denote the complex $\mathcal{O}_S\otimes_C\mathrm{H}(\mathcal{E}(s))$ in

$\mathbb{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_S))$, endowed with the zero differential. Further let h be a splitting of $(\mathcal{E}(s), d(s))$ in $\mathbb{K}(\mathrm{PO}(\boldsymbol{C}))$ such that $h^2=0$ and $hd(s)h=h$. We put $d_0:=d(s)$ and $r:=d-d_0$ for abbreviation. Then by 1.3 (1), the operator $1-[d_0, h]$ induces homotopy equivalences

$$f: \mathcal{L} \longrightarrow (\mathcal{E}, d_0), \quad g: (\mathcal{E}, d_0) \longrightarrow \mathcal{L}$$

in $\mathbb{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_S))$ such that $gf=1$ and $fg=1-[d_0, h]$. It's easy to see that the endomorphism $1+hr$ of $\mathcal{E}$ is invertible in the category $\mathbb{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_{S,s}))$. Further

$$\partial:=gd(1+hr)^{-1}f : \mathcal{L} \longrightarrow \mathcal{L}$$

resp.

$$u:=(1+hr)^{-1}f : \mathcal{L} \longrightarrow \mathcal{E}$$

is a morphism of degree 1 resp. 0 in $\mathbb{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_{S,s}))$ such that $\partial^2=0$ resp. $du=u\partial$ and $\partial(s)=0$ resp. $u(s)=f(s)$. Using a cone argument, one now shows that u is in fact a homotopy equivalence in $\mathbb{K}(\mathrm{PO}_{1,\cdot}(\mathcal{O}_{S,s}))$.

Remarks 4.3. (1) Combining 4.2 with the theory of privileged neighborhoods (in the sense of Malgrange [5]), one obtains a splitting proposition for $\mathcal{E}$ on the section level. For a precise formulation of this "relative splitting criterion", we refer to [3] (II 3.29).

(2) One can use 2.4 and 4.2 to show an improved version of the direct image theorem for proper holomorphic maps. In order to be able to apply the criterion in this situation, one has to use the so called "Satz über die Nullhomotopie des Čech-Komplexes einer Auflösung" ([3] (II 5.3)). In contrast to known proofs, neither an ascending induction on the dimension of the base space nor a descending induction on the dimension of the cohomology is needed. It is possible to generalize these methods to convex and concave situations. I will give the details in a forthcoming paper.

(3) Also, one can use the criteria 2.4 and 4.2 in order to prove coherence theorems for families of elliptic complexes. Again, I will provide the details at another place.

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